J. OPERATOR THEORY
31(1994), 67–82

COISOMETRIC EXTENSION AND FUNCTIONAL CALCULUS FOR PAIRS OF COMMUTING CONTRACTIONS

ALFREDO OCTAVIO

INTRODUCTION AND PRELIMINARIES

In studying the properties of pairs of operators, one of the main concerns is the existence of functional calculi. While most of the work has focused on constructing a several variable version of the Riesz-Dunford functional calculus (cf. [8] and [17]), some attention has been given to a two variable version of the Nagy-Foias functional calculus (cf. [4]). The latter is the main tool in the theory of dual algebras. Even though we will not treat the topic of dual algebras generated by commuting contractions in this paper, it is important to point out that this work sprang from our interest in dual algebras.

The purpose of this paper is twofold: on one hand we aim at presenting a (modest) extension of the functional calculus for pairs of commuting contractions constructed in [4]. On the other hand we are interested in studying joint coisometric extensions of pairs of commuting contractions. Related dilations have been studied by Slocinski in [16]. This two topics are not without relation and we intend to exploit this relation in this and future work.

This paper is intended as background material for a series of papers developing the theory of dual algebras generated by commuting contractions. For this reason we shall present some details central to the developement of the theory, which are available elsewhere in the literature.

We now introduce some notation used throughout the paper. Let $\mathcal{H}$ be a complex, infinite dimensional, separable, Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded, linear operators on $\mathcal{H}$. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{C}$ the complex plane, $\mathbb{D}$ the open unit disk in $\mathbb{C}$, and $\mathbb{T} = \partial\mathbb{D}$ the unit circle. If $A \subseteq \mathbf{C}$ and $n \in \mathbf{N}$, we denote by A^n the Cartesian product of n copies of A. We shall call T^2 the *torus* and $\mathbf{D}^2$ the *bidisk*. The space $L^p(T)$ and $H^p(T), 1 \leqslant \leqslant p \leqslant \infty$, are the usual Lebesgue and Hardy spaces relative to the normalized Lebesgue measure m on $\mathbf{T}$. Let $\mathcal{B}$ be a Banach algebra. We denote by $\mathcal{B}^{(n)}$ the set of n-tuples $a = (a_1, a_2, \ldots, a_n)$ of elements of $\mathcal{B}$, and by $\mathcal{B}^{(n)}_{comm}$ the subset of $\mathcal{B}^{(n)}$ such that a_j commutes with a_j commutes with a_k, for all $j, k = 1, 2, \ldots, n$.

If X is a compact Hausdorff space, we denote by $C(X)$ the Banach algebra of all continuous, complex-valued functions on X under the supremum norm. The closure in $C(\mathbf{T})$ of the algebra of polynomials will be denoted by $A(\mathbf{D})$. The algebra $A(\mathbf{D})$ can be identified with the *disk algebra*, which is the algebra of all bounded analytic functions on $\mathbf{D}$ that have a continuous extension to $\mathbf{T}$.

1. BANDS OF MEASURES AND BOUNDED ANALYTIC FUNCTIONS ON THE BIDISK

We denote by $L^p(\mathbf{T}^2), 1 \leqslant p \leqslant \infty$, the Lebesgue spaces relative to normalized Lebesgue area measure m_2 on the torus $\mathbf{T}^2$. We shall need some facts from [14] about boundary values of analytic functions on the bidisk. The algebra $H^\infty(\mathbf{D}^2)$ of bounded analytic functions on $\mathbf{D}^2$ can be identified with a subspace of $L^2(\mathbf{T}^2)$. We now describe how this identification is achieved. Let $w_j = r_j e^{i\theta_j} \in \mathbf{D}$ and $\lambda_j = e^{i\varphi_j} \in \mathbf{T}$, $j = 1, 2$, let $w = (w_1, w_2)$ and $\lambda = (\lambda_1, \lambda_2)$. The *Poisson kernel* $P_w(\lambda)$ is the product

$$P_w(\lambda) = P_{r_1}(\theta_1 - \varphi_1) P_{r_2}(\theta_2 - \varphi_2),$$

where $P_r(\theta) = \dfrac{(1 - r^2)}{1 - 2r\, cos(\theta) + r^2}$ is the familiar Poisson kernel for the unit disk. The Poisson kernel has the following properties (cf. [14 Section 2.1]):

(a) $P_w(\lambda) > 0,\ \lambda \in \mathbf{T}^2,\ w \in \mathbf{D}^2$.

(b) $\displaystyle\int_{\mathbf{T}^2} P_w(\lambda) \mathrm{d}m_2(\lambda) = 1,\ w \in \mathbf{D}^2$.

(c) $\displaystyle P_w(\lambda) = \sum_{k=(k_1,k_2)\in\mathbf{Z}^2} r_1^{|k_1|} r_2^{|k_2|} e^{ik(\theta-\varphi)}$, where $k \cdot \theta = k_1\theta_1 + k_2\theta_2,\ w \in \mathbf{D}^2$.

If $f \in L^\infty(\mathbf{T}^2)$ its *Poisson integral* is

$$P[f](w) = \int_{\mathbf{T}^2} P_w(\lambda) f(\lambda) \mathrm{d}m_2(\lambda), \qquad w \in \mathbf{D}^2.$$

By using the series expansion in (c) and integrating, we see that $P[f]$ is biharmonic (i.e., harmonic in the two variables w_1 and w_2). Let $H^\infty(\mathbf{T}^2)$ be the set of functions in $L^p(\mathbf{T}^2)$ whose Poisson integral is analytic on $\mathbf{D}^2$. The map $f \mapsto P[f]$ is an isometric linear algebra isomorphism between $H^\infty(\mathbf{T}^2)$ and $H^\infty(\mathbf{D}^2)$. The following lemma describes the properties of $H^\infty(\mathbf{T}^2)$ (cf. [14 Chapter 2] and [4]).

LEMMA 1.1.

(a) *$H^\infty(\mathbb{T}^2)$ is a weak*-closed subalgebra of $L^\infty(\mathbb{T}^2)$.*

(b) *A bounded sequence $\{f_n\}$ in $H^\infty(\mathbb{T}^2)$ weak*-converges to f if and only if $\{P[f_n]\}$ converges pointwise to $P[f]$ on $\mathbb{D}^2$.*

(c) *The polynomials form a sequentially weak*-dense subalgebra of $H^\infty(\mathbb{T}^2)$.*

(d) *The isomorphism $f \mapsto P[f]$ is multiplicative on $H^\infty(\mathbb{T}^2)$.*

A function $\tilde{f} \in H^\infty(\mathbb{D}^2)$ can be recovered by finding the Poisson integral of the boundary values of $\tilde{f}$ (which exist almost everywhere, cf. [14, Section 2.3]).

