J. OPERATOR THEORY
30(1993), 137–147

SOME COMPARABILITY RESULTS IN INDUCTIVE LIMIT C^*-ALGEBRAS

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1. INTRODUCTION

An important problem concerning simple C^*-algebras is to state and to study the counterpart of the classical Murray-von Neumann comparison theory for factors. Hopeful candidates for this seem to be the following comparability questions: given a simple C^*-algebra A and two projections p and q in A, $q \neq 0$, such that $\tau(p) \leqslant \tau(q)$ (resp. $\tau(p) < \tau(q)$) for any trace τ on A, is p subordinate (resp. strictly subordinate) to q in the sense of Murray and von Neumann? Both these problems and some other related ones are posed and deeply studied by B. Blackadar in [4]. Note however that the second comparability question appears to be more reasonable.

Answers for such questions have been obtained for unital (simple) AF-algebras by B. Blackadar [2], for the irrational rotation algebras by M. Rieffel [14], [15], and partial results for the Choi algebras by J. Anderson, B. Blackadar and U. Haagerup [1]. Recently, M.Dădârlat and A. Némethi [10] have got results on such problems for certain simple C^*-inductive limits of finite direct sums of matrix algebras over finite CW-complexes.

A more general comparison theory for positive elements in a C^*-algebra was developed by J. Cuntz [7], J. Cuntz and G. Pedersen [8], B. Blackadar and D. Handelman [6], M. Rørdam [16].

In this paper we consider mainly C^*-algebras A, not necessarily simple, that are inductive limits of finite direct sums of C^*-algebras of the form $C(X, M_n)$, where X is a compact space, and the connecting homomorphisms are injective and unital. Our main result (Theorem 3.7) asserts in particular that if the algebra A is simple and has slow dimension growth in the sense of B. Blackadar, M. Dădârlat and M. Rørdam [5], then A satisfies the second comparability question and has cancellation. We introduce a notation of slow dimension growth for not necessarily simple C^*-algebras which coincides with that given in [5] for simple C^*-algebras. In fact we prove in Theorem 3.7 that both the second comparability question and the cancellation are satisfied by a langer class of C^*-algebras called with relatively large entries (see Definition 3.6). Theorem 3.7 gives a particular affirmative answer to a conjecture of B. Blackadar in [4] (see also 3.1).

We also give a procedure to yield C^*-inductive limits which satisfy the above comparability questions (Proposition 3.13). This result shows in particular that the comparability questions are shape invariant.

It should be emphasized that an important tool in this paper is the use of what we call commutative up to an equivalence on projections diagrams.

2. PRELIMINARIES

In this article we will use only *unital* C^*-algebras. Following [4] we recall first two definitions.

DEFINITION 2.1. Let p and q projections in a C^*-algebra A. Then:

(i) p is *equivalent* to q, written $p \sim q$, if there is u in A such that $u^*u = p$, $uu^* = q$;

(ii) p is *subordinate* to q, written $p \preceq q$, if p is equivalent to a subprojection of q;

(iii) p is *strictly subordinate* to q, written $p \prec q$, if p is equivalent to a proper subprojection of q.

DEFINITION 2.2. A C^*-algebra A satisfies FCQ1 (resp. FCQ2) if whenever p and q are projections in A with $q \neq 0$ and $\tau(p) \leqslant \tau(q)$ (resp. $\tau(p) < \tau(q)$) for any trace τ on A, then $p \preceq q$ (resp. $p \prec q$).

We have to mention that by a trace on A we mean a tracial state on A. If A has no traces then FCQ1 (resp. FCQ2) simply says that $p \preceq q$ (resp. $p \prec q$) for any projections p and q, $q \neq 0$. Note that any unital AF-algebras satisfies FCQ2 but there is a simple AF-algebra which does not satisfy FCQ1 [2], [3, 7.6.2].

For both the fundamental comparability questions FCQ1 and FCQ2 the next notion is convenient.

DEFINITION 2.3. Two homomorphisms $\Phi, \Phi' : A \to B$ of C^*-algebras will be called *equivalent on projections*, if $\Phi(p) \sim \Phi'(p)$ for any projection p in A. A diagram of C^*-algebras and homomorphisms is called *commutative up to an equivalence on projections*, in short EP-*commutative*, if any paths in the diagram starting and ending at the same places provide equivalent on projections homomorphisms.

