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FUNCTIONAL ANALYSIS OF SUBELLIPTIC OPERATORS ON LIE GROUPS

A.F.M. TER ELST and DEREK W. ROBINSON

1. INTRODUCTION

The theory of general subelliptic, or subcoercive, operators H associated with a continuous representation of a Lie group G was developed in three previous papers [7], [8], [1] and the aim of this paper is to establish that these operators have good function-analytic properties. Specifically we examine operators H associated with the left regular representation of G on the L_p-spaces formed with respect to left, or right, Haar measure. We then prove that for a large class of holomorphic functions f one can use complex analysis to define $f(H)$ and the resulting operators are bounded on all the L_p-spaces with $p \in \langle 1, \infty \rangle$.

Holomorphic functional analysis of the type we consider has been developed for a large class of operators, including generators of holomorphic semigroups, by McIntosh and Yagi [14], [15], [18]. Their theory makes essential use of earlier work by Kato [11], [12] and Lions [13] on fractional powers of operators and interpolation spaces. Subsequently, the general theory has been applied to elliptic partial differential operators by Duong [5] and our analysis is comparable to the discussion in Duong's Chapter 7. The main tool is a version of the Calderon-Zygmund approach to singular integration. It is based on interpolation betweeen L_2-estimates and weak L_1-estimates. In our case, however, the required L_2-estimates can be derived for an arbitrary unitary representation. But first we describe the general framework for semigroup generators. Basically we adopt McIntosh's formalism modified in a manner suitable for application to subelliptic operators. For an extension to operators of type ω which are not necessarily one-to-one we refer to [4].

Let H be the generator of a holomorphic semigroup $t \mapsto S_t = e^{-tH}$ on a Banach space $\mathcal{X}$ which is strongly, or weakly*, continuous. Suppose that S is holomorphic in the sector $\Lambda(\theta_a)$ where $0 < \theta_a \leqslant \pi/2$ and

$$\Lambda(\varphi) = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \varphi\}$$

for all $\varphi \in \langle 0, \pi]$. Then if $\theta \in \langle 0, \theta_a\rangle$ there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S_z\| \leqslant M e^{\omega|z|}$ for all $z \in \mathbf{C}$ with $|\arg z| < \theta$. Therefore replacing H by $H + \nu I$ with $\nu \geqslant 0$ suitably large we may assume that S is uniformly bounded uniformly in the sector $\Lambda(\theta)$, i.e., there is an $M \geqslant 1$ such that $\|S_z\| \leqslant M$ for all $z \in \mathbf{C}$ with $|\arg z| \leqslant \theta$. It then follows that the resolvent $(-\lambda I + H)^{-1}$ is defined and satisfies bounds $\|(-\lambda I + H)^{-1}\| \leqslant M|\lambda|^{-1}$ for all non-zero $\lambda \in \mathbf{C}$ with $|\arg \lambda| \geqslant \pi/2 - \theta$. For example, if H is a positive self-adjoint operator on a Hilbert space then the semigroup $t \mapsto S_t = e^{-tH}$ is holomorphic in the open right half-plane $\Lambda(\pi/2)$ and $\|S_z\| \leqslant 1$ for all $z \in \mathbf{C}$ with $|\arg z| \leqslant \pi/2$.

Next for $0 < \varphi \leqslant \pi$ let

$$F_\varphi = \{f : \Lambda(\varphi) \to \mathbf{C} : f \text{ is bounded and holomorphic}\}.$$

Then F_φ is a Banach space with respect to the norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \Lambda(\varphi)\}.$$

It is also convenient to introduce the subspaces

$$\Phi_{\varphi,\delta} = \{f \in F_\varphi : |f(z)| \leqslant c|z|^\delta (1 + |z|)^{-2\delta} \text{ for some } c > 0 \text{ and all } z \in \Lambda(\varphi)\},$$

where $\delta > 0$, and to set

$$\Phi_\varphi = \bigcup_{\delta > 0} \Phi_{\varphi,\delta}.$$

Then if $f \in \Phi_\varphi$ with $\varphi \in \langle \pi/2 - \theta, \pi]$ one can define an operator $f(H)$ by

$$f(H) = (2\pi i)^{-1} \int_{\Gamma_\chi} d\lambda\, f(\lambda)(-\lambda I + H)^{-1} \tag{1}$$

where Γ_χ is the contour determined by the function

$$\Gamma_\chi(t) = \begin{cases} t e^{i\chi} & \text{if } t \in [0, \infty\rangle \\ -t e^{-i\chi} & \text{if } t \in \langle -\infty, 0] \end{cases},$$

with $\pi/2 - \theta < \chi < \varphi$. The integral in (1) is norm-convergent, independent of the particular choice of contour and the operator $f(H)$ is bounded.

If $f \in F_\varphi$ one can use a similar algorithm to define $f(H)$ but the contour integral in (1) is not necessarily norm-convergent and the resulting operator is not necessarily bounded. Therefore $f(H)$ is defined by interpreting the integral in (1) in the strong, or weak*, topology according to whether S is strongly, or weakly*, continuous. The domain of $f(H)$ is then taken to be the subspace of $\mathcal{X}$ on which the integral is convergent. It follows that the operators $f(H)$ constructed in this manner are closed. Now we define H to have a *bounded functional calculus over* F_φ if all the operators $\{f(H) : f \in F_\varphi\}$ are bounded and if $f \mapsto f(H)$ is continuous as a map from the Banach space of holomorphic functions F_φ into the Banach algebra $\mathcal{L}(\mathcal{X})$ of bounded operators on $\mathcal{X}$, i.e., if there is a $c > 0$ such that

$$\|f(H)\| \leqslant c\|f\|_\infty$$

for all $f \in F_\varphi$. For example, if H is again a positive self-adjoint operator on a Hilbert space then it is readily established that H has a bounded functional calculus over F_φ for any $\varphi \in \langle 0, \pi]$; the resulting operators $f(H)$ coincide with the usual functional definition

$$f(H) = \int_0^\infty \mathrm{d}E_H(\lambda) f(\lambda)$$

in terms of the spectral family E_H of H.

McIntosh and Yagi [14], [15], [18] have given a variety of criteria for a generator to have a bounded functional calculus and we will apply some of their results to subelliptic operators. In this latter context the existence of the functional analysis is a property of the representation of the Lie group to which the subelliptic operator is associated. Throughout the sequel we adopt the notation of [16] modified as in [7] and [1]. In Section 2 we first discuss the holomorphic functional analysis for subelliptic operators associated with a general unitary representation. Then in Section 3 we consider similar problems in the context of the left regular representation of the group acting on the L_p-spaces. Finally, in Section 4, we apply our results to the complex interpolation theory of the C^n-subspaces of the unitary representations and the regular representations.

2. UNITARY REPRESENTATIONS

Let $(\mathcal{H}, G, U)$ denote a (continuous) unitary representation of G on the Hilbert space $\mathcal{H}$ and $H = \mathrm{d}U(C)$ the m-th order subelliptic operator corresponding to a subcoercive form C and a fixed algebraic basis for the Lie algebra of G. It follows from Theorem 3.3 of [8] that H is closed on the natural domain $\mathcal{H}'_m$ and it generates a holomorphic semigroup S with a sector of holomorphy $\Lambda(\theta_a)$ which contains a representation independent subsector $\Lambda(\theta_C)$ with $\theta_C \in \langle 0, \pi/2]$. The value of θ_C is determined by the principal coefficients of H and if these coefficients are real and symmetric then S is holomorphic in the open right half-plane, i.e., $\theta_a = \theta_C = \pi/2$. Now if the real part of the zero-order coefficient c_0 of H is sufficiently large the semigroup S is contractive uniformly in a subsector $\Lambda(\theta)$ of the universal sector $\Lambda(\theta_C)$ of holomorphy, i.e., $\|S_z\| \leqslant 1$ for all $z \in \Lambda(\theta)$. This latter statement follows from Theorem 3.3 of [8]. The fifth statement of this theorem establishes that for each $\theta \in \langle 0, \theta_C \rangle$ there is an $\omega \geqslant 0$ such that $\|S_z\| \leqslant e^{\omega|z|}$ for all $z \in \Lambda(\theta)$. Therefore replacing H by $H + \nu I$ with $\nu > \omega/\cos\theta$ the corresponding semigroup is uniformly contractive, and exponentially decreasing, in the subsector $\Lambda(\theta)$. But with this normalization of c_0 it follows that H is accretive and since H is a generator it must then be maximal accretive. Moreover, H is one-to-one. This sufficies, however, to ensure that H has a bounded functional calculus over F_φ for any $\varphi \in \langle \pi/2 - \theta, \pi]$.