The above discussion can be applied to other H^p spaces. For $1 \leqslant p < \infty$, we can define $H^p(\mathbb{T}^2)$ to be the subspace of $L^p(\mathbb{T}^2)$ consisting of all the functions $f \in L^p(\mathbb{T}^2)$ such that $P[f]$ is analytic on $\mathbb{D}^2$. Theorem 2.1.4 of [14] says that a function f in $L^p(\mathbb{T}^2)$ belongs to $H^p(\mathbb{T}^2)$ if its Fourier coefficients defined by

$$\widehat{f}(k) = \int_{\mathbb{T}^2} f(\lambda_1\lambda_2)\overline{\lambda}_1^{k_1}\overline{\lambda}_2^{k_2}\mathrm{d}m_2, \qquad k = (k_1, k_2) \in \mathbb{Z}^2,$$

are zero for every k outside of $\mathbb{Z}^2_+$ ($\mathbb{Z}^2_+ = \{(k_1, k_2) \in \mathbb{Z}^2 : k_1, k_2 \geqslant 0\}$).

NOTATION. In what follows we shall make no notational distinction between a function $h \in H^p(\mathbb{T}^2)(1 \leqslant p \leqslant \infty)$ and its analytic extension to $\mathbb{D}^2$, namely $P[h]$. We will write $h(w)$ to denote the value of the value of the function $P[h]$ at the point $w \in \mathbb{D}^2$.

Observe that $\mathbb{T}^2$ is only a small part of the boundary of $\mathbb{D}^2$. But it is the part that matters the most, since

$$||h||_\infty = \operatorname*{ess\,sup}_{\mathbb{T}^2} |h|, \qquad h \in H^\infty(\mathbb{D}^2).$$

In fact, $\mathbb{T}^2$ is usually called the *distinguished (or Bergman-Shilov) boundary* of $\mathbb{D}^2$, since values of a function on $\mathbb{D}^2$ can be recovered from the boundary values on $\mathbb{T}^2$.

As in the one variable case, we shall denote by $A(\mathbb{D}^2)$ the closure in $C(\mathbb{T}^2)$ of the algebra of polynomials. The algebra $A(\mathbb{D}^2)$ can be identified with the algebra of bounded analytic functions on $\mathbb{D}^2$ that have a continuous extension to $\mathbb{T}^2$. We call $A(\mathbb{D}^2)$ the *bidisk algebra*. The bidisk algebra is a uniform algebra with Shilov boundary $\mathbb{T}^2$ (cf. [9, Chapter 3]).

We follow [4] in the construction of the functional calculus. Other methods of constructing this functional calculus have been presented (cf. [10]) but the approach in [4] seems to be the only one suitable for our desired generalization. A complex, Borel measure μ on $\mathbb{T}^2$ is called an *annihilating measure* for $A(\mathbb{D}^2)$ if $\int_{\mathbb{T}^2} f\mathrm{d}\mu = 0$ for all $f \in A(\mathbf{D}^2)$. The set of all annihilating measures of $A(\mathbf{D}^2)$ will be denote by $A^\perp$. Let X be a measurable space, and denote by $M(X)$ the set of all (finite) complex measures defined on X. A set $\beta \subseteq M(X)$ is called a *band* on X if is satisfies

(a) If $\mu \in \beta$ and $\nu \in M(X)$ is absolutely continuous with respect to μ, then $\nu \in \beta$.

(b) If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence in β with ${}_{n=1}^{\infty}|\mu_n| < \infty$, then

$$\sum_{n=1}^{\infty} \mu_n \in \beta.$$

For more details on bands of measures confer [4]. If β is a band on X we denote by $\beta^\perp$ the set of all complex measures on X singular with respect to every measure in β. Note that both $\beta^\perp$ and $A^\perp$ are bands on $\mathbf{T}^2$. Following [4] we define three additional bands on $\mathbf{T}^2$ as follows:

$$\beta_1 = \{\mu \in M(\mathbf{T}^2) : \mu \text{ is carried by } E \times \mathbf{T}, E \text{ a Borel set, } m(E) = 0\},$$

$$\beta_2 = \{\nu \in M(\mathbf{T}^2) : \nu \text{ is carried by } \mathbf{T} \times F, F \text{ a Borel set, } m(F) = 0\},$$

and

$$\beta_0 = A^\perp \cap \beta_1^\perp \cap \beta_2^\perp.$$

The following is Lemma 2.1 of [4].

LEMMA 1.2. *Let $\mu \in \beta_0$, and let $\{f_n\}$ be a bounded sequence in $A(\mathbf{D}^2)$. Then $f_n \mapsto 0$ pointwise on $\mathbf{D}^2$ if and only if $f_n \mapsto 0$ in the weak* topology of $L^\infty(\mathbf{T}^2, |\mu|)$.*

We conclude this section by showing that $H^\infty(\mathbf{T}^2)$ is the dual of a quotient space. Let $\mathcal{X}$ be a Banach space. The dual of $\mathcal{X}$ will be denoted $\mathcal{X}^*$. If $S \subset \mathcal{X}^*$ we denote by ${}^aS$ the *preannihilator* of S (i.e. ${}^aS = \{x \in \mathcal{X} : y(x) = 0 \text{ for all } y \in S\}$). It is well known that the dual space of $L^1(\mathbf{T}^2)$ can be identified with $L^\infty(\mathbf{T}^2)$. Since $H^\infty(\mathbf{T}^2)$ is a weak*-closed subspace of $L^\infty(\mathbf{T}^2)$ (Lemma 1.1 (a)), we have that $H^\infty(\mathbf{T}^2)$ can be identified with the dual of the space $\dfrac{L^1(\mathbf{T}^2)}{{}^a(H^\infty(\mathbf{T}^2))}$ (cf. [5, Proposition 2.1]). Define $L^1_0(\mathbf{T}^2)$ to be the subspace of $L^1(\mathbf{T}^2)$ consisting of those functions $f \in L^1(\mathbf{T}^2)$ such that $f(-n_1, -n_2) = 0$ for all $(n_1, n_2) \in \mathbf{Z}^2_+ = \{(n_1, n_2) \in \mathbf{Z} : n_1, n_2 \geqslant 0\}$.

