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CHARACTERS AND FACTOR REPRESENTATIONS OF THE UNITARY GROUP OF THE CAR-ALGEBRA

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1. INTRODUCTION

Let $\mathcal{A}$ be the C^*-algebra given by the infinite tensor product of (2×2)-matrix algebras. $\mathcal{A}$ is the UHF algebra with invariant 2^∞ and is also known as the CAR (canonical anticommutation relation) algebra. In principle, the methods developed for the study of inductive limit groups in [4, 5, 9, 16, 17] are applicable to the investigation of the representation theory of the unitary group of $\mathcal{A}$. Let us see what this means in the framework given by Voiculescu [17] for the classification of the finite characters. Let $U(2^\infty)$ be the inductive limit of the unitary groups of the finite tensor products: $\bigotimes_{k=1}^{m} M_2(\mathbb{C})$. The conjugacy classes of $U(2^\infty)$ admit a natural binary operation since any conjugacy class in $\bigotimes_{k=1}^{n} M_2(\mathbb{C})$ by an 2^n-tuple $\gamma = (z_1, z_2, \ldots, z_{2^n})$, with $|z_i| = 1$. Given two such classes, γ and γ', we take their product to be the natural class in $\bigotimes_{k=1}^{n} M_2(\mathbb{C}) \bigotimes \bigotimes_{k=1}^{m} M_2(\mathbb{C})$.

A finite character χ of $U(2^\infty)$ is multiplicative relative to this binary operation. Now, if τ is the restriction of the trace $\mathcal{A}$ to $U(2^\infty)$, then τ is multiplicative because the trace on a tensor product is the product of the traces on the factors. Hence, we have by a theorem of Voiculescu that $\tau^p\overline{\tau}^q$ are also finite characters. This is very different from the behavior of tensoring the fundamental trace of $U(N)$ with itself which is highly reducible. This initial result provided the original motivation for this paper.

The primitive ideal space of $U(2^\infty)$ is parametrized by $\{0, 1, \ldots, \infty\} \times \mathbf{Z}$ and the ideal corresponding to $\tau^p\overline{\tau}^q$ is $(p, p-q)$. Also, unlike $U(\infty)$, the usual inductive limit unitary group, $C^*(U(2^\infty))$ possesses no faithful factor representations. If π is a factor representation whose kernel does not have an infinite invariant, the π is norm continuous and so extends to give a representation of the full unitary group of $\mathcal{A}$. We note that $U(2^\infty)$ has a much smaller primitive ideal space than $U(N)$ which is a full rank integral cone in $\mathbf{Z}^n$.

A particularly pretty result was obtained on the spectral analysis of the branching matrix M for a primitive quotient A with positive signatures. It is easy to show that A is a stationary AF-algebra, so A has a unique trace that corresponds to the largest eigenvalue (Perron eigenvalue) of M. The associated eigenvector gives the dimensions of the irreducible representations of the appropriate size symmetric group. The order structure of $K_0(A)$ can be computed if M can be fully diagonalized [7,10]. Surprinsingly, this can be done, especially since the entries of M are computed by a repeated application of the Littlewood-Richardson rule. The eigenvectors of M are given by the columns of the character table of the symmetric group with all eigenvalues being positive powers of 2. Low rank examples are given in an appendix.

In some loose sense, the representation theory of $U(\infty)$ is an additive theory while that of $U(2^\infty)$ is multiplicative and rigid.

We identified the unimodular subgroup $SU(2^\infty)$ whose Lie algebra consists of all skew-adjoint elements with trace zero. For more general algebras, it would be interesting to compare this definition of the unimodular subgroup with the commutator subgroup, say. Now, we have shown that any factor representation of $SU(2^\infty)$ is norm continuous and factorizable in the sense of Arveson [1]. The notion of factorizable representation is defined in the terms of the exponential e^A of a C^*-algebra A. The states of e^A correspond to holomorphic non-linear states of A. In particular, $SU(2^\infty)$ behaves very much like a compact group.

Finally, we examined the product of two gauge-invariant quasi-free states ω_A of $U(2^\infty)$. We showed it is factorial provided $Tr[A(I-A)] = \infty$. This trace condition has appeared many times as a factor condition. Its importance was first shown by Strătilă and Voiculescu [16] and by Baker [2]. We note that these representations of $U(2^\infty)$ can be viewed as nonlinear quasi-free representations in the sense of [1].

