J. OPERATOR THEORY
31(1994), 229–252

ON THE TYPE OF SECOND QUANTIZATION FACTORS

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INTRODUCTION

Among ITPFI factors, there is a class of algebras which motivated in part the investigation of Araki and Woods in [6]: the so-called second quantization algebras. These are the von-Neumann algebras on the Fock space $e^{\mathcal{H}}$ ($\mathcal{H}$ is a separable complex Hilbert space) generated by the Weyl unitaries with test functions in a closed, real linear subspace of $\mathcal{H}$. Second quantization algebras corresponding to local subspaces (these are the subspaces corresponding to bounded regions of $\mathbf{R}^4$ in the free scalar field, see e.g. [3]) are known to be injective type III_1 factors. Hence, it is natural to ask which type of factors one gets dealing with generic second quantization algebras.

The main result of Section 2 is that all the Powers factors and also some examples of III_0 factors appear as second quantization algebras.

In Sections 3 and 4 we identify the class of second quantization algebras with the Grassmannian of the linear subspaces of $\mathcal{H}$, and therefore we can study the continuity properties of the map which associates with any point in the Grassmannian the type of the second quantization algebra over it. We study also the possibility of approximating in norm topology the factors in a given equivalence class with factors in other classes, and in particular we prove that the class of III_1 factors is the only class of isomorphic factors which is dense in the space of second quantization factors.

1. BACKGROUND

In this section we present the main definitions and known theorems on second quantization algebras.

Let $\mathcal{H}$ be a complex separable Hilbert space. The *symmetric Fock space* over it is

$$e^{\mathcal{H}} = \bigoplus_{n=0}^{\infty} H^{\otimes_s n}$$

where $H^{\otimes_s n}$ is the subspace of the n-th tensor product of $\mathcal{H}$ which is pointwise invariant under the natural action of the permutation group.

The set of *coherent* vectors in e^H consists of the vectors

$$e^h = \bigoplus_{n=0}^{\infty} \frac{h^{\otimes n}}{\sqrt{n!}}.$$

This set turns out to be total in $e^{\mathcal{H}}$ (see e.g. [15] p.32).

There are two important classes of operators acting on $e^{\mathcal{H}}$:
Second quantization operators

$$e^A = \bigoplus_{n=0}^{\infty} A^{\otimes n},$$

where A is a densely defined, closed operator on $\mathcal{H}$, and
Weyl unitaries, which are the range of the map

$$h \to W(h)$$

from $\mathcal{H}$ to the unitaries on $e^{\mathcal{H}}$ defined by

$$W(h)e^0 = \exp\left(-\frac{1}{4}\|h\|^2\right) e^{\frac{i}{\sqrt{2}}h}, \quad h \in \mathcal{H}$$

$$W(h)W(k) = \exp\left(-\frac{i}{2}\mathrm{Im}(h,k)\right) W(h+k) \quad h,k \in \mathcal{H}$$

The vector e^0 is called *vacuum* and the relations in the last equality are called Canonical Commutation Relations. Via the preceding equalities $W(h)$ becomes a well defined, isometric and invertible (with inverse $W(-h)$) operator on the dense set spanned by coherent vectors, and hence it extends to a unitary on $e^{\mathcal{H}}$. Weyl unitaries generate the so-called *second quantization algebras*. With each closed real linear subspace $K \subset \mathcal{H}$ (in the following we shall write $K \leqslant_{\mathbb{R}} \mathcal{H}$), a von-Neumann algebra $\mathcal{R}(K)$ is associated, defined by

$$\mathcal{R}(K) = \{W(h), \quad h \in K\}''.$$

If K is *standard*, i.e. $K + iK$ is dense in $\mathcal{H}$, and $K \cap iK = \{0\}$, a closed, densely defined, antilinear operator s is defined on $\mathcal{H}$ such that

$$s : K + iK \to K + iK$$

$$h + ik \to h - iK.$$

We recall now some known properties of the second quantization algebras and their modular operators.

THEOREM 1.1. [13] *A second quantization algebra $\mathcal{R}(K)$ is in standard form w.r.t. the vacuum if and only if K is standard. In this case $S = e^{s}$, $\Delta = e^{\delta}$, and $J = e^{j}$, where S is the Tomita operator of $(\mathcal{R}(K), e^0)$ and $S = J\Delta^{\frac{1}{2}}$, $s = j\delta^{\frac{1}{2}}$ are the polar decompositions of S and s, respectively.*

THEOREM 1.2. ([1] Theorem 1) *The map $K \to \mathcal{R}(K)$ is an isomorphism of complemented nets, where the complementation of an algebra is its commutant and the complementation of a real subspace K is the simplectic complement $K' = \{h \in \in \mathcal{H} : \text{Im}(h,k) = 0\}$.*

Due to the previous theorem we shall call *factor subspace* a real subspace K for which $K \cap K' = \{0\}$.

The following three theorems were originally stated in a different way, in particular they preceded the Tomita-Takesaki theory. The reformulation in terms of the modular operators is a consequence of Proposition 3.2.

THEOREM 1.3.

(i) *Second quantization factors are* ITPFI *factors.*

(ii) *Second quantization factors are type* I *if and only if $\delta|_{[0,1]}$ is a trace class operator, where $\delta|_{[0,1]}$ is the restriction of the modular operator to the spectral subspace relative to the interval* $[0,1]$.

(iii) *Second quantization factors which are not type* I *are type* III.

Point i) was proved by Araki [1] for the completely diagonalizable case. The extension to the general case is due to Dell'Antonio ([9] Proposition 4) and follows from the Hilbert-Schmidt perturbation invariance stated in the next theorem. We shall describe ITPFI decomposition of second quantization factors later in this section. The second statement is a direct reformulation of Thorem 4 in [1] via Proposition 3.2. The third statement was proven by Araki in [2].

THEOREM 1.4. [9] *Let K_1 and K_2 be two standard factor subspaces with modular operators δ_1 and δ_2, respectively, and define the operators α_1, α_2 as*

$$\alpha_i = \frac{4\delta_i}{(\delta_i - I)^2}, \quad i = 1, 2.$$

Since $1 \notin \sigma(\delta_i)$ (see comment after Theorem 1.6), α_i is well defined.

If $(\alpha_1)^{\frac{1}{2}}$ is unitarily equivalent to the sum of $(\alpha_2)^{\frac{1}{2}}$ plus a Hilbert-Schmidt operator, then $\mathcal{R}(K_1)$ and $\mathcal{R}(K_2)$ are isomorphic von-Neumann algebras.

This result was proved by Dell'Antonio in terms of the operator α introduced in [1]. The equivalence with the above described functional calculus of δ is shown in Proposition 3.2. As a matter of fact, even though the argument in [9] is correct, the original statement ([9] Prop.3) is not completely correct (cf. Remark 2.6 and also [6], p.127).

THEOREM 1.5. [1] [6] [7] *Let K be a standard factor subspace and δ its modular operator. Then $\mathcal{R}(K)$ is completely determined up to isomorphism by the spectral properties of δ, and*

$$\sigma_{ess}(\delta) \subset S(\mathcal{R}(K)).$$

Theorem 2 in [1] and Proposition 3.2 imply that the isomorphism class of $\mathcal{R}(K)$, and therefore the Connes invariant, depends only on the spectral measure and multiplicity of δ. The fact that (an appropriate function of) the points in the essential spectrum of α give rise to points in r_∞ is mentioned in [6], and Connes [7] proved the equality of r_∞ with S (up to 0 and 1). We shall give a proof of the inclusion $\sigma_{ess}(\delta) \subset S(\mathcal{R}(K))$ in Proposition 2.9.

Finally, we mention a theorem which, even though it is not used in the following analysis, motivated in part the study of the subject.