PROPOSITION 1.3. *The space $L^1_0(\mathbf{T}^2)$ is isomorphic to ${}^a(H^\infty(\mathbf{T}^2))$.*

Proof. To every bounded linear functional φ acting on $L^1(\mathbf{T}^2)$ there corresponds a unique function $g \in L^\infty(\mathbf{T}^2)$ related by

$$\varphi(f) = \int_{\mathbf{T}^2} f(\lambda) g(\lambda) \mathrm{d}m_2, \quad f \in L^1(\mathbf{T}^2).$$

If $f \in {}^a(H^\infty(\mathbb{T}^2)) \subset L^1(\mathbb{T}^2)$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$, then

$$\int_{\mathbb{T}^2} f(\lambda)\lambda_1^{n_1}\lambda_2^{n_2}\mathrm{d}m_2 = 0, \quad (n_1, n_2) \in \mathbb{Z}_+^2.$$

But these are the Fourier coefficients $f(-n_1, -n_2)$ of f, and thus by the above definition, $f \in L_0^1(\mathbb{T}^2)$.

On the other hand, if $f \in L_0^1(\mathbb{T}^2)$, then

$$\int_{\mathbb{T}^2} f(\lambda)\lambda_1^{n_1}\lambda_2^{n_2}\mathrm{d}m_2 = \widehat{f}(-n_1, -n_2) = 0, \quad (n_1, n_2) \in \mathbb{Z}_+^2.$$

Since a function $g \in H^\infty(\mathbb{T}^2)$ is the weak*-limit of some sequence of polynomials $\{p_n\}$, we have that for any function $h \in L^1(\mathbb{T}^2)$

$$\int_{\mathbb{T}^2} h(\lambda)p_n(\lambda)\mathrm{d}m_2 \to \int_{\mathbb{T}^2} h(\lambda)g(\lambda)\mathrm{d}m_2.$$

Thus, since $f \in L_0^1(\mathbb{T}^2) \subset L^1(\mathbb{T}^2)$, and

$$\int_{\mathbb{T}^2} f(\lambda)p_n(\lambda)\mathrm{d}m_2 = 0,$$

we have

$$\int_{\mathbb{T}^2} f(\lambda)g(\lambda)\mathrm{d}m_2 = 0$$

for every $g \in H^\infty(\mathbb{T}^2)$. Thus, $f \in {}^a(H^\infty(\mathbb{T}^2))$. ■

By the above proposition we have that the dual of $\dfrac{L^1(\mathbb{T}^2)}{L_0^1(\mathbb{T}^2)}$ is isometrically isomorphic to $H^\infty(\mathbb{T}^2)$.

REMARK 1.4. *The results in this section can be trivially extended to higher dimensions. The proofs remain virtually unchanged.*

In this section we discuss the existence of joint dilations and extensions of certain types for an n-tuple $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\mathrm{comm}}$ of contraction operators (i.e., $\|T_j\| \leqslant 1$ for $j = 1, 2, \ldots, n$).

If $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\mathrm{comm}}$ a *joint dilation* of T is a n-tuple $S = (S_1, S_2, \ldots, S_n) \in \mathcal{L}(\mathcal{K})^{(n)}_{\mathrm{comm}}$ for some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{K}$ has a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{H} \oplus \mathcal{K}_2$ relative to which each S_j, $j = 1, 2, \ldots, n$, has a matrix of the form

$$S_j = \begin{pmatrix} S_{1,1}^{(j)} & S_{1,2}^{(j)} & S_{1,3}^{(j)} \\ 0 & T_j & S_{2,3}^{(j)} \\ 0 & 0 & S_{3,3}^{(j)} \end{pmatrix}.$$

Note that this implies, in particular, that $\mathcal{K}_1 \oplus (0) \oplus (0)$ and $\mathcal{K}_1 \oplus (0) \oplus (0)$ are common invariant subspaces for S, and consequently, $\mathcal{H} = (\mathcal{K}_1 \oplus \mathcal{H}) \ominus \mathcal{K}_1$ is a common semi-invariant subspace for S. In this situation it follows easily that if $m_1, m_2, \ldots, m_n$ are nonnegative integers, then

$$P_{\mathcal{H}} S_1^{m_1} S_2^{m_2} \ldots S_n^{m_n} x = T_1^{m_1} T_2^{m_2} \ldots T_n^{m_n} x, \quad x \in \mathcal{H}.$$

The equation above implies the existence of a joint dilation as we defined it (cf. [15]). Some authors take it as the definition of joint dilation (cf. [11]).

A *joint extension* of T is an n-tuple $B = (B_1, B_2, \ldots, B_n) \in \mathcal{L}(\mathcal{K})^{(n)}_{\text{comm}}$ such that $\mathcal{K} \supseteq \mathcal{H}$, the subspace $\mathcal{H}$ is a common invariant subspace for B, and

$$T_i x = B_i x \quad x \in \mathcal{H},\ i = 1, 2, \ldots, n.$$

If $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ is an n-tuple of contradictions, then a joint dilation (extension) $S = (S_1, S_2, \ldots, S_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ where $\mathcal{K} \supseteq \mathcal{H}$, is said to be a joint unitary (resp., isometric, coisometric) dilation (extension) if each of the operators $S_j,\ j = 1, 2, \ldots, n$, is a unitary (resp., an isometry, a coisometry).

Now, we will define the concept of "minimal" dilation (extension) for an n-tuple of contractions. To see that our definition makes sense in the case of dilations we will need the following

LEMMA 2.1. *If $U = (U_1, U_2, \ldots, U_n) \in \mathcal{L}(\mathcal{K})^{(n)}_{\text{comm}}$ is a joint dilation for $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ and $\mathcal{K}' \supset \mathcal{H}$ is a common invariant subspace for U, then $U|\mathcal{K}' = (U_1|\mathcal{K}', U_2|\mathcal{K}', \ldots, U_n|\mathcal{K}' \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ is a joint dilation of T.*

Proof. It is enough to show that if $\mathcal{H} = \mathcal{N}_1 \ominus \mathcal{N}_2$, where $\mathcal{N}_1 \supset \mathcal{N}_2$ and $\mathcal{N}_j$ is a common invariant subspace for U, $j = 1, 2$, then $\mathcal{H} = (\mathcal{N}_1 \cap \mathcal{K}') \ominus (\mathcal{N}_2 \cap \mathcal{K}')$. To see this we take $x \in \mathcal{H} = \mathcal{N}_1 \ominus \mathcal{N}_2$. Then $x \in \mathcal{N}_1 \cap \mathcal{K}'$, x is orthogonal to $\mathcal{N}_2$, and hence, to $\mathcal{N}_2 \cap \mathcal{K}'$. Therefore, $x \in (\mathcal{N}_1 \cap \mathcal{K}') \ominus (\mathcal{N}_2 \cap \mathcal{K}')$, and

$$\mathcal{H} = \mathcal{N}_1 \ominus \mathcal{N}_2 \subseteq (\mathcal{N}_1 \cap \mathcal{K}') \ominus (\mathcal{N}_2 \cap \mathcal{K}').$$