COMMENT. The unitary group of a general UHF algebra behaves much the same way as for $U(2^\infty)$. We chose the 2^∞ case to simplify notation and to be able to analyze the branching matrices. It would be interesting to work out the representation theory of the universal covering group $U(\widetilde{2^\infty})$. Recall that the fundamental group $\pi_1(U(2^\infty)) \simeq K_0(\mathrm{UHF}(2^\infty)) \simeq \mathbf{Z}[\frac{1}{2}]$, the group of 2-adic integers. $U(\widetilde{2^\infty})$ has additional characters, since the determinant Δ can be defined on it as follows. We first remark though that if we use the classical definition of the determinant relative to a trace in a type II_1 factor, then the determinant of any invertible element is 1. Now, we imbedd $U(2^n)$ into $U(2^{n+1})$ by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Let Δ_n be the 2^n root of the usual determinant function on $U(2^n)$. Then the family of determinants forms a consistent system on the limits of the universal covering groups of $U(2^n)$. We conjecture that the characters for $U(\widetilde{2^\infty})$ are given by $\Delta^k \tau^p \overline{\tau}^q$.

We shall parametrize the unitary dual $U(\widehat{2^N})$ by pairs of Young diagrams $\{\overline{\mu}; \lambda\}$, where $\overline{\mu}$ and λ have no more than 2^N rows. We need to consider the decomposition of representations of $U(2^{N+1})$ restricted to $U(2^N)$. Let $\{\overline{\mu}; \lambda\} \in U(\widehat{2^{N+1}})$. Then, the results of King [11], $\{\overline{\mu}; \lambda\}$ restricted to $U(2^N) \times U(2^N)$ becomes

$$\sum_{\theta,\varphi} \left(\sum_{\rho} \left\{ \overline{\mu/\rho\theta}; \lambda/\rho\varphi \right\} \times \{\overline{\theta}; \varphi\} \right). \tag{0.1}$$

Next, we consider the restriction of

$$U(2^N) \times U(2^N) \downarrow U(2^N),$$

where $V \in U(2^N)$ is identified with $\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \in U(2^{N+1})$. This will amount to taking tensor products in (0.1). We recall the tensor product decomposition:

$$\{\overline{\mu}_1; \lambda_1\} \bigotimes \{\overline{\mu}_2; \lambda_2\} = \sum_{\alpha,\beta} \left\{ \overline{(\mu_1/\alpha) \cdot (\mu_2/\beta)}; (\lambda_1/\beta) \cdot (\lambda_2/\alpha) \right\}.$$

Hence, (0.1) becomes on its restriction to $U(2^N)$:

$$\sum_{\theta,\varphi} \left(\sum_{\rho} \left(\sum_{\alpha,\beta} \left\{ \overline{(\mu/\rho\theta\alpha) \cdot (\theta/\beta)}; (\lambda/\rho\varphi\beta) \cdot (\varphi/\alpha) \right\} \right) \right). \tag{0.2}$$

For positive signatures $\overline{\mu} = 0$, and we write $\{\lambda\}$ for $\{0; \lambda\}$. Here, (0.2) becomes

$$\{\lambda\} \downarrow \sum_{\varphi} \{\lambda/\varphi\} \bigotimes \{\varphi\}. \tag{0.3}$$

We note that all the skew Schur functions and tensor products that occur can be computed by the Littlewood-Richardson rule [12]. Several applications may be required.

Let $\mathcal{A}$ be the CAR-algebra which is the C^*-inductive limit of the complex matrix algebras $M(2^N)$ where $x \in M(2^N)$ is mapped to the 2×2 matrix $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ which can be naturally identified as an element of $M(2^{N+1})$. So, $\mathcal{A}$ is the UHF-algebra with invariant 2^∞. We let $U(\mathcal{A})$ denote the full unitary group of $\mathcal{A}$. Following Strătilă and Voiculescu [15], we can introduce an AF C^*-algebra that supports the unitary representation theory of the inductive limit of compact groups $\varinjlim U(2^N)$, which we denote by $U(2^\infty)$. We let $C^*(U(2^\infty))$ denote this group algebra. Note that $U(2^\infty)$ is a norm dense subgroup of $U(\mathcal{A})$.

PROPOSITION 1. $\mathrm{Prim}(U(2^\infty)) \simeq \{0,1,2,\dots,\infty\} \times \mathbb{Z}$.