The last statement can be inferred from the results of Kato, McIntosh and Yagi. First, if $|H| = (H^*H)^{1/2}$ denotes the usual self-adjoint modulus then $D(|H|) = D(H) = \mathcal{H}'_m$ and

$$\|Hx\| = \|\,|H|x\|$$

for all $x \in \mathcal{H}'_m$. Since H is maximal accretive and $|H|$ is positive self-adjoint it follows from [10] that for each $\gamma \in [0,1]$ one has $D(H^\gamma) = D(|H|^\gamma)$ and there exists $c_\gamma \geqslant 1$ such that

$$c_\gamma^{-1}\|\,|H|^\gamma x\| \leqslant \|H^\gamma x\| \leqslant c_\gamma\|\,|H|^\gamma x\|$$

for all $x \in D(H^\gamma)$. Similar bounds follow by the same argument for the fractional powers of the adjoint H^* of H and then H has a bounded functional analysis over F_φ, for each $\varphi > \pi/2 - \theta_C$, as an immediate consequence of the theorem in Section 8 of [14]. This result can be extended to F_φ with $\varphi > \pi/2 - \theta_a \geqslant \pi/2 - \theta_C$ if one replaces H by $H + \nu I$ with a suitable $\nu \geqslant 0$. The argument is similar to the above. If $\theta \in \langle 0, \theta_a \rangle$ then S satisfies bounds $\|S_z\| \leqslant M e^{\omega|z|}$ for all $z \in \Lambda(\theta)$. Hence after replacing H by $H + \nu I$, with ν sufficiently large, the semigroup is uniformly bounded in the sector $\Lambda(\theta)$. But $H + \nu I$ is again one-to-one and for large enough ν it remains maximal accretive. Therefore the foregoing reasoning still applies.

One can, however, deduce much more from the criteria given by McIntosh and Yagi.

THEOREM 2.1. *Let $H = dU(C)$ be an m-th order subelliptic operator associated with the unitary representation $(\mathcal{H}, G, U)$. Further suppose that the holomorphic semigroup S generated by H has holomorphy sector $\Lambda(\theta_a)$ and that the real part of the zero-order coefficient of C is sufficiently large that S is uniformly bounded in the sector $\Lambda(\theta)$, where $\theta \in \langle 0, \theta_a \rangle$, and contractive on $\mathbf{R}_+$. Then one has the following.*

(i) *The operator H has a bounded functional analysis over F_φ for each $\varphi \in \in \langle \pi/2 - \theta, \pi]$.*

(ii) *The imaginary powers $\{H^{it} : t \in \mathbf{R}\}$ form a strongly continuous group.*

(iii) *If $\varphi \in \langle \pi/2 - \theta, \pi]$ and $f \in \Phi_\varphi$ then there exists $c_f > 0$ such that*

$$\int_0^\infty dt\, t^{-1} \|f(tH)x\|^2 \leqslant c_f \|x\|^2$$

for all $x \in \mathcal{H}$.

(iv) *If $[\mathcal{H}, \mathcal{K}]_\gamma$ $(= [\mathcal{K}, \mathcal{H}]_{1-\gamma})$ denotes the complex interpolation space between $\mathcal{H}$ and the subspace $\mathcal{K}$ then*

$$D(H^\gamma) = [\mathcal{H}, \mathcal{H}'_{nm}]_{\gamma/n} = D((H^*)^\gamma)$$

for all $\gamma \in \langle 0, n\rangle$ and $n \in \mathbf{N}$.

Proof. The main part of the proof is based on the theorem in Section 8 of [14] but since the assumptions do not ensure that H is one-to-one this theorem is not directly applicable. Hence we first make a reduction as in Theorem 3.8 of [4]. But this reduction simplifies considerably in the Hilbert space context.

First, one defines a projection P by the strong limit

$$Px = \lim_{n\to\infty} (I + nH)^{-1} x$$

and then the range of P is the nullspace of H. Moreover, the range of $I - P$ is the closure of the range of H. Now the subspaces $\mathcal{H}_0 = P\mathcal{H}$ and $\mathcal{H}_1 = (I - P)\mathcal{H}$ are both invariant under S. The restriction of S to $\mathcal{H}_0$ is the identity and the restriction $S^{(1)}$ of S to $\mathcal{H}_1$ is a holomorphic semigroup which is uniformly bounded in the sector $\Lambda(\theta)$ and contractive on $\mathbf{R}_+$. Let H_1 denote the generator of $S^{(1)}$. Then H_1 is the restriction of H to $\mathcal{H}_1$ and it is a one-to-one operator with dense domain on $\mathcal{H}_1$, by Theorem 3.8 of [4].

Secondly, one can repeat this reduction with respect to the adjoint semigroup S^* and its generator H^* and the corresponding projection is P^*. Then $\mathcal{H}_1^\dagger = (I - P^*)\mathcal{H}$ is isomorphic to the dual of $\mathcal{H}_1 = (I - P)\mathcal{H}$, the restriction $S^{(1)\dagger}$ of S^* to $\mathcal{H}_1^\dagger$ is the adjoint of $S^{(1)}$ and the generator $\mathcal{H}_1^\dagger$ of $S^{(1)\dagger}$ is the adjoint of H_1.

Now one can apply the theorem in Section 8 of [14] to H_1 on $\mathcal{H}_1$ and its adjoint on $\mathcal{H}_1^\dagger$.

Since H_1 is densely-defined and closed it follows that $D(H_1) = D(|H_1|)$ and $\|H_1\varphi\| = \|\,|H_1|\varphi\|$ for all $\varphi \in D(H_1)$. But since $S^{(1)}$ is contractive on $\mathbf{R}_+$ its generator H_1 is maximal accretive on $\mathcal{H}_1$. Hence it follows from the discussion at the beginning of the section that for each $\gamma \in [0,1]$ one has $D(H_1^\gamma) = D(|H_1|^\gamma)$ and there exists $c_\gamma \geqslant 1$ such that

$$c_\gamma^{-1} \| \, |H_1|^\gamma x \| \leqslant \|H_1^\gamma x\| \leqslant c_\gamma \| \, |H_1|^\gamma x \|$$

for all $x \in D(H_1^\gamma)$. Similar bounds follow by the same argument for the fractional powers of the adjoint $\mathcal{H}_1^\dagger$ of H_1. Then H_1 has a bounded functional analysis over F_φ, on $\mathcal{H}_1$, for each $\varphi > \pi/2 - \theta$, as an immediate consequence of the theorem in Section 8 of [14].

Next one has

$$(\lambda I + H)^{-1} x = \lambda^{-1} P x + (\lambda I + H_1)^{-1} (I - P) x$$

for each $x \in \mathcal{H}$. Therefore

$$f(H)x = f(0)Px + f(H_1)(I - P)x$$

and

$$\|f(H)x\| \leqslant |f(0)| \, \|Px\| + \|f(H_1)\| \, \|(I - P)x\| \leqslant c \|f\|_\infty \|x\|.$$

Thus H has a bounded functional analysis over F_φ, on $\mathcal{H}$, for each $\varphi > \pi/2 - \theta$ and Statement (i) is established.

Statement (ii) follows from applying Statement (i) to the functions $z \mapsto z^{it}$. Continuity of the group $\{H^{it} : t \in \mathbf{R}\}$ is a consequence of the arguments in [14].

Statement (iii) for H_1 on $\mathcal{H}_1$ follows from McIntosh's theorem. But if $f \in \Phi_\varphi$ then $f(tH)x = f(tH_1)(I - P)x$ and the analogous statement follows for H.

Finally, Statement (iv) follows from Theorem $\mathcal{A}$ of Yagi [18]. This theorem applies directly to $H + \varepsilon I$ with $\varepsilon > 0$ but since both sets in the resulting identity are independent of $\varepsilon \geqslant 0$ it then follows for H. ■

Statement (iii) of the theorem only concerns the restriction H_1 of H to $\mathcal{H}_1$ because $f \in \Phi_\varphi$ and hence $f(tH)x = f(tH_1)(I - P)x$. Since H_1 is one-to-one the theorem in Section 8 of [14], as reformulated in Theorem 2.4 of [4], then gives bounds

$$c_f^{-1} \|(I - P)x\|^2 \leqslant \int_0^\infty dt \, t^{-1} \|f(tH)x\|^2 \leqslant c_f \|(I - P)x\|^2$$

for some $c_f > 0$ and all $x \in \mathcal{H}$. This is of interest when applied to the function $z \mapsto z^{1-\gamma} e^{-z}$ where $\gamma \in \langle 0, 1 \rangle$. Then one has

$$c^{-1} \|(I - P)x\|^2 \leqslant \int_0^\infty dt \, t^{-1} (t^{1-\gamma} \|H^{1-\gamma} S_t x\|)^2 \leqslant c \|(I - P)x\|^2$$

for some $c > 0$ and all $x \in \mathcal{H}$, or, equivalently

$$c^{-1}\|H^\gamma x\|^2 \leqslant \int_0^\infty dt\, t^{-1}(t^{1-\gamma}\|HS_t x\|)^2 \leqslant c\|H^\gamma x\|^2$$

for all $x \in D(H^\gamma)$. Now the real interpolation spaces $(\mathcal{H}, D(H))_{\gamma,2;K}$ defined by the Peetre K-method have an equivalent norm

$$x \mapsto \|x\|_\gamma = \|x\| + \left(\int_0^\infty dt\, t^{-1}(t^{1-\gamma}\|HS_t x\|)^2\right)^{1/2}$$

(see [2], Theorem 3.4.2). Hence $D(H^\gamma) = (\mathcal{H}, D(H))_{\gamma,2;K} = [\mathcal{H}, D(H)]_\gamma$ when $D(H^\gamma)$ is equipped with the graph norm. This gives an extension of [6] Theorem 7.2.V.