Thus, we can decompose the space $(\mathcal{N}_1 \cap \mathcal{K}') \ominus (\mathcal{N}_2 \cap \mathcal{K}')$ as $\mathcal{H} \oplus \mathcal{H}'$ (where $\mathcal{H}' = [(\mathcal{N}_1 \cap \mathcal{K}') \ominus (\mathcal{N}_2 \cap \mathcal{K}')] \ominus \mathcal{H}$.) If $x \in (\mathcal{N}_1 \cap \mathcal{K}') \ominus (\mathcal{N}_2 \cap \mathcal{K}')$, we can decompose x as $x = x_1 \oplus x_2$, where $x_1 \in \mathcal{H}$ and $x_2 \in \mathcal{H}'$. To finish the proof it is enough to show that $x_2 = 0$. The vector x_2 is in $\mathcal{N}_1 \cap \mathcal{K}' \subset \mathcal{N}_1$. On the other hand, since $x_2 \in \mathcal{H}'$, then x_2 is orthogonal to $\mathcal{N}_1 \ominus \mathcal{N}_2 = \mathcal{H}$. Thus, $x_2 \in \mathcal{N}_2$. Since $x_2 \in \mathcal{K}'$, we have that $x_2 \in \mathcal{N}_2 \cap \mathcal{K}'$. But x_2, is orthogonal to $\mathcal{N}_2 \cap \mathcal{K}'$. Therefore, $x_2 = 0$. ■

Let $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ be an n-tuple of contractions and let $U = (U_1, U_2, \ldots, U_n) \in \mathcal{L}(\mathcal{K})^{(n)}_{\text{comm}}$ be a joint isometric (resp., a unitary, a coisometric) dilation(extension) of T. We say that U is a *minimal* joint isometric (resp. unitary, coisometric) dilation (extension) if there is no nontrivial subspace $\mathcal{K}' \supset \mathcal{H}$ for U such that $U_j|\mathcal{K}'$, $j = 1, 2, \ldots, n$, is an isometry (resp., a unitary, a coisometry). Note that contrary to the one variable case, all minimal unitary dilations of a pair of contractions are not isomorphic.

The question whether an n-tuple $T = (T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ of contractions has a joint unitary dilation is an important one, and it turns out, rather surprisingly, that the answer depends on n. The following beautiful theorem was proved by Ando in [1] (cf. also [11], Theorem 6.4]).

THEOREM 2.2. *Every pair of commuting contractions has a joint unitary dilation, and thus (Lemma 2.1) a minimal joint unitary dilation.*

On the other hand, we have the following striking example by Parrot [12] (cf. also [11]).

EXAMPLE 2.3. For every $n \geqslant 3$, there exist an n-tuple of commuting contractions which has no joint unitary dilation (cf. EXAMPLE 2.4.3.).

Here we will need not only Theorem 2.2, but the following easy consequence of the proof given by Ando [1].

THEOREM 2.4. *A pair $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ of contractions has a joint coisometric extension, and thus a minimal joint coisometric extension.*

Proof. Let $(V_1, V_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be the joint isometric dilation of (T_1^*, T_2^*) constructed in [1]. In [1] it is shown that $\mathcal{K} \ominus \mathcal{H}$ is a common invariant subspace for (V_1, V_2). Thus, $\mathcal{H}$ is a common invariant subspace for (V_1^*, V_2^*). Define $B_j = V_j^*$, for $j = 1, 2$. We have that B_1 commutes with B_2, the space $\mathcal{H}$ is a common invariant subspace for (B_1, B_2), and $T_j = B_j|\mathcal{H}$ for $j = 1, 2$. Note that a joint isometric dilation $(V_1, V_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ of a pair of contractions is minimal if and only if $\mathcal{K} = \mathcal{K}'$, where $\mathcal{K}'$ is the closed linear span of $V_1^{n_1} V_2^{n_2} \mathcal{H}$ (cf. [11]). Hence, by reducing to $\mathcal{K}'$, and noticing that $B_j|\mathcal{K}'$ is a coisometry, it is clear that we can find a minimal joint coisometric extension. ■

We now take time to explore this phenomenon in some detail. Let $T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a pair of contractions, and $B = (B_1, B_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a joint coisometric extension of T, so that $\mathcal{K} \supset \mathcal{H}$, $\mathcal{H}$ is a common invariant subspace for B, and $B_j|\mathcal{H} = T_j$, $j = 1, 2$. Then, of course, B_1 commutes with B_2, and by the von Neumann-Wold decomposition theorem for isometries, one knows that $\mathcal{K}$ has decompositions $\mathcal{K} = \mathcal{S}_j \oplus \mathcal{R}_j$, $j = 1, 2$, such that $\mathcal{S}_j, \mathcal{R}_j$ are reducing subspaces for B_j, $j = 1, 2$ and $B_j|\mathcal{S}_j = \mathcal{S}_j^*$, $B_j|\mathcal{R}_j = R_j$, $j = 1, 2$, where $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ are backward shift operators of some multiplicity and R_1 and R_2 are unitary operators.

The reader who is familiar with the theory of dual algebras "in one variable" will have no difficulty understanding that a question of primary interest for us is how these decompositions $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ and $\mathcal{K} = \mathcal{S}_2 \oplus \mathcal{R}_2$ are related to each other. In this direction we have the following theorem. For a related result confer [16].

THEOREM 2.5. *Suppose $(B_1, B_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ is a pair of coisometries and that $\mathcal{K}$ has the decomposition $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ relative to which the matrix for B_1 is*

$$B_1 = \begin{pmatrix} S_1^* & 0 \\ 0 & R_1^* \end{pmatrix},$$

where $S_1^* \in \mathcal{L}(\mathcal{S}_1)$ is a backward shift and $R_1 \in \mathcal{L}(\mathcal{R}_1)$ is a unitary operator. Then the matrix of B_2 with respect to the decomposition $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ has the form

$$B_2 = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix},$$

where $A_4 = 0$, $A_3A_3^* = 1$, $A_2A_3^* = 0$, $A_1A_1^* + A_2A_2^* = 1$, $S_1^*A_2 = A_2R_1$, A_3 commutes with R_1, and A_1 commutes with S_1^*. Furthermore, if R_1 has no part of uniform multiplicity $\aleph_0$ or B_1 and B_2 are doubly commuting isometries (i.e., $B_1B_2 = B_2B_1$ and $B_1B_2^* = B_2^*B_1$), then $A_2 = 0$.