Proof. We shall classify the primitive ideals J of $C^*(2^\infty)$ by using the Strătilă-Voiculescu dynamical system (Ω, Γ) associated to this AF-algebra. The primitive ideals are classified by the orbit closures: $\overline{\Gamma \cdot \gamma}, \gamma \in \Omega$. Now, elements γ in Ω are given by a sequence or path γ_n, where $\gamma \in \widehat{U(2^n)}$, such that γ_{n-1} is a summand of γ or is a finite sequence which terminates. We write $\gamma_n = \{\overline{\mu}_n; \lambda_n\}$. It follows from (0.2) that the quantity $|\lambda_n| - |\overline{\mu}_n|$ is independent of n. So, the integer $m = |\lambda_n| - |\overline{\mu}_n|$ is an invariant of the orbit closure of γ. An elementary calculation shows that m is a complete invariant for orbit closures for paths such that either $\mu_n = 0$, for all n or $\lambda_n = 0$, for all n. Moreover, in this case, either $|\lambda_n|$ or $|\mu|$ is the constant m. We now drop this assumption. If either $|\lambda_n|$ or $|\mu_n|$ is eventually constant, then they are both constant. Hence, in the remaining case to be checked, both $|\lambda_n|$ and $|\mu_n|$ are increasing sequences tending to ∞, such that $|\lambda_n| - |\mu_n|$ is the constant m. Moreover, it can be shown that the representations $\{\overline{\mu}_n; \lambda_n\}$ on restriction will eventually contain all summands such that $|\lambda_n| - |\overline{\mu}_n| = |\lambda_k| - |\overline{\mu}_k|$ and $|\lambda_k| \leqslant |\lambda_n|, |\overline{\mu}_k| \leqslant |\mu_n|$, for $k \leqslant n$, because of the presence of the ρ factor in (0.2). ∎

For the sequence $\{\overline{\mu}_m; \lambda_n\}$ or the corresponding primitive ideal $J \in \mathrm{Prim}(U(2^\infty))$, we have the invariants:

$$|J| = \sup\{|\lambda_n| : n \geqslant 1\} \text{ and } m_J = |\lambda_n| - |\overline{\mu}_n|.$$

Note that: $0 \leqslant |J| \leqslant \infty$ and $m_J \in \mathbb{Z}$. We write J as J_m with $m = m_J$. We call J *positive* if it is the orbit closure of a path with only positive signatures. For a positive primitive ideal J, we have that $|J| = m_J$.

Unlike the C^*-algebra for $U(\infty)$ [4], we have:

COROLLARY 2. *$C^*(U(2^\infty))$ has no faithful factor representations.*

Proof. The zero ideal (0) is not a primitive ideal. ∎

PROPOSITION 3.

(i) *Every primitive quotient of $C^*(U(2^\infty))$ by a positive primitive ideal J is simple; in particular, the point $\{J\}$ is closed in* $\mathrm{Prim}(U(2^\infty))$.

(ii) *In general, the closure of the point $\{J\}$ in* $\mathrm{Prim}(U(2^\infty))$ *consists of all*

$$J' \in \mathrm{Prim}(U(2^\infty))$$

such that $0 \leqslant m_{J'} \leqslant m_J$ and $0 \leqslant |J'| \leqslant |J|$.

(iii) *If $|J|$ is finite, then the primitive quotient of $C^*(U(2^\infty))$ is a stationary AF-algebra of finite rank.*

Proof. Let M be the branching matrix for this primitive quotient. It is sufficient to show that there is some power, say M^j, of M which has only positive entries [7,10]. It is an immediate consequence of the simplicity of the quotient that $\{J\}$ is a closed point. To show that the desired power of M exists we can use the branching rule (0.3) where the tensor product is replaced with the Cartan product for convenience of computation and choose $|\varphi| = 1$ there. We then can modify the Young diagram by any choice of one cell. With succesive applications of this rule, we can change any diagram to any other of the same weight. In particular, we have that $M^{2|J|}$ has strictly positive entries. So, (i) holds.

Statement (ii) follows by the last observation in the proof of Proposition 1, while (iii) is an immediate consequence of the definition [7]. ∎

COROLLARY 4. *Let π be a strongly continuous factor representation of $U(2^\infty)$. If $J = \ker(\pi) \in \mathrm{Prim}(U(2^\infty))$ such that $|J| < \infty$, then π is norm continuous. In particular, if J is a positive primitive ideal, then π is norm continuous.*

Proof. We follow the reasoning of Strătilă and Voiculescu [16, p.99] or [3]. Now, π is norm continuous on $U(2^\infty)$ if it is norm continuous on the corresponding maximal torus since the norm of $\pi(x)$ is invariant under conjugation. By assumption, $\pi|U(2^N)$ decomposes into irreducibles whose signature entries are uniformly bounded in N and whose number of non-zero parts is also bounded independently of N. It follows easily from this observation that π is norm continuous. ∎

