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SQUARE ROOTS OF THE CANONICAL SHIFTS

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1. INTRODUCTION

Assume that $N \subset M$ is a pair of II_1 factors with $[M : N] < \infty$. If M is hyperfinite, then N is hiperfinite ([10]) so that M has many $*$-endomorphisms ρ with $\rho(M) = N$. These $*$-endomorphisms do not always give us right informations of the inclusion $N \subset M$. What kind of $*$-endomorphisms σ of M with $\sigma(M) = N$ reflect the inclusion $N \subset M$ canonically? In this paper we give some results coming from this question. In a series of studies on index theory for pairs $N \subset M$ of infinite factors, Longo's canonical endomorphism γ for $N \subset M$ is used to investigate the relation of the inclusion $N \subset M$ [13, 14]. The endomorphism γ is literally canonical for the inclusion $N \subset M$ but the range of it has the relation $M \supset N \supset \gamma(M)$.

The finite case version Γ of the canonical endomorphism γ is introduced by Ocneanu [16] to classify subfactors of the hyperfinite II_1 factor with index less than 4. The endomorphism Γ is called the canonical shift and defined as a $*$-endomorphism on the tower $\{M' \cap M_k\}_{k=1,2,\dots}$ of the relative commutant algebras induced by the tower

$$N \subset M \subset M_1 \subset \cdots \subset M_k = \langle M_{k-1}, e_k \rangle \subset \cdots$$

of Jones basis constructions with the Jones projections $\{e_k\}_{k=1,2,\dots}$ for $N \subset M$ of II_1 factors with finite index. Then canonical shift Γ is extended to the finite von Neumann algebra $A = \pi\left(\bigcup_k \{M' \cap M_k\}\right)''$. The Γ is determined by $M \supset N$ canonically but has the inclusion $A \supset B \supset \Gamma(A)$ for $B = \pi\left(\bigcup_k \{M_1' \cap M_k\}\right)''$. If $M \supset N$ is a pair of hyperfinite II_1 factors with finite index and the finite depth, then the pair $A \supset B$ is antiisomorphic to the pair $M \supset N$ ([21]). Hence Γ can be considered as a $*$-endomorphism of M with the properties that $M \supset N \supset \Gamma(M)$ and $[M : \Gamma(M)] =$

$= [M : N]^2$, if $M \supset N$ has finite depth.

The purpose of this paper is to give a $*$-endomorphism ρ of A with the property that $\rho(A) = B$ and $\rho^2 = \Gamma$ for the pair $N \subset M$ of hyperfinite II_1 factors with index less than 4. We call such a $*$-endomorphism ρ a *square root* of Γ.

If the principal graph of $N \subset M$ is one of $A_n (n \geqslant 4)$, $D_{2n}(n \geqslant 3)$, $E_6, E_8, E_n^{(1)}$ $(n = 6, 7, 8)$, then a square root ρ of Γ is given by

$$\rho(x) = \lim_{k \to \infty} \mathrm{Ad}\, v_k v_{k-1} \cdots v_1(x), \quad (x \in A).$$

Here the unitary $v_k = (q+1)e_k - 1$ is given by the complex number q satisfying the equality: $(q+1)^2 = q[M : N]$. In such cases, A (resp. B) is a factor and includes the factor $R = \{e_i; i \geqslant 2\}''$ (resp. $R_\lambda = \{e_i; i \geqslant 3\}''$). To determine the square roots of Γ we need the commuting square

$$\begin{array}{ccc} A & \supset & B \\ \cup & & \cup \\ R & \supset & R_\lambda . \end{array}$$

Using the property of the dual principal graph for $A \supset R$, we show that if the principal graph of $M \supset N$ is type $A_n (n \geqslant 4)$ or E_6 then the square root of Γ is unique and that if $M \supset N$ is of type $D_{2n}(n \geqslant 3)$ then there are precisely two square roots of Γ.

If the principal graph of $M \supset N$ is A_3 (resp. D_4), we have $M = N \lhd_\alpha Z_2$ (resp. $N \lhd_\alpha Z_3$) with respect to an outer action α. In the case of the crossed product $M = N \lhd_\alpha G$ of II_1 factor N by an outer action α of an abelian group G, the square roots of Γ also exist and they are determined by $\hat{G}$ and the group of isomorphisms from G onto $\hat{G}$.

We investigate $*$-endomorphisms ρ of A satisfying more weak conditions than square roots of Γ and call the property *self-conjugate* (cf. [14]). A square root of Γ is of course self-conjugate. If $M = N \lhd_\alpha G$ for a finite non abelian G, then A can not have any self conjugate $*$-endomorphism.

The canonical shift Γ is a 2-shift ([2]) on the tower of the relative commutant algebras and a square root ρ of Γ is a 1-shift on the same tower, and so we can compute the entropy of those $*$-endomorphisms. They satisfy the following equality:

$$2H(\rho) = H(\Gamma) = \log[M : N].$$

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2. PRELIMINARIES

Throughout this paper, M is a type II_1 factor with the faithful normal trace τ such that $\tau(1) = 1$ and N is a subfactor of M with $[M : N] < \infty$. By iterating the Jones' basic construction [10], we obtain a whole tower of factors with Jones projections $\{e_i\}$:

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset \cdots \subset M_j = \langle M_{j-1}, e_j \rangle \subset \cdots.$$

Let

$$A_k = M' \cap M_k \quad \text{and} \quad B_k = M_1' \cap M_k \quad (k \geqslant 0).$$

The canonical trace τ is extended to the Markov trace on each M_j, which we denote by the same notation τ. The canonical conjugation J_k on the Hilbert space $L^2(M_k, \tau)$ is defined by $J_k(x) = x^*$ for all $x \in M_k$. We denote by M_∞ the factor $\pi\left(\bigcup_j M_j\right)''$, where π is the GNS representation with respect to τ. Let

$$A = M' \cap M_\infty \quad \text{and} \quad B = M_1' \cap M_\infty.$$

The canonical shift Γ is an endomorphism on the von Neumann algebra $M' \cap M_\infty$ which is defined by $\Gamma(x) = J_{n+1} J_n x J_n J_{n+1}$ for $n \geqslant k$ for all $x \in M' \cap M_{2k}$. The definition does not depend on n ([16], see also [3]) and Γ is τ-preserving ([2], [3]). Hence Γ is extended to a τ-preserving $*$-endomorphism of A and satisfies

$$\Gamma(M_l' \cap M_k) = M_{l+2}' \cap M_{k+2}, \qquad \text{for all } l \leqslant k.$$

The Γ is a 2-shift in the sense of [2].

Let $\mathrm{End}(M, \tau)$ be the set of τ-preserving $*$-endomorphisms of M and $\mathrm{Aut}(M)$ the set of automorphisms of M. For any $\rho \in \mathrm{End}(M, \tau)$, $\rho(M)$ is a subfactor of M because τ is faithful and ultra-weakly continuous.

We denote by E_ρ the conditional expectation $E_{\rho(M)}$ of M onto $\rho(M)$.