Proof. Since B_1 commutes with B_2, an elementary matricial calculation shows that $R_1A_4 = A_4S_1^*$, from which we obtain $R_1^nA_4 = A_4S_1^{*n}$, $n \in \mathbf{N}$ and since $\{S_1^{*n}\}$ converges strongly to zero, and R_1 is unitary, it follows immediately that $A_4 = 0$. A similar argument shows that if B_1 and B_2 are doubly commuting isometries, then $A_2 = 0$. Some additional easy matricial calculations now establish the other equations. Last, suppose R_1 has no part of uniform multiplicity $\aleph_0$. Then, it follows easily from the fact that $A_3R_1 = R_1A_3$ and the theory of spectral multiplicity (cf. [6]) that the coisometry A_3 has a decomposition $A_3 = \bigoplus_{n\in\mathbf{N}} A_3^{(n)}$, where $A_3^{(n)}$ is an n-normal operator (cf. [13]). We now need the following lemma.

LEMMA 2.6. *If $A \in \mathcal{L}(\mathcal{H})$ is an n-normal isometry for some $n \in \mathbf{N}$, then A is a unitary operator.*

Proof. One knows (cf. [13]) that A can be identified (up to a *-algebra isomorphism) with a continuous $A(\cdot) : \mathbf{X} \to M_n$, where $\mathbf{X}$ is an extremally disconnected compact Hausdorff space and M_n is the ring of $n \times n$ complex matrices. Thus, for each $x \in \mathbf{X}$, $A(x)$ is an isometry acting on a finite dimensional space, and thus is a unitary matrix. Hence $A(\cdot)A^*(\cdot) = A^*(\cdot)A(\cdot)$, and thus A is a unitary operator. ■

Completion of Proof of Theorem 2.5. It follows easily from this lemma that each $A_3^{(n)}$ in the decomposition $A_3 = \bigoplus_{n\in\mathbf{N}} A_3^{(n)}$, is a unitary operator. Thus, since $A_2A_3^* = 0$, we have $A_2 = 0$, as desired. ■

Is the operator A_2 in the above theorem necessarily zero without the hypothesis on R_1? The answer, unfortunately, is no, as the next example shows.

EXAMPLE 2.7. Let $B_1 \in \mathcal{L}(\mathcal{H})$ be the coisometry with matricial form

$$B_1 = \begin{pmatrix} S_1^* & 0 \\ 0 & R_1 \end{pmatrix},$$

with respect to the decomposition $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$, where $S_1^* \in \mathcal{L}(\mathcal{S}_1)$ is a backward shift and $R_1 \in \mathcal{L}(\mathcal{R}_1)$ is a unitary operator with uniform multiplicity $\aleph_0$. Write $\mathcal{S}_1 = \bigoplus_{n\in\mathbb{N}} \mathcal{M}$, and $\mathcal{R}_1 = \bigoplus_{n\in\mathbb{N}} \mathcal{N}$, for some Hilbert spaces $\mathcal{M}$ and $\mathcal{N}$. The operator S_1^* is defined, relative to this decomposition of $\mathcal{S}_1$, by the (infinite) matrix

$$S_1^* = \begin{pmatrix} 0 & 1_{\mathcal{M}} & 0 & & \cdots \\ & 0 & 1_{\mathcal{M}} & 0 & \cdots \\ & & 0 & 1_{\mathcal{M}} & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

and R_1 is defined, relative to this decomposition of $\mathcal{R}_1$, by the (infinite) matrix

$$R_1 = \begin{pmatrix} U & 0 & & \cdots \\ 0 & U & 0 & \cdots \\ 0 & 0 & U & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix},$$

where U is a (forward) bilateral shift of multiplicity one in $\mathcal{L}(\mathcal{N})$.

The coisometry B_2 will have the form

$$B_2 = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

with respect to the decomposition $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$. We wish to define A_1, A_2 and A_3 so that B_2 is a coisometry commuting with B_1, and a short calculation (cf. Theorem 2.5.) shows that this will be done provied

$$A_1 S_1^* = S_1^* A_1, \tag{1}$$

$$R_1 A_3 = A_3 R_1, \tag{2}$$

$$A_3 A_3^* = 1_{\mathcal{R}_1}, \tag{3}$$

$$A_2 A_3^* = 0, \tag{4}$$

$$A_1 A_1^* + A_2 A_2^* = 1_{\mathcal{S}_1}, \tag{5}$$

and

$$S_1^* A_2 = A_2 R_1. \tag{6}$$

To make (1) valid, we take A_1 to be the operator in $\mathcal{L}\left(\bigoplus_{n\in\mathbf{N}} \mathcal{M}\right)$ defined by the matrix

$$A_1 = \begin{pmatrix} 0 & C & 0 & & \dots \\ & 0 & C & 0 & \dots \\ & & 0 & C & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

where $C \in \mathcal{L}(\mathcal{M})$ is to be determined later. To make (3) valid, we take A_3 to be the operator in $\mathcal{L}\left(\bigoplus_{n\in\mathbf{N}} \mathcal{N}\right)$ defined by the matrix

$$A_3 = \begin{pmatrix} 0 & 1_{\mathcal{N}} & 0 & & \dots \\ & 0 & 1_{\mathcal{N}} & 0 & \dots \\ & & 0 & 1_{\mathcal{N}} & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

Note that (2) also holds with this definition. To make (4) valid, we define A_2 to be the operator in $\mathcal{L}\left(\bigoplus_{n\in\mathbf{N}} \mathcal{N}, \bigoplus_{n\in\mathbf{N}} \mathcal{M}\right)$ given by the matrix

$$A_2 = \begin{pmatrix} D_1 & 0 & 0 & \dots \\ D_2 & 0 & 0 & \dots \\ D_3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $D_j \in \mathcal{L}(\mathcal{N}, \mathcal{M})$, $j \in \mathbf{N}$ is to be defined in order to make (6) valid. For this purpose it is enough to define D_j, $j \in \mathbf{N}$, so that it satisfies $D_j U = D_{j+1}$. We now proceed to do this. Let $\{e_j\}_{j=-\infty}^{\infty}$ be an orthornormal basis for the space $\mathcal{M}$ and let $\{f_j\}_{j=-\infty}^{\infty}$ be an orthonormal basis for the space $\mathcal{N}$. The operator D_k, $k = 1, 2, \dots$, is defined by $D_k f_j = 0$, $j \neq -k$, and $D_k f_j = e_0$, $j = -k$. Finally, the operator $C \in \mathcal{L}(\mathcal{N})$ is defined so that (5) holds by $Ce_j = e_j$, $j \neq 0$, $Ce_0 = 0$.