THEOREM 1.6. *For the free Bose field, the second quantization algebras corresponding to the local subspaces, i.e. the algebras $\mathcal{R}(\mathcal{O})$ associated to open double cones in the Minkowski space, are type* III_1 *factors.*

The proof of this theorem has a long history. Factor property was proven in [3]. It was conjectured in [6] that $\sigma(\alpha)$ for local subspaces contains continuous parts, which implies the corresponding second quantization factors to be the $\mathcal{R}_\infty$ factor, i.e. the unique injective factor of type III_1 after Connes and Haagerup [8,16]. The first complete proof appeared in [19], where the result in [17] for the massless case is extended to massive theories using a result in [12]. We have also a two-lines proof of this result: Dell'Antonio, in order to prove that local algebras associated to double cones are type III, observed that for these algebras α is an unbounded operator, and therefore it is not trace class [9]. As a matter of fact, this means, via Proposition 3.2, that $1 \in \sigma_{ess}(\delta)$ (this fact follows also from the formulas in [14]), and then Proposition 4.5 implies the factor to be III_1.

We give now an argument for the ITPFI decomposition of second quantization factors that will be used later. It turns out to be the reformulation in terms of the modular operators of the original argument in [1].

Let us restrict our attention for a while to standard factor subspaces K with completely diagonalizable δ. We remark that the factor property amounts to the fact that $1 \notin \sigma_p(\delta)$. In fact

$$K \cap K' = \{k \in \mathcal{H} : sk = k = s^* k\} = \{k \in K : \delta k = k\}.$$

If $\{x_n\}_{n \in \mathbb{N}}$ is an orthonormal system of eigenvectors for $\delta|_{[0,1]}$ then $\{jx_n\}_{n \in \mathbb{N}}$ is an orthonormal system of eigenvectors for $\delta|_{(1,\infty)}$ and $\{x_n, jx_n\}_{n \in \mathbb{N}}$ is a basis for $\mathcal{H}$. Now, if $\mathcal{H}_n$ is the two-dimensional complex subspace generated by x_n and jx_n, $\mathcal{H}_n$ reduces δ, j, and s. Thus

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \quad K = \bigoplus_{n=1}^{\infty} K_n, \quad \mathcal{R}(K) = \bigoplus_{n=1}^{\infty \ \{\Omega\}} \mathcal{R}(K_n),$$

where we posed $K_n = K \cap \mathcal{H}_n$, Ω_n represents the vacuum vector in $e^{\mathcal{H}_n}$, and the last equality is a consequence of the tensor product decomposition of a second quantization algebra when its subspace is a direct sum of complex orthogonal components (see e.g. [15, 6]).

Since K_n is two-dimensional, $\mathcal{R}(K_n)$ is type I, and therefore the preceding formula gives the desired decomposition. Due to Theorem 1.4 and Weyl-von-Neumann Theorem [21], each second quantization factor is isomorphic to a second quantization factor with completely diagonal δ. The Hilbert-Schmidt perturbation in Theorem 1.4 can be chosen in such a way that the spectrum of δ is globally preserved and each point (different from 0 or 1) in $\sigma_{ess}(\delta)$ is an eigenvalue with infinite multiplicity. Non diagonal cases can be described also as continuous direct products of type I factors [5].

REMARK 1.7. The infinite tensor product decomposition extends to general real linear subspaces of $\mathcal{H}$. If $K \leqslant_{\mathbb{R}} \mathcal{H}$, let us define (cf. [1])

$$\mathcal{H}_0 = K \cap iK, \quad \mathcal{H}_\infty = (K + iK)^\perp, \quad \mathcal{H}_I = \{k : \delta k = k\}, \quad \mathcal{H}_F = (\mathcal{H}_0 \oplus \mathcal{H}_\infty \oplus \mathcal{H}_I)^\perp.$$

Then $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_F \oplus \mathcal{H}_I \oplus \mathcal{H}_\infty$ and $K = K_0 \oplus K_F \oplus K_I \oplus 0$, where

(i) $K_0 = \mathcal{H}_0$, therefore $\mathcal{R}(K_0) = \mathcal{R}(\mathcal{H}_0) \simeq \mathcal{B}(e^{\mathcal{H}_0})$ is a type I_∞ factor,

(ii) $K_F \overset{\text{def}}{=} K \cap \mathcal{H}_F$ is a standard factor subspace of $\mathcal{H}_F$,

(iii) $K_I \overset{\text{def}}{=} K \cap \mathcal{H}_I$ is a j-invariant standard subspace of $\mathcal{H}_I$, i.e. $\mathcal{R}(K_I)$ is a continuous maximal abelian subalgebra of $\mathcal{B}(e^{\mathcal{H}_I})$.

By this decomposition, $\mathcal{R}(K)$ is the tensor product of its center $\mathcal{R}(K_I)$ with the ITPFI factor $\mathcal{R}(K_0) \otimes \mathcal{R}(K_F)$.

We observe also that, if K is a non standard factor, δ and α are defined on $\mathcal{H}_F$, but Remark 3.3 shows that it is very natural to extend α to $\mathcal{H}_0 \oplus \mathcal{H}_F$ posing $\alpha = 0$ on $\mathcal{H}_0$.

In this way the "if" part of the statement ii) in Theorem 1.3 becomes a consequence of Theorem 1.4 and of point i) of this remark.

2. THE TYPE OF SECOND QUANTIZATION FACTORS

In this section we shall show that III_λ factors appear as second quantization factors for each $\lambda \in [0,1]$.

We remark that all the constructions of second quantization factors of a given type described in this and in the following sections are based on a prescribed behavior of the spectrum of δ. We want to stress that the construction of subspaces K whose δ fulfills the required properties is always possible. More precisely, the following proposition holds.

PROPOSITION 2.1. *Let $\{\lambda_n\}$ be a sequence with values in $[0,1]$ (or, more generally, let μ be a measure with support in $[0,1]$ and $n(x)$ a measurable function defined on $[0,1]$ with values in $\mathbf{N}\cup\{\infty\}$.) Then there exists a standard subspace K such that the sequence of eigenvalues with multiplicity of $\delta|[0,1]$ coincides with $\{\lambda_n\}$ (or, more generally, the spectral measure and multiplicity of $\delta|[0,1]$ coincide, respectively, with μ and n).*

Proof. We consider the measurable vector bundle on $[0,1]$ whose fiber on $x \in [0,1]$ is the space $\mathbf{C}^{n(x)}$ where by $\mathbf{C}^{(\infty)}$ we mean the Hilbert space $l^2(\mathbf{N})$, and then we identify $\mathcal{H}$ with the direct sum of two copies of the Hilbert space $L^2(\mathrm{d}\mu, \mathbf{C}^{n(x)})$ of square integrable sections,

$$\mathcal{H} \equiv L^2(\mathrm{d}\mu, \mathbf{C}^{n(x)}) \oplus L^2(\mathrm{d}\mu, \mathbf{C}^{n(x)}).$$

Then we consider the space $K = \{k_1 \oplus k_2 : C\mathcal{M}_{\sqrt{x}}k_1 = k_2\}$, where C is the complex conjugation on all the fibers and $\mathcal{M}_{\sqrt{x}}$ is the multiplication operator by $\sqrt{x}$ on $L^2(\mathrm{d}\mu, \mathbf{C}^{n(x)})$. It is easy to see that the modular operator δ associated with K coincides with $\mathcal{M}_x \oplus \mathcal{M}_{x^{-1}}$, and therefore the thesis follows. ∎

A result of Störmer [20] shows that the Connes invariant of the weak closure of an asymptotically abelian C^*-algebra (see [10]) in the representation of an invariant factorial state coincides with the spectrum of the modular operator Δ associated with this state.

We shall prove that, if the modular operator δ of a standard factor subspace K satisfies $\sigma(\delta) = \sigma_{\mathrm{ess}}(\delta)$, then second quantization C^*-algebra over K is asymptotically abelian w.r.t. a vacuum preserving action of $\mathbf{Z}$. By Störmers's theorem we get $S(\mathcal{R}(K)) = \sigma(\Delta)$.