The connection with FCQ1 or FCQ2 can be explained by a simple remark. Given an EP-commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ & \alpha \searrow \quad \nearrow \beta & \\ & B & \end{array}$$

where A, B are unital C^*-algebras and α, β are unital and injective homomorphisms we have that if B satisfies FCQ1 (resp. FCQ2) then A satisfies FCQ1 (resp. FCQ2).

Note also that any diagram commutative within homotopy is commutative up to an equivalence on projections.

Next we shall describe an important, and relevant for our purposes, example of an EP-commutative diagram (see [10]). Let us fix two C^*-algebras $A = \bigoplus_{j=1}^{r} C(X_j, M_{n_j})$, $B = \bigoplus_{i=1}^{s} C(Y_i, M_{m_i})$ with all the base spaces X_j and Y_i compact and connected, and a homomorphism $\Phi : A \to B$. Let $\Phi^{(i)}$ be the component of Φ from A into $C(Y_i, M_{m_i})$. Since any finite dimensional $*$-representation of A is a direct sum of some irreducible $*$-represention*s* and of a zero $*$-representation and since each Y_i is connected, there are uniquely determined nonnegative integers $k_{ij} (1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant r)$ with the following properties:

$$h_i = m_i - \sum_{j=1}^{r} k_{ij} n_i \geqslant 0 \quad (1 \leqslant i \leqslant s); \tag{2.4}$$

for any $1 \leqslant i \leqslant s$ and any point y in Y_i there exist the points $x_{j,l}^{(i)}$ in X_j $(1 \leqslant j \leqslant\leqslant r,\ 1 \leqslant l \leqslant k_{ij})$ and a unitary u_i in M_{m_i}, which depend on y, such that:

$$\Phi^{(i)} \left(\bigoplus_{j=1}^{r} f_j \right)(y) = u_i \left(\bigoplus_{j=1}^{r} \left(\bigoplus_{l=1}^{k_{ij}} f_j(x_{j,l}^{(i)}) \right) \bigoplus O_{h_i} \right) u_i^* \tag{2.5}$$

for every f_i in $C(X_j, M_{n_j}) (1 \leqslant j \leqslant r)$.

Consider the finite dimensional C^*-algebras $F(A) = \bigoplus_{j=1}^{r} M_{n_j}$, $F(B) = \bigoplus_{i=1}^{s} M_{m_i}$. The homomorphism Φ induces a homomorphism $F(\Phi) : F(A) \to F(B)$ given by:

$$F(\Phi) \left(\bigoplus_{j=1}^{r} a_j \right) = \bigoplus_{i=1}^{s} \left(\bigoplus_{j=1}^{r} k_{ij} a_j \bigoplus O_{h_i} \right), \quad a_j \in M_{n_j}, \tag{2.6}$$

where, if $k \in \mathbb{Z}_+$ and $a \in M_n$ we put simply $ka = a \oplus a \oplus \cdots \oplus a \in M_{kn}$. In short we say that $F(\Phi)$ is induced by the matrix of nonnegative integers $[k_{ij}]_{i,j}$. Note that if Φ is injective (resp. unital) then $F(\Phi)$ is injective (resp. unital).

Finally, choose arbitrary elements $x_j \in X_j$ $(1 \leqslant j \leqslant r)$, $y_i \in Y_i$ $(1 \leqslant i \leqslant s)$ and let $\varepsilon_1 : A \to F(A)$, $\varepsilon_2 : B \to F(B)$ be the corresponding evaluation maps given by:

$$\varepsilon_1\left(\bigoplus_{j=1}^{r} f_j\right) = \bigoplus_{j=1}^{r} f_j(x_j), \quad f_j \in C(X_j, M_{n_j}),$$

$$\varepsilon_2\left(\bigoplus_{i=1}^{s} g_i\right) = \bigoplus_{i=1}^{s} g_i(y_i), \quad g_i \in C(Y_i, M_{m_i}).$$

Using the above description of Φ and $F(\Phi)$ it follows easily that the diagram:

(2.7)
$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ {\scriptstyle \varepsilon_1}\downarrow & & \downarrow{\scriptstyle \varepsilon_2} \\ F(A) & \xrightarrow[F(\Phi)]{} & F(B) \end{array}$$

is commutative up to an equivalence on projections. Indeed, for any projection p in a C^*-algebra of the form $C(X, M_n)$ with X compact and connected, let us define:

$$\dim p = \dim p(x), \quad x \in X.$$

More generally, for $p = \bigoplus_{j=1}^{r} p_j$ a projection in A we denote

$$\dim p = (\dim p_1, \dim p_2, \ldots, \dim p_r) \in \mathbb{Z}_+^r.$$

By extending these definitions we have $\dim p = \dim \varepsilon_1(p)$ and the EP-commutativity of diagram (2.7) just means that:

(2.8)
$$\dim \Phi(p) = \dim F(\Phi)\varepsilon_1(p)$$

for any projection p in A. In an explicit form, relation (2.8) means:

(2.9)
$$\dim \Phi^{(i)}(p_j) = k_{ij} \dim p_j, \quad 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant r,$$

for all projections $\bigoplus_{j=1}^{r} p_j$ in A.

Note that $F(\Phi\Psi) = F(\Phi)F(\Psi)$.

We end this section with a convention of notation. Whenever $(A_n, \Phi_{m,n})$ is an inductive system of C^*-algebras with connecting homomorphisms $\Phi_{m,n} : A_n \to A_m$ $(1 \leqslant$

$\leqslant n \leqslant m$) we denote $\Phi_n = \Phi_{n+1,n}$ ($n \geqslant 1$). Throughout this paper we shall use only inductive systems with *unital injective connecting homomorphisms.*

3. RESULTS

In [4] B. Blackadar states the following conjecture (see [4, Conjecture 5.2.2]):

CONJECTURE 3.1. *If A is a simple C^*-algebra which is an inductive limit $\varinjlim C(X_n, F_n)$ where the X_n are compact and F_n finite dimensional, then A satisfies* FCQ2 *and has cancellation.*

In what follows we prove in particular that the conjecture is true under some additional assumptions.

Before stating the results in detail we need a few preliminaries and notation. Throughout the first part of this section let $A = \varinjlim(A_n, \Phi_{m,n})$ be the C^*-algebra inductive limit of C^*-algebras A_n of the form:

$$A_n = \bigoplus_{j=1}^{r_n} C(X_{n,j}, M_{[n,j]})$$

with $X_{n,j}$ compact spaces, $[n, j]$ positive integers, and with a system of connecting unital injective homomorphisms $\Phi_{m,n} : A_n \to A_m$ ($1 \leqslant n \leqslant m$). For any $1 \leqslant i \leqslant r_m$ let $\Phi^{(i)}_{m,n}$ denote the component of $\Phi_{m,n}$ from A_n to $C(X_{m,i}, M_{[m,i]})$. Since all $\Phi_{m,n}$ are injective we identity each A_n with a C^*-algebra of A.

The following lemma essentially uses an argument of B. Blackadar (cf. [2, Proof of Proposition 4.1]).

LEMMA 3.2. *Let p and q be projections in some A_n with $\tau(p) < \tau(q)$ for any trace τ on A. Then there is $m \geqslant n$ such that:*

$$\dim \Phi^{(i)}_{m,n}(p)(x) < \dim \Phi^{(i)}_{m,n}(q)(x), \quad x \in X_{m,i},\ 1 \leqslant i \leqslant r_m.$$

Proof. As in [2] we have that under our assumptions there is $m \geqslant n$ such that $\sigma(\Phi_{m,n}(p)) < \sigma(\Phi_{m,n}(q))$ for any trace σ on A_m. The conclusion follows easily if one takes $\sigma = \mathrm{tr} \circ \varepsilon^{(i)}_m$ where $\varepsilon^{(i)}_m$ is the component of ε_m in $M_{[m,i]}$ and tr is the usual trace on $M_{[m,i]}$. ■

As a first simple consequence of this lemma we have:

PROPOSITION 3.3. *If a C^*-algebra A is an inductive limit as mentioned above such that all vector bundles over any base space $X_{n,j}$ are trivial, then A satisfies* FCQ2 *and has cancellation.*

We omit the obvious proof. Note however that the conclusion remains true even in the case when the connecting homomorphisms are not injective. In particular, any C^*-inductive limit of finite direct sums of interval and circle algebras satisfies FCQ2 and has cancellation.