3. ABSOLUTELY CONTINUOUS PAIRS OF CONTRACTIONS

Let $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a pair of contradictions, let $(U_1, U_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a joint minimal unitary dilation of (T_1, T_2) and let E_j be the spectral measure of U_j, $j = 1, 2$. Following [2], we can form the amalgamation (or product) of E_1 and E_2, thus obtaining a projection valued measure E on Borel subsets of the torus. This measure is called the *joint spectral measure* of (U_1, U_2). As is shown in [2],

$$U_1^{n_1} U_2^{n_2} = \int_{\mathbf{T}^2} \lambda_1^{n_1} \lambda_2^{n_2} \mathrm{d}E(\lambda_1 \lambda_2), \quad n_1, n_2 \in \mathbf{N}.$$

The spectral measures E_1, E_2 and E satisfy the following additional properties (cf. [2]):

(a) E_1 commutes with E_2 (i.e., if M_1 and M_2 are Borel subsets of $\mathbb{T}$, then the projections $E_1(M_1)$ and $E_2(M_2)$ commute).

(b) If M_1 and M_2 are Borel subsets of $\mathbb{T}$, then $E(M_1 \times M_2) = E_1(M_1)E_2(M_2)$.

For $x \in \mathcal{K}$, we denote by μ_x the measure on the Borel subsets of $\mathbb{T}^2$ given by

$$\mu_x(M) = \langle E(M)x, x\rangle$$

for Borel subsets $M \subseteq \mathbb{T}^2$. We say that E is *absolutely continuous* with respect to some (finite, Borel, positive) measure $\nu \in M(\mathbb{T}^2)$ if μ_x is absolutely continuous with respect to ν for all $x \in K$. We say that $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ belongs to the class $ACC^{(2)}(\mathcal{H})$ of absolutely continuous pairs of the contractions if the joint spectral measure of (some) minimal unitary dilation is absolutely continuous with respect to some (positive) measure ν in β_0. The following is a reformulation of Lemma 4.3 of [4].

LEMMA 3.1. *If $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ is a pair of completely nonunitary contractions, then $(T_1, T_2) \in ACC^{(2)}(\mathcal{H})$.*

There are several questions one could ask about membership in the class $ACC^{(2)}(\mathcal{H})$. One question is : In the above definition, can we replace "some measure ν in β_0" by "m_2"? In other words, if $(T_1, T_2) \in ACC^{(2)}(\mathcal{H})$, and (U_1, U_2) and E are as above, is E absolutely continuous with respect to m_2? The following example provides a negative answer to this question.

EXAMPLE 3.2. For $\varphi \in H^\infty(\mathbb{T})$, let M_φ denote the multiplication operator defined on $L^2(\mathbb{T})$ by

$$M_\varphi f = \varphi f, \quad f \in L^2(\mathbb{T}).$$

Let $T_1 = T_2 = M_z|H^2(\mathbb{T})$, where z denotes the position function $z = e^{i\theta} \mapsto e^{i\theta}$. One knows that the operator $M_z|H^2(\mathbb{T})$ is a completely nonunitary contraction whose minimal unitary dilation is M_z. The spectral measure of M_z, denoted by E_1, has support $\mathbb{T}$, and is given by

$$E_1(A) = M_{\chi_a},$$

for every Borel subset A of $\mathbb{T}$. Thus, by Lemma 2.3.1, $(T_1, T_2) \in ACC^{(2)}(\mathcal{H})$. What is the support of the amalgamation of E_1 with itself (denoted $E_1 \times E_1$)? A simple calculation (using property (b) above), permits us to conclude that if A_1 and A_2 are disjoint Borel subsets of $\mathbb{T}$, then $(E_1 \times E_1)(A_1 \times A_2) = E_1(A_1)E_1(A_2) = 0$. By the construction of the amalgamation (cf. [2]), this implies that the "diagonal"

$\{(\lambda,\lambda) : \lambda \in \mathbf{T}\}$ contains the support of $E_1 \times E_1$. Thus, $E_1 \times E_1$ is not absolutely continuous with respect to m_2.

Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be absolutely continuous if the spectral measure of its minimal unitary dilation is absolutely continuous with respect to m (normalized Lebesgue measure on $\mathbf{T}$). An important question whose answer has, thus far, eluded us is as follows:

PROBLEM 3.3. *If $(T_1,T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ is a pair of absolutely continuous contractions, is $(T_1,T_2) \in ACC^{(2)}(\mathcal{H})$?*

The following lemma provides us with some additional members of the class $ACC^{(2)}(\mathcal{H})$. Its proof follows easily from the properties of a minimal joint unitary dilation and our construction of a coisometric extension (cf. Theorem 2.4).

LEMMA 3.4.

(a) *If $(T_1,T_2) \in ACC^{(2)}(\mathcal{H})$, then $(T_1^*,T_2^*) \in ACC^{(2)}(\mathcal{H})$.*

(b) *If $(T_1,T_2) \in ACC^{(2)}(\mathcal{H})$, then there is a minimal joint isometric dilation $(V_1,V_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ of (T_1,T_2) such that $(V_1,V_2) \in ACC^{(2)}(\mathcal{K})$.*

(c) *If $(T_1,T_2) \in ACC^{(2)}(\mathcal{H})$, then there is a minimal joint coisometric extension $(B_1,B_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ of (T_1,T_2) such that $(B_1,B_2) \in ACC^{(2)}(\mathcal{K})$.*

4. THE FUNCTIONAL CALCULUS

Let $\mu \in \beta_0$ be a (positive) measure. We define the algebra $H^\infty(\mu+m_2)$ as the weak* closure of the bidisk algebra $A(\mathbf{D}^2)$ in $L^\infty(\mathbf{T}^2,\mu+m_2)$ (the space of essentially bounded functions on $\mathbf{T}^2$ with respect to the measure $\mu+m_2$). The Poisson integral (cf. Section 1),

$$P[f](w) = \int_{\mathbf{T}^2} P_w(\lambda)f(\lambda)\,dm_2(\lambda), \quad w \in \mathbf{D}^2,$$

can be used to define the map $f \mapsto P[f]$, which is an isometric isomorphism from $H^\infty(\mu+m_2)$ onto $H^\infty(\mathbf{D}^2)$ (the onto part follows from the weak*-density of polinomials and Lemma 1.2). The remarks in Section 1 about the relation between $H^\infty(\mathbf{T}^2)$ and $H^\infty(\mathbf{D}^2)$ apply to $H^\infty(\mu+m_2)$. We define a map $\Psi : H^\infty(\mathbf{T}^2) \to H^\infty(\mu+m_2)$ by extending the identity map from $A(\mathbf{D}^2)$ onto itself in such a way as to make Ψ a weak*-continuous algebra isomorphism. The map Ψ is defined in the following way: For $f \in A(\mathbf{D}^2)$, we define $\Psi(f) = f$, for $h \in H^\infty(\mathbf{T}^2)$ we approximate h in the weak*-topology by a (bounded) sequence $\{h_n\}$ in $A(\mathbf{D}^2)$ (cf. Lemma 1.1(c)). For each $w \in \mathbf{D}^2$ the sequence $\{h_n(w)\}$ converges to $h(w) = P[h](w)$ by Lemma 1.1(b). Hence, Lemma 1.2 implies that the sequence $\{h_n\}$ is a weak*-Cauchy sequence in

