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STRONG RESONANCE PROBLEMS FOR ELLIPTIC SEMILINEAR BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

Let A be a linear elliptic partial differential operator on a domain $\Omega \subset \mathbf{R}^n$ such that the boundary value problem

$$Au = \lambda u \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega \tag{1.1}$$

is selfadjoint and has a discrete set of isolated eigenvalues of finite multiplicities bounded from below and no other spectrum (i.e., no essential spectrum). Let λ_ℓ be one of these eigenvalues, and let $f(x,t)$ be a Caratheodory function on $\Omega \times \mathbf{R}$. We say that the boundary value problem

$$Au - \lambda_\ell u = f(x,u) \text{ in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega \tag{1.2}$$

is resonant like at infinity if

$$\liminf f(x,t)/t \leqslant 0 \leqslant \limsup f(x,t)/t, \quad |t| \to \infty. \tag{1.3}$$

There has been considerable research concerning resonant problems beginning with the work of Landesman and Lazer [4] in which sufficient conditions were given for (1.2) to have a solution. In studying such problems, one can differentiate between different degrees of resonance depending on how closely $f(x,t)/t$ approximates 0(cf., e.g., [2,9]). Thus one can consider cases ranging from the situation in which lim sup $|f(x,t)/t|$ is unbounded to the case when $f(x,t) \to 0$ as $|t| \to \infty$ (in [9] we distinguish six categories). It appears that the more closely $f(x,t)$ approximates 0, the more difficult it is to solve (1.2). Following Bartolo-Benci-Fortunato [2] we call the situation

$$f(x,t) \to 0, \quad F(x,t) := \int_0^t f(x,s)ds \text{ bounded as } t \to \infty \tag{1.4}$$

strong resonance. Only a few authors addressed this situation (cf., [2, 13, 9]). In [2], Bartolo-Benci-Fortunato assumed $f(x,t) = f(t)$,

$$tf(t) \to 0 \text{ as } |t| \to \infty \tag{1.5}$$

$$F(t) \leqslant b_0, \quad t \in \mathbf{R} \tag{1.6}$$

$$F(t) \to b_0 \text{ as } |t| \to \infty. \tag{1.7}$$

In [13], Thews assumed that $f(x,t) = f(t)$ is odd. We assume in [9] that

$$\liminf_{\|v\| \to \infty} \int_\Omega F(x,v)dx \geqslant B_0, \quad v \in N(A) \tag{1.8}$$

$$\limsup_{|t| \to \infty} tf(x,t) \leqslant W_1(x) \in L^1(\Omega) \tag{1.9}$$

$$f(x,t) \to 0 \text{ as } |t| \to \infty \tag{1.10}$$

and

$$\min(0, B_1) < 2(c_1 + B_0) \tag{1.11}$$

where

$$B_1 = \int_\Omega W_1(x)dx$$

and c_1 is the infimum of the energy functional corresponding to (1.2) on a subspace.

In the present paper we continue our work on strong resonance. Actually, we allow a slightly more general situation in which

$$|f(x,t)| \leqslant C(|t|^\gamma + 1) \tag{1.12}$$

for some constant $\gamma < 1$. We assume

$$\limsup_{|t| \to \infty} [2F(x,t) - tf(x,t)] \leqslant W_1(x) \in L^1(\Omega) \tag{1.13}$$

$$b_0 \leqslant \int_\Omega F(x,v)\mathrm{d}x, \quad (Av,v) \leqslant \lambda_\ell \|v\|^2 \tag{1.14}$$

and

$$\int_\Omega W_1(x)\mathrm{d}x < 2b_0. \tag{1.15}$$

Since this result has little overlap with the other work mentioned, we believe that much more will have to be done before optimal results can be obtained for the strong resonance case.