The foregoing results apply directly to G acting by left translations L on the Hilbert space $L_2(G; dg)$. But if G is a subgroup of a second Lie group G_1 they can also be applied to the action L of G on $L_2(G_1) = L_2(G_1; dg)$. Alternatively, the argument in the proof of Theorem 3.7 of [9] to obtain the $L_{\hat{2}}$ case from the L_2 can be used to deduce that $D(|H|) = D(H) = L'_{\hat{2};m}(G_1)$. Then, as a result of the Gårding inequality, $D(|H|^\gamma) = D(H^\gamma)$ with equivalent norms if the real part of the zero-order coefficient of C is large enough. The same reasoning applies to the dual operator. Therefore we have following conclusion.

Corollary 2.2. *Let $H = dL(C)$ be an m-th order subelliptic operator associated with the representation of the Lie subgroup G of the Lie group G_1 acting by left translations L on $L_{\hat{2}}(G_1)$. If the real part of the zero-order coefficient of C is sufficiently large then the conclusions of Theorem 2.1. are again valid.*

3. L_p-SPACES

We continue to examine two Lie groups G and G_1 with G a subgroup of G_1 and now consider the representation of G by left translations L on the L_p-spaces, $L_p(G_1)$ and $L_{\hat{p}}(G_1)$, formed with respect to left, and right, Haar measure over G_1. In addition we now assume that G_1 is connected. Our aim is to establish that each m-th order subelliptic operator $H = dL(C)$ asociated with L and a fixed algebraic basis $a_1, \ldots, a_{d'}$ of rank r for the Lie algebra of G has a bounded functional calculus on these spaces if $p \in \langle 1, \infty\rangle$. The crucial case is when $G = G_1$ but it is subsequently useful to consider the more general situation $G \subset G_1$. Since all analysis takes part on the connected component of the identity of G we may as well assume that G is connected.

First note that the interpolating semigroup S generated by the closures $\overline{H}$ on the various spaces has a representation independent subsector of holomorphy $\Lambda(\theta_C)$, by Theorem 2.5 of [8], and in particular S is holomorphic in this subsector on each of the spaces. Therefore, replacing H by $H + \nu I$ with ν sufficiently large, one may assume that S is uniformly bounded on each of the spaces uniformly in a subsector $\Lambda(\theta)$ with $\theta \in \langle 0, \theta_C \rangle$, i.e., one has bounds $\|S_z\|_{p \to p} \leqslant M$, $\|S_z\|_{\widehat{p} \to \widehat{p}} \leqslant M$ for all $z \in \Lambda(\theta)$ and all $p \in [1, \infty]$. Now the main result we prove is the following.

THEOREM 3.1. *Let $H = \mathrm{d}L(C)$ be an m-th order subelliptic operator associated with left translations L by the group G acting on the spaces $L_p(G_1)$, and $L_{\widehat{p}}(G_1)$ and let $\theta \in \langle 0, \theta_C \rangle$. If $p \in \langle 1, \infty \rangle$ then H is closed and there is a $\nu_0 \geqslant 0$, independent of p, such that the operators $H + \nu I$, $\nu > \nu_0$, have a bounded functional analysis over F_φ for each $\varphi \in \langle \pi/2 - \theta, \pi]$.*

Proof. First remark that it is not at all obvious that the operator H are closed but this is establish by [1] Theorem 2.3.

The next step in the proof is to establish the result for all $f \in \Phi_{\varphi,\delta}$ with δ sufficiently large. Then the general result follows by approximation. The proof for the special class of f is based on the $L_{\widehat{2}}(G_1)$-result established in the previous section and a weak $L_{\widehat{1}}(G_1)$-estimate which follows by the methods of singular integration theory. This estimate is the most difficult part of the proof and it is established from analysis of the kernel associated with $f(\overline{H})$. Finally we deduce the L_p-statement from the $L_{\widehat{p}}$-statement. The details of the proof are very similar to the arguments used in [1] to prove Theorem 2.3.

It follows formally from the definition of $f(\overline{H})$ that its action should be given by

$$f(\overline{H}) = L(K_f) = \int_G \mathrm{d}g \, K_f(g) L(g)$$

where the kernel K_f is defined on $G \setminus \{e\}$ by

$$K_f(g) = (2\pi \mathrm{i})^{-1} \int_{\Gamma_\chi} \mathrm{d}\lambda \, f(\lambda) R_{-\lambda}(g) \tag{2}$$

and R_λ denotes the kernel of the resolvent $(\lambda I + \overline{H})^{-1}$. We begin by examining properties of K_f for $f \in \Phi_\varphi$ and for this purpose it suffices to consider the operators acting on the $L_{\widehat{p}}$-spaces over G, so for a moment we consider the case $G_1 = G$.

If one replaces $\overline{H}$ by $\overline{H} + \nu I$ with ν sufficiently large, one may assume that the semigroup S satisfies the bounds $\|S_z\|_{\widehat{p} \to \widehat{p}} \leqslant M$ for all $z \in \Lambda(\theta)$ and all $p \in [1, \infty]$ where the norms are the operator norms on $L_p(G; \mathrm{d}\widehat{g})$. Then one has the resolvent bounds

$$\|(-\lambda I + \overline{H})^{-1}\|_{\widehat{p} \to \widehat{p}} \leqslant M |\lambda|^{-1} \tag{3}$$

for all non-zero $\lambda \in \mathbb{C}$ with $|\arg z| \geqslant \pi/2 - \theta$ and all $p \in [1, \infty]$. In particular this implies that the kernel $R_{-\lambda} \in L_1(G; dg)$ because the L_1-norm is given by

$$\|R_{-\lambda}\|_1 = \|(-\lambda I + \overline{H})^{-1}\|_{\infty\to\infty}.$$

Therefore if $f \in \Phi_\varphi$ with $\varphi \in \langle \pi/2 - \theta, \pi]$ it follows that K_f is well-defined by (2) and $K_f \in L_1(G; dg)$. Then if ν is large enough it follows from Corollary A.2 that $K_f \in L_1^\rho(G; dg)$ where $\rho > 0$ is such that $\Delta_1(g) \leqslant e^{\rho|g|'}$ for all $g \in G$ with $|\cdot|'$ the modulus on G associated with the algebraic basis and Δ_1 the modular function on G_1. Hence K_f is the kernel of the operator $f(\overline{H})$.

Secondly, following Duong [5], we argue that if $f \in \Phi_{\varphi,\delta}$ with $\delta > 2D'/m$ then $K_f \in L_\infty(G; dg)$. The proof of this observation begins by noting that $(I + \overline{H})^{-\delta}$ is bounded on $L_\infty(G; dg)$ for all $\delta > 0$ and hence its kernel $R_{1,\delta} \in L_1(G; dg)$ because

$$\|R_{1,\delta}\|_1 = \|(I + \overline{H})^{-\delta}\|_{\infty\to\infty}.$$

But if $\delta > D'/m$ then $R_{1,\delta} \in L_\infty(G; dg)$ by the resolvent estimates of Theorem A.1 in the appendix. Therefore if $\delta > 2D'/m$ then $(I + H)^{-\delta/2}$ is bounded from $L_2(G; d\widehat{g})$ to $L_\infty(G; dg)$ and one has

$$\|(I + H)^{-\delta/2}\|_{\widehat{2}\to\infty} = \|R_{1,\delta/2}\|_2 \leqslant (\|R_{1,\delta/2}\|_1 \|R_{1,\delta/2}\|_\infty)^{1/2} < \infty.$$

Next by a duality argument $(I + \overline{H})^{-\delta/2}$ is bounded from $L_1(G; d\widehat{g})$ to $L_2(G; d\widehat{g})$. Now since $f \in \Phi_{\varphi,\delta}$ one may choose $f_1 \in L_\infty(G; dg)$ such that

$$f(z) = (1 + z)^{-\delta} f_1(z).$$

But $f_1(H)$ is bounded on $L_2(G; d\widehat{g})$ by Corollary 2.2 and

$$f(\overline{H}) = (I + H)^{-\delta/2} f_1(\overline{H})(I + \overline{H})^{-\delta/2}.$$

Therefore $f(\overline{H})$ is bounded from $L_1(G; d\widehat{g})$ to $L_\infty(G; dg)$. Consequently,

$$\|K_f\|_\infty = \|f(\overline{H})\|_{\widehat{1}\to\infty} < \infty$$

and $K_f \in L_\infty(G; dg)$.