The next lemma follows from well known results for complex vector bundles over compact spaces (see [13]).

LEMMA 3.4. *Let X be a compact space of dimension d and let p, q, r be projections in $C(X, M_n)$.*

(i) *If $\dim q(x) - \dim p(x) \geqslant \max\left\{\frac{1}{2}d, 1\right\}$, for all $x \in X$, then $p \prec q$;*

(ii) *If $pr = 0 = qr$, $p + r \sim q + r$ and $\dim p(x) \geqslant \frac{1}{2}d$, for all $x \in X$, then $p \sim q$.*

Combining Lemma 3.2 and Lemma 3.4 we obtain the next result.

PROPOSITION 3.5. *Let A be an inductive limit as mentioned in the introduction of this section, where each space $X_{n,j}$ has finite dimension. If p and q are projections in A with $\tau(p) < \tau(q)$ for all traces τ on A then there is a positive integer k_0 such that $kp \prec kq$ in $A \oplus A \oplus \cdots \oplus A$ (k times) for any integer $k \geqslant k_0$.*

Proof. Because each projection in A is unitarily equivalent to a projection in some A_n we may suppose, and we shall, that $p, q \in A_1$. By Lemma 3.2 there is $n \geqslant 1$ such that:

$$\dim \Phi_{n,1}^{(j)}(p)(x) < \dim \Phi_{n,1}^{(j)}(q)(x), \quad x \in X_{n,j},\ 1 \leqslant j \leqslant r_n.$$

Let $k_0 = \max_j\{\dim X_{n,j}\}$, denote by X_j the disjoint union of k copies of $X_{n,j}$ and let $p_j = k\Phi_{n,1}^{(j)}(p)$, $q_j = k\Phi_{n,1}^{(j)}(q)$ where $k \geqslant k_0$ is some arbitrary fixed integer. By Lemma 3.4 we get that $p_j \prec q_j$ in $C(X_j, M_{[n,j]})$ for any $1 \leqslant j \leqslant r_n$, hence $k\Phi_{n,1}(p) \prec k\Phi_{n,1}(q)$ in $A_n \oplus A_n \oplus \cdots \oplus A_n$ (k times). The proof is complete. ∎

Note that if in the above proposition A is in addition *strictly unperforated* [4], then A satisfies FCQ2. The conclusion of Proposition 3.5 holds even in the case when $\Phi_{m,n}$ are not injective.

In order to state the main result of the paper we need some notation and a definition.

Fix an inductive system $(A_n, \Phi_{m,n})$ as mentioned in the introduction of this section, with the additional assumption that all spaces $X_{n,j}$ $(1 \leqslant j \leqslant r_n)$ are *connected*. For any $1 \leqslant n \leqslant m$ let us denote by $[k_{ij}^{(m,n)}]$, $1 \leqslant i \leqslant r_m$, $1 \leqslant j \leqslant r_n$, the matrix which induces the homomorphism $F(\Phi_{m,n}) : F(A_n) \to F(A_m)$ as in (2.6). Since $\Phi_{m,n}$ is unital, each row of the matrix $[k_{ij}^{(m,n)}]$ contains at least one nonzero entry.

The unit of $A_n = \bigoplus_{j=1}^{r_n} C(X_{n,j}, M_{[n,j]})$ has the decomposition $\bigoplus_{j=1}^{r_n} e_j^{(n)}$.

DEFINITION 3.6. Let $(A_n, \Phi_{m,n})$ be an inductive system as above.

(i) The inductive system has *slow dimension growth* if for any $n \geqslant 1$ we have:

$$\liminf_{n\leqslant m} \max_i \left\{ \frac{\dim X_{m,i}}{\min\{\dim \Phi_{m,n}^{(i)}(e_j^{(n)}) : 1 \leqslant j \leqslant r_n \text{ and } \dim \Phi_{m,n}^{(i)}(e_j^{(n)}) \neq 0\}} \right\} = 0.$$

(ii) The inductive system has *relatively large entries* if for any $n \geqslant 1$ there exists $m \geqslant n$ such that:

$$\min\{k_{ij}^{(m,n)} : 1 \leqslant j \leqslant r_n \text{ and } k_{ij}^{(m,n)} \neq 0\} \geqslant \frac{1}{2} \dim X_{m,i}$$

for all $1 \leqslant i \leqslant r_m$.