THEOREM 2.2. *Let K be a standard factor subspace such that δ_K is completely diagonalizable and $\dim(\mathcal{H}_\lambda) = \infty$ for each λ in the point spectrum of δ, where $\mathcal{H}_\lambda$ is the eigenspace relative to λ. Then there exists a vacuum preserving, unitarily* *implemented action of* $\mathbf{Z}$ *on the Fock space for which the* C^**-algebra generated by* $\{W(h) : h \in K\}$ *is asymptotically abelian.*

Proof. Since any eigenvalue of δ has infinite multiplicity, we may identify $\mathcal{H}$ with $\mathcal{H}_1 \otimes L^2(\mathbf{Z})$ in such a way that K is identified with $K_1 \otimes_{\mathbf{R}} L^2_{\mathbf{R}}(\mathbf{Z})$ and the eigenvalues of the modular opeartor δ_1 of the inclusion $K_1 \leqslant_{\mathbf{R}} \mathcal{H}_1$ have multiplicity 1. Note that $s \cong s_1 \otimes I$, and the analogous identifications for δ and j hold. Now let us define $U \cong I \otimes T$, where T is the shift on $L^2(\mathbf{Z})$. Since U commutes with s it preserves K, and e^U implements an automorphism of $\mathcal{R}(K)$. Moreover, any second quantization operator preserves the vacuum. Now, if $k \equiv k_1 \otimes f$ and $h \equiv h_1 \otimes g$ are vectors in K,

$$\lim_{n\to\infty}(h, U^n k) = (h_1, k_1) \quad \lim_{n\to\infty}(g, T^n f) = 0$$

and this property extends to all pairs of vectors in K by linearity and density. Then

$$\|[W(h), e^{U^n} W(k) e^{U^{-n}}]\| = \|[W(h), W(U^n k)]\| = 2\left|\sin\left(\frac{1}{2}\operatorname{Im}(h, U^n k)\right)\right| \xrightarrow{n\to\infty} 0$$

and, again by linearity and density, we get the asymptotic abelianness of the generated C^*-algebra w.r.t. the action $n \in \mathbf{Z} \to \operatorname{ad}(e^{U^n})$. ∎

DEFINITION 2.3. Let X be a non-empty subset of $\overline{\mathbf{R}}_+$. Then $g(X)$ will denote the closure in $\mathbf{R}$ of the multiplicative subgroup of $\mathbf{R}_+$ generated by $X - \{0\}$ if $X \neq \{0\}$, and the set $\{0, 1\}$ if $X = \{0\}$.

Now we have to prove the following.

COROLLARY 2.4. *If* K *is a standard factor subspace of* $\mathcal{H}$ *with* $\sigma(\delta) = \sigma_{\text{ess}}(\delta)$, *then*

$$S(\mathcal{R}(K)) = g(\sigma_{\text{ess}}(\delta)).$$

As a consequence,

i) $\mathcal{R}(K)$ *is a* III_1 *factor if* $g(\sigma_{\text{ess}}(\delta)) = \overline{\mathbf{R}}_+$

ii) $\mathcal{R}(K)$ *is a* III_λ *factor if* $g(\sigma_{\text{ess}}(\delta)) = g(\{\lambda\}), \lambda \in (0, 1)$.

Proof. Since $\sigma(\delta) = \sigma_{\text{ess}}(\delta)$, we can find K_1 such that $\sigma(\delta_1) = \sigma_{\text{ess}}(\delta_1) = \sigma(\delta)$, δ_1 is completely diagonalizable each eigenvalue having infinite multiplicity, and $\mathcal{R}(K) \simeq \simeq \mathcal{R}(K_1)$ (Theorem 1.4). Therefore we can apply Theorem 2.2, and Störmer's theorem gives the equality $S(\mathcal{R}(K)) = \sigma(\Delta)$. The remaining part of the theorem is a consequence of the equality $g(\sigma(\delta)) = \sigma(\Delta)$ which is proven in the following Lemma. ∎

LEMMA 2.5. *If* K *is a standard subspace of* $\mathcal{H}$ *then*

$$g(\sigma(\delta)) = \sigma(\Delta).$$

Proof. $\sigma(A^{\otimes n}) = \overline{\sigma(A)^n} \stackrel{\text{def}}{=} \left\{\prod_{i=1}^{n}\lambda_i : \lambda_i \in \sigma(A)\right\}^-$ and $\sigma(A \oplus B) = (\sigma(A) \cup \sigma(B))^-$ for any closed self-adjoint A, B on $\mathcal{H}$. Hence, $\sigma(e^A)$ is the closure of the multiplicative hull of $\sigma(A)$. Since the spectrum of δ is closed w.r.t. the operation of taking the inverse via the relation $\delta_j = j\delta^{-1}$, we get the desired equality. ■

REMARK 2.6. The previous lemma and theorem 1.5 imply that

$$g(\sigma_{\text{ess}}(\delta)) \subset S(\mathcal{R}(K)) \subset g(\sigma(\delta)).$$

As a consequence, when $\sigma(\delta)$ and $\sigma_{\text{ess}}(\delta)$ coincide, the Connes invariant is completely determined, as Corollary 2.4 shows.

The general problem is therefore to understand when the non-essential spectrum modifies the Connes invariant.

For instance, Theorem1.4 shows that a subsequence of eigenvalues of δ which converges rapidly enough to a point $\lambda \in (0,1)$ can be substituted by the eigenvalue λ with infinite multiplicity.

More generally, when the non-essential part of the spectrum of δ can be eliminated by a Hilbert-Schmidt perturbation as described in Theorem 1.4, the results of Corollary 2.4 still hold. Under this hypothesis, we have the following case:

iii) $\mathcal{R}(K)$ is type I_∞ if $g(\sigma_{\text{ess}}(\delta)) = \{0,1\}$.

In fact $1 \in \sigma_{\text{ess}}(\delta)$ is equivalent to the unboundedness of α, and therefore Hilbert-Schmidt perturbations cannot eliminate the non-essential parts of the spectrum of δ around 1. Hence, we have $\sigma_{\text{ess}}(\delta) = 0$. Then the hypothesis corresponds to $\text{tr}(\delta|[0,1] <$ $< +\infty$ and, therefore, iii) follows from 1.3 ii).

On the other hand, Theorem 1.3 implies that if $\delta|_{[0,1]}$ is a compact, non trace class operator we get a type III factor, and we shall show that in this hypothesis it is possible to get III_0 factors (Theorem 2.10) and, more generally, III_λ factors for each $\lambda \in [0,1]$ (Propositions 4.10 and 4.11). The relevance of the non-essential spectrum of δ is illustrated also by Proposition 4.5 and 4.6.

In the following, we shall exhibit explicit examples of III_0 second quantization factors for which the Connes invariant will be calculated making use of the Araki-Wood formula for the *asymptotic ratio set* r_∞ [6].

It is well known that r_∞ is an invariant of the factor [6], and it coincides with the Connes invariant possibly up to 0 or 1 [7].

If M is a type I factor with a cyclic and separating vector Ω, the sequence of eigenvalues with multiplicity of the trace class operator associated with the vector state Ω is denoted by $\text{sp}(\Omega, M)$. It turns out to be the sequence of the eigenvalues lower or equal to one of the modular operator Δ, up to a normalization constant. If

$$M = \bigotimes_{n\in\mathbb{N}}^{\Omega_n} M_n$$

is an ITPFI factor, $\mathrm{sp}(\Omega, M)$ will denote the double-indexed sequence

$$\{\lambda_{nj} : n \in \mathbb{N}, \{\lambda_{nj}\}_{j\in\mathbb{N}} = \mathrm{sp}(\Omega_n, M_n)\}_{n,j\in\mathbb{N}}.$$

It is easy to show the following

PROPOSITION 2.7. *If $M = \mathcal{R}(K)$ is a second quantization factor with completely diagonalizable δ and $\{\lambda_n\}$ is the sequence with multiplicity of the eigenvalues of $\delta|_{[0,1]}$, then*

$$\mathrm{sp}(e^0, \mathcal{R}(K)) = \{\lambda_{nj} \equiv (1-\lambda_n)\lambda_n^{j-1} : j, n \in \mathbb{N}\}.$$

DEFINITION 2.8. Let (M, Ω) be an ITPFI factor, $\{\lambda_{jk}\} = \mathrm{sp}(\Omega, M)$. The asymptotic ratio set $r_\infty(\Omega, M)$ consits of the numbers $x \in \mathbb{R}_+$ for which there exists a sequence of triples $\{I_n, K_n, \varphi_n\}$, where I_n is a finite subset of $\mathbb{N}$, $K_n \subset \mathbb{N}^{I_n}$,and $\varphi_n : K_n \to \mathbb{N}^{I_n}$, with the following properties:

(i) $I_n \cap I_m = \emptyset, \quad n \neq m$

(ii) $(\forall)\mathbf{k} \in K_n, \quad \lambda(\mathbf{k}) \overset{\text{def}}{=} \prod_{j\in I_n} \lambda_{j\mathbf{k}j} \neq 0$

(iii) φ_n is injective and $\varphi_n(K_n) \cap K_n = \emptyset$

(iv) $\sum_{n\in\mathbb{N}} \sum_{\mathbf{k}\in K_n} \lambda(\mathbf{k}) = +\infty$

(v) $\lim_{n\to\infty} \max_{\mathbf{k}\in K_n} \left| x - \frac{\lambda(\varphi_n(\mathbf{k}))}{\lambda(\mathbf{k})} \right| = 0$

We note that for the particular case of the second quantization factors, making use of Proposition 2.6, the condition 2.8v is equivalent to the following (when $x \neq 0$)

(v)′ $\lim_{n\to\infty} \max_{\mathbf{k}\in K_n} \left| \log x - \sum_{j\in I_n} \log \lambda_j(\varphi_n(\mathbf{k})_j - \mathbf{k}_j) \right| = 0$

As a first consequence of the Araki-Wood formula, we shall prove the following Proposition (cf. Theorem 1.5).

PROPOSITION 2.9. *Let K be a factor subspace of $\mathcal{H}$. If $\lambda \in \sigma_{\mathrm{ess}}(\delta)$ then $\lambda \in$ $\in S(\mathcal{R}(K))$.*

Proof. Due to Remark 1.7 we can restrict to the standard case. If λ is equal to 0 or 1 the result is trivial. Otherwise, we may choose an appropriate Hilbert-Schmidt perturbation such that δ becomes completely diagonalizable, λ becomes an eigenvalue with infinite multiplicity, and the isomorphism class of $\mathcal{R}(K)$ does not change (see

Theorem 1.4). Now let λ_{n_p} be the subsequence of the spectral sequence of $\delta|_{[0,1]}$ with value λ. Recalling Proposition 2.6 and Definition 2.7 we set

$$I_p = \{n_p\}$$

$$K_p = \{1\}$$

$$\varphi_p(1) = 2$$

Then i), ii) and iii) are satisfied,

$$\sum_{p\in\mathbf{N}} \sum_{k\in K_p} \lambda(\mathrm{k}) = \sum_{p\in\mathbf{N}}(1-\lambda_{n_p}) = \sum_{p\in\mathbf{N}}(1-\lambda) = +\infty$$

and

$$\lim_{p\to\infty} \max_{k\in K_p} \left| \log\lambda - \sum_{j\in I_n} \log\lambda_j(\varphi_n(\mathrm{k})_j - \mathrm{k}_j) \right| = \lim_{p\to\infty} |\log\lambda - \log\lambda_{n_p}| = 0,$$

therefore the result follows. ∎

We conclude this section with the announced examples of III_0 second quantization factors.

THEOREM 2.10. *Let $\lambda \in (0,1)$, $p \in \mathbf{N}$ and $\{n_k\}_{k\in\mathbf{N}}$ a sequence of positive integers such that*

$$\lim_{k\to\infty} |n_k|_p = 0,$$

where $|m|_p$ is the p-adic modulus[1] *of m. If $\sigma(\delta|_{[0,1]}) = \{\lambda^{n_k},\ k\in\mathbf{N}\}^-$, the eigenvalue λ^{n_k} has finite multiplicity d_k and $\sum_{k=0}^{\infty} d_k\lambda^{n_k} = +\infty$, then $\mathcal{R}(K)$ is a type III_0 factor.*

Proof. By Lemma 2.5 we have $S(\mathcal{R}(K)) \subset g(\lambda^{n_k} : k \geqslant 1)$. Since r_∞ depends asimptotically on the sequence of the eigenvalues of δ and any eigenvalue has finite multiplicity,

$$S(\mathcal{R}(K)) \subset \bigcap_{k_0\in\mathbf{N}} g(\lambda^{n_k} : k \geqslant k_0).$$

By hypothesis,

$$\forall m\in\mathbf{N}\ \exists k_0\in\mathbf{N} : p^m \text{ divides } n_k,\ k\geqslant k_0$$

therefore

$$S(\mathcal{R}(K)) \subset \bigcap_{m\in\mathbf{N}} g(\lambda^{p^m}) = \{0,1\}.$$

[1] The p-adic modulus of an integer m is p^{-q}, where p^q is the greatest integer power of p that divides m.

Since

$$\mathrm{tr}(\delta|_{[0,1]}) = \sum_{k=0}^{\infty} d_k \lambda^{n_k} = +\infty,$$

Theorem 1.3 implies that $\mathcal{R}(K)$ is a type III_0 factor. ■

We shall not prove further results about second quantization III_0 factors. We note only that it is possible to prove that second quantization factors are ITPFI_2 and that the examples in Theorem 2.10 give rise to non-isomorphic factors for different λ and p.

3. THE GRASSMANNIAN OF THE REAL SUBSPACES

We collect in this section some observations on the Grassmannian $\mathcal{G}$ of the real closed subspaces of $\mathcal{H}$, which will be useful in the following section.

A detailed analysis on the relationships among some canonical operators associated with a real linear subspace will follow. On the one hand we think that the "angular" interpretation of those operators is interesting in itself. On the other hand, Proposition 3.2 justifies the formulation of theorems 1.3, 1.4, and Corollary 3.4 will be used in Section 4 to prove the density results.

Since second quantization algebras are labelled by closed real subspaces of $\mathcal{H}$, it is natural to associate the set of these algebras with the Grassmannian $\mathcal{G}$ of such subspaces. From now on, we shall identify $\mathcal{G}$ with the closed subset of $B(\mathcal{H}_{\mathbb{R}})$ consisting of the real projections acting on the real Hilbert space $\mathcal{H}_{\mathbb{R}} = \{\mathcal{H}, \mathrm{Re}(\cdot,\cdot)\}$, hence $\mathcal{G}$ becomes a metric space with the distance

$$\mathrm{dist}(P,Q) = \|P - Q\|,$$

where P and Q are real projections.

We observe that $\mathcal{G}$ decomposes into countably many connected components:

$$\mathcal{G} = \bigoplus_{n\in\mathbb{N}} \mathcal{G}_{n,\infty} \oplus \bigoplus_{n\in\mathbb{N}} \mathcal{G}_{\infty,n} \oplus \mathcal{G}_{\infty,\infty}$$

where $\mathcal{G}_{n,m}$ consists of the subspaces of (real) dimension n and (real) codimension m.

Decomposing an element K of $\mathcal{G}_{n,\infty}$ or $\mathcal{G}_{\infty,n}$, $n \in \mathbb{N}$, according to the decomposition in Remark 1.7, it is easy to see that the subspace K_F is finite dimensional, and therefore $\mathcal{R}(K)$ is always of the form:

(a) Type I_∞ factor

(b) A continuous abelian von-Neumann algebra

(c) The tensor product of case (a) and case (b).

In particular, all the factors in $\mathcal{G}_{n,\infty}$ or $\mathcal{G}_{\infty,n}$ are isomorphic to the infinite injective type I factor. On the contrary, since all the points of $\mathcal{G}_{\infty,\infty}$ are equivalent as real projections, the type of a second quantization factor in $\mathcal{G}_{\infty,\infty}$ depends crucially on the complex structure on $\mathcal{H}$.