Since $K_f \in L_1(G; dg) \cap L_\infty(G; dg)$ it follows automatically that $K_f \in L_2(G; dg)$ and this is crucial for the pricipal estimate.

PROPOSITION 3.2. *There is an $M > 0$ such that*

$$\mu_1(\{g \in G_1 : |(f(\overline{H})u)(g)| > \gamma\}) \leqslant M\|f\|_\infty \|u\|_{\widehat{1}} \gamma^{-1}$$

for all $f \in \Phi_{\varphi,\delta}$ with $\varphi \in \langle \pi/2 - \theta, \pi]$ and $\delta > 2D'/m$ and for all $u \in L_1(G_1;\mu_1)$ where μ_1 denotes the right Haar measure on G_1.

Proof. The action of the operator $f(\overline{H})$ is determined by the kernel $K_f \in L_1 \cap L_\infty$ and we first decompose it into a local part and a global part. Let $\chi : G \to [0,1]$ be a C^∞-function with compact support in a bounded symmetric neighbourhood Ω of the identity e of G and suppose that $\chi = 1$ in a second neighbourhood $\Omega_0 \subset \Omega$ of e. Then one can decompose $f(\overline{H})$ as a sum $T_f + R_f$ where $T_f = L(\chi K_f)$. But it follows from Corollary A.2 in the appendix that one has bounds

$$|K_f(g)| \leqslant a\|f\|_\infty (|g|')^{-D'} e^{-bc^{1/m}|g|'}$$

where $|\cdot|'$ is the canonical modulus on G associated with the algebraic basis used in the definition of the subelliptic operator H and the positive parameter c is a linearly increasing function of the scalar term ν. Thus if ν is sufficiently large $K_f(1-\chi) \in L_1^\rho(G) = L_1(G; e^{\rho|g|'}dg)$ with $\rho > 0$ so large that $\Delta_1(g) \leqslant e^{\rho|g|'}$ for all $g \in G$ where Δ_1 is the modular function of G_1. Since

$$\|R_f\|_{\widehat{p}\to\widehat{p}} \leqslant \|K_f(1-\chi)\|_1^\rho,$$

where the norms of R_f are now relative to $L_{\widehat{p}}$-spaces over G_1, it follows that R_f is a bounded operator on $L_{\widehat{p}}(G_1)$ with norm bounds continuous in f, uniformly in $p \in [1,\infty]$, i.e., one has estimates

$$\|R_f\|_{\widehat{p}\to\widehat{p}} \leqslant c\|f\|_\infty,$$

in particular this is the case if $p = 1$ and $p = 2$. Hence to establish the weak-$L_{\widehat{1}}(G_1)$ estimate on $f(\overline{H})$ it suffices to establish an estimate of this type for T_f. This is achieved in several steps by the reasoning used for an analogous problem in [1].

We begin by considering the action of T_f on the $L_{\widehat{p}}$-spaces over G, i.e., we effectively assume that $G = G_1$. Then the first step is to derive a local version of the required estimate. Hence we introduce the operator S_f from $L_p(\Omega^2;\mu_r)$ to $L_p(\Omega^2;\mu_r)$ by $S_f u = T_f(\chi' u)$ where $\chi' : G \to [0,1]$ is a C^∞-function with support in $\Omega \subset G$ and μ_r is the restriction to Ω^2 of the right Haar measure on G. Then

$$(S_f u)(g) = \int_\Omega d\mu_r(h) k_f(g;h) u(h)$$

where

$$k_f(g;h) = \chi(gh^{-1}) K_f(gh^{-1}) \chi'(h).$$

Now the tactic is to deduce the local result as a corollary of Theorem II.2.4 of [3]. This requires verifying three properties of S_f and k_f. First one needs S_f to be bounded on $L_2(\Omega^2;\mu_r)$. But this follows from Corollary 2.2 since

$$S_f u = T_f(\chi' u) = f(H)(\chi' u) - R_f(\chi' u).$$

Moreover, one has bounds $\|S_f\|_{\hat{2}\to\hat{2}} \leqslant c\|f\|_\infty$. Secondly, one requires that the kernel $k_f \in L_2(\Omega^2 \otimes \Omega^2;\mu_r \otimes \mu_r)$. But this is evident because $K_f \in L_\infty$ and hence k_f is bounded. Finally, one needs an estimate on the variation of K_f in the second variable. But for this it suffices to consider a particular Ω.

LEMMA 3.3. *Let $\eta > 0$ and $\Omega = B'_{2^{-1}\eta} = \{g \in G : |g|' < 2^{-1}\eta\}$. Then there is an $M > 0$, independent of the choice of f, such that*

$$\int_{\Omega(h;h_0)} d\mu_r(g)|k_f(g;h) - k_f(g;h_0)| \leqslant M\|f\|_\infty$$

for all $h, h_0 \in \Omega$ where $\Omega(h;h_0) = \{g \in B'_\eta : |gh_0^{-1}|' > 2|hh_0^{-1}|'\}$.

Proof. The proof is a slight variation of the argument used to establish an analogous property for the sequence of operators T_j in the proof of Theorem 2.2 of [1]. In the estimates on the T_j uniformity in j is essential and in the current context the uniformity in f is crucial. Therefore we repeat some of the details in order to display this uniformity.

First by right invariance of μ_r one has

$$\int_{\Omega(h;h_0)} d\mu_r(g)|k_f(g;h) - k_f(g;h_0)| \leqslant \int_{\Omega_2(h;h_0)} d\mu_r(g)|k_f(gh_0;h) - k_f(gh_0;h_0)|$$

for all $h, h_0 \in \Omega^2$ where $\Omega_2(h;h_0) = \{g \in B'_{2\eta} : |g|' > 2|hh_0^{-1}|'\}$ and where we also denote by μ_r the restriction to $B'_{2\eta}$ of the right Haar measure on G. Now to bound the integrand we choose an absolutely continuous path $\gamma : [0,1] \to G$ from h_0 to h with tangents in the directions $a_1, \ldots, a_{d'}$. Then

$$k_f(g;h) - k_f(g;h_0) = \sum_{i=1}^{d'} \int_0^1 dt\, \gamma_i(t)(A_i k_f)(g;\gamma(t))$$

where the A_i are left derivatives with respect to the second variable. We fix γ such that

$$\int_0^1 dt \left(\sum_{i=1}^{d'} \gamma_i(t)^2\right)^{1/2} \leqslant (1+\varepsilon)|h_0h^{-1}|'$$

where $\varepsilon \in \langle 0, 1\rangle$. Then

$$|h_0\gamma(t)^{-1}|' \leqslant (1+\varepsilon)|h_0 h^{-1}|'$$

for all $t \in [0,1]$. Therefore

$$|k_f(gh_0; h) - k_f(gh_0; h_0)| \leqslant \int_0^1 dt \left(\sum_{i=1}^{d'} \gamma_i(t)^2\right)^{1/2} \left(\sum_{i=1}^{d'} |(A_i k_f)(gh_0; \gamma(t))|^2\right)^{1/2} \leqslant$$

$$\leqslant 2d'|h_0 h^{-1}|' \sup\{|(A_i k_f)(gh_0; \gamma(t))| : i \in \{1, \ldots, d'\},\ t \in [0,1]\}$$

and our next aim is to bound the last factor. But

$$(A_i k_f)(g; h) = (B_i \chi)(gh^{-1}) K_f(gh^{-1}) \chi'(h) + \chi(gh^{-1})(B_i K_f)(gh^{-1})\chi'(h) +$$

$$+ \chi(gh^{-1}) K_f(gh^{-1})(A_i \chi')(h)$$

where B_i denotes the right derivative in the direction a_i. It follows, however, from the estimates on R_λ given in the appendix that

$$|K_f(gh_0 h^{-1})| \leqslant a\|f\|_\infty (|gh_0 h^{-1}|')^{-D'}.$$

Since

$$|gh_0\gamma(t)^{-1}|' \geqslant |g|' - |h_0\gamma(t)^{-1}|' \geqslant |g|' - (1+\varepsilon)|h_0 h^{-1}|' \geqslant 2^{-1}(1-\varepsilon)|g|'$$

for $g \in \Omega_2(h; h_0)$ one then has an estimate

$$|K_f(gh_0\gamma(t)^{-1})| \leqslant a'\|f\|_\infty (|g|')^{-D'}$$

for all $g \in \Omega_2(h; h_0)$ if ε is small enough. Next to handle the right derivative of K_f we note by Lemma 4.3 of [8] one has

$$(B_i\varphi)(g) = (L(g^{-1}) A_i L(g)\varphi)(g) = \sum_{\substack{\beta \in J_r(d') \\ |\beta| \neq 0}} c_{i,\beta}(g)(A^\beta \varphi)(g)$$

where the real-valued functions $c_{i,\beta}$ satisfy the bounds $|c_{i,\beta}(g)| \leqslant M(|g|')^{|\beta|-1} e^{\sigma|g|'}$. But the resolvent bounds in the appendix give bounds

$$|(A^\beta K_f)(g)| \leqslant b\|f\|_\infty (|g|')^{-(D'+|\beta|)}$$

and hence one has

$$|(B_i K_f)(gh_0\gamma(t)^{-1})| \leqslant b'\|f\|_\infty (|gh_0\gamma(t)^{-1}|')^{-(D'+1)} \leqslant b''\|f\|_\infty (|g|')^{-(D'+1)}$$

for all $g \in \Omega_2(h; h_0)$. Combination of these various estimates then gives

$$|(A_i k_f)(gh_0; \gamma(t))| \leqslant a\|f\|_\infty (|g|')^{-(D'+1)}$$

and, consequently,

$$|k_f(gh_0; h) - k_f(gh_0; h_0)| \leqslant a'\|f\|_\infty |h_0 h^{-1}|'(|g|')^{-(D'+1)}$$

for all $g \in \Omega_2(h; h_0)$.