PROPOSITION 5. *Let τ denote the canonical trace of $\mathcal{A}$, so $\tau|U(2^\infty)$ is a finite character. Then $\tau^p\overline{\tau}^q$ is a finite character of $U(2^\infty)$, where $p, q = 0, 1, 2, \ldots$. Moreover, the primitive ideal J corresponding to the character $\tau^p\overline{\tau}^q$ has invariants: $m_J = p - q$ and $|J| = p$.*

Proof. Since the canonical trace of $\mathcal{A}$ is a norm continuous factor trace and since every element of $\mathcal{A}$ is a finite linear combination of unitaries, we must have that:

$$\{\pi_\tau(\mathcal{A})\}'' = \{\pi_\tau(U(\mathcal{A}))\}'' = \{\pi_\tau(U(2^\infty))\}'',$$

where π_τ is the representation associated to τ. In particular, τ is a finite character of $U(2^\infty)$. Note that $\overline{\tau}$ is a positive definite central function since τ is positive definite and $\tau(g^{-1}) = \overline{\tau(g)}$, for $g \in U(2^\infty)$.

By a theorem of Voiculescu [17], to show that $\tau^p\overline{\tau}^q$ is a finite character, it suffices to introduce a binary operation on the conjugacy classes of $U(2^\infty)$. We have described this operation already in the Introduction so we only give a quick review here. The analogue of the product used for $U(\infty)$ works here as well [17]. A conjugacy class in $U(2^\infty)$ is given by a finite sequence $(z_1, \ldots, z_{2^n})$ of complex numbers of modulus 1. The binary operation consists of taking the tensor product of these two finite sequences.

The calculation of the primitive ideal J associated to $\tau^p\overline{\tau}^q$ follows easily from the decomposition of the tensor product: $\pi_1^{\bigotimes p} \bigotimes \overline{\pi}_1^{\bigotimes q}$, where π_1 is the natural representation of $U(2^n)$ on $\mathbb{C}^{2^n}$. From the rules for decomposing a tensor product (see [11] for example), we have that $|J| = p$. (Note that if $q = 0$, this is the only invariant). Moreover, the difference invariant m_J is given by $p - q$, since every irreducible component of $\pi_1^{\bigotimes p}$ has weight p, while $\overline{\pi}_1^{\bigotimes q}$ has weight $-q$. ∎

PROPOSITION 6. *Let $J \in \mathrm{Prim}(U(2^\infty))$ with $|J| < \infty$. Then the primitive quotient $A = C^*(U(2^\infty))/J$ has a unique finite character and no infinite characters.*

Proof. By Proposition 3(c), the primitive quotient A is a stationary AF- algebra of finite rank. But, for such algebras, we know that they have the desired properties [7, 10]. ∎

PROPOSITION 7. *Every finite character t of $U(2^\infty)$ has the form: $\tau^p\overline{\tau}^q$. In particular, every finite character of $U(\mathcal{A})$ has this form as well.*

Proof. By Proposition 6, we may assume that $J \in \mathrm{Prim}(U(2^\infty))$ with $|J| = \infty$. Without loss of generality, we may assume that $J = J_0$, so $|J| = \infty$ and $m_J = 0$. To see this, suppose δ is a finite character with kernel J_m. Then the kernel of $\overline{\tau}^m$ is J_0, for $m > 0$, while for $\tau^m\delta$ is J_0, for $m < 0$. In other words, if any ideal J_m with $|J_m| = \infty$ is the kernel of a character, all are.

Let A be the corresponding primitive quotient. Now, A naturally contains primitive ideals I_k that correspond to the primitive ideals of $C^*(U(2^\infty))$ with arbitrarily large finite invariants. In particular, let M denote the branching matrix for A. So its entries are labeled by pairs of Young diagrams $\{\overline{\mu}; \lambda\}$, with $|\overline{\mu}| - |\lambda| = m (= 0)$, by assumption. The nodes of the Bratteli diagram are labeled by Young diagrams. The ideal I_k is specified by requiring that $\{\overline{\mu}; \lambda\} \in I_k$ if $|\lambda| > k$. The traces of A are given by the eigenvectors $\mathbf{v}$ of M. If the $\{\overline{\mu}; \lambda\}$-entries of $\mathbf{v}$ are 0 for $|\lambda| > k$, then $\mathbf{v}$ is an eigenvector with eigenvalue 4^k and corresponds to the character $\tau^k\overline{\tau}^k$. Hence, a faithful character would have to correspond to an eigenvector with an infinite eigenvalue. ∎