DEFINITION 2.1. A $\rho \in \mathrm{End}(M, \tau)$ is irreducible if $\rho(M)' \cap M = \mathbb{C}1$. The index $\mathrm{Ind}(\rho)$ is $[M : \rho(M)]$. Let ρ have finite index. If there exists a projection $e \in M$ which satisfies that

$$E_\rho(e) = (\mathrm{Ind}\, \rho)^{-1} 1, \quad \rho^2(M) = \{e\}' \cap \rho(M)$$

then ρ is said to be *basic* and such a projection e is called a *basic projection* for ρ. If

$$M = \sigma\text{–weak closure of } \bigcup_j (\rho^j(M)' \cap M)$$

then ρ is said to have the *generating property*. If ρ is basic and satisfy the generating property, ρ is said to be *standard*. The ρ has finite depth if the dimensions of the center of $\rho^k(M)' \cap M$ are bounded. The *principal graph* (resp. *dual principal graph*) for ρ is the principal graph (resp. dual principal graph) for the pair $M \supset \rho(M)$. A $\sigma \in \mathrm{End}(A, \tau)$ is called a *square root* of Γ if $\sigma(A) = B$ and $\sigma^2 = \Gamma$.

We need the following properties ([2]) of basic endomorphisms:

PROPOSITION 2.2. *Let $\rho \in \mathrm{End}(M, \tau)$. Then*
(1) *ρ is basic if and only if M is the basic extension for $\rho(M) \supset \rho^2(M)$.*
(2) *Assume that ρ is basic. Then there exists a $\rho_\infty \in \mathrm{Aut}(M_\infty)$ which satisfies*

$$\rho_\infty(e_k) = e_{k-1}, \quad \rho_\infty(M_k) = M_{k-1} \quad \text{and} \quad \rho_\infty|M = \rho.$$

Hence the principal graph of ρ is same with the dual principal graph of ρ.
(3) *If $\rho \in \mathrm{End}(M, \tau)$ is basic, then the entropy $H(\rho)$ of ρ satisfies*

$$H(\rho) = \lim_{k \to \infty} \frac{H(\rho^k(M)' \cap M)}{k}.$$

LEMMA 2.3. *Let $\rho \in \mathrm{End}(M, \tau)$ satisfies the generating property, then ρ is a shift in the sense of Powers ([24]):*

$$\bigcap_j \rho^j(M) = \mathbb{C}1.$$

Proof. Let $x \in \bigcap_j \rho^j(M)$. For any $\varepsilon > 0$, there are an integer k and an $x_k \in \rho^k(M)' \cap M$ which satisfies that $\|x - x_k\|_2 < \varepsilon$. Let E_k be the conditional expectation of M onto $\rho^k(M)' \cap M$. Since M is a factor, we have $E_k(x) = \tau(x)1$, so that $\|x - \tau(x)1\|_2 < \varepsilon$. ■

Here we give some examples of basic but not standard $*$-endomorphisms and standard $*$-endomorphisms, which have a key role in the last section.

EXAMPLE 2.4. A $*$-endomorphism of a hyperfinite II_1 factor is defined in [5, Section 4.4]. We review it because we need the notations in below. Let $C_0 \subset B_0$ be a pair of finite dimensional von Neumann algebras. Let $[C_0 \to B_0]$ be the inclusion matrix for $C_0 \subset B_0$. Let $\beta = \|[B_0 \to C_0]\|^2$. Assume $2 \leqslant \beta \leqslant 4$. Then the Bratteli diagram for $C_0 \subset B_0$ is one of the Coxeter graphs $\mathcal{G}$ of types $A_n, D_n, E_n, A_n^{(1)}, D_n^{(1)}, E_n^{(1)}$. Let q be a number with $\beta = 2 + q + q^{-1}$. Then normalized Markov trace τ of modulus β for $C_0 \subset B_0$ has an extension τ to the fundamental extension algebra $B_1 = \langle B_0, e_0 \rangle$, where e_0 is the projection from $L^2(B_0, \tau)$ onto $L^2(C_0, \tau)$. By the induction, we have the projection e_j from $L^2(B_{j-1},\tau)$ onto $L^2(B_{j-2},\tau)$, $B_j = \langle B_{j-1}, e_{j-1}\rangle$ and the normalized Markov trace τ with modulus β on B_j for all j. Let $v_j = (q+1)e_j - 1$ and $C_1 = v_0 B_0 v_0^{-1}$. Let C_j be the algebra generated by $\{C_{j-1}, e_{j-1}\}$. Let π be the GNS representation of $\bigcup_j B_j$ with respect to τ and put

$$B(\mathcal{G}) = \pi\left(\bigcup_j B_j\right)''.$$

The $*$-endomorphism Φ is defined for all $x \in B(\mathcal{G})$ by

$$\Phi(x) = \lim_{k\to\infty} v_0 v_1 \cdots v_k x v_k^* \cdots v_1^* v_0^*.$$

The subfactor $\pi\left(\bigcup_j C_j\right)''$ of $B(\mathcal{G})$ coinsides with $\Phi(B(\mathcal{G}))$, which we denote by $C(\mathcal{G})$.

LEMMA 2.5. *The $\Phi \in \mathrm{End}(B(\mathcal{G}),\tau)$ is basic but does not satisfy the generating property if* $\dim C_0 \geqslant 2$.

Proof. We show that $p = e_0$ is a basic projection for Φ. Put $B = B(\mathcal{G})$ and $C = C(\mathcal{G})$. By the Markov property, $\tau(E_{B_0}(p)x) = \tau(x)/\beta$ for all $x \in B_0$. Hence $E_{B_0}(p) = 1/\beta$, so that $E_{C_1}(p) = 1/\beta$. Since the commuting square condition $E_{C_1} = E_C E_{B_1}$ is satisfied, we have that

$$E_C(p) = E_C E_{B_1}(p) = E_{B_1} E_C(p) = E_{C_1}(p) = (1/\beta)1.$$

By the definition, $\Phi(v_j) = v_{j+1}$ for all j. Hence

$$\Phi(C_{j+1}) = \{\Phi(C_1), v_2, v_3, \ldots, v_{j+1}\}''.$$

Since $v_j v_{j+1} v_j = v_{j+1} v_j v_{j+1}$ and $v_j v_{j+i} = v_{j+i} v_j$ for all $i \geqslant 2$ and $j \geqslant 0$, we have $xv_0 = v_0 x$ for all $x \in \Phi(C_1) = v_1 C_1 v_1^{-1}$, which implies that $\Phi(C_{j+1}) \subset \{p\}' \cap C$. Therefore $\Phi^2(B) \subset \{p\}' \cap C$. On the other hand, by [19] $\{p\}' \cap C$ is a subfactor of C which satisfies $[C : \{p\}' \cap C] = [B : C]$ because $E_C(p) = [B : C]^{-1}1$. The pair $\Phi^2(B) \subset \{p\}' \cap C$ satisfies

$$[C : \{p\}' \cap C] = [B : C] = [\Phi(B) : \Phi(C)] = [C : \Phi^2(B)].$$

Hence

$$\Phi^2(B) = \{p\}' \cap C = \{p\}' \cap \Phi(B).$$

Thus Φ is a basic $*$-endomorphism of B with a basic projection e_0. By the definition, $\Phi(x) = x$ for all $x \in C_0$. Hence by Lemma 2.3, Φ does not satisfy the generating property if $\dim C_0 \geqslant 2$.