The method we have used is a new variation of the mountain pass lemma. The situation concerns an orthogonal decomposition $H = M \oplus N$ of a Hilbert space into closed subspaces, $\dim N < \infty$. We assume that there is a continuously differentiable functional G on H satisfying for some $R > 0$

$$G(v) \leqslant m_0, \quad v \in N, \ \|v\| = R \tag{1.16}$$

$$G(v) \leqslant m_1, \quad v \in N, \ \|v\| \leqslant R \tag{1.17}$$

$$G(w) \geqslant m_2, \quad w \in M, \ \|w\| \geqslant R \tag{1.18}$$

$$G(w) \geqslant m, \quad w \in M \tag{1.19}$$

If $m_0 < m$, then the standard mountain pass arguments can be used to find stationary points (or approximate stationary points). However, if $m_0 > m$, the usual theory does not work. Our method is to find a mapping $\varphi_0(v)$ of the set $v \in N, \|v\| = R$ into H so that φ_0 links M and satisfies $G(\varphi_0(v)) \leqslant m$. We then replace the set $v \in N, \|v\| = R$ by the set into which it is mapped by φ_0. In general, this cannot be done. We have been able to do this when $m_2 > m_0$ and there is a $\tau < 1$ and an $\varepsilon > 0$ such that

$$(G'(u), u) \leqslant \tau \|u\| \ \|G'(u)\| \tag{1.20}$$

holds for all $u \in H$ such that $\|u\| = R$ and $G(u) \leqslant m_0 + \varepsilon$. Details of the method are given in the next section. The application to the problem (1.2) is given in Section 3.

2. AN ABSTRACT THEOREM

We now present an abstract theorem in Hilbert space which will be used in finding our solutions. Let

$$H = M \oplus N, \ \ M \neq H, M \neq \{0\}, \dim N < \infty \tag{2.1}$$

be an orthogonal decomposition of a Hilbert space H into closed subspaces. Define

$$B_R = \{u \in H| \ \|u\| \leqslant R\}$$

$$\hat{B}_R := \{u \in H| \ \|u\| \geqslant R\}$$

$$\partial B_R := \{u \in H| \ \|u\| = R\}$$

Let G be a continuously differentiable functional on H. We assume that there is an $R > 0$ for which the following assumptions hold:
I. $[(a)]G(v) \leqslant m_0, v \in \partial B_R \cap N \quad [(b)]G(v) \leqslant m_1, v \in B_R \cap N$
II. $[(a)]G(w) \geqslant m_2 > m_0, w \in \hat{B}_R \cap M \quad [(b)]m := \inf_M G > -\infty$
III. There are a $\tau < 1$ and an $\varepsilon_0 > 0$ such that

$$(G'(u), u) \leqslant \tau \|u\| \, \|G'(u)\| \tag{2.2}$$

holds for all $u \in \partial B_R$ satisfying

$$G(u) \leqslant m_0 + \varepsilon_0. \tag{2.3}$$

We let Ψ denote the set of all positive nonincreasing functions ψ on $[0, \infty)$ such that

$$\int_1^\infty \psi(t) \mathrm{d}t = \infty. \tag{2.4}$$

We have

THEOREM 2.1. *Under the above hypotheses, for each $\psi \in \Psi$ there are a constant $c, m \leqslant c \leqslant m_1$ and a sequence $\{u_k\} \subset H$ satisfying*

$$G(u_k) \to c, \quad G'(u_k)/\psi(\|u_k\|) \to 0. \tag{2.5}$$

Proof. Assume first that $m \leqslant m_0$. If the theorem were false, there would be an $\varepsilon > 0$ and a $\psi \in \Psi$ such that

$$\psi(\|u\|) \leqslant \|G'(u)\| \tag{2.6}$$

holds for all u in the set

$$Q_0 := \{u \in H | m - 3\varepsilon \leqslant G(u) \leqslant m_1 + 3\varepsilon\}. \tag{2.7}$$

In fact, if no such $\varepsilon > 0, \psi \in \Psi$ exists, then for each k, ψ we can find a $u_k \in H$ such that

$$m - (1/k) \leqslant \psi(\|u_k\|)/k$$

(note that $\psi/k \in \Psi$). For a subsequence there is a c satisfying $m \leqslant c \leqslant m_1$ such that (2.5) holds. We may assume that $3\varepsilon < \min[\varepsilon_0, m_2 - m_0]$. Let

$$Q := \{u \in Q_0 | m - 2\varepsilon \leqslant G(u) \leqslant m_1 + 2\varepsilon\}$$

$$Q_1 := \{u \in Q | m - \varepsilon \leqslant G(u) \leqslant m_1 + \varepsilon\}$$