Finally, setting $s = |h_0 h^{-1}|'$, one has

$$\int_{\Omega(h;h_0)} d\mu_r(g)|k_f(g;h) - k_f(g;h_0)| \leqslant$$

$$\leqslant a'\|f\|_\infty \sup_{s \leqslant \eta} \int_{2s \leqslant |g|' \leqslant 2\eta} d\mu_r(g) s(|g|')^{-(D'+1)} \leqslant M\|f\|_\infty$$

where $M > 0$ is a finite constant independent of f. The last estimate is established as in [1]. ■

The operator S_f, defined with $\Omega = B'_{2^{-1}\eta}$ acts on the spaces $L_p(B'_\eta; \mu_r)$. Therefore one can now apply Theorem II.2.4 of [3] to S_f and deduce that it satisfies a weak-$L_{\hat{1}}$ estimate. But since the $L_{\hat{2}}$-bound on S_f and the bounds of Lemma 3.3 are uniform for f with $\|f\|_\infty \leqslant 1$ it follows that the resulting $L_{\hat{1}}$-estimate is also uniform, i.e., one has an $M > 0$ such that

$$\mu_r(\{g \in G : |(S_f u)(g)| > \gamma\}) \leqslant M\|f\|_\infty \|u\|_{\hat{1}} \gamma^{-1}$$

for all $f \in \Phi_{\varphi,\delta}$ with $\varphi \in \langle \pi/2 - \theta, \pi]$ and $\delta > 2D'/m$ for all $u \in L_1(B'_{2\eta}; \mu_r)$.

At this stage one can remove the assumption that $G = G_1$ by the reasoning of [1]. The basic idea is that $G_1 \sim G \times \mathbf{R}^{d_1 - d}$ locally and the right Haar measure μ_1 on G_1 corresponds to the product of μ_r and Lebesgue measure on $\mathbf{R}^{d_1 - d}$. Repetition of the arguments of [1] then give the local weak-$L_{\hat{1}}(G_1)$ estimates

$$\mu_1(\{g \in G_1; |(L(\chi K_f)u)(g)| > \gamma\}) \leqslant c\|f\|_\infty \|u\|_{\hat{1}} \gamma^{-1}$$

for all $f \in \Phi_{\varphi,\delta}$ with $\varphi \in \langle \pi/2 - \theta, \pi]$ and $\delta > 2D'/m$ and for all $u \in L_1(G_1; \mu_1)$ with support in a ball of radius $4^{-1}\eta$ centred on the identity if η is small enough. Then, however, one can use right invariance to extend the bounds to all $u \in L_1(G_1, \mu_1)$ with support in a ball of radius $4^{-1}\eta$ centred at an arbitrary point $h \in G_1$. Moreover, the bounds are uniform in h. Finally, the bounds can be extended to global bounds by use of Lemma 2.4 of [1]. One then has

$$\mu_1(\{g \in G_1; |(T_f u)(g)| > \gamma\}) \leqslant c\|f\|_\infty \|u\|_{\hat{1}} \gamma^{-1}$$

for all $f \in \Phi_{\varphi,\delta}$ with $\varphi \in \langle\pi/2-\theta,\pi]$ and $\delta > 2D'/m$ and for all $u \in L_1(G_1;\mu_1)$. These bounds combined with the earlier L_1-bounds on R_f immediately yield the bounds of Proposition 3.2. ■

Now we can complete the proof of Theorem 3.1.

If ν is sufficiently large $f(H+\nu I)$ is defined and satisfies a bound

$$\|f(H+\nu I)\|_{\hat{2}\to\hat{2}} \leqslant c\|f\|_\infty$$

on $L_{\hat{2}}(G_1)$ for all $f \in F_\varphi$, with c independent of f, by Corollary 2.2. If, however, $f \in \Phi_{\varphi,\delta}$ with $\delta > 2D'/m$ then $f(\overline{H}+\nu I)$ satisfies the weak-$L_{\hat{1}}(G_1)$ estimate of Proposition 3.2. Therefore $f(H+\nu I)$ is bounded on $L_{\hat{p}}(G_1)$ for all $p \in \langle 1,2]$ and $f \in \Phi_{\varphi,\delta}$ with $\delta > 2D'/m$ by interpolation and in addition one has bounds

$$\|f(H+\nu I)\|_{\hat{p}\to\hat{p}} \leqslant c\|f\|_\infty .$$

This result then extends to all $f \in F_\varphi$ by McIntosh's convergence theorem, [14] Section 5. Since similar arguments apply to the formal adjoint $H^\dagger$ of H the desired boundedness properties of $f(H+\nu I)$ on $L_{\hat{p}}(G_1)$ for $p \in \langle 2,\infty\rangle$ follow by duality.

Finally boundedness on the $L_p(G_1)$-spaces follows from boundedness on the $L_{\hat{p}}(G_1)$-spaces as follow. Since $\Delta_1^{1/p} A_i \Delta_1^{-1/p} = A_i - p^{-1} b_i I$ we consider the m-th order subcoercive form $\widehat{C}$ by

$$\widehat{C} = \sum_{\alpha \in J_m(d')} \sum_{\substack{\gamma \in J_m(d') \\ (\beta,\gamma)\in Lb(\alpha)}} c_\alpha (-p)^{-|\gamma|} b^\gamma .$$

Let $H = dL(C)$ and $\widehat{H} = dL(\widehat{C})$. Arguing as in the proof of Lemma 2.1 of [1] one obtains that

$$\Delta_1^{1/p}(\lambda I + H)^{-1}\varphi = (\lambda I + \widehat{H})^{-1}\Delta_1^{1/p}\varphi$$

for all $\varphi \in C_c^\infty(G_1)$. Moreover, the previous results apply to $\widehat{H}$. So if $f \in F_{\varphi,\delta}$ and $\varphi \in C_c^\infty(G_1)$ one obtains

$$\begin{aligned}\|f(H)\varphi\|_p &= \|\Delta_1^{1/p} f(H)\varphi\|_{\hat{p}} = \|f(\widehat{H})\Delta_1^{1/p}\varphi\|_{\hat{p}} \leqslant \\ &\leqslant c\|f\|_\infty \|\Delta_1^{1/p}\varphi\|_{\hat{p}} = c\|f\|_\infty \|\varphi\|_p .\end{aligned}$$

By McIntosh's convergence theorem and density the theorem follows. ■

Now one easily obtains two of the three other conclusions of Theorem 2.1.

COROLLARY 3.4. *Let $H = dL(C)$ be an m-th order subelliptic operator associated with the left translations L by the group G acting on the spaces $L_p(G_1)$ where*

$p \in \langle 1, \infty \rangle$. If the real part of the zero-order coefficient of C is sufficiently large then one has the following.

(i) The imaginary powers $\{H^{it} : t \in \mathbf{R}\}$ form a strongly continuous group.

(ii) $D(H^\gamma) = [L_p, L'_{p;nm}]_{\gamma/n} = D((H^*)^\gamma)$ for all $\gamma \in \langle 0, n \rangle$ and $n \in \mathbf{N}$.

Similar statements are valid on the $L_{\hat{p}}(G_1)$-spaces.

Proof. Statement (i) follows from Theorem 3.1 by the same argument as in Section 8 of [14].

In Theorem 2.3 of [1] the equality $L'_{p;nm} = D(H^n)$ has been proved for all $n \in \mathbf{N}$ if the real part of the zero-order coefficient of C has a sufficiently large value, depending on n. Then by Theorem 1.15.3 of [17] and Statement (i) on has

$$D(H^\gamma) = [L_p, D(H^n)]_{\gamma/n} = [L_p, L'_{p;nm}]_{\gamma/n} = D((H^*)^\gamma) \tag{4}$$

for all $\gamma \in \langle 0, n \rangle$ and $n \in \mathbf{N}$. The proof on the $L_{\hat{p}}$-spaces is similar. ∎

The interpolation properties (4) immediately gives the following comparison of domains for different subcoercive operators.