EXAMPLE 2.6. Let R be the hyperfinite II_1 factor generated by the sequence $\{e_i, i \geqslant 1\}$ of projections satisfying the Jones relations for $\lambda \in (0, 1/4] \cup \cup\{(4\cos^2(\pi/n))^{-1}; n \geqslant 3\}$:

$$\begin{cases} e_j e_i = e_i e_j, & \text{if } |i-j| \neq 1 \\ e_i e_j e_i = \lambda e_i, & \text{if } |i-j| = 1. \end{cases}$$

The $\sigma \in \mathrm{End}(R, \tau)$ defined by $\sigma(e_j) = e_{j+1}$ for all $j \geqslant 1$ is basic and the projection e_1 is a basic projection for σ. Since $\sigma^j(R)' \cap R \supset \{e_1, e_2, \ldots, e_{j-1}\}''$, σ satisfies the generating property. Hence σ is standard.

Furthermore if $\lambda > 1/4$, σ has the form

$$\sigma(x) = \lim_{k \to \infty} v_1 v_2 \cdots v_k x v_k^* \cdots v_2^* v_1^*$$

for all $x \in R$, where q is the number with $\lambda(q + q^{-1} + 2) = 1$ and $v_k = (q+1)e_k - 1$.

Throughout this paper, we denote this endomorphism σ of R by σ_λ.

EXAMPLE 2.7. The typical example of standard endomorphisms is the canonical shift Γ of the factor $A = M' \cap M_\infty$ for the pair $M \supset N$ of type II_1 factors with $[M : N] < \infty$ and finite depth. In fact, Γ is basic by [2] and, since $\Gamma(A_j) = M_2' \cap M_{j+2}$, it satisfies the generating property by the following:

$$\left(\bigcup_j ((\Gamma^j(M' \cap M_\infty))' \cap M' \cap M_\infty)\right)'' \supset \pi\left(\bigcup_j (M_{2j} \cap M')\right)''.$$

If $M \supset N$ has finite depth, then Γ has the generating property ([19]). Hence Γ is standard.

The following proposition is fundamental in our main results and assures that basic $*$-endomorphisms have parallel properties as braided endomorphisms in the sense of [15].

LEMMA 2.8. *Assume that $\rho \in \mathrm{End}(M, \tau)$ is basic and e is a basic projection for ρ. Then the sequence $\{e_j = \rho^{j-1}(e); j = 1, 2, \ldots\}$ satisfies the Jones relations for $\lambda = (\mathrm{Ind}\,\rho)^{-1}$.*

Proof. By [2; Proposition 22], a basic projection e satisfies that $e\rho(e)e = = E_{\rho^2}(\rho(e)e) = E_\rho(e)e = \lambda e$. Hence $(\rho(e)e\rho(e))^2 = \lambda\rho(e)e\rho(e)$ and so $\rho(e)e\rho(e) = = \lambda\rho(e)$ by the relation that $\|\rho(e)e\rho(e) - \lambda\rho(e)\|_2^2 = 0$. Since ρ is a $*$-endomorphism and $\rho^2(M) = \{e\}' \cap \rho(M)$, the sequence $(e_j)_j$ satisifies the conditions. ■

A $\rho \in \mathrm{End}(M, \tau)$ with finite index is said a *braided* endomorphism ([15]) if there is a unitary u in $\rho^2(M)' \cap M$ which satisfies the braiding relation $u\rho(u)u = \rho(u)u\rho(u)$.

Proposition 2.9. Let $\rho \in \mathrm{End}(M,\tau)$ satisfy $\mathrm{Ind}\,\rho \leqslant 4$. *If ρ is basic, then ρ is a braided endomorphism.*

Proof. Assume that ρ is basic. Let e be a basic projection of ρ. Let take a q with $\mathrm{Ind}(\rho) = 2 + q + q^{-1}$. Let $u = (q+1)e - 1$. Then $u \in \rho^2(M)' \cap M$. Since $e\rho(e)e = \mathrm{Ind}(\rho)^{-1}e$ and $\rho(e)e\rho(e) = \mathrm{Ind}(\rho)^{-1}\rho(e)$ by Lemma 2.8, the pair (ρ, u) satisfies the braiding relation. ■

3. SELF CONJUGATE $*$-ENDOMORPHISMS

In this section, we study $*$-endomorphisms of the relative commutant algebra A for the inclusion $N \subset M$ with respect to relations of basic endomorphisms. In this section we assume that A is a factor. For a $\sigma \in \mathrm{End}(A,\tau)$, let

$$\mathrm{Aut}(A,\sigma) = \{\theta \in \mathrm{Aut}(A) : \theta(\sigma^n(A)) = \sigma^n(A) \text{ for all } n\}.$$

LEMMA 3.1. *The following three statements are equivalent for $\rho, \sigma \in \mathrm{End}(A,\tau)$;*
(1) *there exists a $\theta \in \mathrm{Aut}(A,\sigma)$ with $\rho = \sigma \cdot \theta$,*
(2) *there exists a $\theta' \in \mathrm{Aut}(\sigma(A),\sigma)$ with $\rho = \theta' \cdot \sigma$,*
(3) *$\sigma^n(A) = \rho^n(A)$ for all n.*

Proof. It is clear that (1)⇒(3) and (2)⇒(3). Put $\theta = \sigma^{-1}\rho$ and $\theta' = \rho\sigma^{-1}$, then we have (3)⇒(1) and (3)⇒(2). ■

DEFINITION 3.2. Two $\rho, \sigma \in \mathrm{End}(A,\tau)$ are *equivalent* if they satisfies one of the conditions in Lemma 3.1. The conjugate $\overline{\rho}$ of ρ with the property $\rho(A) = B$ is defined by

$$\overline{\rho} = \rho^{-1}\Gamma.$$

A $\rho \in \mathrm{End}(A,\tau)$ with $\rho(A) = B$ is *self conjugate* (cf. [14]) if ρ is equivalent to $\overline{\rho}$.

Proposition 3.3. (1) *If $\sigma \in \mathrm{End}(A,\tau)$ is a square root of Γ, then σ is self--conjugate.*

(2) *If $\rho \in \mathrm{End}(M,\tau)$ is basic and $\rho(M) = N$, then the restriction of ρ_∞^{-1} to A is self-conjugate and satisfies*

$$\rho_\infty^{-1}(M_k' \cap M_l) = M_{k+1}' \cap M_{l+1} \text{ for all } k \leqslant l.$$

Proof. (1) It is obvious by the definitions.

(2) If ρ is basic, then $\rho_\infty(M_i' \cap M_j) = M_{i-1}' \cap M_{j-1}$, for all $0 \leqslant i \leqslant j$, by Proposition 2.2. Therefore the restriction of ρ_∞^{-1} to A satisfies

$$\rho_\infty^{-1}(A) = B \text{ and } \rho_\infty^{-k}(A) = M_k' \cap M_\infty = (\rho_\infty \Gamma)^k(A) \quad \text{for all } k. \qquad \blacksquare$$

A von Neumann subalgebra D of a von Neumann algebra C is called normal in C if $(D' \cap C)' \cap C = D$. If $[M : N] \leqslant 4$, then by Skau's lemma ([5]) $\{e_i; i \leqslant 2\}' \cap M_\infty = M$. Hence M is normal in M_∞. If $M = N \rtimes_\alpha G$ for an outer action α of G of a factor N, then M is also normal in M_∞ (cf. see [24]). If $N \subset M$ is an inclusion of hyperfinite II_1 factors with finite index, then M is normal in M_∞ if and only if $N \subset M$ is *strongly amenable* in the sense of Popa ([22]). When an inclusion $N \subset M$ has finite index, $N \subset M$ is called *extremal* if $E_{M' \cap M_1}(e_1) = [M : N]^{-1}1$ ([23]).