$$Q_2 := H/Q \text{ and}$$

$$\eta(u) = \mathrm{d}(u, Q_2)/[\mathrm{d}(u, Q_1) + \mathrm{d}(u, Q_2)].$$

Note that

$$Q_0 \subset \hat{H} := \{u \in H | G'(u) \neq 0\}.$$

For each $\alpha < 1 - \tau$ there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{H}$ into itself such that

$$\|Y(u)\| \leqslant 1, (G'(u), Y(u)) \geqslant \alpha \|G'(u)\|, u \in \hat{H} \tag{2.8}$$

and

$$(Y(u), u) < 0, u \in Q_0 \cap \partial B_R \tag{2.9}$$

(cf., e.g.,[9]). Let $\sigma(t, v)$ be the (unique) solution of

$$\sigma'(t) = -\eta(\sigma(t))Y(\sigma(t)), t \in \mathbf{R}, \sigma(0) = v \tag{2.10}$$

for $v \in \partial B_R \cap N$. Here we make use of the fact that $\eta(u)Y(u)$ is locally Lipschitz on the whole of H. We observe that $\sigma(t, v)$ never enters B_R. In fact we have

$$\mathrm{d}\|\sigma(t, v)\|^2/\mathrm{d}t = 2(\sigma, \sigma') = -2\eta(\sigma)(\sigma, Y(\sigma)).$$

Moreover, every point of ∂B_R is the center of a neighborhood in which $\eta(u)(u, Y(u)) \leqslant \leqslant 0$. Hence $\mathrm{d}\|\sigma(t, v)\|^2/\mathrm{d}t \geqslant 0$ whenever $\sigma(t, v) \in \partial B_R$. We also have

$$\mathrm{d}(G(\sigma(t))/\mathrm{d}t = (G'(\sigma), \sigma') = -\eta(\sigma)(G'(\sigma), Y(\sigma)) \leqslant -\alpha\eta(\sigma)\|G'(\sigma)\| \leqslant 0. \tag{2.11}$$

Thus

$$G(\sigma(t_2, v)) \leqslant G(\sigma(t_1, v)) \leqslant m_0, \quad t_1 < t_2. \tag{2.12}$$

In particular, this shows that $\sigma(t,v)$ can never intersect M. For it cannot intersect that part of M inside B_R, and hypothesis II(a) shows that it cannot intersect that part of M in $\hat{B}_R$. Let T satisfy

$$\alpha \int_R^{R+T} \psi(t)dt \geqslant 2\varepsilon.$$

If there is a $t_1 < T$ such that $\sigma(t_1,v) \notin Q_1$, then

$$G(\sigma(T,v)) \leqslant G(\sigma(t_1,v)) \leq m - \varepsilon$$

in view of (2.12). On the other hand, if $\sigma(t,v) \in Q_1$ for $0 \leqslant t \leqslant T$, then (2.8) and (2.11) imply

$$G(\sigma(T,v)) - G(v) \leqslant -\alpha \int_0^T \|G'(\sigma(t,v))\|dt \leqslant$$

$$\leqslant -\alpha \int_0^T \psi(\|\sigma(t,v)\|)dt \leqslant -\alpha \int_0^T \psi(\|v\| + t)dt =$$

$$= -\alpha \int_0^T \psi(R+t)dt = -\alpha \int_R^{R+T} \psi(\tau)d\tau < -2\varepsilon.$$

Hence

$$G(\sigma(T,v)) < m - \varepsilon, \quad v \in \partial B_R \cap N. \tag{2.13}$$