A C^*-algebra A is said to be with slow dimension growth (resp. relatively large entries) if there exists an inductive system $(A_n, \Phi_{m,n})$ which has slow dimension growth (resp. relatively large entries) such that $A = \varinjlim(A_n, \Phi_{m,n})$.

Obviously, slow dimension grouth implies relatively large entries.

Note that when the C^*-algebra A is simple, the above notion of slow dimension growth agrees with that introduced in [5].

In [5] it is proved that if a simple unital C^*-algebra A has slow dimension growth then the stable rank of A is one, that is, the invertible elements in A are norm dense in A. The first instance of this result was obtained in [11], [12] under the nonessential assumption that the dimensions of the base spaces $X_{n,j}$ are bounded.

THEOREM 3.7. *If a C^*-algebra A has relatively large entries, then A satisfies* FCQ2 *and has cancellation.*

Proof. Fix an inductive system $(A_n, \Phi_{m,n})$ with relatively large entries such that $A = \varinjlim(A_n, \Phi_{m,n})$.

Let us consider two fixed projections p and q in A with $\tau(p) < \tau(q)$ for any trace τ on A. We may suppose that $p, q \in A_1$. By Lemma 3.2 there is $n \geqslant 1$ such that $\dim \Phi_{n,1}^{(j)}(p) < \dim \Phi_{n,1}^{(j)}(q)$ for all $1 \leqslant j \leqslant r_n$. We fix n and denote $\delta_j = \dim \Phi_{n,1}^{(j)}(q) - \dim \Phi_{n,1}^{(j)}(p)$, $1 \leqslant j \leqslant r_n$. Since $\delta_j \geqslant 1$, the relation (2.9) yields:

$$\dim \Phi_{m,1}^{(i)}(q) - \dim \Phi_{m,1}^{(i)}(p) = \sum_{j=1}^{r_n} k_{ij}^{(m,n)} \delta_j \geqslant \sum_{j=1}^{r_n} k_{ij}^{(m,n)} \tag{3.8}$$

for all $m \geqslant n$ and $1 \leqslant i \leqslant r_m$.

But A has relatively large entries, hence there is $m \geqslant n$ such that $\min\{k_{ij}^{(m,n)} : 1 \leqslant j \leqslant r_n \text{ and } k_{ij}^{(m,n)} \neq 0\} \geqslant \frac{1}{2}\dim X_{m,i}$ for all $1 \leqslant i \leqslant r_m$. It follows that:

$$\sum_{j=1}^{r_n} k_{ij}^{(m,n)} \geqslant \frac{1}{2}\dim X_{m,i} \tag{3.9}$$

for all $1 \leqslant i \leqslant r_m$.

Combining (3.8) and (3.9) we obtain

$$\dim \Phi_{m,1}^{(i)}(q) - \dim \Phi_{m,1}^{(i)}(p) \geqslant \max\left\{\frac{1}{2}\dim X_{m,i}, 1\right\}.$$

Lemma 3.4 now implies $\Phi_{m,1}^{(i)}(p) \prec \dim \Phi_{m,1}^{(i)}(q)$ in $C(X_{m,i}, M_{[m,i]})$ for any $1 \leqslant i \leqslant r_m$, hence $p \prec q$ in A_m. In conclusion, A satisfies FCQ2.

Now we want to prove that A has cancellation. It is enough to consider three projections p, q, r in A_1 such that $pr = 0 = qr$ and $p + r \sim q + r$ in A_1 and to show that $p \sim q$ in some A_m. Moreover, we may assume, and we shall, that p, q, r belong to the first term $C(X_{1,1}, M_{[1,1]})$ of the direct sum $A_1 = \bigoplus_{j=1}^{r_1} C(X_{1,j}, M_{[1,j]})$.

