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ENDOMORPHISMS OF $\mathcal{B}(\mathcal{H})$ AND CUNTZ ALGEBRAS

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1. PRELIMINARIES

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the von Neumann algebra of all bounded linear operators on $\mathcal{H}$. The term endomorphism will be reserved to denote a $*$-homomorphism of $\mathcal{B}(\mathcal{H})$ into itself. Since the Calkin algebra does not have nonzero representations on a separable Hilbert space, an endomorphism of $\mathcal{B}(\mathcal{H})$ is normal; hence if it is nonzero, it must be one to one. An endomorphism may preserve the identity operator on $\mathcal{H}$, in which case it will be called *unital*. An extreme case of nonunital behavior of an endomorphism α is one for which $\alpha^k(I) \to 0$ in the strong operator topology as $k \to \infty$. Such α will be called *completely nonunital*. The natural concept of equivalence between two endomorphisms is called *conjugacy*.

DEFINITION 1.1. Two endomorphisms, α_1 of $\mathcal{B}(\mathcal{H}_1)$ and α_2 of $\mathcal{B}(\mathcal{H}_2)$, are said to be *conjugate* if there exists an isomorphism $\theta : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ such that $\theta \circ \alpha_1 = \alpha_2 \circ \theta$. In this case we will use the notation $\alpha_1 \overset{c}{\sim} \alpha_2$.

Note that θ is implemented by a unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\alpha_1(A) = W^{-1}\alpha_2(WAW^{-1})W \quad A \in \mathcal{B}(\mathcal{H}_1),$$

so conjugacy corresponds to spatial equivalence, and thus it preserves spatial properties of endomorphisms. A weaker notion of equivalence, namely *outer conjugacy*, is important in developing an index theory for endomorphisms.

DEFINITION 1.2. Two endomorphisms α_1, α_2 are *outer conjugate* if there is an automorphism γ of $\mathcal{B}(\mathcal{H}_2)$ such that α_1 is conjugate to $\alpha_2 \circ \gamma$.

Powers observed in [11] that the commutant of the image of a unital endomorphism is a factor of type I_n for n a positive integer or ∞, and called this n the multiplicity (or index) of the endomorphism. He showed that this index is a complete outer conjugacy invariant for unital endomorphisms.

In [2] Arveson pointed out by viewing an endomorphism α as a normal representation of $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$, that there exists a collection $\{V_j\}_{j=1}^n$ of isometries with mutually orthogonal ranges, such that

$$\alpha(A) = \sum_{j=1}^{n} V_j A V_j^* \quad A \in \mathcal{B}(\mathcal{H}).$$

Thus, attention was drawn again upon such collections and in particular upon the C^*-algebras they generate.

THE CUNTZ ALGEBRAS $\mathcal{T}_n$. Suppose that n is a positive integer or ∞ and $\{v_j\}_{j=1}^n$ is a collection of isometries on a Hilbert space which satisfy

$$\sum_{j=1}^{n} v_j v_j^* < I.$$

This condition is equivalent to saying that the ranges of the isometries are mutually orthogonal and they do not span the whole space. The C^*-algebra generated by such a collection depends only on n (up to canonical isomorphism) and not on the particular choice of the generators. If $n = 1$ this is a classical result of Coburn [5]; the case $2 \leqslant n \leqslant \infty$ was proved by Cuntz in [6, 7]. The isomorphism class of $C^*(\{v_j\}_{j=1}^n)$ will be denoted by $\mathcal{T}_n$, which by abuse will also denote a particular representative. If $n < \infty$ the projection $I - \sum_{j=1}^{n} v_j v_j^*$ is in $\mathcal{T}_n$ and generates an ideal $\mathcal{J}_n$, isomorphic to the compact operators on a separable, infinite dimensional Hilbert space. $\mathcal{T}_1$ is the well known Toeplitz C^*-algebra and the corresponding quotient is (isomorphic to) the C^*-algebra of continuous functions on the circle. For $2 \leqslant n < \infty$, the ideal $\mathcal{J}_n$ is maximal in $\mathcal{T}_n$ and the corresponding quotient is Cuntz's $\mathcal{O}_n$, the simple C^*-algebra generated by n isometries having range projections adding up to the identity operator. In the case $n = \infty$, $\mathcal{T}_n$ itself is simple, and in fact it is isomorphic to $\mathcal{O}_\infty$.

REMARK 1.3. As a consequence of Cuntz's results, whenever $\{V_j\}_{j=1}^n$ are n isometries on a Hilbert space $\mathcal{H}$, satisfying $\sum_j V_j V_j^* \leqslant I$, there is a unique representation π of $\mathcal{T}_n$ on $\mathcal{H}$ such that $\pi(v_j) = V_j$ for $j = 1, 2, \ldots, n$.

If $n < \infty$ and $\sum_j V_j V_j^* = I$, the representation factors through $\mathcal{O}_n$ and thus it can be thought of as a representation of $\mathcal{O}_n$.

It will be convenient to include here for further reference a brief summary of some basic aspects of the structure of $\mathcal{T}_n$. The orthogonality of the ranges of the isometries makes it possible to define an inner product on their closed linear span $\mathcal{E} = \overline{\text{span}}\{v_j\}_{j=1}^n$ via $\langle x, y\rangle I = y^*x$, where x and y are in $\mathcal{E}$. Since the Hilbert space norm and the operator norm coincide, $\mathcal{E}$ becomes a Hilbert space inside $\mathcal{T}_n$. The isometries in $\mathcal{E}$ are the unit vectors and $\{v_j\}_{j=1}^n$ is an orthonormal base.

Along the same lines, let $\mathcal{W}_k$ denote the set of products of k isometries among the generators $\{v_j\}_{j=1}^n$ and let $\mathcal{E}^k$ be the closed linear span of $\mathcal{W}_k$, then a similar argument shows that $\mathcal{E}^k$ is a Hilbert space and $\mathcal{W}_k$ is an orthonormal basis. Moreover, the mapping $e_1 \otimes \cdots \otimes e_k \mapsto e_1 \cdots e_k$ extends to a unitary operator from $\mathcal{E}^{\otimes k}$ onto $\mathcal{E}^k$.

The restriction of a representation π of $\mathcal{T}_n$ to $\mathcal{E}$ satisfies $\pi(y)^*\pi(x) = \langle x, y\rangle I$ for $x, y \in \mathcal{E}$ and, of course, determines the representation. It is a bit surprising that this condition alone is enough to force a map π from a Hilbert space $\mathcal{E}$ into $\mathcal{B}(\mathcal{H})$ to be linear and bounded (in fact, isometric) and thus to extend to a representation of $\mathcal{T}_n$. This was observed by Arveson using the positivity of $(\pi(\lambda x + \mu y) - \lambda\pi(x) - \mu\pi(y))^*$ $(\pi(\lambda x + \mu y) - \lambda\pi(x) - \mu\pi(y))$ for all scalars λ and μ, and the sesquilinearity of the inner product on $\mathcal{E}$.

For each n let $\mathcal{T}_n$ denote a particular representative of its isomorphism class, and let $\{v_j\}_{j=1}^n$ be a fixed set of generating isometries. $\mathcal{E}$ denotes the Hilbert space of dimension n generated by $\{v_j\}_{j=1}^n$, which is an orthonormal basis for $\mathcal{E}$. The subspace $\mathcal{E}$ generates $\mathcal{T}_n$ as a C^*-algebra and $\langle x, y\rangle I = y^*x$ for $x, y \in \mathcal{E}$.

An important consequence of the fact that the condition $\sum_{j=1}^n v_j v_j^* < I$ determines the isomorphism class of the C^*-algebra generated by the isometries v_j is that whenever U is a unitary operator on $\mathcal{E}$ there exists a unique automorphism γ_U of $\mathcal{T}_n$ such that $\gamma_U(x) = Ux$ for all $x \in \mathcal{E}$. The reason for this is that U transforms one collection of isometries satisfying Cuntz's condition into another, so the C^*-algebras generated are isomorphic. These automorphisms are called *quasifree* in [1], they satisfy $\gamma_U(\mathcal{E}) = \mathcal{E}$ and, in fact, they are characterized by this condition.

For further reference, a few facts about the C^*-algebras $\mathcal{O}_n$ are listed here. $\mathcal{O}_n$ is the C^*-algebra generated by a collection of isometries $\{V_j\}_{j=1}^n$ such that $\sum_{j=1}^n V_j V_j^* = I$. For $n = 2, 3, \ldots, \infty$, $\mathcal{O}_n$ is simple and depends only on n and not on the choice of the generators. The quasifree automorphisms γ_U with U of the form λI are called *gauge automorphisms*, and denoted γ_λ. The group $\{\gamma_\lambda : \lambda \in \mathbf{T}\}$ is the *gauge group* of automorphisms of $\mathcal{O}_n$. The fixed point algebra $\{x \in \mathcal{O}_n : \gamma_\lambda(x) = x\ (\forall)\ \lambda \in \mathbf{T}\}$ of this action is denoted by $\mathcal{F}_n$. It is an important tool in studying the structure of $\mathcal{O}_n$

owing to the following result:

PROPOSITION 1.4.(CUNTZ) [6]. *If v is a fixed isometry in $\mathcal{E}$, the elements of the form*

$$x = \sum_{j=-m}^{-1} v^{*|j|} f_j + f_0 + \sum_{j=1}^{m} f_j v^j \quad m \geqslant 0,\ f_j \in \mathcal{F}_n \ \text{for}\ |j| \leqslant m,$$

are dense in $\mathcal{O}_n$. The map $x \mapsto f_0$ is well defined and extends to a (faithful, completely positive) conditional expectation $\Phi : \mathcal{O}_n \to \mathcal{F}_n$ of norm 1; moreover,

$$\Phi(x) = \int_0^{2\pi} \gamma_\lambda(x) \frac{d\lambda}{2\pi}, \quad x \in \mathcal{O}_n.$$

Denote by $\mathcal{D}_k$ the closed linear span of products rs^*, where $r, s \in \mathcal{W}_k$. Then $\mathcal{D}_k$ is a C^*-subalgebra of $\mathcal{T}_n$ isomorphic to the compact operators on $\mathcal{E}^{\otimes k}$. In fact, the unitary operator betwenn $\mathcal{E}^{\otimes k}$ and $\mathcal{E}^k$ determined by $e_1 \otimes \cdots \otimes e_k \mapsto e_1 \ldots e_k$ implements a spatial equivalence between $\mathcal{K}(\mathcal{E}^{\otimes k})$ and $\mathcal{D}_k$. More specifically, the isomorphism is determined by extending the map:

$$\bigotimes_{i=1}^{k} \langle \cdot, f_i \rangle e_i \mapsto e_1 \ldots e_k f_k^* \ldots f_1^*$$

to an isomorphism from $\mathcal{K}(\mathcal{E}^{\otimes k})$ onto $\mathcal{D}_k$ where $e_i, f_i \in \mathcal{E}$, $\langle \cdot, f_i \rangle e_i$ denotes the usual rank-one operator and $\mathcal{D}_k$ acts on $\mathcal{E}^k$ by multiplication on the left.