Corollary 3.5. *Let C_1 and C_2 be subcoercive forms of order m_1 and m_2, both of step r. If $H_i = dL(C_i)$ are the operators on $L_p(G_1)$ with $p \in \langle 1, \infty \rangle$ associated with the algebraic basis of rank r of the Lie algebra of the subgroup $G \subseteq G_1$ then*

$$D((\nu I + H_1)^{t/m_1}) = D((\nu I + H_2)^{t/m_2})$$

for all $t \in [0, \infty\rangle$. Similar identities are valid on the $L_{\hat{p}}(G_1)$-spaces.

4. INTERPOLATION OF C^n-SUBSPACES

The complex interpolation properties of subcoercive operators give some interesting results for interpolation between the C^n-subspaces associated with a general set $a_1, \ldots, a_{d'}$ of elements of the Lie algebra $\mathfrak{g}$ of G. It is no longer necessary to assume that $a_1, \ldots, a_{d'}$ is an algebraic basis.

One can again derive results for unitary representations or the regular representations on the L_p-spaces with $p \in \langle 1, \infty \rangle$. First recall that $(\mathcal{H}, \mathcal{K})_{\gamma,q;K}$ denotes the real interpolation spaces defined by the Peetre K-method.

Proposition 4.1. *Let $(\mathcal{H}, G, U)$ be a unitary representation of the connected Lie group G and $\mathcal{H}'_n$ the C^n-subspaces associated with a family $a_1, \ldots, a_{d'}$ of elements of the Lie algebra $\mathfrak{g}$ of G. Then*

$$\mathcal{H}'_k = [\mathcal{H}'_j, \mathcal{H}'_l]_{(k-j)/(l-j)}.$$

for $j, k, l \in \mathbf{N}_0$ with $j < k < l$ and with the convention $\mathcal{H}'_0 = \mathcal{H}$.

More generally

$$(\mathcal{H}'_j, \mathcal{H}'_l)_{(\gamma-j)/(l-j),2;K} = [\mathcal{H}'_j, \mathcal{H}'_l]_{(\gamma-j)/(l-j)}$$

for $\gamma \in \langle j, l \rangle$.

Proof. We may assume that $a_1, \ldots, a_{d'}$ are linearly independent.

Let G' denote the connected subgroup of G with Lie algebra $\mathfrak{g}'$ generated by the subbasis $a_1, \ldots, a_{d'}$ and let H be the sublaplacian associated with the subbasis. Now we can apply Theorem 2.1 to the representation $(\mathcal{H}, G', U')$, where $U' = U|G'$, and to the subcoercive operator H. Then if ν is large enough the operator $(\nu I + H)$ has a bounded functional calculus. Therefore $(\nu I + H)^{is}$ is uniformly bounded if $s \in [-1, 1]$ and by Theorem 1.15.3 of [17] one has

$$[D((\nu I + H)^{\delta}), D((\nu I + H)^{\gamma})]_{\theta} = D((\nu I + H)^{\delta(1-\theta)+\gamma\theta})$$

for $0 \leqslant \delta < \gamma < \infty$ and $0 < \theta < 1$. But $\mathcal{H}'_n = D((\nu I + H)^{n/2})$ by [8], Theorem 3.3. Therefore

$$\mathcal{H}'_k = D((\nu I + H)^{k/2}) =$$

$$= D((\nu I + H)^{j/2}), D((\nu I + H)^{l/2})]_{(k-j)/(l-j)} = [\mathcal{H}'_j, \mathcal{H}'_l]_{(k-j)/(l-j)}$$

for $j < k < l$. Moreover, if $j < \gamma < l$ then

$$(\mathcal{H}'_j, \mathcal{H}'_l)_{(\gamma-j)/(l-j),2;K} = (\mathcal{H}, \mathcal{H}'_l)_{\gamma/l,2;K} = D((\nu I + H)^{\gamma/2}) =$$

$$= D((\nu I + H)^{j/2}), D((\nu I + H)^{l/2})]_{(\gamma-j)/(l-j)} = [\mathcal{H}'_j, \mathcal{H}'_l]_{(\gamma-j)/(l-j)}$$

where the first two identifications follow from [6] Theorem 3.2 and Lemma 7.1. ∎

The proposition establishes that the real interpolation space with $q = 2$ play a distinguished role for unitary representations. It is unclear whether there are analogous identifications for all the complex interpolation spaces associated with the C^n--structure of the regular representations. Nevertheless the first conclusion of Proposition 4.1 can be extended to these representations acting on the L_p-, or $L_{\widehat{p}}$-, spaces over G, with $p \in \langle 1, \infty \rangle$, by a similar argument based on Theorem 3.1.

COROLLARY 4.2. *Let $a_1, \ldots, a_{d'}$ be elements of the Lie algebra $\mathfrak{g}$ of the connected Lie group G and $L'_{p;n}(G)$, $L'_{\widehat{p;n}}(G)$ the corresponding C^n-subspaces. Then*

$$L'_{p;k}(G) = [L'_{p;j}(G), L'_{p;l}(G)]_{(k-j)/(l-j)}$$

if $j < k < l$ and $p \in \langle 1, \infty \rangle$. Similar identities are valid on the $L_{\widehat{p}}(G)$-spaces.

It is possible that complex interpolation results similar to the first statement of Proposition 4.1, or to the statement of Corollary 4.2, are valid for general representations but the above proof which relies on special regularity properties of the unitary representations and the regular representations does not shed any light on this question.

A. RESOLVENT ESTIMATES

Let S be the semigroup generated by the closure of the m-th order subelliptic operator $H = dU(C)$ determined by the subcoercive form $C = (c_\alpha)_{\alpha \in J_m(d')}$ with coefficients $c_\alpha \in \mathbf{C}$. Then for $\nu \in \mathbf{C}$ with the real part $\operatorname{Re}\nu$ sufficiently large the fractional powers $(\nu I + \overline{H})^{-\delta}$ of the resolvent are defined for all $\delta > 0$ by the Laplace transforms

$$(\nu I + \overline{H})^{-\delta} = \Gamma(\delta)^{-1} \int_0^\infty dt\, e^{-\nu t} t^{\delta - 1} S_t. \tag{5}$$

Since S satisfies bounds $\|S_t\| \leqslant M e^{\omega t}$ for some $M \geqslant 1$, some $\omega \in \mathbf{R}$ and all $t \geqslant 0$ it follows that the resolvents are indeed defined and satisfy norm bounds

$$\|(\nu I + \overline{H})^{-\delta}\| \leqslant c(\operatorname{Re}\nu - \omega)^{-\delta}$$

whenever $\operatorname{Re}\nu > \omega$. More generally if $\alpha \in J(d')$ with $|\alpha| < m\delta$ one has the bounds

$$\|A^\alpha(\nu I + \overline{H})^{-\delta}\| \leqslant c(\operatorname{Re}\nu - \omega)^{-(m\delta - |\alpha|)/m} \tag{6}$$

on the left derivatives of the resolvents.

It follows from the definition of subcoercivity [7] that to each subcoercive form C there corresponds an angle $\theta_C \in \langle 0, \pi/2 \rangle$, determined by the principal coefficients of C, such that each of the forms $e^{i\theta}C = (e^{i\theta}c_\alpha)_{\alpha \in J_m(d')}$, $\theta \in \langle 0, \theta_C \rangle$, is subcoercive. Hence the closures of the operators $e^{i\theta}H = dU(e^{i\theta}C)$, $\theta \in \langle 0, \theta_C \rangle$, generate continuous semigroups S^θ which give a holomorphic extension of S to the sector $\Lambda(\theta_C) = \{z \in \mathbf{C} : |\arg z| < \theta_C\}$ by the identification $S_{e^{i\theta}t} = S^\theta_t$ for $t > 0$. If $\theta \in \langle 0, \theta_C \rangle$, the holomorphic extension automatically satisfies bounds $\|S_z\| \leqslant M e^{\omega|z|}$ for all $z \in \Lambda(\theta)$ with $M \geqslant 1$ and $\omega \in \mathbf{R}$ independent of z, but dependent on θ. Consequently, the fractional powers $(\nu I + \overline{H})^{-\delta}$ of the resolvent are defined for all $\nu \in \mathbf{C}$ such that $\operatorname{Re}(e^{i\varphi}\nu) > \omega$ for some $\varphi \in \langle -\theta, \theta \rangle$ by

$$(\nu I + \overline{H})^{-\delta} = (e^{i\varphi}\nu I + e^{i\varphi}\overline{H})^{-\delta} e^{i\delta\varphi} =$$

$$= \Gamma(\delta)^{-1} e^{i\delta\varphi} \int_0^\infty dt\, e^{-e^{i\varphi}\nu t} t^{\delta-1} S_t^\varphi .$$

Now set

$$\Gamma(\varphi;\omega) = \Gamma_+(\varphi;\omega) \cup \Gamma_-(\varphi;\omega)$$

where $\Gamma_\pm$ denote the half-planes

$$\Gamma_\pm(\varphi;\omega) = \{z \in \mathbf{C} : \mathrm{Re}(z e^{\pm i\varphi}) > \omega\}.$$

Then the resolvents and their fractional powers are defined by the above method for all $\nu \in \Delta(\theta;\omega)$ where

$$\Delta(\theta;\omega) = \bigcup_{\varphi\in[0,\theta]} \Gamma(\varphi;\omega).$$

Moreover, one has bounds, analogous to (6),

$$\|A^\alpha(\nu I + \overline{H})^{-\delta}\| \leqslant c\rho(\nu;\Delta)^{-(m\delta-|\alpha|)/m} \tag{7}$$

for all $\nu \in \Delta(\theta;\omega)$ where $\rho(\nu;\Delta)$ denotes the distance of ν to the boundary of Δ.