PROPOSITION 8. *$U(2^\infty)$ has no infinite characters.*

Proof. We use to introduce a ring multiplication between representations so we can apply the results of Wassermann [18, 3, Appendix]. Given the unitary groups

$U(2^M)$ and $U(2^N)$, there is a natural subgroup G of $U(2^{M+N})$ formed by the subgroup generated in the tensor product matrix algebra $\bigotimes_{k=1}^{M} M_2(\mathbb{C}) \otimes \bigotimes_{k=1}^{N} M_2(\mathbb{C})$. Note that the unitary group of this full tensor is $U(2^{M+N})$. Given representations π_1 and π_2 of $U(2^M)$ and $U(2^N)$, respectively, they naturally give a representation π_3 of G [11, p. 444]. We can then form their product by inducing the representation π_3 from G up to $U(2^{M+N})$. This multiplication will induce a multiplication on a certain algebraic completion of K_0-groups of the primitive quotients A of $C^*(U(2^\infty))$, just as in the $U(\infty)$ case, to make a ring (see the Appendix to [3]). If the product of two non-zero positive elements of this ring is non-zero, then we know that A admits no faithful infinite character. Now, let A be a primitive quotient by an ideal J_m with $|J| = \infty$, since we know the result already when $|J_m| < \infty$. Let π_1 be given by the diagrams $\{\overline{\mu}_1, \lambda_1\}$ and π_2 by $\{\overline{\mu}_2, \lambda_2\}$. By assumption, we know $|\overline{\mu}_i| - |\lambda_i| = m$, for $i = 1, 2$. We can avoid the use of the difficult decomposition sum with alternating terms in [11, 6.10] for mixed tensor representations by multiplying by an appropriate power of the determinant. We can make a further simplifying assumption by assuming that π_1 and π_2 are representations of the same unitary group $U(2^N)$, say, since we are working not with representations themselves by their images in the completed K_0-group of A. In this case, the appropriate power of the determinant to multiply either π_1 or π_2 is 2^N, which is the maximal number of columns for a $U(2^N)$ irreducible representation. By Frobenius reciprocity, to show that the product is non-zero it suffices to verify that there is a representation π of $U(2^{N+1})$ whose restriction to G contains π_3. If we were working just with representations in the postive signatures, this follows at once by the identity (6.1) in [11]. The mixed tensor case now follows from identity (6.8) in [11] since the determinant of $U(2^{N+1})$ restricts to give the 2^N- th power of the determinant on each $U(2^N)$. ∎

COMMENTS. 1.The ring multiplication on $K_0(A)$ where A is a primitve quotient in the positive signatures can be explicitly computed in low rank cases since the evaluation of (6.2) in [11] requires the decomposition of the tensor product of irreducible representations of the symmetric group. Extensive tables tabulating this decomposition exist.

2. The asymptotic methods of Kerov and Vershik [9] can also be applied to $U(2^\infty)$. For $U(\infty)$, we consider a normalized character restricted to elements of the form $(z, 1, 1, \ldots)$. In other words, we restrict a character to the canonical copy of $U(1)$ in $U(\infty)$. For $U(2^\infty)$, the corresponding elements are $(z, z, \ldots)$. In particular,

$$\frac{\{\lambda_N\}_{2^N}(z, z, \ldots, z)}{\dim(\lambda_N)} = z^{|\lambda_N|}.$$

Hence, for the sequence of normalized characters in positive signatures to possess a limit, we must have that the size of the diagrams eventually stabilizes. This reasoning can be extended to arbitrary sequences of characters by using the recent notion of rational Schur functions. This gives a statistical interpretation of the difference invariant m_J of a primitive ideal. Further, if we restrict the character to the canonical copy of $U(2)$ in $U(2^N)$, there appears a more complicated asymptotic behavior which we hope to investigate in the future.

We would also like to mention what happens if we consider τ^k as $k \to \infty$. The limit exists in the sense that it converges to the trivial character δ_0 of $U(2^\infty)$ which is zero everywhere except taking the value 1 at the identity. But δ_0 is not a continuous character.