If $\mathcal{A}_k$ is defined to be $\mathcal{D}_0 + \mathcal{D}_1 + \cdots + \mathcal{D}_k$, then $\mathcal{A}_k \subset \mathcal{A}_{k+1}$ and $\mathcal{F}_n$ is the norm closure of the union of the $\mathcal{A}_k$. In the tensor product picture presented by the isomorphism above, the embedding corresponds to

$$\mathbf{C}I_k + \mathcal{K}(\mathcal{E}) \otimes I_{k-1} + \cdots + \mathcal{K}(\mathcal{E}^{\otimes k}) \hookrightarrow \mathbf{C}I_{k+1} + \mathcal{K}(\mathcal{E}) \otimes I_k + \cdots + \mathcal{K}(\mathcal{E}^{\otimes k+1})$$
$$x \mapsto x \otimes I_1$$

where I_j denotes the identity on $\mathcal{E}^{\otimes j}$.

If $n = \dim \mathcal{E}$ is finite $\mathcal{A}_k$ coincides with $\mathcal{D}_k$, which is isomorphic to the algebra of $n^k \times n^k$ matrices. Since the embedding is the canonical one, $\mathcal{F}_n$ is the UHF algebra of pure type n^∞. The study of $\mathcal{F}_\infty$ is more involved since $\mathcal{F}_\infty$ is approximately finite but not a UHF algebra. However, it can be seen as a C^*-subalgebra of the infinite tensor product of unital C^*-algebras $(\mathcal{K}(\mathcal{E}) + \mathbf{C}I)^{\otimes\infty}$, and it turns out that much of what is true for representations of UHF algebras also holds for a certain class of representations of $\mathcal{F}_\infty$.

2. ENDOMORPHISMS AND $\mathcal{T}_n$

This section collects the basic results linking the representation theory of the C^*-algebras $\mathcal{T}_n$ to the study of endomorphisms of $\mathcal{B}(\mathcal{H})$. The starting point is Arveson's observation about the relevance of n-tuples of isometries with orthogonal ranges combined with the fact that such n-tuples determine representations of $\mathcal{T}_n$.

THEOREM 2.1. *If π is a nondegenerate representation of $\mathcal{T}_n$ on $\mathcal{H}$ then*

$$\alpha(A) = \sum_{j=1}^{n} \pi(v_j)A\pi(v_j)^* \quad A \in \mathcal{B}(\mathcal{H}) \tag{1}$$

defines an endomorphism α of $\mathcal{B}(\mathcal{H})$. Conversely, every endomorphism of $\mathcal{B}(\mathcal{H})$ arises in this fashion, for some n ($1 \leqslant n \leqslant \infty$), and some representation π of $\mathcal{T}_n$.

Furthermore, the set $E = \{T \in \mathcal{B}(\mathcal{H}) : \alpha(A)T = TA, \ (\forall)\ A \in \mathcal{B}(\mathcal{H})\}$ is a Hilbert space relative to the inner product given by $T^*S = \langle S, T\rangle I$, and π establishes a Hilbert space isomorphism between $\mathcal{E}$ and E. In particular $\pi(\mathcal{E}) = E$.

This is Proposition 2.1 of [2] complemented with Remark 1.3.

The notation $\alpha = \mathrm{Ad}_\pi$ will be used to denote the relation (1) between a representation of $\mathcal{T}_n$ and the endomorphisms of $\mathcal{B}(\mathcal{H})$ it implements. It will soon become apparent that Ad_π does not depend on the particular choice of an orthonormal base for $\mathcal{E}$.

If π is a representation of $\mathcal{T}_n$, and $\alpha = \mathrm{Ad}_\pi$, then $\alpha^k(I)$ is a decreasing sequence of projections. If $\mathcal{W}_k$ denotes, as before, the words of length k on the isometries (i.e. the products of k elements chosen among the v_j's), then $\mathcal{W}_k$ is an orthonormal basis for $\mathcal{E}^k$, and whenever $\xi \in \mathcal{H}$,

$$\alpha^k(I)\xi = \sum_{s \in \mathcal{W}_k} \pi(s)I\pi(s)^*\xi \in \overline{\pi(\mathcal{E}^k)\mathcal{H}}.$$

While, for $r \in \mathcal{W}_k$,

$$\pi(r)\xi = \sum_{s \in \mathcal{W}_k} \pi(s)\pi(s)^*\pi(r)\xi = \alpha^k(I)\pi(r)\xi$$

because $\pi(s)^*\pi(r) = \delta_{rs}I$. Hence $\alpha^k(I)$ is precisely the projection onto the subspace $\overline{\pi(\mathcal{E}^k)\mathcal{H}}$. Unital and completely non-unital endomorphisms can now be characterized in terms of the associated representations:

Ad_π is unital if and only if $\overline{\pi(\mathcal{E})\mathcal{H}} = \mathcal{H}$, and

Ad_π is completely nonunital (i.e. $\mathrm{Ad}_\pi^k(I) \to 0$ strongly as $k \to \infty$) if and only if $\overline{\pi(\mathcal{E}^k)\mathcal{H}} \searrow (0)$ as $k \to \infty$.

The isometries determine the endomorphism, but the converse is not true; modifying a representation by a quasifree automorphism of $\mathcal{T}_n$ does not change the associated endomorphism. This is all that can happen, as the following proposition shows.

PROPOSITION 2.2. *Suppose that π and σ are nondegenerate representations of $\mathcal{T}_m$ and $\mathcal{T}_n$ respectively. Then $\mathrm{Ad}_\pi = \mathrm{Ad}_\sigma$ if and only if $m = n$ and $\pi = \sigma \circ \gamma_U$ for some unitary operator U in $\mathcal{E}$.*

Proof. By the last assertion of Theorem 2.1, if the endomorphisms coincide then $\pi(\mathcal{E}_m) = E = \sigma(\mathcal{E}_n)$, and since π and σ are one to one on $\mathcal{E}$, then $m = n$. From now on let $\mathcal{E} = \mathcal{E}_m = \mathcal{E}_n$. The map $U = \pi^{-1} \circ \sigma$ is well defined on $\mathcal{E}$, and is a unitary operator. By definition of U, it follows that $\sigma = \pi \circ \gamma_U$ on $\mathcal{E}$, hence on all of $\mathcal{T}_n$.

Conversely, if $\sigma = \pi \circ \gamma_U$ for some unitary operator U on $\mathcal{E}$, then $\pi(\mathcal{E}) = \sigma(\mathcal{E})$. Thus the image of the identity operator is the same under both Ad_π and Ad_σ. By Theorem 2.1 $(\mathrm{Ad}_\pi(A) - \mathrm{Ad}_\sigma(A))T = T(A - A) = 0$ whenever $T \in \pi(\mathcal{E}) = \sigma(\mathcal{E})$ and $A \in \mathcal{B}(\mathcal{H})$. Thus $\mathrm{Ad}_\pi(A) - \mathrm{Ad}_\sigma(A)$ vanishes on $\pi(\mathcal{E})\mathcal{H}$, which is the range of $\mathrm{Ad}_\pi(I) = \mathrm{Ad}_\sigma(I)$.

This implies that $\mathrm{Ad}_\pi(A) - \mathrm{Ad}_\sigma(A) = (\mathrm{Ad}_\pi(A) - \mathrm{Ad}_\sigma(A))\mathrm{Ad}_\pi(I) = 0$, so $\mathrm{Ad}_\pi = = \mathrm{Ad}_\sigma$. ∎

As a consequence, the endomorphism Ad_π does not depend on the particular orthonormal basis of $\mathcal{E}$ used in (1); besides, the multiplicity $n_\alpha = \dim \mathcal{E}$ is uniquely determined by the endomorphism, as it is the multiplicity of the identity representation in the endomorphism seen as a representation of $\mathcal{B}(\mathcal{H})$. This extends Powers definition of multiplicity to the nonunital case. However, if we are to include nonunital endomorphisms in our considerations, this multiplicity index is not enough to characterize endomorphisms up to outer conjugacy. For this, it is necessary to consider another nonnegative integer associated with an endomorphism α. Let ν_α denote the dimension of $I - \alpha(I) = I - \sum_{j=1}^{n} V_j V_j^*$, which measures by how much α fails to be unital. This will be called the *deficiency* of α. A slight modification of the proof of Theorem 2.4 in [11] is enough to show that the pair (n_α, ν_α) is a complete outer conjugacy invariant for α.

PROPOSITION 2.3. *The endomorphisms α and β are outer conjugate if and only if $n_\alpha = n_\beta$ and $\nu_\alpha = \nu_\beta$.*

Proof. Assume without loss of generality that α and β act on the same Hilbert space $\mathcal{H}$. If the endomorphisms are outer conjugate, then $\alpha = \beta \circ \theta$ for some automorphism θ of $\mathcal{B}(\mathcal{H})$. Since $\theta(I) = I$, it follows that $\nu_\alpha = \nu_\beta$. It is also clear that

$n_\alpha = n_\beta$, because they are the multiplicities of the identity representation of $\mathcal{B}(\mathcal{H})$ in α and β.

If $\nu_\alpha = \nu_\beta$, let S be a partial isometry on $\mathcal{H}$ having initial projection $I-\alpha(I)$ and final projection $I-\beta(I)$. Define $W = \sum_{j=1}^{n} T_j V_j^* + S$, where the V_j's are isometries implementing α and the T_j's are isometries implementing β. Then W is a unitary operator on $\mathcal{H}$, which can be verified directly by computing $W^*W = \sum_{j=1}^{n} V_j T_j^* \sum_{i=1}^{n} T_i V_i^* + S^*S =$ $= \sum_{i=1}^{n} V_j V_j^* + S^*S = \alpha(I) + S^*S = I$, and analogously, $WW^* = I$. Furthermore, a similar calculation shows that $T_j = WV_j$, hence $\beta(A) = W\alpha(A)W^*$. This implies that α and β are outer conjugate. ∎

Going back to the conjugacy relation, it is now posible to characterize it in terms of representations of $\mathcal{T}_n$.