$H^\infty(\mu+m_2)$. Let $\widetilde{h} \in H^\infty(\mu+m_2)$ be the weak*-limit of the sequence $\{h_n\}$ in $H^\infty(\mu+m_2)$. We define $\Psi(h)=\widetilde{h}$. The map Ψ is clearly multiplicative since it is multiplicative on $A(\mathbb{D}^2)$. Since sequences are enough to determine weak* continuity (cf. [5, Theorem 2.3]), we have the following:

PROPOSITION 4.1. *If $\mu \in \beta_0$, then there is a weak*-homeomorphism Ψ from $H^\infty(\mu+m_2)$ onto $H^\infty(\mathbb{T}^2)$. Furthermore, Ψ is an algebra isomorphism.*

Let $\mathcal{A}_{T_1,T_2}$ denote the dual algebra generated by (T_1,T_2) (i.e., the smallest weak*-closed algebra with identity containing T_1 and T_2) (cf. [3]). It is well known that $\mathcal{A}_{T_1,T_2}$ can be identified with the dual of a quotient space (cf. [3]). We denote this quotient space by Q_{T_1,T_2}.

Finally, the following theorem constructs a functional calculus and lists its properties. This theorem is a modest generalization of Theorem 4.4 of [4].

THEOREM 4.2. *If $(T_1,T_2) \in ACC^{(2)}(\mathcal{H})$, then there is an algebra homomorphism $\Phi_{T_1,T_2}: H^\infty(\mathbb{T}^2) \to \mathcal{A}_{T_1,T_2}$ with the following properties:*

(a) *$\Phi_{T_1,T_2}(1)=I_{\mathcal{H}}$, $\Phi_{T_1,T_2}(w_1)=T_1$, $\Phi_{T_1,T_2}(w_2)=T_2$, where w_1 and w_2 denote the coordinate functions.*

(b) *$\|\Phi_{T_1,T_2}(h)\| \leqslant \|h\|_\infty$, for all $h \in H^\infty(\mathbb{T}^2)$.*

(c) *Φ_{T_1,T_2} is weak* continuous. (i.e., continuous when both H^∞ and $\mathcal{A}_{T_1,T_2}$ are given the corresponding weak*-topologies).*

(d) *The range of Φ_{T_1,T_2} is weak*-dense in $\mathcal{A}_{T_1,T_2}$.*

(e) *There is a bounded, linear, one-to-one map*

$$\Phi_{T_1,T_2}: Q_{T_1,T_2} \to \frac{L^1(\mathbb{T}^2)}{L^1_0(\mathbb{T}^2)}$$

with $\varphi^*_{T_1,T_2}=\Phi_{T_1,T_2}$.

(f) *If Φ_{T_1,T_2} is an isometry, then it is a weak*-homeomorphism onto $\mathcal{A}_{T_1,T_2}$ and φ_{T_1,T_2} is an isometry onto $\dfrac{L^1(\mathbb{T}_2)}{L^1_0(\mathbb{T}^2)}$.*

Proof. Parts (a), (b), and (c) were proved in [4, Theorem 4.4]. For the sake of completeness we present an outline of the proof. Let $(U_1,U_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a joint minimal unitary dilation of (T_1,T_2) whose joint spectral measure E is absolutely continuous with respect to some (positive) measure $\mu \in \beta_0$. By Proposition 4.1, it sufficies to construct a homomorphism with domain $H^\infty(\mu+m_2)$ instead of $H^\infty(\mathbb{T}^2)$. Let $h \in H^\infty(\mu+m_2)$. For all $x,y \in \mathcal{H}$, the complex measure $\mu_{x,y}(\cdot)=\langle E(\cdot)x,y\rangle$ is absolutely continuous with respect to μ. It is not hard to see that the function

$$(x,y) \mapsto \int_{\mathbb{T}^2} h(\lambda_1,\lambda_2)\mathrm{d}\mu_{x,y}$$

is a bounded sesquilinear functional, and hence defines a unique operator, which we denote by $h(T_1, T_2)$, such that

$$\langle h(T_1,T_2)x,y\rangle = \int_{\mathbf{T}^2} h(\lambda_1,\lambda_2)\mathrm{d}\mu_{x,y}, \quad x,y\in\mathcal{H}.$$

We define our homomorphism by $\Phi_{T_1,T_2}(h) = h(T_1,T_2)$. Easy computations imply (a) and (b) above and the linearity of $h \mapsto h(T_1,T_2)$. That Φ is multiplicative follows from the fact that it is multiplicative on polynomials. If $\{h_n\}$ is a sequence in $H^\infty(\mu+ +m_2)$ that weak*-converges to some $h \in H^\infty(\mu+m_2)$, by the definition of weak*-topology and the Radon-Nikodim theorem, for any measure ν which is absolutely continuous with respect to $\mu+m_2$ we have that

$$\int_{\mathbf{T}^2} h_n \mathrm{d}\nu \to \int_{\mathbf{T}^2} h\mathrm{d}\nu, \quad \text{as } n\to\infty.$$

Since $\mu_{x,y}$ is absolutely continuous with respect to $\mu+m_2$, we have that for all $x,y\in\mathcal{H}$

$$\langle h_n(T_1,T_2)x,y\rangle = \int_{\mathbf{T}^2} h_n(\lambda_1,\lambda_2)\mathrm{d}\mu_{x,y} \to \int_{\mathbf{T}^2} h(\lambda_1,\lambda_2)\mathrm{d}\mu_{x,y} = \langle h_n(\lambda_1,\lambda_2)x,y\rangle,$$

as $n\to\infty$. Thus, the sequence $\{h_n(T_1,T_2)\}$ converges to $h(T_1,T_2)$ in the weak operator topology. Since on bounded sets the weak operator topology and the weak*-topology coincide, part (c) follows from the fact that sequences are enough to determine weak*-continuity (cf. [5], Theorem 2.3]). Part (d) follows from the weak*-density of the polynomials $p(T_1,T_2)$ in $\mathcal{A}_{T_1,T_2}$. Part (e) is a consequence of (c) and the fact that a linear map between Banach spaces is weak*-continuous if, and only if, it is adjoint of a bounded linear map (see [5], Proposition 2.5]). For part (f) it is enough to show that Φ_{T_1,T_2} has trivial kernel and norm-closed range by virtue of [5, Theorem 2.7], and the hypothesis makes the verification trivial. ■

We will need the following statement in our future work.