Define

$$\varphi_0(v) = \sigma(T,v), \quad v \in \partial B_R \cap N. \tag{2.14}$$

Then $\varphi_0(v)$ is a continous map of $\partial B_R \cap N$ into H such that any continuous map φ of $B_R \cup N$ into H which satisfies

$$\varphi(v) = \varphi_0(v), v \in \partial B_R \cap N \tag{2.15}$$

must satisfy

$$\varphi(B_R \cap N) \cap M \neq \vee. \tag{2.16}$$

To see this let P be the orthogonal projection of H onto N and let $\varphi_t(v)$ be any continuous map of $B_R \cap N$ into H such that

$$\varphi_t(v) = \sigma(t,v), \quad v \in \partial B_R \cap N.$$

Since $P\sigma(t,v) \neq 0$ for $v \in \partial B_R \cap N$ and $0 \leqslant t \leqslant T$, the Brouwer degree $i(P\varphi_t, B_R \cap N, 0)$ is defined and satisfies

$$i(P\varphi_t, B_R \cap N, 0) = i(P, B_R \cap N, 0) = 1.$$

Hence (2.16) holds. Let S denote the set of all continuous maps φ of $B_R \cap N$ into H which satisfy (2.15), and define

$$c := \inf_{\varphi \in S} \max_{v \in B_R \cap N} G(\varphi(v)). \tag{2.17}$$

Then (2.16) and hypothesis II(b) imply $c \geqslant m$. We shall show that $c \leqslant m_1$. If there did not exist a sequence satisfying (2.5), there would be a $\delta > 0$ and a $\psi \in \Psi$ such that (2.6) holds for all u in the set

$$Q_0' := \{u \in H \mid G(u) - c \leqslant 3\delta\}.$$

If necessary, we reduce δ so that it satisfies $3\delta < \varepsilon$. Let

$$Q' := \{u \in Q_0' \mid |G(u) - c| \leqslant 2\delta\}$$

$$Q_1' := \{u \in Q' \mid |G(u) - c| \leqslant \delta\}$$

$Q_2' := H/Q'$ and let $\eta_1(u)$ be defined for the Q_j' in the same way $\eta(u)$ was defined for the Q_j. Let $Y(u)$ be any locally Lipschitz continuous map satisfying (2.8) (there is no need for it to satisfy (2.9)), and let $\sigma_1(t,u)$ be the solution of

$$\sigma'(t) = -\eta_1(\sigma(t))Y(\sigma(t)), t \in \mathbf{R}, \sigma(0) = u.$$

By (2.8) we have

$$\|\sigma_1(t,u) - u\| \leqslant |t| \tag{2.18}$$

and

$$\mathrm{d}G(\sigma_1(t,u))/\mathrm{d}t = (G'(\sigma_1), \sigma_1') = -\eta_1(\sigma_1)(G'(\sigma_1), Y(\sigma_1)) \leqslant \tag{2.19}$$

$$\leqslant -\alpha\eta_1(\sigma_1)\|G'(\sigma_1)\| \leqslant 0$$

Thus

$$G(\sigma_1(t_2,u)) \leqslant G(\sigma_1(t_1,u)), \quad t_1 < t_2.$$

In view of the definition (2.17) of c, there is a $\varphi \in S$ such that

$$G(\varphi(v)) < c + \delta, \quad v \in B_R \cap N. \tag{2.20}$$

Let

$$M = \max_{B_R \cap N} \|\varphi(v)\| \tag{2.21}$$

and pick T so that

$$\alpha \int_{M}^{M+T} \psi(t)\mathrm{d}t > 2\delta. \tag{2.22}$$

For any $v \in B_R \cap N$, if there is a $t_1 < T$ such that $\sigma_1(t_1, \varphi(v)) \notin Q_1'$ then (2.20) implies

$$G_1(\sigma(T, \varphi(v))) \leqslant G(\sigma_1(t_1, \varphi(v))) < c - \delta.$$

On the other hand, if $\sigma_1(t, \varphi(v)) \in Q_1'$ for $0 \leqslant t \leqslant T$, then (2.19) gives

$$G(\sigma_1(T, \varphi(v))) - G(\varphi(v)) \leqslant -\alpha \int_0^T \|G'(\sigma_1(t, \varphi(v)))\|\mathrm{d}t \leqslant$$

$$\leqslant -\alpha \int_0^T \psi(\|\sigma_1(t, \varphi(v))\|)\mathrm{d}t \leqslant -\alpha \int_0^T \psi(M + t)\mathrm{d}t =$$

$$-\alpha \int_M^{M+T} \psi(\tau)\mathrm{d}\tau < -2\delta.$$

Thus

$$G(\sigma_1(T, \varphi(v))) < c - \delta, \quad v \in B_R \cap N. \tag{2.23}$$

Let

$$\varphi_1(v) = \sigma_1(T, \varphi(v)), \quad v \in B_R \cap N. \tag{2.24}$$

Then $\varphi_1 \in S$ since $\eta_1(u) = 0$ for $u \notin Q'$ and $\varphi_0(v) \notin Q'$ for $v \in \partial B_R \cap N$. But then (2.23) contradicts (2.17). Hence the conclusion of the theorem must hold.