PROPOSITION 2.4. *Suppose that π and σ are nondegenerate representations of $\mathcal{T}_m$ and $\mathcal{T}_n$ respectively. Then $\mathrm{Ad}_\pi \overset{c}{\sim} \mathrm{Ad}_\sigma$ if and only if $m = n$ and there exists a unitary operator U on $\mathcal{E}$ such that the representations $\pi \circ \gamma_U$ and σ are unitarily equivalent.*

Proof. Let $\mathcal{H}_\pi$ and $\mathcal{H}_\sigma$ be the Hilbert spaces corresponding to π and σ respectively. The endomorphisms are conjugate if and only if there is a unitary operator $W : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that

$$\sum_{j=1}^{n} \sigma(v_j)A\sigma(v_j)^* = W\sum_{j=1}^{n} \pi(v_j)(W^{-1}AW)\pi(v_j)^*W^{-1} =$$
$$= \sum_{j=1}^{n} (W\pi(v_j)W^{-1})A(W\pi(v_j)W^{-1})^*$$

for all A in $\mathcal{B}(\mathcal{H}_\sigma)$. That is, if and only if σ and $\mathrm{Ad}_W \circ \pi$ give rise to the same endomorphism. By Proposition 2.2 this occurs if and only if $m = n$ and there is a unitary U on $\mathcal{E}$ such that $\sigma = \mathrm{Ad}_W \circ \pi \circ \gamma_U$. ∎

We are thus led to define a new equivalence relation which captures the conjugacy of endomorphisms at the level of representations of Cuntz algebras.

DEFINITION 2.5. Two representations π and σ of $\mathcal{T}_n$ are *quasifree-equivalent*, (to be denoted $\pi \overset{qf}{\sim} \sigma$) if there exists a unitary operator U on $\mathcal{E}$ such that $\pi \circ \gamma_U$ is unitarily equivalent to σ.

These preliminary results make it possible to study properties and relations between endomorphisms by studying the corresponding properties and relations of the corresponding representations of the C^*-algebras $\mathcal{T}_n$. The first step in this direction is to characterize the unital and completely nonunital behaviours in terms of representations of $\mathcal{T}_n$.

In [2] Arveson defined singular and nonsingular (later renamed essential) continuous product systems and produced a decomposition of an arbitrary continuous product system into an essential part and a singular part. Arveson's decomposition (Proposition 1.14 of [2]) is analogous to the Wold decomposition for isometries on an infinite dimensional Hilbert space, which states that every isometry is unitarily equivalent to a direct sum of a unitary part and an isometry of pure type (which is necessarily a multiple of the unilateral shift). His definition can be adapted to the present situation in order to obtain a Wold-type decomposition for representations of $\mathcal{T}_n$, and consequently, for endomorphisms of $B(\mathcal{H})$.

DEFINITION 2.6. A representation π of $\mathcal{T}_n$ on $\mathcal{H}$ is *essential* if $\overline{\pi(\mathcal{E})\mathcal{H}} = \mathcal{H}$, and it is *singular* if $\overline{\pi(\mathcal{E}^k)\mathcal{H}} \searrow (0)$ as $k \to \infty$.

Arveson's result can now be restated as a theorem for representations of $\mathcal{T}_n$.

THEOREM 2.7. *If π is a nondegenerate representation of $\mathcal{T}_n$ on a separable Hilbert space $\mathcal{H}$, then there exists a unique decomposition $\pi = \pi_e + \pi_s$ such that Ad_{π_e} is unital and Ad_{π_s} is nonunital. Moreover, this decomposition is central.*

A sketch of the proof, adapted from [2] follows. Let

$$\mathcal{H}_e = \bigcap_{k \geqslant 1} \overline{\pi(\mathcal{E}^k)\mathcal{H}}$$

and verify that $\mathcal{H}_e$ is invariant under both $\pi(\mathcal{E})$ and $\pi(\mathcal{E})^*$, therefore under $\pi(\mathcal{T}_n)$. The corresponding subrepresentation π_e clearly satisfies $\overline{\operatorname{span}}\, \pi_e(\mathcal{E})\mathcal{H} = \mathcal{H}$. If $\mathcal{H}_s$ is defined to be $\mathcal{H}_e^\perp$, then it is also invariant under $\pi(\mathcal{T}_n)$, and the corresponding subrepresentation, π_s, is singular. Uniqueness follows from the definition of $\mathcal{H}_e$.

If $\mathcal{W}_k$ is, as before, the collection of words of length k on the v_j's, and $\sum_{s \in \mathcal{W}_k} \pi(ss^*)$ denotes the strong limit of the net $\left\{ \sum_{s \in F} \pi(ss^*) : F \text{ a finite subset of } \mathcal{W}_k \right\}$, then

$$\left(\sum_{s \in \mathcal{W}_k} \pi(ss^*) \right) \mathcal{H} = \overline{\pi(\mathcal{E}^k)\mathcal{H}} \searrow \mathcal{H}_e, \tag{2}$$

so $\sum_{s \in \mathcal{W}_k} \pi(ss^*)$ converges strongly to P_e, the projection corresponding to $\mathcal{H}_e$, therefore the decomposition is central. ∎

REMARK 2.8. If n, the dimension of $\mathcal{E}$, is finite, then $p_0 = 1 - \sum_{j=1}^{n} v_j v_j^*$ is a minimal projection in $\mathcal{T}_n$ which generates an ideal $\mathcal{J}_n$ isomorphic to the compact operators. The condition of π being essential is equivalent to $\pi(p_0) = 0$ hence to π having $\mathcal{J}_n$ as its kernel; while singularity of π corresponds to π being nondegenerate when restricted to $\mathcal{J}_n$. These situations are customarily referred to as π being $\mathcal{J}_n$--singular and $\mathcal{J}_n$-essential *respectively*, so we have a most unfortunate reversal of the terminology.

If π is a cyclic representation, with a cyclic vector Ω, then $P_e\Omega$ is cyclic for π_e and $P_s\Omega$ is cyclic for π_s because the decomposition is central. In this case π is essential if and only if $P_s\Omega = 0$, and singular if and only if $P_e\Omega = 0$.

This observation enables us to characterize the states which give rise to singular and to essential representations. Suppose first ω is a state of $\mathcal{T}_n$ and define $\alpha^*\omega$ by

$$\alpha^*\omega(x) = \sum_{j=1}^{n} \omega(v_j x v_j^*) \quad x \in \mathcal{T}_n.$$

Then $\alpha^*\omega$ is a positive linear functional of norm at most 1 on $\mathcal{T}_N$. Note that by iteration, $\alpha^{*k}\omega(x) = \sum_{s\in\mathcal{W}_k} \omega(sxs^*)$. If n is finite, then α^* is just the adjoint map of the C^*-algebra endomorphism $\alpha : x \mapsto \sum_j v_j x v_j^*$ of $\mathcal{T}_n$. Notice that the above definition of α^* circumvents the problem of not being able to define $x \mapsto \sum_{j=1}^{\infty} v_j x v_j^*$ on $\mathcal{T}_\infty$. In this case α^* is not the adjoint map of an endomorphism of the C^*-algebra $\mathcal{T}_\infty$.

COROLLARY 2.9. *Let ω be a state of $\mathcal{T}_n$ and let π denote its GNS representation. Then π is essential if and only if $\|\alpha^{*k}\omega\| = 1$ for all $k \geqslant 0$, and π is singular if and only if $\|\alpha^{*k}\omega\| \to 0$ as $k \to \infty$.*

Proof. Denote by Ω the cyclic unit vector corresponding to ω. Then

$$\|\alpha^{*k}\omega\| = \alpha^{*k}\omega(I) = \sum_{s\in\mathcal{W}_k} \langle \pi(ss^*)\Omega, \Omega\rangle =$$

$$= \left\langle \sum_{s\in\mathcal{W}_k} \pi(ss^*)\Omega, \Omega \right\rangle \to \langle P_e\Omega, \Omega\rangle = \|P_e\Omega\|^2 \quad \text{as } k \to \infty.$$

It follows that $\|\alpha^{*k}\omega\| = 1$ for all $k \geqslant 0$ if and only if $\|P_e\Omega\| = 1$, and that $\|\alpha^{*k}\omega\| \to 0$ if and only if $P_e\Omega = 0$. ∎

Thus, a state ω will be called essential or singular according to its associated GNS representation being such.

REMARK 2.10. Since $\alpha^{*k}\omega(I) = \sum_{s\in\mathcal{W}_k} \omega(ss^*)$ it is possible to characterize essential and singular states in terms of their restrictions to the subalgebras $\mathcal{D}_k = \overline{\operatorname{span}}\,\mathcal{W}_k\mathcal{W}_k^*$. A state ω is essential if and only if $\|\omega\lceil\mathcal{D}_k\| = 1$ for all k; it is singular if and only if $\|\omega\lceil\mathcal{D}_k\| \to 0$ as $k \to \infty$.