We now show that the branching matrices M for the primitive quotients by positive primitive ideals can be completely analyzed spectrally. With this information, it is possible to completely describe the order structure on the K_0-group of the quotient. What is particularly surprising is that the eigenvectors for M are given by the columns of the character table for the symmetric group while the eigenvalues are powers of 2. These matrices fail to be symmetric already when $|J| \geqslant 6$. We discovered this result empirically by analyzing the table of branching matrices given below. Subsequently, D. Zeilberger kindly provided the proof given below. In the following Proposition, we use the notation from MacDonald [12]. Note that s_λ denotes the Schur function associated to the Young diagram λ.

PROPOSITION 9. *Let* $m_{\lambda\nu} = \langle \sum s_{\lambda/\mu} s_\mu, s_\nu \rangle$, *where* $|\lambda| = |\nu| = N$, *then*

$$\sum_\nu m_{\lambda,\nu} \chi_\rho^\nu = \chi_\rho^\lambda 2^r(\rho)$$

where $r(\rho)$ *is the number of rows of* ρ.

Proof. We need several identites from [12]:

$$\sum_\mu s_{\lambda/\mu}(x) s_\mu(y) = s_\lambda(x, y) \tag{8.1}$$

$$p_\rho = \sum_\nu \chi_\rho^\nu s_\nu \tag{8.2}$$

$$s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho \tag{8.3}$$

Also, we note that relative to the inner product $\langle \cdot, \cdot \rangle$, we have: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ $\langle p_\lambda, z_\mu^{-1} p_\mu \rangle = \delta_{\lambda,\mu}$.

Now, we let

$$s_\lambda(x) = s_\lambda(x_1, x_1, x_2, x_2, \ldots) = \sum_\rho z_\rho^{-1}\chi_\rho^\lambda p_\rho(x_1, x_1, x_2, x_2, \ldots). \tag{8.4}$$

But $p_m(x_1, x_1, x_2, x_2, \ldots) = x_1^m + x_1^m + x_2^m + x_m^2 + \cdots = 2p_m(x_1, x_2, x_3, \ldots)$. In particular, we have: $p_\rho(x_1, x_1, x_2, x_2, \ldots) = p_{\rho_1}(x_1, x_1, \ldots)\cdots p_{\rho_r}(x_1, x_1, \ldots) = 2^{r(\rho)}p_\rho$. Now, we observe that if $\sum_\mu s_{\lambda/\mu}s_\mu = \tilde{s}_\lambda$, then we have that $m_{\lambda\nu} = \langle \tilde{s}_\lambda, s_\nu\rangle$. By (8.1) with $y = x$, we see that:

$$\tilde{s}_\lambda(x_1, x_2, x_3, \ldots) = s_\lambda(x, x) =$$

$$= s_\lambda(x_1, x_2, x_3, \ldots, x_1, x_2, x_3, \ldots) = s_\lambda(x_1, x_1, x_2, x_2, \ldots)$$

So, we obtain

$$\tilde{s}_\lambda(x_1, x_2, \ldots) = s_\lambda(x_1, x_1, x_2, x_2, \ldots). \tag{8.5}$$

Now, we get:

$$\sum_\nu m_{\lambda,\nu} = \sum_\nu \langle \tilde{s}_\lambda, s_\nu\rangle \chi_\rho^\nu = \langle \tilde{s}_\lambda, \sum_\nu \chi_\rho^\nu s_\nu\rangle = \langle \tilde{s}_\lambda, p_\rho\rangle.$$

So,

$$\tilde{s}_\lambda = \sum_\rho z_\rho^{-1}2^{r(\rho)}\chi_\rho^\lambda p_\rho.$$

Hence, $\langle \tilde{s}_\lambda, p_\rho\rangle = 2^{r(\rho)}\chi_\rho^\lambda$, by the orthogonality of p_ρ. But $\sum_\nu m_{\lambda,\nu}\chi_\rho^\lambda = \langle \tilde{s}_\lambda, p_\rho\rangle$. ∎

COMMENT. An important special case is given by the classical Weyl duality between unitary and symmetric representations. By decomposing powers of the canonical trace of $U(2^N)$, we find the entries of the Perron eigenvector must be given by the dimensions of the irreducible representations of the symmetric group $S(k)$, where k is the power of the trace.

Let D be a diagonal operator on a separable Hilbert space H relative to an orthonormal basis $\{e_j\}_{j=1}^\infty$ such that $0 \leqslant D \leqslant I$. Let ω_D be the associated gauge-invariant quasi-free state on the CAR-algebra $\mathcal{A}$.