Assume next that $m_0 < m$. In this case we need not go through the first part of the proof to find φ_0 but merely take $\varphi_0(v) = v$ on $\partial B_R \cap N$ and take $3\delta < m - m_0$. We then proceed as before.

Finally, we show that $c \leqslant m_1$. To see this let $\sigma(t, v)$ be the solution of (2.10) for each $v \in B_R \cap N$. Then (2.11) implies that

$$G(\sigma(T, v)) \leqslant G(v) \leqslant m_1, \quad v \in B_R \cap N.$$

Since $\sigma(T,v)$ satisfies (2.15), it is in S. Hence (2.17) gives

$$c \leqslant \max_{v \in B_R \cap N} G(\sigma(T,v)) \leqslant m_1.$$

This completes the proof. ∎

3. SEMILINEAR BOUNDARY VALUE PROBLEMS

Let A be a selfadjoint operator on $L^2(\Omega)$, where Ω is a (bounded or unbounded) domain in $\mathbb{R}^n$. Let $f(x,t)$ be a Caratheodory function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ (measurable in x for every t and continuous in t for almost every x). We are concerned with finding solutions of

$$Au = f(x,u), \quad u \in D(A). \tag{3.1}$$

A function $u \in D := D(|A|^{1/2})$ will be called a semistrong solution of (3.1) if

$$a(u,v) = (f(x,u),v), \quad v \in D \tag{3.2}$$

where

$$a(u,v) = (Au,v), \quad u,v \in D, \quad a(u) = a(u,u)$$

and

$$(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x, \quad \|u\|^2 = (u,u).$$

In many situations it can be shown that a semistrong solution of (3.1) is a solution. We shall avoid this question and refer to a semistrong solution as a solution. We make the following assumptions:

(A) A has no essential spectrum in $(-\infty, 0]$ and

$$N := \oplus_{\lambda \leqslant 0} N(A-\lambda) \tag{3.3}$$

is finite dimensional. Moreover, if $u \in N(A)$ and $u \not\equiv 0$, then $u \neq 0$ a.e.

(B) There is a number p such that $1 \leqslant p \leqslant 2$ and

$$|f(x,t)| \leqslant V(x)^p |t|^{p-1} + V(x)W(x) \tag{3.4}$$

where $V(x) > 0$ is such that multiplication by $V(x)$ is a compact operator from D to $L^p(\Omega)$, $W \in L^{p'}(\Omega)$, $p' = p/(p-1)$. We take

$$\|u\|_D^2 := a(u) + K\|u\|^2$$

where K is such that $A + K \geqslant 1$.

(C) If we put

$$F(x,t) := \int_0^t f(x,s)\mathrm{d}s$$

and

$$H(x,t) := 2F(x,t) - tf(x,t)$$

then we assume

$$b_0 \leqslant \int_\Omega F(x,v)\mathrm{d}x, v \in N \tag{3.5}$$

$$\limsup_{|t|\to\infty} H(x,t) \leqslant W_1(x) \in L^1(\Omega) \tag{3.6}$$

$$H(x,t) \leqslant W_2(x) \in L^1(\Omega), \quad t \in \mathbf{R} \tag{3.7}$$

and

$$b_1 := \int_\Omega W_1(x)\mathrm{d}x < 2b_0. \tag{3.8}$$

We have

THEOREM 3.1. *Under hypotheses* (A)–(C), *problem* (3.1) *has a semistrong solution.*

Proof. We show that the hyporheses of Theorem 2.1 are satisfied. We take D as a Hilbert space H and put

$$G(u) := a(u) - 2\int_\Omega F(x,u)\mathrm{d}x. \tag{3.9}$$

It follows that $G \in C^1(D, \mathbf{R})$ and that

$$(G'(u), v)_D = 2a(u,v) - 2(f(x,u),v), \quad u, v \in D \tag{3.10}$$

(cf., e.g., [11]). We take

$$N = \bigoplus_{\lambda \leqslant 0} N(A - \lambda), \quad M = D \cap N^\perp. \tag{3.11}$$