THE FOCK REPRESENTATION. In [8] Evans introduced representations of the C^*-algebras $\mathcal{T}_n$ on full Fock space, analogues to the Fock representation of the CAR algebra on antisymmetric Fock space. Specifically, for $1 \leqslant n \leqslant \infty$ let $\mathcal{E}$ be an n-dimensional Hilbert space and denote by $\mathsf{F}_\mathcal{E}$ the full Fock space over $\mathcal{E}$, that is,

$$\mathsf{F}_\mathcal{E} = \bigoplus_{k=0}^{\infty} \mathcal{E}^{\otimes k},$$

where $\mathcal{E}^{\otimes 0}$ is a one dimensional Hilbert space spanned by the unit vector Ω_0 and $\mathcal{E}^{\otimes k}$ is a Hilbert space tensor product of k copies of $\mathcal{E}$, for each $k = 1, 2, \ldots$. The Fock representation of $\mathcal{T}_n$ is the representation obtained by letting $\mathcal{E}$ act on $\mathsf{F}_\mathcal{E}$ by *left creation* operators. More explicitly, for $e \in \mathcal{E}$ the operator $\varphi(e)$ on $\mathsf{F}_\mathcal{E}$ is defined by

$$\begin{aligned} \varphi(e) : e_1 \otimes \cdots \otimes e_k &\mapsto e \otimes e_1 \otimes \cdots \otimes e_k \\ \mathcal{E}^{\otimes k} &\to \mathcal{E}^{\otimes k+1} \quad k \geqslant 1, \\ \text{and } \varphi(e)\Omega_0 &= e. \end{aligned} \tag{3}$$

A simple computation shows that $\varphi(y)^*\varphi(x) = \langle x, y\rangle I$ for $x, y \in \mathcal{E}$, hence $\{\varphi(v_j)\}_{j=1}^n$ is a collection of isometries with orthogonal ranges. In fact, if P_0 denotes the rank one projection onto the subspace spanned by Ω_0, then

$$\sum_{j=1}^{n} \varphi(v_j)\varphi(v_j)^* = I - P_0.$$

Thus φ extends to a representation of $\mathcal{T}_n$ on $\mathsf{F}_\mathcal{E}$, which will also be denoted by φ. It is not hard to see that φ is irreducible and that Ω_0 is cyclic, so that the state ω_0 induced by Ω_0 is pure. This state vanishes on the nontrivial products formed by the v_j's and their adjoints and is in fact characterized by this property. Since the subspace $\overline{\varphi(\mathcal{E}^k)\mathsf{F}_\mathcal{E}} = \mathcal{E}^{\otimes k} \oplus \mathcal{E}^{\otimes k+1} \oplus \cdots$ of $\mathsf{F}_\mathcal{E}$ shrinks to the zero subspace as $k \to \infty$, the Fock representation is singular, and the endomorphism of $\mathcal{B}(\mathsf{F}_\mathcal{E})$ it implements is completely nonunital, with multiplicity index $n = \dim\mathcal{E}$ and deficiency index $\nu = \dim P_0 = 1$.

When $n = 1$, $\mathsf{F}_\mathcal{E}$ can be naturally identified with ℓ^2 and $\varphi(v_1)$ with the unilateral shift. This observation suggests the following theorem, which states that also for $n > 1$ the Fock representation is the fundamental building block with which all singular representations of $\mathcal{T}_n$ are made, leaving to the essential representations the role of the unitary part in the Wold decomposition.

An important difference between the present situation and the study of continuous product systems is that the singular component of a representation of $\mathcal{T}_n$ is always a multiple of a fixed representation, while for continuous product systems, Arveson has produced in [3] different singular states which are not quasi-equivalent to each other.

THEOREM 2.11. *A nondegenerate representation of $\mathcal{T}_n$ on a separable Hilbert space is singular if and only if it is unitarily equivalent to a multiple of the Fock representation, in which case its multiplicity is $\nu = \dim[\pi(\mathcal{E})\mathcal{H}]^\perp$.*

Proof. The multiples of the Fock representation are clearly singular. To prove the converse, assume π is a singular representation. Let $\alpha = \mathrm{Ad}_\pi$ be the associated endomorphism and let $\mathcal{N}_0 = [\pi(\mathcal{E})\mathcal{H}]^\perp$. Then $\mathcal{N}_0$ is the subspace of $\mathcal{H}$ corresponding to the projection $(I - \alpha(I))$. For $k \geqslant 0$ define

$$\mathcal{N}_k = \overline{\pi(\mathcal{E}^k)\mathcal{N}_0} = \alpha^k(I - \alpha(I))\mathcal{H}, \tag{4}$$

and note that $\mathcal{N}_k \perp \mathcal{N}_l$ whenever $k \neq l$. By singularity of π, $\alpha^k(I)$ tends strongly to 0, so that

$$I = (I - \alpha(I)) + (\alpha(I) - \alpha^2(I)) + (\alpha^2(I) - \alpha^3(I)) + \cdots$$

in the sense of strong convergence, thus

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{N}_k = \bigoplus_{k=0}^{\infty} \overline{\pi(\mathcal{E}^k)\mathcal{N}_0}.$$

Let φ denote the Fock representations of $\mathcal{T}_n$ on $\mathbf{F}_\mathcal{E}$, and for each $k \geqslant 0$ define a map on elementary tensors

$$\begin{aligned} U_k : e_1 \otimes \cdots \otimes e_k \otimes \xi &\mapsto \pi(e_1 \cdots e_k)\xi \\ \mathcal{E}^{\otimes k} \otimes \mathcal{N}_0 &\to \mathcal{N}_k. \end{aligned} \tag{5}$$

Since for $e_j, f_j \in \mathcal{E}$, $f_j^* e_j = \langle e_j, f_j\rangle I$, it follows that

$$\begin{aligned} \langle \pi(e_1 \cdots e_2)\xi, \pi(f_1 \cdots f_k)\eta\rangle &= \langle \pi(f_k^* \cdots f_1^* e_1 \cdots e_k)\xi, \eta\rangle = \\ &= \prod_{j=1}^{k} \langle e_j, f_j\rangle\langle \xi, \eta\rangle = \\ &= \langle e_1 \otimes \cdots \otimes e_k \otimes \xi, f_1 \otimes \cdots \otimes f_k \otimes \eta\rangle. \end{aligned} \tag{6}$$

Hence each U_k extends to a unitary operator from $\mathcal{E}^{\otimes k} \otimes \mathcal{N}_0$ onto $\mathcal{N}_k$ for each $k = 0, 1, 2, \ldots$. The direct sum of these unitaries is a unitary operator

$$U : \mathbf{F}_\mathcal{E} \otimes \mathcal{N}_0 = \bigoplus_{k=1}^{\infty} \mathcal{E}^{\otimes k} \otimes \mathcal{N}_0 \to \bigoplus_{k=0}^{\infty} \mathcal{N}_k = \mathcal{H}$$

and it only remains to check that U interwines π on $\mathcal{H}$ with $(\varphi \otimes I_0)$ on $\mathsf{F}_\mathcal{E} \otimes \mathcal{N}_0$, for which elementary tensors suffice. Let $x, e_1, \ldots, e_k \in \mathcal{E}$ and $\xi \in \mathcal{N}_0$, then

$$\pi(x)U(e_1 \otimes \cdots \otimes e_k \otimes \xi) = \pi(xe_1 \cdots e_k)\xi =$$
$$= U(x \otimes e_1 \otimes \cdots \otimes e_k \otimes \xi) = U(\varphi(x)(e_1 \otimes \cdots \otimes e_k) \otimes I_0\xi =$$
$$= U(\varphi \otimes I_0)(x)(e_1 \otimes \cdots \otimes e_k \otimes \xi)$$

So π coincides with $U(\varphi \otimes I_0)U^{-1}$ on $\mathcal{E}$, hence on all of $\mathcal{T}_n$. ∎

Arveson's decomposition together with Theorem 2.11 now yield the Wold Decomposition for Representations of $\mathcal{T}_n$: *A nondegenerate representation of $\mathcal{T}_n$ on a separable Hilbert space is equal to the sum of an essential representation and a multiple of the Fock representation, with multiplicity $\nu = \dim[\pi(\mathcal{E})\mathcal{H}]^\perp = \operatorname{rank}(I - \mathrm{Ad}_\pi(I))$.*

One would like to have a concept of direct sum that, applied to two endomorphisms α and β of $\mathcal{B}(\mathcal{H}_\alpha)$ and of $\mathcal{B}(\mathcal{H}_\beta)$, respectively, produced another endomorphism θ of $\mathcal{B}(\mathcal{H}_\alpha \oplus \mathcal{H}_\beta)$ such that $\theta(A \oplus B) = \alpha(A) \oplus \beta(B)$ whenever $A \in \mathcal{B}(\mathcal{H}_\alpha)$ and $B \in \mathcal{B}(\mathcal{H}_\beta)$. Unfortunately, not only this is not always possible: such θ exists only if α and β have the same multiplicity, but when possible it may fail to be determined even up to conjugacy. There exist pairs θ, θ' of endomorphisms of $\mathcal{B}(\mathcal{H}_\alpha \oplus \mathcal{H}_\beta)$ such that

$$\theta(A \oplus B) = \alpha(A) \oplus \beta(B) = \theta'(A \oplus B)$$

yet θ and θ' are not conjugate. Such a pair can be constructed as follows. Start with an irreducible representation π of $\mathcal{T}_n$ and a quasifree automorphism γ_U of $\mathcal{T}_n$ such that $\pi \circ \gamma_U$ is disjoint from π (e.g. see examples at the end of Section 4). Consider the representations $\pi \oplus \pi$ and $\pi \oplus \pi \circ \gamma_U$; the endomorphisms they induce are not conjugate because the factor representation $\pi \oplus \pi$ cannot be quasifree-equivalent to $\pi \oplus \pi \circ \gamma_U$ which is not a factor representation. However, both "decompose" as $\mathrm{Ad}_\pi + \mathrm{Ad}_\pi$.

In view of this, there is little hope that a result will carry over from subrepresentations of $\mathcal{T}_n$ to endomorphisms. However, if two endomorphisms are conjugate, the same holds separately, as a consequence of the uniqueness of the Wold decomposition, for their unital components and for their completely nonunital components. Since the Wold decomposition has the added feature that the singular part is always a multiple of the Fock representation, the converse holds, and the components obtained determine the original endomorphism up to conjugacy.

THEOREM 2.12. *Two endomorphisms have conjugate unital parts and conjugate completely nonunital parts if and only if they are conjugate themselves.*

Proof. That conjugate endomorphisms have conjugate unital and completely nonunital parts is easy to see from the proof of the Wold decomposition theorem, one just restricts the isomorphism implementing the conjugacy to each component. To prove the converse, assume Ad_π and Ad_σ are two endomorphisms such that $\pi_s \overset{qf}{\sim} \sigma_s$ and $\pi_e \overset{qf}{\sim} \sigma_e$. Then π and σ have the same multiplicity n and deficiency ν, and π_e is unitarily equivalent to $\sigma_s \circ \gamma_U$ for some quasifree automorphism γ_U. Since also π_s is unitarily equivalent to $\sigma_s \circ \gamma_U$ because they are both singular and have the same deficiency, it follows that π is unitarily equivalent to $\sigma \circ \gamma_U$. ■

In view of the previous discussion, the study of conjugacy classes of endomorphisms of a given multiplicity index n can be reduced to the separate study of the completely nonunital ones (which are characterized up to conjugacy by their deficiency indices), and the unital ones, which present a much richer structure, as they are indexed by the quasifree-equivalence classes of essential representations of $\mathcal{T}_n$. In the case of finite n these correspond to nondegenerate representations of the quotient C^*-algebra $\mathcal{O}_n$. If $n = \infty$, $\mathcal{T}_n$ is simple and equal to $\mathcal{O}_n$, but the representations involved are only the essential ones, i.e. those for which $\sum_{j=1}^{n} \pi(v_j v_j^*) = I$, a condition more restrictive than nondegeneracy. The remaining of this work is thus devoted to study essential representations of $\mathcal{O}_n$ for $2 \leqslant n \leqslant \infty$.