THEOREM 3.4. *Assume that M is hyperfinite. If $N \subset M$ is strongly amenable and a $\rho \in \mathrm{End}(A, \tau)$ is self-conjugate, then we have*

$$\rho(M_i' \cap M_j) = M_{i+1}' \cap M_{j+1}, \quad \textit{for all } j \geqslant i.$$

Furthermore, if $N \subset M$ is extremal, then a self-conjugate ρ is standard and

$$2H(\rho) = H(\Gamma).$$

Proof. First we show that if ρ is self-conjugate then $\rho(M_i' \cap M_\infty) = M_{i+1}' \cap M_\infty$ for all $i \geqslant 0$. By the definition, $\rho(M' \cap M_\infty) = M_1' \cap M_\infty$. Assume that $\rho(M_i' \cap M_\infty) = M_{i+1}' \cap M_\infty$ for all $i \leqslant k-1$. Then

$$\rho^k(A) = \rho^{-1}\Gamma\rho^{k-1}(A) = \rho^{-1}\Gamma(M_{k-1}' \cap M_\infty) = \rho^{-1}(M_{k+1}' \cap M_\infty).$$

Hence

$$\rho(M_k' \cap M_\infty) = \rho^{k+1}(A) = M_{k+1}' \cap M_\infty.$$

If M is normal in M_∞, then M_j is normal in M_∞ for all j. Hence we have for all $i \leqslant j$

$$\rho(M_i' \cap M_j) = \rho(M_i' \cap M_\infty \cap (M_j' \cap M_\infty)') =$$
$$= M_{i+1}' \cap M_\infty \cap (M_{j+1}' \cap M_\infty)' = M_{i+1}' \cap M_{j+1}.$$

Since

$$\begin{array}{ccc} M_1' \cap M_\infty & \subset & M_\infty \\ \cup & & \cup \\ M_1' \cap M_2 & \subset & M_2 \end{array}$$

satisfy the commuting square condition [22],

$$E_{\rho(A)}(e_2) = E_{M_1' \cap M_\infty}(e_2) = E_{M_1' \cap M_\infty} E_{M_2}(e_2) =$$

$$= E_{M_1' \cap M_2}(e_2) = [M : N]^{-1} 1 = \mathrm{Ind}(\rho)^{-1} 1.$$

Since ρ satisfies $\rho^j(A) = M_j' \cap M_\infty$ for all j, $\rho(B) = M_2' \cap M_\infty = \{e_2\}' \cap M_1' \cap M_\infty$. It implies that e_2 is a basic projection for ρ. Also ρ has the generating property by the relation:

$$\left(\bigcup_j (\rho^j(A)' \cap A)\right)'' = \left(\bigcup_j ((M_j' \cap M_\infty)' \cap M_\infty \cap M')\right)'' \supset \pi \left(\bigcup_j (M_j \cap M')\right)'' = A.$$

Therefore ρ is standard.

Put $k_j = j$. Then ρ satisfies the conditions (1) and (2) for 1-shift in [2]. Hence

$$H(\rho) = \lim_{k \to \infty} \frac{H(A_k)}{k}.$$

On the other hand, by [2]

$$H(\Gamma) = \lim_{k \to \infty} \frac{H(A_{2k})}{k}.$$

Hence we have $2H(\rho) = H(\Gamma)$. ■

The above Theorem holds for an inclusion of non hyperfinite II_1 factors if M is normal in M_∞.

REMARK. If the principal graph and the dual principal graph for an inclusion $N \subset M$ are different, then there does not exist any self conjugate $\rho \in \mathrm{End}(A, \tau)$. For example, consider $M = N \rtimes_\alpha G$ by an outer action α of a non abelian finite group G. Then M is normal in M_∞. Since G is non abelian, the dual principal graph for $N \subset M$ is diferent from the principal graph. Hence any $\rho \in \mathrm{End}(A, \tau)$ is not self-conjugate by Theorem 3.4.

4. THE SQUARE ROOT OF THE CANONICAL SHIFT

In this section, we obtain the square roots of the canonical shift of the pair of hyperfinite type II_1 factors with index smaller than 4 and Jones pairs.

Let $\mathcal{G}$ be one of the Coxeter graphs $A_n, D_{2n}, E_6, E_8, E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$. Let $M = B(\mathcal{G})$ and $N = C(\mathcal{G})$ in Example 2.4. For convenience sake, we denote the projection e_j implementing the conditional expectation of B_j onto B_{j-1} in Example 2.4 by e_{-j}. Also we denote by e_n $(n \geqslant 1)$ the projection for the pair $M_{n-1} \supset M_{n-2}$

and $v_j = qe_j - (1 - e_j)$ for all $j \in \mathbf{Z}$. Let ρ be the $*$-endomorphism Φ of M onto N in Example 2.4. Then ρ is basic by Lemma 2.5. Hence ρ has the extension (which we denote by the same notation ρ) to M_∞ by Proposition 2.2. The following lemma is essentially contained in the example in [5, Section 4.4] and the existence of a square root of the canonical shift is really depend on the result of the following lemma.

LEMMA 4.1. *Under the above conditions*

$$\rho(x) = \mathrm{Ad}(v_k v_{k-1} \cdots v_0 \cdots v_{-(j-1)})(x), \quad \text{for } x \in \rho^j(M)' \cap M_k \ (j \geqslant 0,\ k \geqslant 1).$$

Proof. The unitaries $\{v_i\}_i$ satisfies $v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}$ for all integers i. Hence for all i,

$$\rho(e_i) = e_{i-1} = \lim_{j \to \infty} \mathrm{Ad}\, v_i v_{i-1} \cdots v_j(e_i).$$

By the definition,

$$\rho(x) = \lim_{j \to \infty} \mathrm{Ad}\, v_0 \cdots v_{-j}(x), \quad (x \in M_0 = M).$$

The algebra $\left\{ \sum_i x_i e_1 y_i : x_i, y_i \in M_0 \right\}$ is dense in M_1 with respect to the σ-strong topology and ρ is a $*$-endomorphism of M_1 onto M_0. Hence we have

$$\rho(y) = \lim_{j \to \infty} \mathrm{Ad}\, v_1 v_0 \cdots v_{-j}(y), \quad (y \in M_1)$$

because $v_1 \rho(x) v_1^* = \rho(x)$ for all $x \in M$. By a similar method we have

$$\rho(x) = \lim_{j \to \infty} \mathrm{Ad}(v_k v_{k-1} \cdots v_0 \cdots v_{-j}(x)), \quad (x \in M_k).$$