If P and Q are two projections on a real Hilbert space with range M, N, respectively, the angle between M and N is defined (cf. [4]) as the symmetric positive operator $\theta_{M,N}$ acting on M such that

$$\cos\theta_{M,N} = |QP|\Big|_{M}$$

$$\|\theta_{M,N}\| \leqslant \frac{\pi}{2}.$$

Since $\mathcal{H}$ has also a complex Hilbert space structure, we can associate with a real projection P with range K the angle between K and iK, i.e. the operator

$$\theta = \arccos|PiP|\Big|_{K}.$$

Moreover, if K is a standard factor subspace, a unique, closed, injective operator φ (see [1]) from K to the orthogonal $K^{\perp}$ is defined by the property

$$h + \varphi(h) \in \mathrm{i}K, \quad h \in K.$$

We begin to study the relations among $\theta, \alpha = \varphi^*\varphi$ and the modular operator δ in the easiest non trivial case, i.e. when $\mathcal{H}$ is two-dimensional.

If K is a standard factor subspace of $\mathbf{C}^2$, then K has real dimensions 2, and hence it is possible to choose two real orthonormal vectors y^+, y^- in K with

$$\cos\theta \overset{\text{def}}{=} \mathrm{Im}(y^+, y^-) > 0,$$

and it is easy to see that

$$|PiP|\Big|_{K} = \cos\theta I\Big|_{K},$$

Now the vectors

$$\begin{cases} x^+ = \left(2\cos\dfrac{\theta}{2}\right)^{-1}(y^+ - \mathrm{i}y^-) \\ x^- = \left(2\sin\dfrac{\theta}{2}\right)^{-1}(y^+ + \mathrm{i}y^-) \end{cases}$$

form a complex orthonormal basis in $\mathcal{H}$, with respect to which Tomita operators satisfy

$$\jmath = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \delta = \begin{pmatrix} \tan^2\dfrac{\theta}{2} & 0 \\ 0 & \cot^2\dfrac{\theta}{2} \end{pmatrix}$$

where C is the complex conjugation.

With similar computations we can show that

$$\alpha = \tan^2\theta \cdot I\Big|_K$$

and

$$\theta_{K,K'} = \left(\frac{\pi}{2} - \theta\right) \cdot I\Big|_K .$$

Now we shall see that the relations established in the two-dimensional case hold in full generality.

We observe that a symmetric operator defined on a standard K can be extended by complex linearity to a densely defined operator on $\mathcal{H}$, but this extension is not self-adjoint in general. Yet, for the operators we are studying, the following property holds:

PROPOSITION 3.1. *Let K be a standard factor subspace. Then the previously defined operators α, θ and $\theta_{K,K'}$ extend to self-adjoint operators on $\mathcal{H}$. In this case, spectral measure and multiplicity are preserved.*

The proposition above allows us to use the same symbols for these operators and their complex linear extensions. Now we can state the following

PROPOSITION 3.2. *Let K be a standard factor subspace. Then the following relations hold:*

$$\cos\theta = \frac{|\delta - I|}{\delta + I}$$
$$\theta_{K,K'} = \left(\frac{\pi}{2} - \theta\right)$$
$$\alpha = \tan^2\theta = \frac{4\delta}{(\delta - 1)^2}$$
$$P = \frac{1}{\delta + 1} + j\frac{\delta^{\frac{1}{2}}}{\delta + 1}$$

Proof of Propositions 3.1 and 3.2. The complex extensions of the symmetric operator θ is self-adjoint if and only if θ verifies

$$\mathrm{Im}(x, \theta y) = \mathrm{Im}(\theta x, y) \quad x, y \in K.$$

But

$$\mathrm{Im}(x, \theta y) = -\mathrm{Re}(x, i\theta y) = -\mathrm{Re}(x, PiP\theta y) \quad x, y \in K.$$

Then self-adjointness follows from the fact that θ commutes with PiP because θ is a functional calculus of $|PiP|$ and PiP commutes with its transpose $-PiP$.

Now let E be a spectral projection relative to the complex linear θ. Since the complex linear θ is self-adjoint and preserves K, it commutes with s, s^*, j and δ, and therefore it commutes with the real projections on K, $\mathrm{i}K$ and K'. Then E has the same commutation properties, its restriction to K is a spectral projection for the real linear θ and commutes with the real α and $\theta_{K,K'}$. Hence, we need only to prove the propositions on the fiber of the direct integral decomposition of $\mathcal{H}$ over the spectrum of (the complex linear) θ. On such a space α and the angular operators are constant, while j and δ have a matricial form like in the two-dimensional case. Then the thesis follows from the $\mathbf{C}^2$ example. ■

REMARK 3.3. We note that θ is a well-defined complex linear operator on $(K + +\mathrm{i}K)^-$ in the general case. Therefore, if K is a factor, i.e. $\frac{\pi}{2}$ is not an eigenvalue of θ, Proposition 3.2 extends α to $(K + \mathrm{i}K)^-$. In particular, α turns out to be 0 on $K \cap \mathrm{i}K$.

As we said, Propositions 3.1 and 3.2 allowed us to state Theorems 1.3 and 1.4 in terms of modular operators. On the other hand, a Corollary on the continuity of the map $K \to \delta$ immediately follows.

COROLLARY 3.4. *Let $\{K_n\}_{n=0}^{\infty}$ be a sequence of standard factor subspaces with projections P_n such that all the modular conjugations j_n and all the spectral projections $\mathcal{X}_{[0,1]}(\delta_n)$ coincide, $j_n \equiv j$ and $\mathcal{X}_{[0,1]}(\delta_n) \equiv E$. Then the following are equivalent:*
(i) $\lim_{n\to\infty} P_n = P_0$
(ii) $\lim_{n\to\infty} E\delta_n = E\delta_0$.

4. TOPOLOGY ON THE GRASSMANNIAN, CLOSED AND DENSE SETS

In this section we study the topological properties of the classification of second quantization factors, i.e. we restrict our attention to the open submanifold $\mathcal{G}_F \subset \mathcal{G}_{\infty,\infty}$ which consists of factor subspaces[2]. To this end, we consider a map ψ on $\mathcal{G}_F$ which assigns to each subspace K a number in $[0, 1]$. This number summarizes all the information contained in the essential spectrum of δ which contributes to identify the Connes invariant (if K is not standard, we always mean δ relative to the inclusion $K_p \leqslant_{\mathbf{R}} \mathcal{H}_F$ in the decomposition of Remark 1.7).

More precisely, ψ will allow us to include the results of Proposition 2.9 and 4.5, and the fact that this is all the contribution given by the essential spectrum, in the statement (a) of Theorem 4.2.

[2] We note that $\mathcal{G}_F$ is not dense in $\mathcal{G}_{\infty,\infty}$, in fact the set $\{K \leqslant_{\mathbf{R}} \mathcal{H} : 1 \notin \sigma_{ess}(\delta)$ and 1 is an eigenvalue with multiplicity one$\}$ is an open set of non-factors.

Then, recalling Definition 2.3, let $\psi : \mathcal{G}_F \to [0,1]$ be the function defined as follows:

$$\psi(K) = \begin{cases} \lambda & \text{if } g(\sigma_{\text{ess}}(\delta)) = g(\lambda)\ \lambda \in (0,1) \text{ and } 1 \notin \sigma_{\text{ess}}(\delta) \\ 0 & \text{if } \sigma_{\text{ess}}(\delta) = 0 \\ 1 & \text{if } g(\sigma_{\text{ess}}(\delta)) = \overline{\mathbb{R}}_+ \text{ or } 1 \in \sigma_{\text{ess}}(\delta). \end{cases} \tag{4.1}$$

Let us introduce in $[0,1]$ the partial ordering given by

$$0 \preceq a \preceq 1 \quad a \in [0,1]$$

$$a \preceq b\ a, b \in (0,1), \quad a = b^n,\ n \in \mathbb{N}$$

and note that if $a, b \in (0,1)$, $a \preceq b \Leftrightarrow g(a) \subseteq g(b)$.

Now we may state the main theorem of this section:

THEOREM 4.2.

(a) $\psi^{-1}(\lambda)$ *contains* III_μ *factors if and only if* $\lambda \preceq \mu$

(b) $\psi^{-1}(\{\mu \in [0,1] : \mu \preceq \lambda\})$ *is closed in* $\mathcal{G}_F \in [0,1]$

(c) *The sets* $\psi^{-1}(\{\mu : \lambda \prec \mu \prec 1\})$, $\lambda \in [0,1)$ *and the set of* III_1 *factors are dense in* $\mathcal{G}_F$.