3. UNITAL ENDOMORPHISMS AND $\mathcal{O}_n$

An endomorphism α of $\mathcal{B}(\mathcal{H})$ is unital if and only if $\alpha = \mathrm{Ad}_\pi$ for an essential representation π of $\mathcal{O}_n$ on $\mathcal{B}(\mathcal{H})$, where n is Powers' multiplicity index of α. Suppose than π is essential and $\alpha = \mathrm{Ad}_\pi$ is defined as before by $\alpha(A) = \sum_{j=1}^{n} \pi(v_j) A \pi(v_j)^*$, it is possible to characterize two relevant α-invariant subalgebras of $\mathcal{B}(\mathcal{H})$ in terms of π itself.

PROPOSITION 3.1. *Suppose π is an essential representation of $\mathcal{O}_n$ on $\mathcal{H}$, and let $\alpha = \mathrm{Ad}_\pi$; it follows that*

i) $\{A \in \mathcal{B}(\mathcal{H}) : \alpha(A) = A\} = \pi(\mathcal{O}_n)'$, *and*

ii) $\bigcap_{k \geqslant 0} \alpha^k(\mathcal{B}(\mathcal{H})) = \pi(\mathcal{F}_n)'$

Proof. To prove i) first note that $A \in \pi(\mathcal{O}_n)'$ implies $\alpha(A) = A$ because A commutes with $\pi(v_j)$ for $j = 1, 2, \ldots, n$, in which case $\alpha(A) = A \sum_{j=1}^{n} \pi(v_j)\pi(v_j)^* = AI = A$. The reverse inclusion follows from the fact that if $\alpha(A) = A$, then $\alpha(A^*) = A^*$ as well, so both A and A^* commute with $\pi(\mathcal{E})$ by the last assertion in Theorem 2.1. This, in turn, implies that A commutes with $\pi(\mathcal{E}) \cup \pi(\mathcal{E})^*$ hence with $\pi(\mathcal{O}_n)$.

In order to prove ii) let $k \geqslant 0$ and consider $r, s \in \mathcal{W}_k$. Repeated application of Theorem 2.1 yields $\alpha^k(A)\pi(r) = \pi(r)A$ for $A \in \mathcal{B}(\mathcal{H})$, and also $\alpha^k(A^*)\pi(s) = \pi(s)A^*$ so that $\alpha^k(A)\pi(rs^*) = \pi(rs^*)\alpha^k(A)$. Thus $\alpha^k(\mathcal{B}(\mathcal{H})) \subseteq \pi(\mathcal{W}_k\mathcal{W}_k^*)'$.

Conversely, suppose $A \in \pi(\mathcal{W}_k\mathcal{W}_k^*)'$, and let $B = \pi(s_0^*)A\pi(s_0)$ where s_0 is any element in $\mathcal{W}_k$. Then

$$\alpha^k(B) = \sum_{s\in\mathcal{W}_k} \pi(s)\pi(s_0^*)A\pi(s_0)\pi(s)^* = \sum_{s\in\mathcal{W}_k} \pi(ss_0^*)A\pi(s_0s^*) =$$

$$= \sum_{s\in\mathcal{W}_k} \pi(ss_0^*)\pi(s_0s^*)A = \sum_{s\in\mathcal{W}_k} \pi(ss^*)A = \alpha^k(I)A = A$$

so $A \in \alpha^k(\mathcal{B}(\mathcal{H}))$. Thus $\alpha^k(\mathcal{B}(\mathcal{H})) = \pi(\mathcal{W}_k\mathcal{W}_k^*)'$ for every $k \geqslant 0$ and ii) follows from the fact that $\mathcal{F}_n$ is the C^*-algebra generated by $\bigcup_{k\geqslant 0} \mathcal{W}_k\mathcal{W}_k^*$. ■

When one (or both) of these subalgebras is trivial the endomorphism is of a more elementary nature, in that it does not act trivially (or as an automorphism) on a nontrivial subalgebra of $\mathcal{B}(\mathcal{H})$.

DEFINITION 3.2. An endomorphism α of $\mathcal{B}(\mathcal{H})$ is *ergodic* if $\{A \in \mathcal{B}(\mathcal{H}) : \alpha(A) = A\} = \mathbb{C}I$, and it is *strongly ergodic* (a *shift* in Powers' terminology) if $\bigcap_{j\geqslant 0} \alpha^j(\mathcal{B}(\mathcal{H})) = \mathbb{C}I$.

Thus $\alpha = \mathrm{Ad}_\pi$ is ergodic if and only if π is irreducible, and it is strongly ergodic if and only if $\pi\restriction\mathcal{F}_n$ is irreducible. A first step to understand general endomorphisms of $\mathcal{B}(\mathcal{H})$ is to study irreducible representations of $\mathcal{O}_n$ and, in particular, representations of $\mathcal{O}_n$ whose restrictions to $\mathcal{F}_n$ are irreducible, modulo the quasifree-equivalence relation. In order to include infinite multiplicity in the study, it is necessary to develop a better understanding of the fixed point algebra $\mathcal{F}_\infty$.

ESSENTIAL STATES OF $\mathcal{F}_\infty$. Specifically, we need to carry out an analysis of essential states of $\mathcal{F}_\infty$ which enables us to decide when a state is primary (i.e. a factor state) and when two primary states are quasi-equivalent, in terms similar to those given by Powers in [10] for states of UHF algebras. Along these lines, we will prove that product states are primary, and use the shift α^* on essential states of $\mathcal{F}_n$ to characterize quasi-equivalence of factor states. The main results are stated in a form which is valid for $2 \leqslant n \leqslant \infty$, provided that when $n = \infty$ the states involved are assumed to be essential.

Let $\mathcal{E}$ be an infinite dimensional Hilbert space, and for each $i \in \mathbb{N}$ let $\mathcal{K}_i$ be an isomorphic copy of the compact operators on $\mathcal{E}$. Construct the C^*-algebraic tensor product of the unital C^*-algebras $\widetilde{\mathcal{K}}_i = \mathcal{K}_i + \mathbb{C}I$, that is, let

$$\mathcal{C} = \bigotimes_{i=1}^{\infty} \widetilde{\mathcal{K}}_i.$$

For each $j = 1, 2, \dots$ the subalgebra $\mathcal{D}_j$ generated by the elements $T_1 \otimes \cdots \otimes T_j \otimes \otimes I \otimes I \otimes \cdots$ for $T_i \in \mathcal{K}_i$ and $0 \leqslant i \leqslant j$ is naturally isomorphic to $\mathcal{K}(\mathcal{E}^{\otimes j})$. Letting $\mathcal{A}_j = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2 + \cdots + \mathcal{D}_j$, $\mathcal{F}_\infty$ can be seen as the C^*-algebraic direct limit of the sequence$\{\mathcal{A}_j\}_{j=0}^{\infty}$.

Any state ω on this direct limit is determined by its restrictions to the sequence of subalgebras $\{\mathcal{A}_j\}$. If ω is essential, then $\omega\restriction\mathcal{D}_j$ has norm one because it is supported on the compacts $\mathcal{D}_j \cong \mathcal{K}(\mathcal{E}^{\otimes j})$ at the j^{th} level for each $j = 1, 2, \dots$, so there is a positive $\Omega_j \in \mathcal{D}_j$ with $\operatorname{tr}\Omega_j = 1$ such that

$$\omega(T) = \operatorname{tr}(\Omega_j T) \quad T \in \mathcal{A}_j. \tag{7}$$

The sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ satifies a *coherence* condition:

$$\operatorname{tr}(\Omega_{j+1}(T \otimes I)) = \operatorname{tr}(\Omega_j T) \quad T \in \mathcal{A}_j, \tag{8}$$

and indeed, any sequence $\Omega_j \in \mathcal{D}_j$ with $\operatorname{tr}\Omega_j = 1$, satisfying (8) gives rise to an essential state of $\mathcal{F}_\infty$ via (7). The same coherent sequence naturally determines a state $\breve{\omega}$ of $\mathcal{C} = \bigotimes_{j=1}^{\infty} \widetilde{K}_j$, which extends ω and is locally normal in the sense that $\breve{\omega}\restriction\mathcal{D}_j$ is a regular (normal) state of $\mathcal{D}_j \cong \mathcal{K}(\mathcal{E}^{\otimes j})$. This is the unique state extension of ω to $\mathcal{C}$, because at each level $\omega\restriction\mathcal{A}_j$ extends uniquely to $\bigotimes_{i=1}^{j} \widetilde{K}_i$. Thus, any essential state ω of $\mathcal{F}_\infty$ has a unique extension to a state $\breve{\omega}$ of $\mathcal{C}$, which is locally normal.

Let F be a finite subset of $\mathbb{N}$, and let $\mathcal{K}_F$ denote the C^*-subalgebra $\bigotimes_{i\in F} \mathcal{K}_i$ generated by $\mathcal{K}_i$ with $i \in F$, where $\mathcal{K}_i$ has been identified with its image in the tensor product $\mathcal{C}$. In other words, $\mathcal{K}_F$ is generated by the elementary tensors $\bigotimes_{i=1}^{\infty} A_i$ where $A_i \in \mathcal{K}_i$ if $i \in F$ and $A_i = I$ otherwise. Thus $\mathcal{K}_{\{1,\dots,j\}}$ coincides with $\mathcal{D}_j$ and $\mathcal{K}_F = \mathbb{C}I$ if $F = \emptyset$.

LEMMA 3.3. *Let $(\pi, \mathcal{H}, \Omega)$ be the GNS triple associated with $\breve{\omega}$. Then π is faithful and $\pi(\mathcal{F}_\infty)' = \pi(\mathcal{C})'$.*

Proof. π is faithful because for each $j \geqslant 1$ its restriction to $\mathcal{D}_j$ is isometric. The inclusion $(\supseteq)$ in the second assertion is obvious. To prove $(\subseteq)$ suppose $T \in \pi(\mathcal{F}_\infty)'$ and let F be any finite subset of $\mathbb{N}$. Since T commutes with $\mathcal{K}_{\{1,\dots,\max F\}} \subset \mathcal{F}_\infty$ then T commutes with $\mathcal{K}_F$, and therefore T commutes with $\pi(\mathcal{C})$. ■

From this it follows that Ω is cyclic for the action of $\pi(\mathcal{F}_\infty)$, because the subspace $\overline{\operatorname{span}}\,\pi(\mathcal{F}_\infty)\Omega$ is invariant under $\pi(\mathcal{C})$ and contains Ω, hence it is all of $\mathcal{H}$. As a consequence, $\pi\restriction\mathcal{F}_\infty$ is unitarily equivalent to the GNS representation of ω and $\pi(\mathcal{F}_\infty)'' = \pi(\mathcal{C})''$ so that ω is primary (pure) if and only if $\breve{\omega}$ is primary (pure).