On the other hand $\rho^i(e_0) = e_{-i} \in \rho^j(M)$ for $1 \leqslant j \leqslant i$. Therefore ρ has the form in the statement. ■

PROPOSITION 4.2. *Assume that $\mathcal{G}$ is neither of type A_3 or D_4. Then the $\rho^{-2} = \Gamma$ on $A = M' \cap M_\infty$.*

Proof. By the method of [20], all basic extension algebras M_k are realized as the factor acting on $L^2(M, \tau)$. Let $\gamma_0(a) = J_0 a J_0$ for all $a \in M$. Then we have

$$\Gamma(x) = J_{k+1} J_k x J_k J_{k+1} = \rho^{-(k+1)} \cdot \gamma_0 \cdot \rho \cdot \gamma_0 \cdot \rho^k(x),$$

for all $x \in M' \cap M_{2k}$. In order to prove $\Gamma(x) = \rho^{-2}(x)$ for all $x \in M' \cap M_{2k}$, it is sufficient to show

$$\gamma_0 \rho \gamma_0(x) = \rho^{-1}(x) \quad \text{for all } x \in \rho^k(M)' \cap M_k.$$

Let $x \in \rho^k(M)' \cap M_k$, then Lemma 4.1 implies the following two relations:

$$\rho^{-1}(x) = \mathrm{Ad}(v^*_{-(k-2)} \cdots v^*_0 \cdots v^*_{k+1})(x)$$

and

$$\gamma_0 \cdot \rho \cdot \gamma_0(x) = \mathrm{Ad}(\gamma_0(v_k) \cdots \gamma_0(v_0) \cdots \gamma_0(v_{-(k-1)})).$$

On the other hand, $\mathcal{G}$ is the principal graph of $B(\mathcal{G}) \supset C(\mathcal{G})$ ([1, 4]). (In [4], we proved this fact if $\mathcal{G}$ is one of A_n, D_{2n}, E_6 and E_8. Let $\mathcal{G}$ be one of $E_j^{(1)}$ $(j = 6, 7, 8)$, then there exists a pair $M \supset N$ with the principal graph $\mathcal{G}$ ([5]). On the other hand, $\mathcal{G}$ has only one biunitary connection by [11]. Hence by a similar method in [4] we can prove that for such a $\mathcal{G}$.) By the shape of the graph $\mathcal{G}$,

$$M'_{j-2} \cap M_j = \mathbb{C}e_j \oplus \mathbb{C}(1 - e_j)$$

for all j. Since

$$M_{j-2} = J_0(\rho^{j-2}(M)' \cap B(L^2(M)))J_0 = \gamma_0(\rho^{j-2}(M)') \quad \text{for all } j$$

we have

$$\gamma_0(e_j) \in \rho^j(M)' \cap \rho^{j-2}(M) = \mathbb{C}e_{-j+2} \oplus \mathbb{C}(1 - e_{-j+2}),$$

for all j. Comparing the value of the trace of those two projections, we have

$$\gamma_0(e_j) = e_{-j+2}.$$

Hence for all j

$$\gamma_0(v_j) = v^*_{-(j-2)}.$$

This implies $\rho^{-1}(x) = \gamma_0 \cdot \rho \cdot \gamma_0(x)$ for all $x \in \rho^k(M)' \cap M_k$. ∎

Now we study $*$-endomorphisms for the pair given by crossed products, in order to obtain a square root of Γ for the pair $M \supset N$ with the principal graph of A_3 or D_4.

Let N be a finite factor, G a finite abelian group and α an outer action of G on N. Let $M = N \rtimes_\alpha G$. We denote the canonical unitary in M which corresponds $g \in G$ by $u_0(g)$. The action $\hat{\alpha}$ is defined by $\hat{\alpha}_\gamma(u_0(g)a) = \langle \gamma, g \rangle u_0(g)a$, $(\gamma \in \hat{G},\ g \in$ $\in G,\ a \in N)$. Then the automorphism $\hat{\alpha}_\gamma$ induces an unitary $u_1(\gamma)$ on $L^2(M, \tau)$. The factor generated by M and $\{u_1(\gamma); \gamma \in \hat{G}\}$ is the basic extension algebra M_1 and the projection from $L^2(M, \tau)$ onto $L^2(N, \tau)$ is:

$$e_1 = \sum_{\gamma \in \hat{G}} \frac{u_1(\gamma)}{|G|}.$$

Iterating this method, we have the sequence $\{u_{2i-1}(\gamma); \gamma \in \hat{G}\}_{i\geqslant 1}$ and $\{u_{2i}(g); g \in \in G\}_{i\geqslant 1}$ of unitaries, which satisfy

$$M_{2i} = \{M_{2i-1}, u_{2i}(G)\}'' \text{ and } M_{2i+1} = \{M_{2i}, u_{2i+1}(\hat{G})\}''.$$

For a $\chi \in \hat{G}$ we put

$$e_{2i}^{\chi} = \sum_{g\in G} \langle \chi, g\rangle^{-1} \frac{u_{2i}(g)}{|G|}$$

$$e_{2i+1}^{\chi} = \sum_{g\in G} \langle \chi, g\rangle \frac{u_{2i+1}(\psi(g))}{|G|}.$$

Here ψ is an isomorphism from G onto $\hat{G}$. We remark that e_{2i+1}^{χ} does not depend on the choice of ψ. The projection e_i^{χ} satisfies the property (J) for all i.

LEMMA 4.3. *For all $i \geqslant 1$, the following hold:*

$$J_{2i}u_{2i}(g)J_{2i} = u_{2i+2}(g), \quad \Gamma(u_{2i}(g)) = u_{2i+2}(g)$$

$$J_{2i+1}u_{2i+1}(\gamma)J_{2i+1} = u_{2i+3}(\gamma), \quad \Gamma(u_{2i+3}(\gamma)) = u_{2i+5}(g).$$

Proof. For the sake of simplicity, we denote $u_{2i-1}, u_{2i}, u_{2i+1}$ and u_{2i+2} by v_1, v_2, v_3 and v_4, respectively. Let e be a Jones projection for the inclusion $M_{2i} \supset \supset M_{2i-2}$. Remark that $v_2(g) \in M'_{2i-2} \cap M_{2i}$, for all $g \in G$. Using the Fourier expansion with respect to the orthonormal basis in M_{2i} module M_{2i-2}, we have

$$av_2(g)^* = \sum_{t\in\hat{G}} \sum_{h\in G} v_1(t)v_2(h)ev_2(hg)^*v_1(t)^*a,$$

for all $a \in M_{2i}$. It implies that

$$J_{2i}v_2(g)J_{2i} = \sum_{t\in\hat{G}} \sum_{h\in G} v_1(t)v_2(h)ev_2(hg)^*v_1(t)^*$$

for all $g \in G$. By the formula in [20], e is given by

$$e = \frac{1}{|G|^3} \sum_{s,p\in\hat{G}} \sum_{k,l\in G} v_3(s)v_2(k)v_4(l)v_3(p) =$$

$$= \frac{1}{|G|^2} \sum_{t\in\hat{G}} \sum_{k\in G} \langle t, k\rangle v_2(k)v_3(t)v_4(k)$$