(d) *The set of type* I *factors*[3] *is a dense subset of* $\psi^{-1}(0)$.

As an immediate Corollary we get

COROLLARY 4.3. *The class* III_1 *second quantization factors is the only class of isomorphic factors which is dense in* $\mathcal{G}_F$.

Remark 4.4. Theorem 1.5 shows that the non III_1 factors should be looked for when $\sigma_{\text{ess}}(\delta)$ is contained in $g(\lambda)$ for some $\lambda \in (0,1)$, and therefore δ is completely diagonalizable. In this case the Connes spectrum is completely determined by the sequence Λ of the eigenvalues of $\delta|[0,1]$. The definition of the Araki-Wood invariant shows that it is "superadditive" w.r.t. Λ, i.e. if Λ_i is the sequence associated to a subspace K_i and the sequence associated to K is $\bigcup_i \Lambda_i$ then

$$S(\mathcal{R}(K) \supset \bigcup_i S(\mathcal{R}(K_i)).$$

[3] The idea of the density of type I factors in the class of second quantization factors is linked to the split property of an inclusion of local algebras, and more precisely to the posibility of constructing a family $\{F_n,\ n \in \mathbb{Z}\}$ of type I factors interpolating the inclusion $A \subset B$, such that $A = \bigcap F_n$, $B = \bigcup F_n$ [11]. The corresponding family of projections in the one particle space has no limit points in norm topology, because the distance of any couple of included projections is 1. Nevertheless, things change in strong topology, and in fact we think that in this case the outlined argument could be used to prove the density of the class of type I factors.

In the following propositions we shall explicitly calculate the Connes invariant for given sequences Λ. These results will allow us to prove part a) of Theorem 4.2.

PROPOSITION 4.5. *Let $\Lambda = \{\lambda_n\}$ be the sequence of the eigenvalues of $\delta|[0,1]$ associated with a factor subspace K. If Λ converges to 1, then $S(\mathcal{R}(K)) = \overline{\mathbf{R}}_+$. As a consequence, any second quantization factor for which $1 \in \sigma_{ess}(\delta)$ is III_1.*

Proof. First note that the second part of the Proposition is a consequence of the first part, Theorem 1.4, and Remark 1.7. Now choose $x \in (0,1)$ and set $p_n = \left[\frac{\log x}{\log \lambda_n}\right]$ where $[\cdot]$ is the integer part. Then

$$|\log x - p_n \log \lambda_n| = |\log \lambda_n| \left| \frac{\log x}{\log \lambda_n} - p_n \right| \leqslant |\log \lambda_n| \to 0,$$

and therefore $\lambda_n^{p_n} \to x$. According to the definition 2.8, set

$$I_n = \{n\}$$

$$K_n = \bigcup_{k \text{ even}} \{kp_n + 1, \dots, (k+1)p_n\}$$

$$\varphi_n : m \in K_n \to m + p_n.$$

Properties (i), (ii) and (iii) of 2.8 are obviously satisfied. Note that

$$\sum_{k \in K_n} \lambda(k) = \sum_{k \in K_n} (1-\lambda_n)\lambda_n^{k-1} = (1-\lambda_n) \sum_{k \text{ even}} \sum_{j=1}^{p_n} \lambda_n^{kp_n + j - 1} =$$

$$= (1-\lambda_n) \sum_{k \text{ even}} \lambda_n^{kp_n} \frac{1-\lambda_n^{p_n}}{1-\lambda_n} = \frac{1-\lambda_n^{p_n}}{1-\lambda_n^{2p_n}} = \frac{1}{1+\lambda_n^{p_n}} \to \frac{1}{1+x}.$$

As a consequence,

$$\sum_{n \in \mathbf{N}} \sum_{k \in K_n} \lambda(k) = +\infty.$$

Finally,

$$\lim_{n \to \infty} \max_{k \in K_n} \left| x - \frac{\lambda(\varphi_n(k))}{\lambda(k)} \right| = \lim_{n \to \infty} \max_{k \in K_n} |x - \lambda_n^{p_n}| = 0.$$

By the arbitrariness of x, we have $S(\mathcal{R}(K)) = \overline{\mathbf{R}}_+$. ∎

PROPOSITION 4.6. *Let $\Lambda = \{\lambda_n\}$ be the sequence of the eigenvalues of $\delta|_{[0,1]}$ associated with a factor subspace K. If Λ converges to $\lambda \in (0,1)$ and*

$$\exists r > 0 : \sum_{n=1}^{\infty} \exp\left(-\frac{r}{|\lambda - \lambda_n|}\right) = +\infty$$

then $S(\mathcal{R}(K)) = \overline{\mathbb{R}}_+$.

Proof. By means of small perturbations, reordering, and taking subsequences, we can always restrict to the case

$$\lambda_{2n} \equiv \lambda, \quad 0 < \lambda_{2n+1} \uparrow \lambda.$$

Then pose

$$\varepsilon_n = \log \lambda - \log \lambda_{2n-1}$$

$$I_n = \{2n-1, 2n\},$$

and note that properties 2.8 i, ii are automatically satisfied.
Choose $x > 1$ and consider the function

$$l(x) = \lim_{n\to\infty} \max_{\mathbf{k}\in K_n} \left| \log x - \sum_{j\in I_n} \log \lambda_j (\varphi_n(\mathbf{k})_j - \mathbf{k}_j) \right|.$$

In order to fulfill condition 2.8 v′ and since

$$l(x) = \lim_{n\to\infty} \max_{\mathbf{k}\in K} |\log x - (\varphi_n(\mathbf{k})_{2n-1} - \mathbf{k}_{2n-1}) \log \lambda_{2n-1} - (\varphi_n(\mathbf{k})_{2n} - \mathbf{k}_{2n}) \log \lambda| =$$

$$\lim_{n\to\infty} \max_{\mathbf{k}\in K} |\log x - (\varphi_n(\mathbf{k})_{2n-1} - \mathbf{k}_{2n-1} + \varphi_n(\mathbf{k})_{2n} - \mathbf{k}_{2n}) \log \lambda + (\varphi_n(\mathbf{k})_{2n-1} - \mathbf{k}_{2n-1})\varepsilon_n|,$$

we look for functions φ_n satisfying

$$\left.\begin{array}{l} \text{a) } \varphi_n(\mathbf{k})_{2n-1} - \mathbf{k}_{2n-1} + \varphi_n(\mathbf{k})_{2n} - \mathbf{k}_{2n} = 0 \\ \text{b) } \varphi_n(\mathbf{k})_{2n-1} - \mathbf{k}_{2n-1} = \left[\dfrac{\log x}{\varepsilon_n}\right] \end{array}\right. \qquad \forall\, \mathbf{k} \in K_n.$$

In fact in this case, since $\varepsilon_n \to 0$,

$$l(x) = \lim_{n\to\infty} \left| \log x - \left[\frac{\log x}{\varepsilon_n}\right] \varepsilon_n \right| = 0.$$

If we pose $p_n = \left[\dfrac{\log x}{\varepsilon_n}\right]$, and we define

$$K_n = \mathbb{N} \times J_n \text{ where } J_n = \{mp_n + j : m \geqslant 0 \text{ even},\ 1 \leqslant j \leqslant p_n\}$$

$$\varphi_n(j_1, j_2) = (j_1 + p_n, j_2 - p_n),$$

conditions a) and b) are fulfilled.
Observe also that

$$\varphi_n(K_n) = \{n \in \mathbb{N} : n > p_n\} \times \{mp_n + j : m \geqslant 0 \text{ odd},\ 1 \leqslant j \leqslant p_n\},$$

therefore $\varphi_n(K_n) \cap K_n = \emptyset$, i.e. property 2.8 iii is satisfied.
In order to check condition 2.8 iv, observe that

$$\sum_{k \in K_n} \lambda(k) = \sum_{\substack{k_1 \in \mathbf{N} \\ k_2 \in J_n}} (1-\lambda_{2n-1})(1-\lambda_{2n}) \lambda_{2n-1}^{k_1-1} \lambda_{2n}^{k_2-1} =$$

$$= (1-\lambda_{2n-1})(1-\lambda) \left(\sum_{k=1}^{\infty} \lambda_{2n-1}^{k-1} \right) \left(\sum_{m=0}^{\infty} \sum_{j=1}^{p_n} \lambda^{(2m+1)p_n+j-1} \right) = \frac{\lambda^{p_n}}{1+\lambda^{p_n}}.$$