Lemma 3.4. *For each finite subset F of $\mathbf{N}$, the restriction of π to $\mathcal{K}_F$ is nondegenerate.*

Proof. Let e_{ij} be a system of matrix units for $\mathcal{K}_F$; it suffices to show that $\sum_{i=1}^{m}\pi(e_{ii})$ is weakly convergent to the identity I on $B(\mathcal{H})$. If G is a finite subset of $\mathbf{N}$ containing F and $x \in \mathcal{K}_G$, then $x^*\sum_{i=1}^{m} e_{ii}x \to x^*x$ σ-weakly in $\mathcal{K}_G$, and since the restriction $\breve{\omega}\restriction\mathcal{K}_G$ is regular, then $\breve{\omega}\left(x^*\sum_{i=1}^{m} e_{ii}x\right)$ tends to $\breve{\omega}(x^*x)$ as $m \to \infty$. In terms of the cyclic vector Ω this means

$$\left\langle \sum_{i=1}^{m} \pi(e_{ii})\pi(x)\Omega, \pi(x)\Omega \right\rangle \to \langle \pi(x)\Omega, \pi(x)\Omega\rangle \quad \text{as } m \to \infty.$$

Since Ω is cyclic and $\bigcup\{\mathcal{K}_G : G \text{ finite and } F \subseteq G \subset \mathbf{N}\}$ is norm-dense in $\mathcal{C}$, the polarization identity yields weak convergence. ■

Proposition 3.5. *A state ω of $\mathcal{F}_\infty$ is primary if and only if given $\varepsilon > 0$ and $m_0 \in \mathbf{N}$ there exists $m \in \mathbf{N}$ such that for $j \geqslant 0$*

$$|\omega(AB) - \omega(A)\omega(B)| \leqslant \varepsilon\|A\|\,\|B\| \tag{9}$$

whenever $A \in \mathcal{K}_{\{1,\dots,m_0\}}$ and $B \in \mathcal{K}_{\{m,\dots,m+j\}}$.

Proof. By the previous remarks ω is primary if and only if $\breve{\omega}$ is primary. Define a family of von Neumann algebras indexed by the finite subsets of $\mathbf{N}$ by letting

$$\mathcal{M}_F = \pi(\mathcal{K}_F)''. \tag{10}$$

Since $\pi(\mathcal{K}_F)$ is nondegenerate, von Neumann's bicomutant theorem shows that $\mathcal{M}_F$ coincides with the weak-closure of $\pi(\mathcal{K}_F)$, therefore each $\mathcal{M}_F$ is a type I factor containing the identity I of $B(\mathcal{H})$, $\mathcal{M}_F$ and $\mathcal{M}_G$ commute with each other whenever $F \cap G = \emptyset$, and $\mathcal{M}_{F\cup G}$ is generated by $\mathcal{M}_F \cup \mathcal{M}_G$ as a von Neumann algebra.

Therefore the C^*-subalgebra of $B(\mathcal{H})$ generated by the family $\{\mathcal{M}_F\colon F \text{ finite and } F \subset \mathbf{N}\}$, and the state $\breve{\omega}$ are in the conditions of Theorem 2.6.10 in [4]. It is easy to adapt the statement of that theorem to the present situation in which the index set is the collection of finite subsets of $\mathbf{N}$, and conclude that $\breve{\omega}$ is primary if and only if for any $m_0 \in \mathbf{N}$ and any $\varepsilon > 0$ there exists some $m \in \mathbf{N}$ such that for any $j \geqslant 0$, the inequality

$$|\breve{\omega}(AB) - \breve{\omega}(A)\breve{\omega}(B)| \leqslant \varepsilon\|A\|\,\|B\|$$

is satisfied whenever $A \in \mathcal{M}_{\{1,\dots,m_0\}}$, $B \in \mathcal{M}_{\{m,\dots,m+j\}}$. Since the state $\breve{\omega}$ restricted to $\mathcal{M}_F$ is normal, it suffices that the above condition be satisfied for compact operators $A \in \mathcal{K}_{\{1,\dots,m_0\}}$, and $B \in \mathcal{K}_{\{m,\dots,m+j\}}$ for every $j \geqslant 0$, hence the criterion may be stated, as in (9), in terms of ω itself. ∎

In particular, as expected, an essential *product* state ω of $\mathcal{F}_\infty$ is primary, the reason being that $\omega(AB) = \omega(A)\omega(B)$ for $A \in \mathcal{K}_{\{1,\dots,m_0\}}$ and $B \in \mathcal{K}_{\{m,\dots,m+j\}}$ provided that $m > m_0$ and $j \geqslant 0$. Furthermore, one can obtain the following characterization of quasi equivalence of essential primary states of $\mathcal{F}_n$ in terms of α^*-shifted states.

PROPOSITION 3.6. *If ω_1 and ω_2 are two essential factor states of $\mathcal{F}_n$, then*

$$\omega_1 \overset{q}{\sim} \omega_2 \ \text{ if and only if } \ \|\alpha^{*j}(\omega_1 - \omega_2)\| \to 0 \ \text{ as } \ j \to \infty. \tag{11}$$

REMARK 3.7. For finite dimensional $\mathcal{E}$, α^* is the adjoint of α and

$$\|\alpha^{*j}(\omega_1 - \omega_2)\| = \|(\omega_1 - \omega_2)\restriction\alpha^j(\mathcal{F}_n)\|$$

so Proposition 3.6 is a restatement of Theorem 2.7 of [10] (see also [9, Thm. 12.3.2.]). The point here is that it also applies to essential states of $\mathcal{F}_\infty$.

Proof. Denote by α the endomorphism of $\bigotimes_{j=1}^{\infty} \widetilde{K}_j$ induced by the standard right--shift: $T_1 \otimes \cdots \otimes T_j \mapsto I \otimes T_1 \otimes \cdots \otimes T_j$. In general, ω_1 is quasi-equivalent to ω_2 if and only if $\frac{1}{2}(\omega_1 + \omega_2)$ is primary, which happens if and only if $\frac{1}{2}(\breve{\omega}_1 + \breve{\omega}_2)$ is primary. Theorem 2.6.11 of [4] applies here so $\frac{1}{2}(\breve{\omega}_1 + \breve{\omega}_2)$ is primary if and only if

$$\|(\breve{\omega}_1 - \breve{\omega}_2)\restriction\alpha^j(\mathcal{C})\| \to 0 \quad \text{as } j \to \infty.$$

The norm of this restriction is determined by the compacts at the j^{th} level, $\mathcal{D}_j = \mathcal{K}_{\{1,\dots,j\}}$. Since ω is an essential state of $\mathcal{F}_\infty$, $\alpha^*\omega = (\breve{\omega} \circ \alpha)\restriction\mathcal{F}_\infty$, therefore

$$\|(\breve{\omega}_1 - \breve{\omega}_2) \circ \alpha^j\| = \|\alpha^{*j}(\omega_1 - \omega_2)\|$$

which completes the proof. ∎

4. STRONGLY ERGODIC ENDOMORPHISMS

Early in the short history of endomorphisms of type I factors, R. Powers proved that a shift with a pure normal invariant state is conjugate to another shift if and only if the other shift also has a pure normal invariant state [11, Thm. 2.2]. The proof uses a C^*-algebra isomorphic to $\mathcal{B}(\mathcal{E})^{\otimes\infty}$ which is naturally embedded in $\mathcal{B}(\mathcal{H})$ once the shift has been given; here $\mathcal{E}$ is a Hilbert space of dimension equal to the multiplicity of the shift being considered. This C^*-algebra is $\mathcal{F}_n$ if $\dim\mathcal{E} < \infty$, and it contains $\mathcal{F}_\infty$ if $\dim\mathcal{E} = \infty$. In both cases it is weakly dense in $\mathcal{B}(\mathcal{H})$, and the restriction of a vector state of $\mathcal{B}(\mathcal{H})$ to $\mathcal{F}_n$ is an essential pure state. Such considerations suggest that one ought to consider the states of $\mathcal{F}_n$ that arise from vector states of $\mathcal{B}(\mathcal{H})$ via the representation implementing a strongly ergodic endomorphism in order to study conjugacy.

More specifically, suppose α is a strongly ergodic endomorphism of $\mathcal{B}(\mathcal{H})$ and let $\pi : \mathcal{O}_n \to \mathcal{B}(\mathcal{H})$ be a representation implementing α (i.e. $\alpha = \mathrm{Ad}_\pi$). Because of strong ergodicity, the image of $\mathcal{F}_n$ under π is weakly dense in $\mathcal{B}(\mathcal{H})$; so if Ω is a unit vector in $\mathcal{H}$, then $\omega(x) = \langle\pi(x)\Omega, \Omega\rangle$ for $x \in \mathcal{F}_n$ defines a pure state. This state is essential by the assumption that α be unital. Since

$$\alpha^*\omega(x) = \sum_j \omega(v_j x v_j^*) = \sum_j \langle\pi(x)\pi(v_j^*)\Omega, \pi(v_j^*)\Omega\rangle,$$

$\alpha^*\omega$ is a convex linear combination of vector states in an irreducible representation, therefore $\alpha^*\omega$ is quasi-equivalent to ω.

DEFINITION 4.1. An essential state ω of $\mathcal{F}_n$is *quasi-invariant* if it is quasi-equivalent to $\alpha^*\omega$.

Thus, if a representation of $\mathcal{O}_n$ implements a strongly ergodic endomorphism, the restriction to $\mathcal{F}_n$ of a vector state is quasi-invariant.