Hence

$$J_{2i}v_2(g)J_{2i} = \frac{1}{|G|} \sum_{t\in\hat{G}} \sum_{h\in G} \langle t, h^{-1}\rangle v_3(t)v_4(g) = u_{2i+2}(g) = v_4(g)$$

for all $g \in G$. Let ξ_0 be the cyclicic vector for M_{2i+1} in $L^2(M_{2i+1}, \tau)$. Then

$$\Gamma(u_{2i}(g))x\xi_0 = J_{2i+1}u_{2i+2}(g)x^*\xi_0 = J_{2i+1}\alpha_g(x^*)\xi_0 = \alpha_g(x)\xi_0 = u_{2i+2}(g)x\xi_0$$

for all $x \in M_{2i+1}$, because $u_{2i}(g) \in M' \cap M_{4i}$. Hence $\Gamma(u_{2i}(g)) = u_{2i+2}(g)$ for all $g \in G$, and similarly $\Gamma(u_{2i+1}(\gamma)) = u_{2i+3}(\gamma)$ for all $\gamma \in \hat{G}$. ■

THEOREM 4.4. *Let ρ be an isomorphism of A onto B. Then $\rho^2 = \Gamma$ if and only if there exist an isomorphism ψ from G onto $\hat{G}$ and a $\chi \in \hat{G}$ which satisfy that*

$$\rho(u_{2i}(g)) = \langle \chi, g\rangle u_{2i+1}(\psi(g)), \quad \textit{for all } g \in G \tag{1}$$

$$\rho(u_{2i+1}(\psi(g))) = \langle \chi, g\rangle^{-1} u_{2i+2}(g), \quad \textit{for all } g \in G \tag{2}$$

Proof. The factor $M_j' \cap M_\infty$ is generated by the family $\{u_{2i}(g), u_{2i+1}(\gamma); g \in \in G, \gamma \in \hat{G}, i \geqslant j+2\}$. Let ψ be an isomorphism from G onto $\hat{G}$. Remark that ψ satisfies $\langle\psi(h), g\rangle = \langle h, \psi(g)\rangle$, for all $h, g \in G$. Since $u_i(s)u_j(t) = u_j(t)u_i(s)$ for all $s, t \in G \cup \hat{G}$ if $|i-j| \neq 1$ and $u_{2i+1}(\gamma)u_{2i}(g) = \langle\gamma, g\rangle u_{2i}(g)u_{2i+1}(\gamma)$ for all $i, g \in G$ and $\gamma \in \hat{G}$, the map ρ defined by (1) and (2) is extended to a $*$-endomorphism of $M' \cap M_\infty$. By Lemma 4.1, $\Gamma(u_{2i}(g)) = u_{2i+2}(g) = \rho^2(u_{2i}(g))$ for all $g \in G$, $i \geqslant 1$ and $\Gamma(u_{2i-1}(\gamma)) = u_{2i+1}(\gamma) = \rho^2(u_{2i-1}(\gamma))$ for all $\gamma \in \hat{G}$, $i \geqslant 1$. Hence Γ and $\rho \in \mathrm{End}(A, \tau)$ satisfy $\Gamma = \rho^2$.

Conversely, let ρ be a square root of Γ. Remark that M is normal in M_∞. Hence by Proposition 3.3 and Theorem 3.4, $\rho(M_i' \cap M_j) = M_{i+1}' \cap M_{j+1}$, for all $0 \leqslant i \leqslant j$. Remark that M_{2i} is generated by $\{u_{2i}(g); g \in G\}$ and M_{2i-1} and that $\mathrm{Ad}\, u_{2i}(g)$ is an outer automorphism of M_{2i-1} for all $g \in G$. The unitary $\rho(u_{2i}(g)) \in M_{2i+1}$ preserves M_{2i} glovally invariant for all $g \in G$. Hence for a $g \in G$ there is a unitary $v \in M_{2i}$ and a $\psi(g) \in \hat{G}$ with $\rho(u_{2i}(g)) = vu_{2i+1}(\psi(g))$. On the other hand,

$$\rho(\{u_{2i}(g); g \in G\}'') = \rho(M_{2i-2}' \cap M_{2i}) = \{u_{2i+1}(\gamma); \gamma \in \hat{G}\}'',$$

which implies that $v \in M_{2i} \cap \{u_{2i+1}(\gamma) : \gamma \in \hat{G}\}''$. Hence v is a scalar. Since ρ is a $*$-endomorphism, we have a $\chi_i \in \hat{G}$ and an isomorphism ψ_i from G onto $\hat{G}$ with

$$\rho(u_{2i}(g) = \langle \chi_i, g\rangle u_{2i+1}(\psi_i(g)), \quad (g \in G).$$

Similarly for all i we have a $h_i \in G$ and an isomorphism φ from $\hat{G}$ onto G with $\rho(u_{2i+1}(\gamma)) = \langle\gamma, h_i\rangle u_{2i+2}(\varphi_i(\gamma))$ for all $\gamma \in \hat{G}$. Since

$$u_{2i+2}(g) = \Gamma(u_{2i}(g)) = \rho(\langle\chi_i, g\rangle u_{2i+1}(\psi_i(g))) =$$

$$= \langle \chi_i, g\rangle\langle \psi_i(g), h_i\rangle u_{2i+2}(\varphi(\psi(g)))$$

for all $g \in G$, and $\{u_{2i+2}(g); g \in G\}$ is orthonormal family with respect to τ, we have that $\varphi = \psi^{-1}$ and $\langle \chi_i, g\rangle = \langle \psi_i(g), h_i\rangle^{-1}$. Similarly, we have

$$\langle \chi_{i+j}, g\rangle u_{2(i+j)+1}(\psi_{i+j}(g)) = \langle \chi_i, g\rangle u_{2(i+j)+1}(\psi_i(g))$$

for all i, j and $g \in G$. Hence $\chi_i = \chi_j$ and $\psi_i = \psi_j$ for all i and j. Thus there are a $\chi \in \hat{G}$ and an isomorphism ψ from G onto $\hat{G}$ which satisfy the conditions (1) and (2) for all i and $g \in G$. ∎

We denote the $\chi \in \hat{G}$ and the isomorphism ψ from G onto $\hat{G}$ with the properties (1) and (2) by χ_ρ and ψ_ρ. It is obvious that the correspondence: $\rho \to (\chi_\rho, \psi_\rho)$ is one to one.

COROLLARY 4.5. *Let $M = N \lhd_\alpha G$, for an outer action of a finite abelian group G on a finite factor N. Then there exists an one to one correspondence between the set of square roots of Γ and $\hat{G} \times \mathrm{Iso}(G, \hat{G})$, where $\mathrm{Iso}(G, \hat{G})$ is the set of isomorphisms of G onto $\hat{G}$.*

COROLLARY 4.6. *Let $M \supset N$ be one of the following pairs:*

(1) $M \supset N$ is Jones pair $R \supset R_\lambda$.

(2) $M = N \lhd_\alpha G$ of II_1 factor N by an outer action α of an abelian finite group G.

(3) M is hyperfinite and $[M : N] < 4$.

(4) M is hyperfinite and the principal graph of $M \supset N$ is one of $E_n^{(1)}$ for $n = 6, 7, 8$. Then there exists a square root σ of the canonical shift Γ.