Clearly $\dim p = \dim q$ so that $p = 0$ or $q = 0$ imply $p \sim q$. Assume in what follows that $p \neq 0 \neq q$ and fix $m \geqslant 1$ satisfying for any $1 \leqslant j \leqslant r_1$, $1 \leqslant i \leqslant r_m$:

$$\text{either } k_{ij}^{(m,1)} = 0 \text{ or } k_{ij}^{(m,1)} \geqslant \frac{1}{2}\dim X_{m,i}. \tag{3.10}$$

Let $\Phi_{m,1}(p) = \bigoplus_{i=1}^{r_m} p_i$ and $\Phi_{m,1}(q) = \bigoplus_{i=1}^{r_m} q_i$ be the decomposition of p and q in $A_m = \bigoplus_{i=1}^{r_m} C(X_{m,i}, M_{[m,i]})$. By (2.9) one has:

$$\dim p_i = k_{i1}^{(m,1)} \dim p, \ \dim q_i = k_{i1}^{(m,1)} \dim q \tag{3.11}$$

for any $1 \leqslant i \leqslant r_m$.

If $k_{i1}^{(m,1)} = 0$ then $p_i = 0 = q_i$ hence $p_i \sim q_i$. When $k_{i1}^{(m,1)} \neq 0$ we obtain:

$$\dim p_i = \dim q_i \geqslant k_{i1}^{(m,1)} \geqslant \frac{1}{2}\dim X_{m,i}.$$

By Lemma 3.4 one gets $p_i \sim q_i$ in $C(X_{m,i}, M_{[m,i]})$. Thus $p \sim q$ in A_m. The proof is complete. ∎

REMARKS 3.12.

i) From Theorem 3.7 we can obtain a result of M. Dădârlat and A. Némethi in the case of unital connecting homomorphisms (cf. [10, Proposition 2.2.1., Corollary

2.2.4.]). In [10] the conclusions of Theorem 3.7 were obtained under the assumptions that $X_{n,j}$ are finite connected CW-complexes with $\{\dim X_{n,j}\}$ bounded and A is simple and unital, but the connecting homomorphisms are not necessarily unital. In contrast with [10] our proof avoids any K-theoretical argument.

ii) Theorem 3.7 together with a result in [9] imply that any C^*-crossed product $C(\mathbf{T}^n)\rtimes G$, where G is a dense torsion subgroup of the n-torus $\mathbf{T}^n$ $(n \geqslant 1)$, which acts on $\mathbf{T}^n$ by rotations, satisfies FCQ2 and has cancellation. In particular, any finite tensor product of Bunce-Deddens algebras satisfies FCQ2 and has cancellation.

We end this section with a rather general result which is related to a remark in Section 2 and, in particular, shows that the comparability questions are shape invariant in an appropriate sense (see [10]).

PROPOSITION 3.13. *Let* $A = \varinjlim(A_n, \Phi_{m,n})$ *and* $B = \varinjlim(B_n, \Psi_{m,n})$, *where* A_n, B_n *are arbitrary unital* C^**-algebras and the connecting homomorphisms* $\Phi_{m,n}$, $\Psi_{m,n}$ *are injective and unital.*

Suppose that there exists an EP-commutative diagram with unital and injective homomorphisms α_n *and* β_n $(n \geqslant 1)$*:*

(3.14)

$$\begin{array}{ccccccccccccccc}
A_1 & & \overset{\Phi_1}{\rightarrow} & & A_2 & & \overset{\Phi_2}{\rightarrow} & & A_3 & & \overset{\Phi_3}{\rightarrow} & & A_4 & & \overset{\Phi_4}{\rightarrow}\cdots \\
& \alpha_1\searrow & & \beta_1\nearrow & & \alpha_2\searrow & & \beta_2\nearrow & & \alpha_3\searrow & & \beta_3\nearrow & & \alpha_4\searrow & \\
& & B_1 & & \overset{\Psi_1}{\rightarrow} & & B_2 & & \overset{\Psi_2}{\rightarrow} & & B_3 & & \overset{\Psi_3}{\rightarrow} & & \cdots
\end{array}$$

Then, A *satisfies* FCQ1 *(resp.* FCQ2*) if and only if* B *satisfies* FCQ1 *(resp.* FCQ2*).*

Proof. It suffices to prove only the "if" part. We identify first each A_n (resp. B_n) with a C^*-subalgebra of A (resp. B). Let us note then the following fact:

(3.15) given a trace σ on B, there is a trace τ on A such that $\sigma(\alpha_n(p)) = \tau(p)$ for any $n \geqslant 1$ and all projections p in A_n.