PROPOSITION 10. *Let $p(V) = \omega_D(V)$, where $V \in U(2^\infty)$. If $\mathrm{Tr}[D(I-D)] = \infty$, then $[p(V)]^2$ is factorial.*

Proof. It is a little surprising that the same factor condition arises here as in [2] or [16]. Now, we begin by recalling our version of the factor condition relative to the dynamical system (X, G, μ) attached to certain AF-algebras [4, 15]:

$$\lim_{N\to\infty} \sum_{i=1}^r |\mu_{(k+1,N)}(D_i) - \mu_{(1,N)}(D_i)| = 0.$$

In particular, we assume that $X = \varprojlim X_{(1,n)}$, where $X_{(i,j)}, i < j$, is the $(j-i+1)$--product of some fixed finite set F. Let μ_n be a probability measure on F. Then $\mu_{(k+1,N)} = \bigotimes_{n=k+1}^{N} \mu_n$ be a measure on $X_{(i+1,N)}$ and we let μ denote the limit measure on X. Choose a partition of F into r subsets $F_1, \dots, F_r$. We introduce the group of path permutations by having two paths γ and γ' being able to be permuted one onto the other if there exists an index n such that $\gamma(n)$ and $\gamma'(n)$ belong to the same subset F_i, say, and they agree for all larger indicies. By $\mu_{(k+1,N)}(D_i)$ we mean the measure of the set of finite paths that originate at index $k+1$ and whose terminal node at index N lies in the subset F_i.

In our case, $|F| = 4, r = 2$ and the indices are identified with the partitions of 2: (1^2) and (2). The Bratteli diagram is determined by the decompositions: $\Lambda^2(\mathbf{C}^{2^{N+1}}) \downarrow 3\Lambda^2(\mathbf{C}^{2^N}) \oplus S^2(\mathbf{C}^{2^N})$ and $S^2(\mathbf{C}^{2^{N+1}}) \downarrow \Lambda^2(\mathbf{C}^{2^N}) \oplus 3S^2(\mathbf{C}^{2^N})$.

Let $\{e_\alpha\}$ be the standard orthonormal basis for the full exterior algebra $\Lambda(\mathbf{C}^N)$ relative to the basis vectors $\{e_j\}_{j=1}^N$. Here, α is identified with a finite strictly increasing sequence: $1 \leqslant i_1 < i_2 < \cdots \leqslant N$. Suppose $De_j = p_j e_j$. Then $\omega_D(V) = \mathrm{Tr}[D^{(N)}V]$, where $D^{(N)}$ is the induced operator on the exterior algebra. So, we have:

$$p(V) = \omega_D\left(\sum_\alpha z_\alpha e_\alpha\right) = \sum_\alpha t_\alpha^{(N)} z_\alpha,$$

where $t_\alpha^{(N)} = p_{i_1} p_{i_2} \cdots p_{i_j} \cdot \Pi\{(1-p_i) : i \neq i_1, \dots, i_j, 1 \leqslant i \leqslant N\}$ [16].

Now, the difference, for fixed k, $|\mu_{(k+1,N)}(D_{1^2}) - \mu_{(1,N)}(D_{1^2})| = \sum_\alpha t_\alpha^{(N)} t_\beta^{(N)}$, where either α or β contains an index from the set $\{1, 2, \dots, k\}$. We shall denote the set of all indices α that have an index $\leqslant k$ by I_N. Since $\mathrm{Tr}[D(I-D)] = \infty$, we know that

$$\lim_{N\to\infty} \sum_{\gamma \in I_N} t_\gamma^{(N)} = 0.$$

We next make the estimate:

$$\sum_{\alpha,\beta \in I_N} t_\alpha^{(N)} t_\beta^{(N)} \leqslant \left(\sum_{\gamma \in I_N} t_\gamma^{(N)}\right)^2 + 2 \sum_{\gamma \in I_N} t_\gamma^{(N)}.$$

This follows because if we restrict both α and β to belong to I_N, then

$$\sum_{\alpha,\beta \in I_N} t_\alpha^{(N)} t_\beta^{(N)} \leqslant \left(\sum_{\gamma \in I_N} t_\gamma^{(N)}\right)^2.$$

In the other case, we restrict α to belong to I_N while β does not. Then, for fixed α,

$$\sum_{\beta\in I_N} t_\alpha^{(N)} t_\beta^{(N)} \leqslant t_\alpha^{(N)} \left(\sum_{\beta\in I_N} t_\beta^{(N)} \right) \leqslant t_\alpha^{(N)}.$$