Proof. (1) If $\lambda \geqslant 1/4$, then $A \supset B$ is isomorphic to the pair $\{e_i : i \geqslant 2\}'' \supset \{e_i : i \geqslant 3\}''$ by Skau's lemma. Hence the $\sigma \in \mathrm{End}(A, \tau)$ defined by $\sigma(e_i) = e_{i+1}$ is the square root of Γ. If $\lambda < 1/4$, then the drived tower is precisely the sequence of fixed point algebras for the tensor product action of the torus T on $\bigotimes_k M_2(\mathbb{C})$ $(k \geqslant 0)$ [5]. Hence the endomorphism of A, which shifts one factor in the infinite tensor product to the right, is a square root of Γ.

(2) This is clear by Corollary 4.5.

(3) By the results ([5, 7, 9, 11, 16, 17, 21]) of the classification for subfactors of the hyperfinite II_1 factor with index less than 4, it is sufficient to prove the statement in the case where the principal graph of $M \supset N$ is one of $A_n (n \geqslant 3)$, $D_{2n} (n \geqslant 4), E_6, E_8$. If the principal graph is A_n $(n \geqslant 3)$, the subfactor is unique up to conjugacy ([11, 16, 17, 21]). Hence we have a square root of Γ by (1). If the principal graph is D_{2n} $(n \geqslant 2)$, then the subfactor is unique up to conjugacy ([11,16,17]) so that $M \supset N$ is isomorphic to $B(\mathcal{G}) \supset C(\mathcal{G})$ for the Coxeter graph $\mathcal{G}$ of D_{2n} $(n \geqslant 2)$. In the case of

$n \geqslant 3$, $\sigma = \rho^{-1}$ in Proposition 4.2 is a square root of Γ. If $n = 2$, it is obvious by (2). If the principal graph of $M \supset N$ is E_6 (resp. E_8), then $M \supset N$ is either isomorphic or anti-isomorphic to $B(\mathcal{G}) \supset C(\mathcal{G})$ of the coxeter graph $\mathcal{G}$ of type E_6 (resp. E_8) ([9, 11, 17]). Hence they have a square root of Γ by Proposition 4.2.

(4) Kawahigashi ([11]) proved that each Coxeter graph of type $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ has only one biunitary connection. Hence $M \supset N$ is isomorphic to $B(\mathcal{G}) \supset C(\mathcal{G})$ ([17]). By Proposition 4.2, $M \supset N$ has a square root of Γ. ■

Next we give a characterization for a $\rho \in \mathrm{End}(A, \tau)$ to be a square root of Γ. By the fact that $e_i \in M_{i-2}' \cap M_i$ for all $i \geqslant 2$, we have the following commuting square coming from $M \supset N$:

$$\begin{array}{ccc} A = M' \cap M_\infty & \supset & B = M_1' \cap M_\infty \\ \cup & & \cup \\ R = \{e_i : i \leqslant 2\}'' & \supset & R_\lambda = \{e_i : i \geqslant 3\}''. \end{array}$$

Here $\lambda = [M : N]^{-1}$. We denote the tower of basic constructions for $R \subset A$ as follows:

$$R \subset A \subset A_1(R) = \langle A, e_R \rangle \subset A_2(R) \subset \cdots.$$

LEMMA 4.7. *Let $M \supset N$ be a pair with $2 \neq [M : N] \leqslant 4$. If $\dim(M' \cap M_2) = 2$, then a self conjugate σ satisfies that*

$$\sigma(e_i) = e_{i+1} \quad \text{for all } i \geqslant 2.$$

Proof. First we remark that M is normal in M_∞. In fact, it is clear that $(M' \cap M_\infty)' \cap M_\infty \supset M$. On the other hand, since $[M : N] < 4$, we have $(M' \cap M_\infty)' \cap M_\infty \subset \{e_2, e_3, \ldots\}' \cap M_\infty = M$ by Skau's lemma. Hence M is normal in M_∞. By Theorem 3.4, $\dim(M_{i-1}' \cap M_i) = \dim M' \cap M_2 = 2$ for all i. Since $e_i \in M_{i-2}' \cap M_i$ for all i, we have $\{e_i\}'' = M_{i-2}' \cap M_1$. By Proposition 3.4,

$$\sigma(\{e_i\}'') = \sigma(M_{i-2}' \cap M_i) = (M_{i-1}' \cap M_{i+1}) = \{e_{i+1}\}''$$

It implies that $\sigma(e_i) = e_{i+1}$ or $\sigma(e_i) = 1 - e_{i+1}$ for all i. Hence $\sigma(e_i) = e_{i+1}$ because σ is trace preserving. ■

REMARK. Under the conditions of Lemma 4.7, two self conjugate $\sigma, \rho \in \mathrm{End}(A, \tau)$ satisfy that $\sigma(R) = R_\lambda = \rho(R)$ and $\sigma(x) = \rho(x)$ for all $x \in R$. This fact is a key in the characterization for square roots of Γ.

A square root of Γ is self-conjugate. Hence we have:

COROLLARY 4.8. *If the principal graph of $M \supset N$ is one of Coxeter graphs A_n $(n \geqslant 3), D_{2n}$ $(n \geqslant 3), E_6, E_8, E_i^{(1)}$ $(i = 6, 7, 8)$ and A_∞, then a square root σ of Γ must satisfy $\sigma(e_i) = e_{i+1}$ for all $i \geqslant 2$.*

COROLLARY 4.9. *If the principal graph for $M \supset N$ is of type A_n $(n > 3)$, then there exists a unique square root of Γ.*

Proof. If $M \supset N$ is of type A_n $(n > 3)$, then there exists a square root σ of Γ by Corollary 4.6. The σ satisfies $\sigma(e_i) = e_{i+1}$ for all $i \geqslant 2$ by Corollary 4.8. Since $M \supset N$ is of type A_n, we have $A = R = \{e_i; i \geqslant 2\}''$. Therefore the square root of Γ is unique. ∎

PROPOSITION 4.10. *Assume that $M \supset N$ satisfies the conditions in Lemma 4.7. Then the number of square roots of Γ is smaller than the cardinal number of projections $p \in A' \cap A_2(R)$ with $\tau(p)[A : R] = 1$. Here τ is the Markov trace of modulus $[A : R]^{-1}$ for the pair $R \subset A$.*

Proof. Since R is a subfactor of A with finite index ([5]), there exists a projection $e \in A$ which satisfies that $\{e\}' \cap R = P$ is a subfactor of R and $A = \{R, e\}''$ is isomorphic to the basic extension $\langle R, e_P^R \rangle$ for the inclusion $R \supset P$ via an isomorphism φ with $\varphi(x) = x$ $(x \in R)$ and $\varphi(e) = e_P^R$ ([10]). Here e_P^R is the projection of $L^2(R)$ onto $L^2(P)$. Let σ and ρ be square roots of Γ. Then σ and ρ are self-conjugate. Hence $\sigma(x) = \rho(x)$ for all $x \in R$ by Lemma 4.7. This implies that $\sigma(P) = \rho(P)$. Assume that $\sigma \neq \rho$. Then $\sigma(e) \neq \rho(e)$ because $\left\{ \sum_i x_i e y_i ; x_i, y_i \in R \right\}$ is a dense $*$-subalgebra of A ([19]). Two projections $\sigma(e)$ and $\rho(e)$ are contained in $\sigma(P)' \cap \sigma(A) = \rho(P)' \cap \rho(A)$ and $\tau(\sigma(e)) = \tau(\rho(e)) = [A : R]^{-1}1$. Since $P' \cap A$ is isomorphic to $A' \cap A_2(R)$, we have the conclusion. ∎