Next remark that the semigroup S has a kernel K which has a holomorphic extension to the sector $\Lambda(\theta_C)$ and which satisfies "Gaussian bounds" uniformly throughout each of the subsectors $\Lambda(\theta)$ with $\theta \in \langle 0, \theta_C\rangle$. Specifically for each $\alpha \in J(d')$ there exist $a, b > 0$ and $\omega \in \mathbf{R}$ such that

$$|(A^\alpha K_z)(g)| \leqslant a|z|^{-(D'+|\alpha|)/m} e^{\omega|z|} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}} \tag{8}$$

for all $z \in \Lambda(\theta)$ and $g \in G$. Consequently the operators $(\nu I + \overline{H})^{-\delta}$ have kernels $R_{\nu,\delta}$ determined by the semigroup kernel K and the appropriate Laplace transform, e.g.,

$$R_{\nu,\delta}(g) = \Gamma(\delta)^{-1} \int_0^\infty dt\, e^{-\nu t} t^{\delta-1} K_t(g) \tag{9}$$

for $\mathrm{Re}\,\nu > \omega$. But as one has bounds on the derivatives of K one can derive bounds on the corresponding derivatives of the $R_{\nu,\delta}$ for all $\nu \in \Delta(\theta,\omega)$. One finds

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a \int_0^\infty dt\, e^{-\rho t} t^{-1+\beta} e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \tag{10}$$

for all $g \in G$ and $\nu \in \Delta(\theta;\omega)$ where $\beta = (m\delta - D' - |\alpha|)/m$ and $\rho = \rho(\nu;\Delta)$. Estimating as in the proof of Theorem III.6.7 in [16] these bounds can be reformulated as follows.

THEOREM A.1. *For each $\alpha \in J(d')$ and $\theta \in \langle 0, \theta_C \rangle$ there exist $a, b > 0$ and $\omega \in \mathbf{R}$, such that*

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a\rho^{(D'+|\alpha|-m\delta)/m} F_{|\alpha|,\delta}(\rho^{1/m}|g|')e^{-b\rho^{1/m}|g|'} \tag{11}$$

for all $g \in G \setminus \{e\}$ and all $\nu \in \Delta(\theta; \omega)$ where $\rho = \rho(\nu; \Delta)$ and

$$F_{k,\delta}(x) = \begin{cases} x^{-(D'+k-m\delta)} & \text{if } D'+k > m\delta \\ 1 + \log^+ x^{-1} & \text{if } D'+k = m\delta \\ 1 & \text{if } D'+k < m\delta \end{cases}$$

with $\log^+ y = \log y$ if $y \geqslant 1$ and $\log^+ y = 0$ if $y \leqslant 1$.

Proof. Before beginning the proof we briefly comment on the structure of the bounds (11). First the factor $\rho^{(D'+|\alpha|-m\delta)/m}$ is a reflection of the behaviour given in the norm bounds (6) of the dependence of the resolvent derivatives on ρ. The remaining terms depend only on the rescaled distance $g \mapsto \rho^{1/m}|g|'$. Secondly, the factor $F_{k,\delta}(\rho^{1/m}|g|')$ gives the small distance behaviour and this depends on the relative size of the local dimension, the order of the operator, etc. Thirdly, the exponential factor $e^{-b\rho^{1/m}|g|'}$ dictates the large distance behaviour and this is dimension independent.

The proof consists of estimating the bounds (10) and there are three distinct cases to consider, $\beta < 0$, $\beta > 0$ and $\beta = 0$. We examine them in this order.

Case 1. $\beta < 0$. A change of integration variable in (10) gives

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a(|g|')^{m\beta} \int_0^\infty dt\, t^{-1+\beta} e^{-f(t)} \tag{12}$$

where

$$f(t) = \rho(|g|')^m t + bt^{-1/(m-1)}.$$

In particular one has the estimate

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a'(|g|')^{-(D'+|\alpha|-m\delta)}$$

for all $g \in G$ with

$$a' = a \int_0^\infty dt\, t^{-1+\beta} e^{-bt^{-1/(m-1)}}$$

independent of $\nu = \Delta(\theta; \omega)$. This establishes the required estimate for $\rho^{1/m}|g|' \leqslant 1$.

Next we consider the case $\rho^{1/m}|g|' \geqslant 1$. Since f has a unique minimum at the point $t = ((m-1)\rho(|g|')^m/b)^{-(m-1)/m}$ the integral in (12) can be estimated in two parts, $I_<$ the integral over $\langle 0, t_0]$ and $I_>$ the integral over $[t_0, \infty\rangle$ where $t_0 = (\rho^{1/m}|g|')^{-(m-1)}$. One has

$$I_> \leqslant \int_{t_0}^{\infty} dt\, t^{-1+\beta} e^{-\rho(|g|')^m t}$$

and

$$I_< \leqslant \int_0^{t_0} dt\, t^{-1+\beta} e^{-bt^{-1/(m-1)}}.$$

Therefore

$$I_> \leqslant e^{-\rho(|g|')^m t_0} \int_{t_0}^{\infty} dt\, t^{-1+\beta} \leqslant -\beta^{-1} t_0^{\beta} e^{-\rho(|g|')^m t_0}.$$

But if n is any integer greater than $-(m-1)\beta$ then one has bounds

$$t_0^{\beta}(\rho^{1/m}|g|')^n \leqslant n!\varepsilon^{-n} e^{\varepsilon\rho^{1/m}|g|'}$$

for each $\varepsilon > 0$. Moreover, $\rho(|g|')^m t_0 = \rho^{1/m}|g|'$. Consequently, by choosing $\varepsilon = 2^{-1}$, one obtain bounds

$$I_> \leqslant a e^{-2^{-1}\rho^{1/m}|g|'}$$

for all $g \in G$.

Next consider $I_<$. A change of integration variable $t \to t^{-(m-1)}$ gives

$$I_< \leqslant (m-1) \int_{t_1}^{\infty} dt\, t^{\gamma} e^{-bt}$$

where $t_1 = t_0^{-1/(m-1)} = \rho^{1/m}|g|'$ and $\gamma = (m-1)(1-\beta) - m > -1$. Suppose $\gamma \geqslant 0$. If n is any integer greater than γ, then $t^{\gamma} \leqslant t^n \leqslant n!\varepsilon^{-n} e^{\varepsilon t}$ for all $t \geqslant t_1 \geqslant 1$ and $\varepsilon > 0$. Hence setting $\varepsilon = b/2$ one obtains bounds

$$I_< \leqslant 2^n (m-1) n! b^{-n} \int_{t_1}^{\infty} dt\, e^{-bt/2} \leqslant a e^{-b'\rho^{1/m}|g|'}$$

for suitable $a, b' > 0$. Secondly, suppose $\gamma \in \langle -1, 0\rangle$. Then

$$I_< \leqslant (m-1) \int_{t_1}^{\infty} dt\, e^{-bt} = (m-1) b^{-1} e^{-bt_1}.$$

Thus one again obtains bounds

$$I_< \leqslant a e^{-b'\rho^{1/m}|g|'}.$$

Since similar bounds were already established for $\rho^{1/m}|g|' \leqslant 1$ this completes the discussion of Case 1.

Case 2. $\beta > 0$. Since $\rho t = b((|g|')^m t_0^{-1})^{1/(m-1)}$ if $t = b^{(m-1)/m}|g|'\rho^{-(m-1)/m}$ we set $t_0 = \rho^{-(m-1)/m}|g|'$ and consider the integral in (10) in two parts, $I_<$ the integral over $\langle 0, t_0]$ and $I_>$ the integral over $[t_0, \infty\rangle$. Then

$$I_< \leqslant e^{-b((|g|')^m t_0^{-1})^{1/(m-1)}} \int_0^{t_0} dt\, t^{-1+\beta} e^{-\rho t} \leqslant$$

$$\leqslant e^{-b\rho^{1/m}|g|'} \int_0^{\infty} dt\, t^{-1+\beta} e^{-\rho t}.$$

But the latter integral is finite for $\rho > 0$, because $\beta > 0$, and by a change of variables $s = \rho t$ one obtains bounds

$$I_< \leqslant a\rho^{(D'+|\alpha|-m\delta)/m} e^{-b\rho^{1/m}|g|'}$$

as desired.