Suppose now that $\alpha_1 = \mathrm{Ad}_{\pi_1}$ and $\alpha_2 = \mathrm{Ad}_{\pi_2}$ are two strongly ergodic endomorphisms which are conjugate. Then by Proposition 2.4 π_2 is unitarily equivalent to $\pi_1 \circ \gamma_U$ for some unitary operator U on $\mathcal{E}$. Since the restrictions to $\mathcal{F}_n$ of $\pi_1 \circ \gamma_U$ and π_2 are irreducible, whenever ω_1 and ω_2 are vector states for π_1 and π_2, their restrictions to $\mathcal{F}_n$ are quasifree-equivalent. Summarizing, if $\alpha = \mathrm{Ad}_\pi$ is a strongly ergodic endomorphism of $\mathcal{B}(\mathcal{H})$, and if $\Omega \in \mathcal{H}$ is a unit vector, then $\omega(x) = \langle\pi(x)\Omega, \Omega\rangle$ defines an *essential, quasi-invariant, pure* state of $\mathcal{F}_n$. Furthermore, if α_1 and α_2 are conjugate, then ω_1 and ω_2 are quasifree-equivalent.

The state $\alpha^*\omega$ may fail to be pure even when ω is. The next lemma introduces another state related to ω which is as pure as ω and which is useful in giving a characterization of quasi-invariance in terms of unitary equivalence.

LEMMA 4.2. *Suppose ω is a state of $\mathcal{F}_n$ and define $\omega'(x) = \omega(v_1^* x v_1)$ for $x \in \mathcal{F}_n$. Then ω' is pure if and only if ω is pure. Moreover, ω is quasi-invariant if and only if it is unitarily equivalent to ω'.*

Proof. It is clear that ω' is a state on $\mathcal{F}_n$. To prove it is pure, consider $\mathcal{F}_n$ embedded in $(\mathbb{C}I + \mathcal{K}(\mathcal{E}))^{\otimes\infty}$, where $\dim\mathcal{E} = n$. Separating the first factor in the tensor product, the state ω' appears as the tensor product of two pure states. In fact, $\omega' = \omega_{v_1} \otimes \omega$ corresponding to the decomposition $\mathcal{F}_n \cong (\mathbb{C}I+\mathcal{K}(\mathcal{E}))\otimes(\mathbb{C}I+\mathcal{K}(\mathcal{E}))^{\otimes\infty}$, where ω_{v_1} is the vector state of $\overline{\mathcal{K}}(\mathcal{E})$ corresponding to $v_1 \in \mathcal{E}$. Thus ω' is pure if and only if ω is pure. For $n = \infty$ the argument applies to the unique extension $\tilde{\omega}$ of ω obtained in the discussion preceding Lemma 3.3. A simple computation shows that $\alpha^*(\omega') = \omega$, so the second assertion follows from the fact that quasi-equivalence of factor states of $\mathcal{F}_n$ is an asymptotic property, Proposition 3.6. Both ω and ω' being pure, quasi-equivalence can be replaced with unitary equivalence. ■

The following theorem characterizes the situation in which there exists a lifting of an irreducible representation of $\mathcal{F}_n$ to a representation of $\mathcal{O}_n$ on the same Hilbert space. It generalizes Powers' construction of a strongly ergodic endomorphism, showing that the notion of quasi-invariance is enough to characterize strong ergodicity.

THEOREM 4.3. *The GNS representation of $\mathcal{F}_n$ associated with an essential pure state extends to a representation of $\mathcal{O}_n$ on the same Hilbert space if and only if the state is quasi-invariant. This extension is unique up to a gauge automorphism.*

Proof. If the extension exists, the discussion at the beginning of the section shows that the state is quasi-invariant.

Let ω be an essential quasi-invariant pure state of $\mathcal{F}_n$ and let π be the associated GNS representation on the Hilbert space $\mathcal{H}$, with cyclic vector Ω. Suppose π_1 extends π, then

$$\pi_1(v_j) = \pi(v_j v_1^*)\pi_1(v_1) \quad j = 1, 2, \ldots, n,$$

and

$$\pi_1(v_1)\pi(x)\Omega = \pi(v_1 x v_1^*)\pi_1(v_1)\Omega.$$

Since the v_j's generate $\mathcal{O}_n$ and $\pi(\mathcal{F}_n)\Omega$ is dense in $\mathcal{H}$, the extension π_1 is completely determined by $\Omega' = \pi(v_1)\Omega$.

For $x \in \mathcal{F}_n$, $\langle\pi(x)\Omega', \Omega'\rangle = \langle\pi(v_1^* x v_1)\Omega, \Omega\rangle = \omega'(x)$, and thus Ω' is determined uniquely up to a scalar multiple of modulus one because π is irreducible. If π_2 is another extension, then $\pi_2(v_1)\Omega = \lambda\Omega'$ for some $\lambda \in \mathbf{T}$ and this implies $\pi_1 = \pi_2 \circ \gamma_\lambda$.

To prove existence, suppose ω is quasi-invariant, by the preceding lemma ω is unitarily equivalent to ω', so there is a unit vector $\Omega' \in \mathcal{H}$ such that $\omega'(x) = \omega(v_1^* x v_1) = \langle\pi(x)\Omega', \Omega'\rangle$ for all x in $\mathcal{F}_n$. Define V_1 on vectors of the form $\pi(x)\Omega$ by

$$V_1 : \pi(x)\Omega \mapsto \pi(v_1 x v_1^*)\Omega' \quad x \in \mathcal{F}_n, \tag{12}$$

then observe that

$$
\begin{aligned}
&\langle V_1\pi(x)\Omega, V_1\pi(y)\Omega\rangle = \langle \pi(v_1 x v_1^*)\Omega', \pi(v_1 y v_1^*)\Omega'\rangle = \\
(13)\qquad &= \langle \pi(v_1 y^* v_1^* v_1 x v_1^*)\Omega', \Omega'\rangle = \omega'(v_1 y^* x v_1^*) = \\
&= \omega(y^* x) = \langle \pi(x)\Omega, \pi(y)\Omega\rangle \quad x, y \in \mathcal{F}_n.
\end{aligned}
$$

Vectors of the form $\pi(x)\Omega$ are dense in $\mathcal{H}$, so the map V_1 defined above extends to an isometry on $\mathcal{H}$, which will also be called V_1.

From setting $x = y = I$ above, it follows that $V_1\Omega = \pi(v_1 v_1^*)\Omega'$ is a unit vector. Since $\pi(v_1 v_1^*)$ is a projection and Ω' has modulus one, it follows that $V_1\Omega = = \pi(v_1 v_1^*)\Omega' = \Omega'$. Fix now $x \in \mathcal{F}_n$, then $V_1\pi(x)\pi(y)\Omega = V_1\pi(xy)\Omega = \pi(v_1 x y v_1^*)\Omega' = = \pi(v_1 x v_1^*)\pi(v_1 y v_1^*)\Omega' = \pi(v_1 x v_1^*)V_1\pi(y)\Omega$ for all y in $\mathcal{F}_n$, so

$$
(14)\qquad V_1\pi(x) = \pi(v_1 x v_1^*)V_1 \quad x \in \mathcal{F}_n.
$$

From this, it is clear that $V_1 V_1^* \leqslant \pi(v_1 v_1^*)$. To prove the reverse inequality, note that

$$
\begin{aligned}
&\pi(v_1 v_1^*)\pi(x)\Omega' = \pi(v_1 v_1^*)\pi(x)\pi(v_1 v_1^*)V_1\Omega = \\
&= \pi(v_1(v_1^* x v_1)v_1^*)V_1\Omega = V_1\pi(v_1^* x v_1)\Omega \quad x \in \mathcal{F}_n.
\end{aligned}
$$

Since Ω' is cyclic, this yields $\pi(v_1 v_1^*) \leqslant V_1 V_1^*$, and it follows that $V_1 V_1^* = \pi(v_1 v_1^*)$.

Once V_1 has been chosen there is only one possible choice for the remaining isometries, i.e.:

$$
(15)\qquad V_j = \pi(v_j v_1^*)V_1 \quad j = 1, 2, \ldots, n.
$$

Computing

$$
\begin{aligned}
V_j^* V_j &= V_1^*\pi(v_1 v_j^*)\pi(v_j v_1^*)V_1 = V_1^*\pi(v_1 v_j^* v_j v_1^*)V_1 = \\
&= V_1^*\pi(v_1 v_1^*)V_1 = V_1^*(V_1 V_1^*)V_1 = I
\end{aligned}
$$

shows that V_j is an isometry for each $j = 1, 2, \ldots, n$. In addition they form an essential Cuntz system because

$$
\begin{aligned}
&\sum_j V_j V_j^* = \sum_j \pi(v_j v_1^*)V_1 V_1^* \pi(v_1 v_j^*) = \\
(16)\qquad &= \sum_j \pi(v_j v_1^* v_1 v_1^* \pi v_1 v_j^*) = \sum_j \pi(v_j v_j^*) = I,
\end{aligned}
$$

by the assumption that ω, and hence π, is essential. Since $\{V_j\}_{j=1}^n$ is an essential Cuntz system of isometries on $\mathcal{H}$, there exists an essential representation $\tilde{\pi}$ of $\mathcal{O}_n$ on $\mathcal{H}$ such that $\tilde{\pi}(v_j) = V_j$ for $j = 1, 2, \ldots, n$.

It still remains to show that $\tilde{\pi}\lceil\mathcal{F}_n = \pi$, for which it suffices to prove that the two representations coincide on elements of the form rs^*, where both r and s are in $\mathcal{W}_k$ for $k = 0, 1, \ldots, n$. This is enough because such elements generate $\mathcal{F}_n$. The proof is by induction on the length of the words.