COROLLARY 4.11. *If $M \supset N$ is a pair of hyperfinite II_1 factors with the principal graph E_6, then there exists the unique square root of Γ.*

Proof. If the principal graph of $M \supset N$ is the Coxeter graph E_6, then there exists a square root of Γ by Theorem 4.6. On the other hand, $A \supset R$ is isomorphic to the pair of factors with index $3 + \sqrt{3}$, which is constructed from the Coxeter graph E_6 by the method in [5, Section 4.5]. Okamoto ([18]) showed that the principal graph for the pair is the following:

On the other hand, Haagerup proved that if a finite depth subfactor has index $3+\sqrt{3}$ then the dual principal graph must be the above graph. The graph means that there exists only one projection $p \in A' \cap A_2(R)$ with $\tau(p) = [A : R]^{-1}1$. The pair $M \supset N$ satisfies the conditions in Lemma 4.8. Hence a square root of Γ for $M \supset N$ must be the endomorphism in the proof of Theorem 4.6 by Proposition 4.10. ∎

LEMMA 4.12. *Let $\sigma \in \mathrm{End}(A, \tau)$. If a subfactor R of A containes a projection e with $E_{\sigma(A)}(e) = (\mathrm{Ind}(\sigma))^{-1}1$, then σ satisfies*

$$E_R(\sigma(u)) = 0$$

for all unitaries $u \in A$ with the property $E_R(u) = 0$.

Proof. It is sufficient to prove $\tau(x\sigma(u)) = 0$ for all $x \in R$. By the property of e, we have $E_{\sigma(R)}(e) = E_{\sigma(R)}E_{\sigma(A)}(e) = (\mathrm{Ind}(\sigma))^{-1}1$. It implies that R is the basic extension of $\sigma(R) \supset \{e\}' \cap \sigma(R)$ ([19]). Hence $\left\{\sum_i a_i e b_i; a_i, b_i \in \sigma(R)\right\}$ is a dense $*$-subalgebra in R. We have

$$\tau(aeb\sigma(u)) = \tau(eb\sigma(u)a) = \tau(E_{\sigma(M)}(e)b\sigma(u)a) =$$

$$= (\mathrm{Ind}\,\sigma)^{-1}\tau(b\sigma(u)a) = (\mathrm{Ind}\,\sigma)^{-1}\tau(u\sigma^{-1}(ab)) = 0 \text{ for all } a, b \in \sigma(R).$$

This implies $\tau(x\sigma(u)) = 0$ for all $x \in R$. ∎

PROPOSITION 4.13. *Assume $A = R \rtimes_\alpha G$ of a factor R by an outer action α of a finite abelian group G. Let $\rho \in \mathrm{End}(A, \tau)$ be irreducible. Then there is a one to one correspondence between $\hat{G}$ and*

$$\Sigma = \{\sigma \in \mathrm{End}(A, \tau); \rho(A) = \sigma(A), \text{ and } \rho(y) = \sigma(y) \text{ for all } y \in R\},$$

via $\sigma = \rho_\gamma$ for the ρ_γ defined by the below $(*)$.

If $G = Z_2$ and there is a projection $e \in R$ with $E_{\rho(R)}(e) = (\mathrm{Ind}\,\rho)^{-1}1$, then $\rho^2 = \sigma^2$, for all $\sigma \in \Sigma$.

Proof. Let $u(g)$ be the canonical unitary in A corresponding $g \in G$. For a $\gamma \in \hat{G}$, let ρ_γ be the $*$-isomorphism from A onto $\rho(A)$ defined by

$$(*) \qquad \rho_\gamma(x) = \sum_{g \in G} \langle \gamma, g \rangle \rho(u(g)E_R(u(g)^* x)), \quad \text{for all } x \in A.$$

Then $\rho_\gamma(x) = \rho(x)$, for all $x \in R$, that is $\rho_\gamma \in \Sigma$.

Conversely, let $\sigma \in \Sigma$. Then

$$\rho(u(g))\rho(a)\rho(u(g))^* = \rho(u(g)au(g)^*) = \sigma(u(g)au(g)^*) = \sigma(u(g))\sigma(a)\sigma(u(g))^*$$

for all $a \in R$, $g \in G$. Hence $\rho(u(g))^*\sigma(u(g))) \in R' \cap A = C1$. We put

$$\gamma(g)1 = \rho(u(g))^*\sigma(u(g)), \quad \text{for all } g \in G.$$

Then we have $\gamma \in \hat{G}$ and for all $x \in A$

$$\sigma(x) = \sigma\left(\sum_g u(g)E_R(u(g)^*x)\right) = \sum_g \gamma(g)\rho(u(g))\rho(E_R(u(g)^*x)) = \rho_\gamma(x).$$

If $\rho_\gamma = \rho_\chi$ for $\gamma, \chi \in \hat{G}$, then $\langle\gamma, g\rangle\rho(u(g)) = \rho_\gamma(u(g)) = \rho_\chi(u(g)) = \langle\chi, g\rangle\rho(u(g))$ for all $g \in G$. It implies that $\gamma = \chi$.

For a $\gamma \in \hat{G}$, put $\sigma = \rho_\gamma$. Since $E_R(u(g)) = 0$ for all $g \in G$, it follows by Lemma 4.12 that $E_R(\rho(u(g))) = 0$ for all $1 \neq g \in G$. Assume that $G = \mathbf{Z}_2$, then there exists a $z \in R$ such that $\rho(u) = uz$, where $u = u(g)$ for the generator $g \in G$. Then for an $x \in A$,

$$\sigma(x) = \rho(E_R(x)) - \rho(u)\rho(E_R(u(x))) = \rho(E_R(x)) - ua\rho(E_R(ux)).$$

Hence for all $x \in A$,

$$\sigma^2(x) = \rho^2(E_R(x)) + \rho(ua\rho(E_R(ux))) = \rho^2(E_R(x)) + \rho^2(ux) = \rho^2(x). \qquad \blacksquare$$

COROLLARY 4.14. *If $M \supset N$ has the principal graph D_{2n} ($n \geqslant 3$), then there exist precisely two square roots of Γ.*

Proof. Let the principal graph of $M \supset N$ be the Coxeter graph D_{2n} ($n \geqslant 3$). Then we have a square root of Γ by Corollary 4.6. Remark that σ is irreducible because $A \supset B$ is isomorphic to $M \supset N$ [21]. If ρ is a square root of Γ, then $\sigma(x) = \rho(x)$ for all $x \in R$ by Lemma 4.7. On the other hand, $A \supset B$ is the simultaneous crossed products $R \lhd_\alpha \mathbf{Z}_2 \supset R_\lambda \lhd_\alpha \mathbf{Z}_2$ by an outer action α of $\mathbf{Z}_2$. Hence we have precisely two square roots of Γ by Proposition 4.13. $\blacksquare$

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