Alternatively

$$I_> \leqslant \int_{t_0}^{\infty} dt\, t^{-1+\beta} e^{-\rho t} \leqslant e^{-2^{-1}\rho t_0} \int_0^{\infty} dt\, t^{-1+\beta} e^{-2^{-1}\rho t}$$

and the bounds follow from another change of variables.

Case 3. $\beta = 0$. Now (10) gives

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a \int_0^{\infty} dt\, t^{-1} e^{-f(t)}$$

where $f(t) = \rho t + b((|g|')^m t^{-1})^{1/(m-1)}$. Therefore, integrating by parts, one has

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a \int_0^{\infty} dt \log t f'(t) e^{-f(t)} =$$

$$= a \int_0^{\infty} dt \log(\rho t) f'(t) e^{-f(t)} = a t_0 \int_0^{\infty} dt \log(\rho t_0 t) f'(t_0 t) e^{-f(t_0 t)},$$

with $t_0 = \rho^{1/m-1}|g|'$. Now we divide the integral into a first part over the interval $\langle 0, 1]$ and a second part over $[1, \infty\rangle$. Then

$$I_< \leqslant t_0 \int_0^{1} dt \log(\rho t_0 t) f'(t_0 t) e^{-b'(\rho^{1/m}|g|')t^{-1/(m-1)}}.$$

and a change of variable $t \to t^{-(m-1)}$ gives

$$I_{<} \leqslant t_0(m-1)\int_1^\infty dt\, t^{-m}\log(\rho t_0 t^{-(m-1)})f'(t_0 t^{-(m-1)})e^{-b\rho^{1/m}|g|'t}.$$

But $f'(t_0t^{-(m-1)}) = \rho(1 - b(m-1)^{-1}t^m)$, so $|f'(t_0t^{-(m-1)})| \leqslant c\rho t^m$ for some $c > 0$, uniformly in $t \geqslant 1$. Therefore one has an estimate

$$I_{<} \leqslant c_1|\rho t_0\log(\rho t_0)|\int_1^\infty dt\, e^{-b\rho^{1/m}|g|'t} + c_2\rho t_0\int_1^\infty dt\, e^{-b\rho^{1/m}|g|'t}\log t$$

and this immediately yields a bound

$$I_{<} \leqslant c_3(1 + |\log(\rho^{1/m}|g|')|)e^{-b'\rho^{1/m}|g|'}.$$

Next consider

$$I_{>} = t_0\int_1^\infty dt\log(\rho t_0 t)f'(t_0t)e^{-f(t_0t)} \leqslant t_0\int_1^\infty dt\log(\rho t_0t)f'(t_0t)e^{-\rho t_0 t} =$$

$$= t_0\int_1^\infty dt\log(\rho t_0t)f'(t_0t)e^{-b\rho^{1/m}|g|'t}.$$

But now $f'(t_0t) = \rho(1 - b(m-1)^{-1}t^{-m/(m-1)})$. Therefore one has estimates

$$I_{>} \leqslant c_1|\rho t_0\log(\rho t_0)|\int_1^\infty dt\, e^{-b\rho^{1/m}|g|'t} + c_2\rho t_0\int_1^\infty dt\, e^{-b\rho^{1/m}|g|'t}\log t \leqslant$$

$$\leqslant c_1'|\log(\rho t_0)|e^{-b\rho^{1/m}|g|'t} + c_2\rho t_0\int_1^\infty dt\log t e^{-b\rho^{1/m}|g|'t}.$$

Moreover

$$\log t = \log(\rho t_0t) - \log(\rho t_0) \leqslant \rho t_0 t - 1 - \log(\rho t_0) \leqslant$$
$$\leqslant \varepsilon^{-1}e^{\varepsilon\rho t_0 t} + 1 + |\log \rho t_0|$$

for all $\varepsilon > 0$. Hence choosing ε sufficiently small one obtains a bound

$$I_{>} \leqslant c_3(1 + |\log(\rho^{1/m}|g|')|)e^{-b\rho^{1/m}|g|'}.$$

Therefore combining these estimates one finds bounds

$$|(A^\alpha R_{\nu,\delta})(g)| \leqslant a(1 + |\log(\rho^{1/m}|g|')^{-1}|)e^{-b\rho^{1/m}|g|'} \tag{13}$$

for all $g \in G$. Finally if $x \leqslant 1$ then $\log x^{-1} = \log^+ x^{-1}$ but if $x \geqslant 1$ then

$$1 + |\log x^{-1}| = 1 + \log x \leqslant x \leqslant \varepsilon^{-1} e^{\varepsilon x}$$

for all $\varepsilon > 0$. Therefore by redefining the values of a and b the bounds (13) can be reexpressed in the desired form. ∎

The zero-order coefficient c_0 of C contributes a multiplicative factor $t \mapsto e^{-c_0 t}$ to the semigroup S. Therefore the values of a and b in the kernel estimates (8) can be assumed to be independent of c_0 and ω can be taken as a linearly decreasing function of $\operatorname{Re} c_0$. Thus if $\operatorname{Re} c_0$ is sufficiently large the parameter ω is negative and the kernel K is exponentially decreasing. Moreover, $\Delta(\theta;\omega) = \Gamma(\theta;\omega)$. Now if f is bounded and holomorphic in the sector $\Lambda(\varphi)$ with $\varphi \in \langle \pi/2 - \theta, \pi]$ one can define the kernel K_f on $G \setminus \{e\}$ by

$$K_f(g) = (2\pi i)^{-1} \int_{\Gamma_\chi} d\nu \, f(\nu) R_{-\nu}(g) \tag{14}$$

where $R_\nu = R_{\nu,1}$ denotes the kernel of the resolvent $(\nu I + \overline{H})^{-1}$ and the contour Γ_χ is determined by the function

$$\Gamma_\chi(t) = \begin{cases} t e^{i\chi} & \text{if } [0,\infty\rangle \\ -t e^{-i\chi} & \text{if } t \in \langle -\infty, 0], \end{cases}$$

with $\chi \in \langle \pi/2 - \theta, \varphi\rangle$. The bounds of the theorem then give bounds on K_f and its derivatives.

Corollary A.2. *Let $\alpha \in J(d')$, $\theta \in \langle 0, \theta_C\rangle$ and $\varphi \in \langle \pi/2, \pi]$. If the real part of the zero-order coefficient c_0 of C is sufficiently large then there exist $a, b > 0$ independent of c_0 and a $c > 0$ linearly-dependent on* $\operatorname{Re} c_0$ *such that*

$$|(A^\alpha K_f)(g)| \leqslant a \|f\|_\infty (|g|')^{-(D'+|\alpha|)} e^{-b c^{1/m} |g|'}$$

for all $f \in F_\varphi$ and all $g \in G \setminus \{e\}$.

Proof. By elementary geometry one obtains bounds $\rho = \rho(\nu, \Delta) \geqslant |\omega| + \tau|\nu|$ for $\nu \in \Gamma_\chi$ with $\tau > 0$. In particular $\rho^{1/m} \geqslant 2^{-1}(|\omega|^{1/m} + \tau^{1/m}|\nu|^{1/m})$ and $\rho^{1/m} \geqslant \tau^{1/m}|\nu|^{1/m}$. Now there are three cases to consider corresponding to the three cases of the theorem.

If $D' + |\alpha| > m$ then the theorem gives bounds

$$|(A^\alpha R_\nu)(g)| \leqslant a(|g|')^{-(D'+|\alpha|-m)} e^{-b'|\nu|^{1/m}|g|'} e^{-b''|\omega|^{1/m}|g|'}$$

with $b' = 2^{-1}b\tau^{1/m}$ and $b'' = 2^{-1}b$. Therefore

$$|(A^\alpha K_f)(g)| \leqslant 2a\|f\|_\infty(|g|')^{-(D'+|\alpha|)}e^{-b''|\omega|^{1/m}|g|'}\int_0^\infty d\nu(|g|')^m e^{-b'|\nu|^{1/m}|g|'}$$

and the desired bounds follow immediately. The other two cases $D' + |\alpha| = m$ and $D' + |\alpha| < m$ are very similar and we omit the details. ∎

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A.F.M. TER ELST
Department of Mathematics and Comp. Sc.,
Eindhoven University of Technology,
P.O. Box 513,
5600 MB Eindhoven,
Netherlands.

DEREK W. ROBINSON
Centre for Mathematics
and its Applications,
School of Mathematical Sciences,
Australian National University,
Canberra, ACT 0200,
Australia.

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