Since the following relations hold for large n,

$$p_n \approx \frac{\log x}{\varepsilon_n}, \quad \varepsilon_n \approx \frac{\lambda - \lambda_{2n-1}}{\lambda}$$

and $\lambda^{p_n} \to 0$, we get

$$\sum_{n \in \mathbf{N}} \sum_{k \in K_n} \lambda(k) = +\infty \Leftrightarrow \sum_{n \in \mathbf{N}} \lambda^{p_n} = +\infty \Leftrightarrow \sum_{n \in \mathbf{N}} \exp\left(\frac{\lambda \log \lambda \log x}{\lambda - \lambda_n} \right) = +\infty$$

Finally, since

$$\sum_{n=1}^{\infty} \exp\left(- \frac{r}{|\lambda - \lambda_n|} \right) \leqslant \sum_{n=1}^{\infty} \left(- \frac{\rho}{|\lambda - \lambda_n|} \right) \quad 0 < \rho \leqslant r,$$

condition 2.8 iv is satisfied for each x in $(0, \exp(r|\lambda \log \lambda|^{-1})]$, which implies $S(\mathcal{R}(K)) = \overline{\mathbf{R}}_+$. ■

PROPOSITION 4.7. *Let* $\Lambda = \left\{ \frac{1}{n} \right\}$ *be the sequence of the eigenvalues of* $\delta|_{[0,1]}$ *associated with a factor subspace* K. *Then* $S(\mathcal{R}(K)) = \overline{\mathbf{R}}_+$.

Proof. First we set

$$I_n = \{k_n, m_n\}$$

$$K_n = \{(1,2)\}$$

$$\varphi_n((1,2)) = (2,1)$$

which implies conditions 2.8 ii, iii. Then we observe that

$$\lim_{n \to \infty} \max_{\mathbf{k} \in K_n} \left| \log x - \sum_{j \in I_n} \log \lambda_j (\varphi_n(\mathbf{k})_j - \mathbf{k}_j) \right| = \lim_{n \to \infty} \left| x - \frac{m_n}{\mathbf{k}_n} \right|$$

therefore, setting $m_n = [k_n x]$ and choosing $k_n \to \infty$, condition 2.8 v is satisfied. Now we prove that, if $x \geqslant 2$, it is possible to construct k_n such that conditions 2.8 i, iv are satisfied.

We shall look for $k_n \in J_n \stackrel{\text{def}}{=} \{2n-1, 2n\}$. Since k_n is increasing and $x \geqslant 2$,

$$\begin{matrix} k_i \neq k_j \\ m_i \neq m_j \end{matrix} \quad \text{when } i \neq j$$

therefore, if

$$(*) \qquad \{k_n : n \in \mathbb{N}\} \cap \{m_n : n \in \mathbb{N}\} = \emptyset,$$

we get condition 2.8 i.

But the set $\{m_n : n \in \mathbb{N}\}$ is contained in $\{[nx] : n \in \mathbb{N}\}$ and, since $x \geqslant 2$, the set $\{[nx] : n \in \mathbb{N}\} \cap J_n$ contains at most one element. Therefore condition $(*)$ is fulfilled if we choose $k_n \neq \{[nx] : n \in \mathbb{N}\}$.

Finally observe that

$$\sum_{\mathbf{k} \in K_n} \lambda(\mathbf{k}) = \left(1 - \frac{1}{k_n}\right)\left(1 - \frac{1}{m_n}\right)\frac{1}{m_n};$$

hence, by the definition of k_n and m_n,

$$\sum_{n \in \mathbb{N}} \sum_{\mathbf{k} \in K_n} \lambda(\mathbf{k}) \approx \sum_{n \in \mathbb{N}} \frac{1}{n} = +\infty,$$

i.e. $x \in S(\mathcal{R}(K))$. By the arbitrariness of x we get the thesis. ■

PROPOSITION 4.8. *For each $\lambda \in (0,1)$ there exists a factor subspace K such that $\sigma_{\text{ess}}(\delta) = \emptyset$ and $S(\mathcal{R}(K)) = g(\lambda)$.*

Proof. We prove this Proposition for $\lambda = \frac{1}{2}$. Examples for a generic λ can be obtained with slight modifications.
In particular, we show that if $\Lambda = \{2^{-[\log_2 n]}\}$ is the sequence of the eigenvalues of $\delta|[0,1]$ associated with a factor subspace K, then $S(\mathcal{R}(K)) = g\left(\frac{1}{2}\right)$.
Let us define the two sequences

$$\begin{aligned} a_n &= n + \frac{2}{3}\left(4^{[\log_4(3n-2)]} - 1\right) \\ b_n &= a_n + 4^{[\log_4(3n-2)]} \end{aligned}$$

and set

$$\begin{aligned} I_n &= \{a_n, b_n\} \\ K_n &= \{(1,2)\} \\ \varphi((1,2)) &= (2,1). \end{aligned}$$

Conditions 2.8 ii, iii are obviously satisfied and a straighforward calculation shows that $I_n \cap I_m = \emptyset$ if $n \neq m$. Moreover,

$$[\log_2 a_n] = 2[\log_4(3n-2)]$$

$$[\log_2 b_n] = 2[\log_4(3n-2)] + 1$$

$$\sum_{\mathbf{k}\in K_n} \lambda(\mathbf{k}) = (1 - 2^{-[\log_2 a_n]})(1 - 2^{-[\log_2 b_n]})2^{-[\log_2 b_n]}$$

hence, in particular,

$$\frac{\lambda(\varphi_n(\mathbf{k}))}{\lambda(\mathbf{k})} = 2.$$

Therefore, to show that $2 \in S(\mathcal{R}(K))$, we have to verify only condition 2.8 iv. But

$$\sum_{n\in\mathbf{N}} \sum_{\mathbf{k}\in K_n} \lambda(\mathbf{k}) \approx \sum_{n\in\mathbf{N}} 2^{-[\log_2 b_n]} \approx$$

$$\approx \sum_{n\in\mathbf{N}} 2^{-2[\log_4(3n-2)]-1} \approx \frac{1}{2}\sum_{n\in\mathbf{N}} \frac{1}{3n-2} = +\infty.$$

On the other hand, $S(\mathcal{R}(K)) \subseteq g(\sigma(\delta)) = g\left(\frac{1}{2}\right)$ and, therefore, $S(\mathcal{R}(K)) = g\left(\frac{1}{2}\right)$. ■

Proof of the Theorem 4.2, part a. Let K be a subspace giving rise to a III_μ factor and such that $\psi(K) = \lambda$. We want to prove that $\lambda \preceq \mu$.

If $\lambda = 0$ or $\mu = 1$ the result is obvious.

If $\lambda \in (0,1)$ then $g(\lambda) \subseteq S(\mathcal{R}(K))$ and therefore $g(\lambda) \subseteq g(\mu)$, i.e. $\lambda \preceq \mu$.

If $\lambda = 1$ Theorem 1.5 and Proposition 4.5 imply that $\mathcal{R}(K)$ is III_1.

On the other hand, let $\lambda \preceq \mu$. We have to exhibit K such that $\mathcal{R}(K)$ is III_μ and $\psi(K) = \lambda$.

If $\lambda = 1$ then $\mu = 1$ and we may choose K as in 4.5.

If $\lambda \in (0,1)$ and $\mu = \lambda$ Corollary 2.4ii gives the desired subspace.

If $\lambda \in (0,1)$ and $\mu = 1$ we take the example in Proposition 4.6.

If $\lambda \in (0,1)$ and $\lambda \prec \mu \prec 1$, we consider a sequence $\lambda_n \to 0$ like the one which generates a III_μ factor in Proposition 4.8 and choose a subspace K such that $\sigma(\delta|[0,1])$ consits exactly of the sequence λ_n and of the point λ with infinite multiplicity, which implies $\psi(K) = \lambda$. Then $\mu \in S(\mathcal{R}(K))$ and $\sigma(\delta) \subset g(\mu)$ by construction and, therefore $\mathcal{R}(K)$ is III_μ.