For $k = 0$, it is obvious that $\tilde{\pi}(rs^*) = I = \pi(rs^*)$ whenever $r, s \in \mathcal{W}_k$. Suppose equality holds for words up to length k, and let $r, s \in \mathcal{W}_k$ so that $v_i r$ and $v_j s$ are in $\mathcal{W}_{k+1}$. Then

$$\begin{aligned}\tilde{\pi}(v_i rs^* v_j^*) &= \tilde{\pi}(v_i)\tilde{\pi}(rs^*)\tilde{\pi}(v_j^*) = \pi(v_i v_1^*)V_1\pi(rs^*)V_1^*\pi(v_1 v_j^*) = \\ &= \pi(v_i v_1^*)\pi(v_1 rs^* v_1^*)V_1V_1^*\pi(v_1 v_j^*) = \\ &= \pi(v_i v_1^* v_1 rs^* v_1^*)\pi(v_1 v_1^*)\pi(v_1 v_j^*) = \pi(v_i rs^* v_j^*).\end{aligned} \tag{17}$$

So $\tilde{\pi}(x) = \pi(x)$ whenever $x \in \mathcal{F}_n$. ∎

COROLLARY 4.4. *If π_1 and π_2 are representations of $\mathcal{O}_n$ such that their restrictions to $\mathcal{F}_n$ are irreducible and quasifree-equivalent, then π_1 and π_2 are quasifree--equivalent themselves.*

Proof. There is a unitary operator W from $\mathcal{H}_1$ to $\mathcal{H}_2$ and a quasifree automorphism γ_U of $\mathcal{O}_n$ such that the representations π_1 and $\mathrm{Ad}_W \circ \pi_2 \circ \gamma_U$ coincide on $\mathcal{F}_n$. Thus both are extensions to $\mathcal{O}_n$ of the same irreducible representation of $\mathcal{F}_n$. By the uniqueness part of Theorem 4.3 this implies that they differ by a gauge automorphism γ_λ. That is, $\pi_1 = \mathrm{Ad}_W \circ \pi_2 \circ \gamma_{\lambda U}$ on $\mathcal{O}_n$, so π_1 is quasifree equivalent to π_2. ∎

THEOREM 4.5. *The extension procedure of Theorem 4.3 establishes a bijection between the conjugacy classes of strongly ergodic endomorphisms of index n and the quasifree-equivalence classes of essential quasi-invariant pure states of $\mathcal{F}_n$.*

Given an essential quasi-invariant pure state of $\mathcal{F}_n$, Theorem 4.3 yields a representation of $\mathcal{O}_n$. This representation gives rise to an endomorphism which is strongly ergodic because its restriction to $\mathcal{F}_n$ is irreducible. Thus the given state appears as the restriction of a vector state to $\mathcal{F}_n$. Corollary 4.4 then ensures that quasifree equivalent states give rise to conjugate endomorphisms. To obtain the inverse map, given a strongly ergodic endomorphism, take the restriction to $\mathcal{F}_n$ of any vector state in any representation of $\mathcal{O}_n$ implementing the endomorphism. The discussion preceding Theorem 4.3 shows that conjugate endomorphisms give rise to quasifree-equivalent states by this process. ∎

It is interesting to note that Theorem 4.3 yields a pure extension of a quasi-invariant pure state ω from $\mathcal{F}_n$ to $\mathcal{O}_n$ via $\tilde{\omega}(x) = \langle\tilde{\pi}(x)\Omega, \Omega\rangle$, for $x \in \mathcal{O}_n$. The next proposition shows that this extension is very far from being gauge-invariant.

PROPOSITION 4.6. *If ω and $\tilde{\omega}$ are as in Theorem 4.3 then $\tilde{\omega} \circ \gamma_\lambda$ is a pure extension of ω for each $\lambda \in \mathbf{T}$; furthermore,*

$$\tilde{\omega} \circ \gamma_\lambda \overset{u}{\sim} \tilde{\omega} \circ \gamma_\mu \quad \text{if and only if} \quad \lambda = \mu.$$

Proof. Since γ_λ is an automorphism of $\mathcal{O}_n$ leaving $\mathcal{F}_n$ pointwise fixed, it is plain that each $\tilde{\omega} \circ \gamma_\lambda$ is a pure extension of ω. The last assertion of the proposition reduces to proving that $\tilde{\omega} \circ \gamma_\lambda \overset{u}{\sim} \tilde{\omega}$ if and only if $\lambda = 1$.

Assume $\tilde{\omega} \circ \gamma_\lambda \overset{u}{\sim} \tilde{\omega}$. Then there must be a unit vector $\xi \in \mathcal{H}$ such that

$$\langle \tilde{\pi}(x)\xi, \xi \rangle = \tilde{\omega} \circ \gamma_\lambda(x) = \langle \tilde{\pi}(\gamma_\lambda(x))\Omega, \Omega \rangle \quad x \in \mathcal{O}_n.$$

Since $\gamma_\lambda(x) = x$ when $x \in \mathcal{F}_n$, it follows that the vector states obtained from ξ and Ω coincide, because they are equal on the weakly dense subalgebra $\tilde{\pi}(\mathcal{F}_n)$ of $\mathcal{B}(\mathcal{H})$. Hence the state $\tilde{\omega}$ is actually equal to the state $\tilde{\omega} \circ \gamma_\lambda$.

Choose $x \in \mathcal{F}_n$ such that $\langle \tilde{\pi}(v_1)\Omega, \tilde{\pi}(x)\Omega \rangle = \tilde{\omega}(x^* v_1) \neq 0$; such an x exists because $\tilde{\pi}(\mathcal{F}_n)\Omega$ is dense in $\mathcal{H}$. Since

$$\gamma_\lambda(x^* v_1) = \gamma_\lambda(x^*)\gamma_\lambda(v_1) = x^* \lambda v_1 = \lambda x^* v_1,$$

it follows that $\tilde{\omega} \circ \gamma_\lambda(x^* v_1) = \lambda \tilde{\omega}(x^* v_1)$, which equals $\tilde{\omega}(x^* v_1)$ only if $\lambda = 1$, because $\omega(x^* v_1) \neq 0$. ∎

This method of extending quasi-invariant pure states of $\mathcal{F}_n$ to pure states of $\mathcal{O}_n$ can be generalized to periodic states, that is, to states ω which are quasi equivalent to $\alpha^{*p}\omega$ for some $p \geqslant 1$. However, all the extensions of a given state obtained in this fashion give rise to conjugate endomorphisms.

EXAMPLES OF STRONGLY ERGODIC ENDOMORPHISMS. As a rule, product states are the main source of eamples, so we now specialize the discussion to such states. Let $\mathcal{E}$ be a separable Hilbert space and view $\mathcal{F}_n$ as embedded in $(\mathbf{C}I + \mathcal{K}(\mathcal{E}))^{\otimes\infty}$. Consider states of the form $\omega = \bigotimes_{i=1}^{\infty} \omega_{f_i}$,where $\{f_i\}_{i=1}^{\infty}$ is a sequence of unit vectors in $\mathcal{E}$, and $\{\omega_{f_i}\}$ is the corresponding sequence of vector states of $\mathcal{K}(\mathcal{E})$.

If U is a unitary operator on $\mathcal{E}$, the quasifree automorphism γ_U transforms a pure product state according to

$$\left(\bigotimes_{i=1}^{\infty} \omega_{f_i}\right) \circ \gamma_U = \bigotimes_{i=1}^{\infty} \omega_{U f_i}.$$

It follows from Proposition 3.6 that the necessary and sufficient condition for unitary equivalence between $\bigotimes_{i=1}^{\infty} \omega_{f_i}$ and $\bigotimes_{i=1}^{\infty} \omega_{g_i}$ is that the infinite series $\sum_i (1 - |\langle f_i, g_i \rangle|)$ be convergent. Thus the states $\bigotimes_{i=1}^{\infty} \omega_{f_i}$ and $\bigotimes_{i=1}^{\infty} \omega_{g_i}$ are quasifree-equivalent if and only if there is a unitary operator U on $\mathcal{E}$ such that

$$\sum_i (1 - |\langle f_i, Ug_i\rangle|) < \infty. \tag{18}$$

Furthermore, α^* acts by shifting the factors in the tensor product one place to the left so $\bigotimes_{i=1}^{\infty} \omega_{f_i}$ is quasi-invariant if and only if

$$\sum_i (1 - |\langle f_i, f_{i-1}\rangle|) < \infty.$$

In order to characterize quasi invariance, denote by θ_i the angle between the subspaces $\mathbb{C}f_i$ and $\mathbb{C}f_{i-1}$. Thus $|\langle f_i, f_{i-1}\rangle| = \cos\theta_i$, and the state determined by the sequence $\{f_i\}$ is quasi-invariant if and only if the series $\sum_i \theta_i^2$ converges.

1. Assume $f_i = f \in \mathcal{E}$ for all $i \geqslant 1$, then $\bigotimes_{i=1}^{\infty} \omega_{f_i}$ is an *invariant* pure product state. All pure invariant states are of this form, [11, Theorem 2.4]. If $g_i = g \in \mathcal{E}$ for all $i \geqslant 1$, then $\bigotimes_{i=1}^{\infty} \omega_{g_i}$ is another such state, and there clearly exists a unitary U on $\mathcal{E}$ which sends g to f, so every term of the series (18) vanishes. Thus, all shifts having pure normal invariant states are in the same conjugacy class, which is a restatement of Theorem 2.3 of [11].

2. Let v_1 and v_2 be two orthogonal unit vectors in $\mathcal{E}$, fix $\delta \in [0, \pi/2]$ and consider a sequence $\{\theta_i\}$ in $[-\pi, \pi]$ satisfying $\sum_i \theta_i^2 < \infty$, and such that the set of limit points of $s_n = \sum_{i=1}^{n} \theta_i$ is the interval $[0, \delta]$. Let $f_i = \cos(s_i)v - \sin(s_i)v_2$. Since $\sum_i \theta_i^2 < \infty$, the state determined by the f_i's is quasi-invariant. Moreover if $\delta \neq \delta'$, the state corresponding to δ cannot be quasi-free equivalent to the one corresponding to δ'. The reason is that the two sets of limit points are not congruent via a unitary transformation of $\mathcal{E}$, therefore the distance between corresponding terms of the sequences f_i and Uf_i' does not tend to zero, causing the series $\sum_i (1 - |\langle f_i, Uf_i'\rangle|)$ to diverge. Thus the endomorphisms obtained from different values of δ are not conjugate to each other and we have uncountably many conjugacy classes of strongly ergodic endomorphisms.

3. Let $s \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and let $f_n = \cos(n^{-s})v_1 + \sin(n^{-s})v_2$. Since

$$\theta_{n+1} = n^{-s} - (n+1)^{-s} = n^{-s}[1 - (1 + 1/n)^{-s}] = O(n^{-(s+1)}),$$

$\sum_n \theta_n^2 < \infty$ hence the corresponding state is quasi-invariant. Assume $s' \neq s$. It is clear that if the unitary operator U does not fix the unique limit point of the sequence, then the series $\sum_i (1 - |\langle f_n, U f_n' \rangle|)$ hopelessly diverges. So suppose U fixes the one--dimensional subspace corresponding to $v_1 = \lim f_n = \lim f_n'$. For such U the angle between the subspaces $\mathbf{C} f_n$ and $\mathbf{C} U f_n'$ is at least $n^{-s} - n^{-s'}$. Since $\sum_n (n^{-s} - n^{-s'})^2$ diverges, different values of s give nonconjugate shifts.

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