Moreover, this argument extends to the difference: $|\mu_{(k+1,N)}(D_{(2)}) - \mu_{(1,N)}(D_{(2)})|$, since we never made use of the restriction that $\alpha \neq \beta$ that holds for the case (1^2) case. ∎

We shall define the special unitary group $SU(\mathcal{A})$. If we try to define the determinant Δ on $U(\mathcal{A})$ by an inductive limit, we find that consistency forces Δ on $U(2^n)$ to be the 2^n-th root of the usual determinant on $U(2^n)$. In particular, Δ cannot be defined on $U(\mathcal{A})$ by only on its universal covering group. On the other hand, if we require that the determinant be 1, this value is consistent with the imbeddings giving $U(2^\infty)$, so can construct $SU(2^\infty)$ and define $SU(\mathcal{A})$ as the closure of $SU(2^\infty)$ in $U(\mathcal{A})$.

An alternative definition is to define $SU(\mathcal{A})$ in terms of its Lie algebra. Let su$(\mathcal{A})$ denote the Lie algebra of all skew-adjoint elements in $\mathcal{A}$ with trace zero. Then let $SU(\mathcal{A})$ be the subgroup of $U(\mathcal{A})$ generated by the exponentiation of $su(\mathcal{A})$ (compare with Cuntz [6]). This definition allows us to consider the analogues of the other classical groups [8]. In any case, $SU(\mathcal{A})$ is a natural group associated to $\mathcal{A}$. The techniques of proof of Corollary 4 immediately yield the result:

PROPOSITION 11. *Every factor representation of $SU(\mathcal{A})$ is norm-continuous.*

COMMENT. It would be interesting to use the technique of Pickrell [14] to attempt to classify the characters or representations realized on a separable Hilbert space of the unitary group of the finite hyperfinite factor. We also note that a general results about norm continuous representations of the unitary groups of C^*-algebras are given in [13].

In [1], Arveson introduced the exponential e^A of a C^*-algebra A. Note that the elements of A are finite linear combinations of unitaries. Now, e^A is the direct sum of C^*-algebras $A^{(n)}$, where $A^{(n)}$ is the subalgebra of $A^{\bigotimes n}$ generated by the elementary tensors $a \otimes a \otimes a \otimes \cdots \otimes a$, $a \in A$. It is easy to see that the unitary group of $A^{(n)}$ corresponds to the n-th fold tensor product of the canonical self-representation of $U(A)$ in the algebra A. These observations have immediate application to the UHF algebra $\mathcal{A}$.

We now state:

PROPOSITION 12. *The components $A^{(n)}$ of the exponential of the algebra UHF(2^∞) are isomorphic to the primitive quotients of $C^*(U(2^\infty))$ by the ideal J_n in positive signatures.*

Appendix

Table of Branching Matrices

	2	1^2
2	3	1
1^2	1	3

eigenvalues: 4, 2

	3	21	1^3
3	4	2	0
21	2	6	2
1^3	0	2	4

eigenvalues: 8, 4, 2

	4	31	2^2	21^2	1^4
4	5	3	1	0	0
31	3	9	3	4	0
2^2	1	3	6	3	1
21^2	0	4	3	9	3
1^4	0	0	1	3	5

eigenvalues: 16, 8, 4^2, 2

	5	41	32	31^2	2^21	21^3	1^5
5	6	4	2	0	0	0	0
41	4	12	6	6	2	0	0
32	2	6	12	6	6	2	0
31^2	0	6	6	14	6	6	0
2^21	0	2	6	6	12	6	2
21^3	0	0	2	6	6	12	4
1^5	0	0	0	0	2	4	6

eigenvalues: 32, 16, 8^2, 4^2, 2

	6	51	42	41^2	3^2	321	2^3	31^3	2^21^2	2^21^2	1^6
6	7	5	3	1	0	0	0	0	0	0	0
51	4	15	8	9	3	4	0	0	0	0	0
42	3	9	18	9	6	12	3	3	1	0	0
41^2	0	8	9	19	3	12	1	9	3	0	0
3^2	1	3	6	3	10	8	1	1	3	0	0
321	0	4	12	12	8	28	8	12	12	4	0
2^3	0	0	3	1	1	8	10	3	6	3	1
31^3	0	0	3	9	1	12	3	19	9	8	0
2^21^2	0	0	1	3	3	12	6	9	18	9	3
21^4	0	0	0	0	0	4	3	9	8	15	4
1^6	0	0	0	0	0	0	0	1	3	5	7

eigenvalues: $64, 32, 16^2, 8^3, 4^3, 2$

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