If $\lambda = 0$, Theorem 2.10 gives III_0 factors, Proposition 4.7 gives III_1 factors, and Proposition 4.8 gives III_λ factors for each λ in $(0,1)$. ■

Proof of Theorem 4.2, part b. The result is obvious if $\lambda = 1$.

If $\lambda \in [0,1)$, we have to show that, given a sequence of subspaces K_n in $\mathcal{G}_F$ with $\lim K_n = K$, $\psi(K_n) \leqslant \lambda$ implies $\psi(K) \leqslant \lambda$, i.e. $\sigma_{\text{ess}}(\delta_n) \subset g(\lambda) - \{1\}$ for each n implies $\sigma_{\text{ess}}(\delta) \subset g(\lambda) - \{1\}$. Now a theorem in [18], p. 208 shows that, if A, B are self-adjoint elements in a C^*-algebra then

$$\sigma(A+B) \subset \sigma(A) + [-\|B\|, +\|B\|]$$

therefore, if $A_n \to A$ are self-adjoint elements in a C^*-algebra with $\sigma(A_n) \subset \Omega$ for each $n \in \mathbf{N}$ and Ω is a closed set, it follows that

$$\sigma(A) \subset \sigma(A_n) + [-\|A - A_n\|, +\|A - A_n\|] \quad (\forall) n \in \mathbf{N},$$

and therefore $\sigma(A) \subset \Omega$.

Finally let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the projection on the Kalkin algebra and consider the sequence $A_n = \pi(\delta_n|_{[0,1]})$. By Corollary 3.4 $A_n \to A = \pi(\delta|_{[0,1]})$, and, by hypothesis,

$$\sigma(A_n) = \sigma_{\text{ess}}(\delta_n|_{[0,1]}) \subset g(\lambda) - \{1\} \quad (\forall) n \in \mathbf{N}.$$

Therefore

$$\sigma_{\text{ess}}(\delta|_{[0,1]}) = \sigma(A) \subset g(\lambda) - \{1\}.$$

Since $\sigma_{\text{ess}}(\delta)$ is symmetric w.r.t. the inverse operation we get the thesis. ∎

The remaining parts of Theorem 4.2 are obvious consequences of the following three Propositions.

PROPOSITION 4.9. *The set $\psi^{-1}(\{\mu : \lambda \prec \mu \prec 1\})$ is dense in $\mathcal{G}_F$ for all $\lambda \in (0,1)$.*

PROPOSITION 4.10. *The set of* III$_1$ *factors is dense in $\mathcal{G}_F$.*

PROPOSITION 4.11. *Type* I *factors are dense in $\psi^{-1}(0)$ and for each $\lambda \in [0,1]$ there exists a dense subset of $\psi^{-1}(0)$ which consits of only* III$_\lambda$ *factors.*

Proof of 4.9. We choose $\lambda \in (0,1)$ and a standard factor K and then we consider the functions

$$f_n : \mathbf{R}_+ \to \left\{ \frac{k}{n} : 1 \leqslant k \leqslant n^2 \right\}$$

$$f_n(x) = \begin{cases} \dfrac{k}{n} & \text{if } k-1 < nx \leqslant k \text{ and } x \leqslant n \\ n & \text{if } x > n \end{cases}$$

and the functions $g_n : (0,1) \to \{\lambda^{\frac{k}{n}} : 1 \leqslant k \leqslant n^2\}$

$$g_n(x) = \lambda^{f_n(\log_\lambda x)}.$$

We note that $g_k(x) \to x$ uniformly.

Now the operator $g_n(\delta|_{[0,1]})$ has a finite spectrum, hence there exists an eigenvalue $\lambda_0 \stackrel{\text{def}}{=} \lambda^{\frac{k_0}{n}}$ with infinite multiplicity. Then we choose a decomposition of the eigenspace $\mathcal{H}_{\lambda_0}$ in a direct sum with infinite dimensional direct summands,

$$\mathcal{H}_{\lambda_0} - \mathcal{H}_a \oplus \mathcal{H}_b$$

and define the sequence d_n of operators on the spectral subspace $\mathcal{H}_{[0,1]}$ of δ relative to the interval $[0,1]$,

$$d_n = \begin{cases} g_n(\delta|_{[0,1]}) & \text{on } \mathcal{H}_{\lambda_0}^{\perp} \\ \lambda^{\frac{k_0}{n}} \cdot I & \text{on } \mathcal{H}_a \\ \lambda^{\frac{k_0-1}{n}} \cdot I & \text{on } \mathcal{H}_b. \end{cases}$$

Observe that the following properties hold:

(a) $d_n \to \delta|_{[0,1]}$ uniformly

(b) $g(\sigma(d_n)) - g(\sigma_{\text{ess}}(d_n)) = g(\lambda^{\frac{1}{n}})$.

Finally, we consider the sequence of operators

$$\delta_n = \begin{cases} d_n & \text{on } \mathcal{H}_{[0,1]} \\ j d_n^{-1} j & \text{on } \mathcal{H}_{[0,1]}^{\perp} \end{cases}$$

and the sequence of spaces $K_n\{k \in \mathcal{H} : j\delta_n^{-\frac{1}{2}}k = k\}$, and note that, since $j\delta_n j = \delta_n^{-1}$, Corollary 3.4 and (a) implies that $K_n \to K$, while (b) implies $\psi(K_n) = \lambda^{\frac{1}{n}}$. ■

Proof of 4.10. We want to approximate any factor subspace K with III_1 factors. To this aim we choose $\lambda \in \sigma_{\text{ess}}(\delta|_{[0,1]})$ and consider the spectral subspace $\mathcal{H}_{[a_n,b_n]}$ relative to the interval $[a_n, b_n]$ where

$$a_n = \max\left(0, \lambda - \frac{1}{n}\right)$$

$$b_n = \min\left(1, \lambda + \frac{1}{n}\right).$$

Since $\mathcal{H}_{[a_n,b_n]}$ is infinite dimensional, we may identify it with $L^2([a_n, b_n])$ and then we define the operator $\mathcal{M}_n$ as the multiplication operator by the function x on the interval $[a_n, b_n]$.

Then let us define

$$d_n = \begin{cases} \mathcal{M}_n & \text{on } \mathcal{H}_{[a_n,b_n]} \\ \delta & \text{on } \mathcal{H}_{[a_n,b_n]}^{\perp} \cap \mathcal{H}_{[0,1]} \end{cases}$$

Finally, define δ_n and K_n as in previous Proposition and observe that $d_n \to \delta|_{[0,1]}$ and $\sigma(\delta_n) \supset [a_n, b_n]$, i.e. $\{\mathcal{R}(K_n)\}_{n\in\mathbb{N}}$ is a sequence of III_1 factors approximating $\mathcal{R}(K)$. ■

Proof of 4.11. Let K be a factor subspace such that $\sigma_{ess}(\delta) = \{0\}$, λ_k is the sequence of eigenvalues (with multiplicity) of $\delta|_{[0,1]}$, and e_k the corresponding sequence of eigenvectors.

Then we take a sequence $\lambda'_k \to 0$ giving rise to a III_λ factor (as in Theorem 2.10 for $\lambda = 0$, Proposition 4.8 for $\lambda \in (0,1)$ and Proposition 4.7 for $\lambda = 1$), or a type I factor (λ_n summable as in Theorem 1.3 ii).

Finally we define d_n on $\mathcal{H}_{[0,1]}$

$$d_n e_k = \begin{cases} \lambda_k e_k & \text{if } k \leqslant n \\ \lambda'_k e_k & \text{if } k > n \end{cases}$$

and the proof goes on as in the preceding Propositions. ■

Acknowledgements. We wish to thank Prof. S. Doplicher and Prof. R. Longo for useful discussions and comments, and Prof. R. Hermann and Prof. A. Ocneanu for the warm hospitality at the Pennsylvania State University where the last version of this paper was written.

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Received January 10, 1992; revised August 2, 1992.