Proposition 4.4. *Let $(T_1,T_2)\in ACC^{(2)}(\mathcal{H})$, let $\mathcal{M}$ be a common invariant subspace for (T_1,T_2), and write $\widehat{T}_j = T_j|\mathcal{M}$, $j=1,2$. Then there is an algebra homomorphism $\Phi_{\widehat{T}_1,\widehat{T}_2} : H^\infty(\mathbf{T}^2)\to\mathcal{A}_{\widehat{T}_1,\widehat{T}_2}$ with properties* (a), (b), (c), (d), (e), *and* (f) *of Theorem 4.2.*

Proof. The matricial representations of T_1 and T_2,

$$T_1 = \begin{pmatrix} \widehat{T_1} & * \\ 0 & * \end{pmatrix}, \quad T_2 = \begin{pmatrix} \widehat{T_2} & * \\ 0 & * \end{pmatrix},$$

with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}'$ (where $\mathcal{M}' = \mathcal{H} \ominus \mathcal{M}$), imply that for any polynomial p we have:

$$p(T_1, T_2) = \begin{pmatrix} p(\widehat{T}_1, \widehat{T}_2) & * \\ 0 & * \end{pmatrix}.$$

By taking weak*-limits, for $h \in H^\infty(\mathbb{T}^2)$ we can define the operator $h(\widehat{T}_1, \widehat{T}_2)$ to satisfy

$$h(T_1, T_2) = \begin{pmatrix} h(\widehat{T}_1, \widehat{T}_2) & * \\ 0 & * \end{pmatrix}.$$

Define $\Phi_{\widehat{T}_1, \widehat{T}_2}(h) = h(T_1, T_2)$. Properties (a), (b) and (c) follow from Theorem 4.2. Properties(d), (e) and (f) can be proved in the same way as in the proof of Theorem 4.2. ■

The possible absence of a joint unitary dilation for 3 or more commuting contractions in $\mathcal{L}(\mathcal{H})$ is a serious disadvantage for the development of this theory. We now note, in addition, that Theorem 4.2 does not extend to the case of 3 or more operators. In particular, part (b) cannot hold even in the case of polynomials. Such an example can be found in [7].

For an n-tuple $(T_1, T_2, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^{(n)}_{\text{comm}}$ of contractions, we may ask the question whether there is a constant $K > 0$ such for every polynomial p the equation

$$\|p(T_1, T_2, \ldots, T_n)\| \leqslant K\|p\|_\infty,$$

is satisfied. Varopoulos has shown in [18] that no constant K could work for all values of n. Still the question remains open of whether for a fixed value of n (say, $n = 3$) such a K exists, and whether for three fixed commuting contractions T_1, T_2, and T_3, there is a $K(T_1, T_2, T_3) > 0$ such that

$$\|p(T_1, T_2, T_3)\| \leqslant K(T_1, T_2, T_3)\|p\|_\infty.$$

The functional calculus developed in Theorem 4.2 is a generalization to two variables of the (one variable) functional calculus developed by Nagy and Foiaş in [11]. The properties (d), (e) and (f) are also generalizations of properties satisfied by the Nagy-Foiaş functional calculus (cf. [5, Theorem 3.2]). If $\sigma_T(T_1, T_2) \subset \mathbb{D}^2$, then the Taylor functional calculus and the functional calculus of Theorem 4.2 coincide. This is a consequence of the uniqueness of the Taylor functional calculus (cf. [8]) and the fact that $\mathcal{A}_{T_1, T_2}$ is contained in the double commutant of $\{T_1, T_2\}$.

In a future paper we will use these results to start a systematic study of dual algebras generated by commuting contractions.

This work is part of the author's Ph. D. thesis at the University of Michigan under the direction of professor Carl Pearcy.

REFERENCES

1. ANDO, T., On a pair of commutative contractions, *Acta Sci. Math. (Szeged)*, **24**(1963), 88–90.
2. BERBERIAN, S. K., *Notes on spectral theory*, Van Nostrand, Princeton, New Jersey, 1966.
3. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., *Dual algebras with applications to invariant subspaces and dilation theory*, CMBS Regional Conf. Ser. in Math., no 56, Amer. Math. Soc. Serie, Providence, R. I., 1985.
4. BRIEM, E.,; DAVIE, A. M.; ØKSENDAL, B. K., Functional calculus for commuting contractions, *J. London Math. Soc. (2)*, **7**(1973), 709–718.
5. BROWN, S.; CHEVREAU, B.; PEARCY, C., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
6. CONWAY, J. B., *A first course in functional analysis*, Springer-Verlag, New York, 1985.
7. CRABB, M. J.; DAVIE, A. M., Von Neumann's inequality for Hilbert space operators, *Bull. London Math. Soc.*, **7**(1975), 49–50.
8. CURTO, R. E., Applications of several complex variables to multiparameter spectral theory, in *Surveys of recent results in operator theory*, v.II, Logman Publ. Co., London, 1988, pp. 25–90.
9. GAMELIN, T. W., *Uniform algebras*, Chelsea, New York, 1969.
10. KOSIEK, M.; PTAK, M., Reflexivity of N-tuples of contractions with rich joint left essential spectrum, *Integral Equations Operator Theory*, **13**(1990), 395–420.
11. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, 1970.
12. PARROT, S., Unitary dilations for commuting contractions, *Pacific J. Math.*, **34**(1970), 481–490.
13. PEARCY, C., A complete set of unitary invariants for operators generating finite W^*-algebras of type I, *Pacific J. Math.*, **12**(1962), 1405–1417.
14. RUDIN, W., *Function theory on polydisks*, Benjamin, New York, 1969.
15. SARASON, D. E., On spectral sets having connected complement, *Acta Sci. Math. (Szeged)*, **26**(1965), 289–299.
16. SLOCIŃSKI, M., Isometric dilation of doubly commuting contractions and related models, *Bull. Acad. Sci. Pol.*, **25**(1977), 1233–1242.
17. TAYLOR, J. L., The analytic functional calculus for several commuting operators, *Acta math.*, **125**(1970), 1–38.
18. VAROPOULOS, N., Sur une inégalité de von Neumann, C. R. Acad. Sci. Paris, Sér. I. Math. **277**(1973), 19–22.

ALFREDO OCTAVIO
Departamento de Matematicas,
I. V. I. C., Caracas
Venezuela.

Received November 4, 1991; revised October 27, 1992.