Indeed, let us consider for each $n \geqslant 1$ a state $\Theta_n : A \to \mathbf{C}$ which extends the state $\sigma \circ \alpha_n : A_n \to \mathbf{C}$. Since the sequence $(\|\Theta_n\|)_{n\geqslant 1}$ is bounded, there is a weak*-convergent subsequence $(\Theta_{n_k})_{k\geqslant 1}$ having the weak* limit τ. Since $\tau(a) = \lim_{k\to\infty} \sigma\alpha_{n_k}(a)$ for any $a \in \bigcup_{n\geqslant 1} A_n$ we clearly have that τ is a trace on A. Take now a projection p in some A_n. The diagram (3.14) is EP-commutative, hence, for any $n_k \geqslant n$, we have $\sigma(\alpha_n(p)) = \sigma(\Psi_{n_k,n}\alpha_n(p)) = \sigma(\alpha_{n_k}\Phi_{n_k,n}(p)) = \Theta_{n_k}(\Phi_{n_k,n}(p)) = \Theta_{n_k}(p)$. Therefore $\sigma(\alpha_n(p)) = \lim_{k\to\infty} \Theta_{n_k}(p) = \tau(p)$, so (3.15) follows.

Of course, a similar result holds when A (resp. B) is replaced with B (resp. A). Consequently, A has traces if and only if B has traces.

As a second step in the proof let us show that:

(3.16) if B satisfies FCQ1 (resp. FCQ2) then A satisfies FCQ1 (resp. FCQ2).

CASE 1. Assume that A and B have traces. Let p and q be two projections in A, $q \neq 0$, with $\psi(p) \leqslant \psi(q)$ (resp. $\psi(p) < \psi(q)$) for all traces ψ on A. We may suppose that p and q are in A_1. Since $\beta_1\alpha_1(q) \sim q$ in A_2 we get that $\alpha_1(q)$ is a nonzero projection in B_1. By (3.15) it follows that $\sigma(\alpha_1(p)) \leqslant \sigma(\alpha_1(q))$ (resp. $\sigma(\alpha_1(p)) < \sigma(\alpha_1(q))$ for any trace σ on B, hence $\alpha_1(p) \preceq \alpha_1(q)$ (resp. $\alpha_1(p) \prec \alpha_1(q)$) in B. By a standard argument we may suppose that $\Psi_1\alpha_1(p) \preceq \Psi_1\alpha_1(q)$ (resp. $\Psi_1\alpha_1(p) \prec \Psi_1\alpha_1(q)$) in B_2. Applying β_2 we obtain $\beta_2\Psi_1\alpha_1(p) \preceq \beta_2\Psi_1\alpha_1(q)$ in A_3, so that, by our assumption concerning the diagram (3.14), we find $p \preceq q$ in A_3, hence $p \preceq q$ in A. Thus FCQ1 is clear. For FCQ2 it is enough to remark that $p \preceq q$ and $\psi(p) < \psi(q)$ for all traces ψ lead to $p \prec q$. The proof of Case 1 is complete. ∎

CASE 2. A and B have no traces. Start with p and q projections in A, $q \neq 0$. As above we may assume that $p, q \in A_1$. Then $\alpha_1(q)$ is a nonzero projection in B_1 and since B satisfies FCQ1 (resp. FCQ2) we may suppose that $\Psi_1\alpha_1(p) \preceq \Psi_1\alpha_1(q)$ (resp. $\Psi_1\alpha_1(p) \prec \Psi_1\alpha_1(q)$) in B_2.

Arguing as above we may suppose that $p \preceq q$ in A_3. If B satisfies FCQ2, then, since β_2 is injective, it follows that $\beta_2\Psi_1\alpha_1(p) \prec \beta_2\Psi_1\alpha_1(q)$, or, the diagram (3.14) being EP-commutative, $p \prec q$ in A_3. The proof is complete. ∎

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Received June 10, 1991.