By hypothesis (A), N is finite dimensional. In particular, the norms $||u||_D$ and $||u||$ are equivalent on N. Let $\varepsilon > 0$ be such that

$$b_1 + \varepsilon < 2b_0. \tag{3.12}$$

By (3.5) and (3.11)

$$G(v) \leqslant -2\int_\Omega F(x,v)\mathrm{d}x \leqslant -2b_0, \quad v \in N. \tag{3.13}$$

Let λ_+ be the smallest positive point in $\sigma(A)$ (there is one since 0 is not in $\sigma_e(A)$). Then

$$\|w\|_D^2 \leqslant (1 + K\lambda_+^{-1})a(w), \quad w \in M. \tag{3.14}$$

Hence by (3.3)

$$G(w) \geqslant a(w) - C\int_\Omega (|Vw|^p + W|Vw|)\mathrm{d}x \geqslant \tag{3.15}$$

$$\geqslant a(w) - C'(\|Vw\|^p + \|W\|_{p'}\|Vw\|_p) \to \infty \text{ as } \|w\|_D \to \infty$$

since $p < 2$. Thus (3.13) and (3.15) show that hypothesis I and II of Section 2 hold with $m_0 = m_1 = -2b_0$ and R sufficiently large. We now show that hypothesis III holds as well with τ any number satisfying $0 < \tau < 1$ and $\varepsilon_0 = \varepsilon$.

Let N' be the orthogonal complement of the nullspace $N(A)$ of A in the subspace N. Thus $N = N(A) \oplus N'$. Since $|a(v)|^{1/2}$ is a norm on N', it is equivalent to $\|v\|_D$ on this subspace. I claim that for all R sufficiently large the inequality (2.2) holds for all $u \in B_R$ satisfying (2.3). If this were not the case, there would be a sequence $\{u_k\} \subset H$ such that

$$\|u_k\|_D \to \infty, \; G(u_k) \leqslant m_0 + \varepsilon \tag{3.16}$$

and

$$(G'(u_k), u_k)_D > \tau\|u_k\|_D \, \|G'(u_k)\|_D. \tag{3.17}$$

By (3.9) and (3.10) we have

$$(G'(u), u)_D = 2G(u) + 2\int_\Omega H(x,u)\mathrm{d}x. \tag{3.18}$$

Thus

$$(G'(u_k), u_k)_D \leqslant 2(m_0 + \varepsilon + b_2). \tag{3.19}$$

Hence, the only way (3.17) can happen is if

$$\|u_k\|_D \, \|G'(u_k)\|_D \leqslant C. \tag{3.20}$$

Let $u_k = v_k + v_{0k} + w_k$, where $v_k \in N', v_{0k} \in N(A)$ and $w_k \in M$. Then

$$(G'(u_k), w_k)_D = 2a(w_k) - 2(f(x, u_k), w_k).$$

Consequently by (3.3) and (3.4)

$$2a(w_k) \leqslant C + C' \int_\Omega (|Vu_k|^{p-1}|Vw_k| + W|Vw_k|)dx \leqslant \tag{3.21}$$

$$\leqslant C + C''(\|u_k\|_D^{p-1} + 1)\|w_k\|_D.$$

Similarly

$$2|a(v_k)| \leqslant C + C'''(\|u_k\|_D^{p-1} + 1)\|v_k\|_D. \tag{3.22}$$

Let $t_k = \|u_k\|_D, \tilde{u}_k = u_k/t_k$ Then $\|\tilde{u}_k\|_D = 1$ and

$$\|\tilde{u}_k + \tilde{w}_k\|_D \to 0$$

by (3.21) and (3.22) since $p < 2$ Hence

$$\|\tilde{v}_{0k}\|_D^2 = K\|\tilde{v}_{0k}\|^2 \to 1.$$

Thus there is a renamed subsequence such that $\tilde{v}_k \to 0, \tilde{w}_k \to 0$ and $\tilde{v}_{0k} \to \tilde{v}_0$ in $L^2(\Omega)$ and a.e. Since $\tilde{v}_0 \not\equiv 0$ we have $\tilde{v}_0 \neq 0$ a.e. and

$$|u_k|/t_k = |\tilde{u}_k| \to |\tilde{v}_0| \neq 0 \text{ a.e.}$$

Thus $|u_k| \to \infty$ a.e. This implies

$$\limsup_{k\to\infty} H(x, u_k(x)) \leqslant W_1(x) \text{ a.e.}$$

in view of (3.6). Consequently

$$\limsup_{k\to\infty} \int_\Omega H(x, u_k)dx \leqslant b_1 \tag{3.23}$$

and

$$\limsup_{k\to\infty}(G'(u_k), u_k)_D \leqslant 2(m_0 + \varepsilon + b_1) < 0 \tag{3.24}$$

by (3.18) and (3.11). This contradicts (3.17). Hence hypothesis II is satisfied for any R sufficiently large. We may now apply Theorem 2.1 to conclude that for each $\psi \in \Psi$

there is a $c \leqslant m_1$ that and a sequence satisfying (2.5). If we take $\psi(t) = 1/(1+t)$, then

$$(1 + \|u_k\|_D)\|G'(u_k)\|_D \to 0 \tag{3.25}$$

and (3.20) holds. If $t_k = \|u_k\|_D \to \infty$, then the argument following (3.20) implies that (3.23) holds. By (3.18) and (3.12)

$$\limsup_{k\to\infty}(G'(u_k), u_k)_D \leqslant 2(c + b_1) < 0 \tag{3.26}$$

since $c \leqslant m_1 = m_0 = -2b_0$. But this contradicts (3.25). Hence the t_k are bounded. Thus there is a renamed subsequence such that $u_k \to u$ weakly in D and a.e. in Ω. Hence (cf., e.g., [11])

$$(G'(u_k), v) = 2a(u_k, v) - 2(f(x, u_k), v) \to 2a(u, v) - 2(f(x, u), v) = (G'(u), v)$$

for all $v \in D$. Thus u is a solution of (3.2). ■

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REFERENCES

1. AMBROSETTI, A.; RABINOWITZ, P. H., Dual variational methods in critical point theory and applications, *J. Functional Anal.*, 14(1973), 349–381.
2. BARTOLO, P.; BENCI, V.; FORTUNATO, D., Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity, *Nonlinear Anal. Th. Meth. Appl.*, 7(1983), 981–1012.
3. DE FIGUEIREDO, D. G., Positive solutions of some classes of semilinear elliptic problems, *Proc. Symp. Pure Math.*, 45(1986), 371–379.
4. LANDESMAN, E. M.; LAZER, A. C., Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.*, 19(1970), 609–623.
5. NIRENBERG, L., Variational and topological methods in nonlinear problems, *Bull. Amer. Math. Soc.*, 4(1981), 267–302.
6. NIRENBERG, L., Variational methods in nonlinear problems, *Lectures Notes in Math.*, Vol. 1365, Springer-Verlag, 1988, 100–119.
7. RABINOWITZ, P. H., Variational methods for nonlinear eigenvalue problems, Eigenvalues of Nonlinear Problems (G. Prodi, Ed.) *C.I. M.E. Ed. Cremonese, Roma* 1975, 141–195.
8. RABINOWITZ, P. H., Minimax methods in critical point theory with applications to differential equations, *Conf. Board of Math. Sci. Conf. Reg. Ser. in Math.*, No. 65, Amer. Math. Soc. 1986.
9. SCHECHTER, M., Nonlinear elliptic boundary value problems at strong resonance, *Amer. J. Math.*, 112(1990), 439–460.
10. SCHECHTER, M., A variation of the mountain pass lemma and applications, *J. London Math. Soc.*
11. SCHECHTER, M., Mountain pass and mountain cliff theorems

12. Tarafdar, E., An approach to nonlinear elliptic boundary value problems, *J. Austral. Math. Soc.*, **34**(1983), 316–335.
13. Thews, K., Nontrivial solutions of elliptic equations at resonance, *Proc R. Soc. Edinb.*, **85A**(1980), 119–129.

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