J. OPERATOR THEORY
32(1994), 47–75

EQUIVALENCE CLASSES OF SUBNORMAL OPERATORS

JAMES ZHIJIAN QIU

Communicated by Norberto Salinas

ABSTRACT. Let G be a bounded simply connected domain with harmonic measure ω and let $P^2(\omega)$ be the closure in $L^2(\omega)$ of $\mathcal{P}$, the set of analytic polynomials. Let S_ω be the operator defined by $S_\omega f = zf$ for each $f \in P^2(\omega)$. We characterize all subnormal operators similar or quasisimilar to S_ω and we describe the unitary equivalence class of S_ω. We make the assumption in this study that G is a normal domain (we say G is normal if $\mathcal{P}$ is dense in the Hardy space $H^1(G)$). Some examples are given to show that the normality of G is necessary. We also give some characterizations of a domain (i.e., a connected open subset in the plane) that is the image of a weak-star generator of $H^\infty(\mathbf{D})$.

KEYWORDS: *Similarity, quasisimilarity, Carleson measures, perfectly connected domains.*

AMS SUBJECT CLASSIFICATION: Primary 47B20; Secondary 30H05, 30E10, 46E15.

1. INTRODUCTION

Throughout this article we tacitly assume that

G is a bounded normal domain with harmonic measure ω.

That is, we assume that $\mathcal{P}$ is dense in the Hardy space $H^1(G)$. Normal domains are characterized in [24] (see Theorem 0.3 in Section 2). Jordan domains, Carathéodory domains, are normal domains. A crescent bounded by two circles is an example of non-Caratheodory normal domains [2].

Let H_1 and H_2 be Hilbert spaces and let A and B be bounded operators on H_1 and H_2, respectively. The operators A and B are unitarily equivalent if there is an isometric isomorphism U of H_1 onto H_2 such that $UA = BU$. The operators are similar if there exists an invertible operator X from H_1 to H_2 such that $XA = BX$. A weaker equivalence relation among operators is quasisimilarity. We say A and B are quasisimilar if there are injective and dense-range bounded operators $X\colon H_1 \to H_2$ and $Y\colon H_2 \to H_1$ such that $XA = BX$ and $YB = AY$. We denote these equivalence relations by $\cong$, $\simeq$ and $\sim$, respectively.

Let μ be a finite positive measure with compact support. We use the symbol S_μ to denote the operator defined by $S_\mu f = zf$ for each $f \in P^2(\mu)$, where $P^2(\mu)$ is the closure of $\mathcal{P}$ in $L^2(\mu)$.

In this paper we study the following problem: When is a subnormal operator unitarily equivalent, similar, or quasisimilar to S_ω. In 1973, W. Clary answered the latter two questions for the unilateral shift [6]. In 1979, W. Hastings extended Clary's quasisimilarity theorem to an isometry of finite cyclic multiplicity [15]. A decade later, J. McCarthy [18] extended Clary's quasisimilarity result to a rationally cyclic shift operator R_σ on $R^2(K,\sigma)$, where σ is the harmonic measure on a compact set K and $R(K)$ is a hypo-Dirichlet algebra.

In this paper we extend Clary's results in another direction. We generalize both his similarity and quasisimilarity results to S_ω. We also characterize all operators that are unitarily equivalent to S_ω.

To state our main results, we need the notion of a Carleson measure on a simply connected domain. We call a positive measure τ on G a *Carleson measure* if there is a positive constant c such that for every $t \in [1,\infty)$

$$\|p\|_{L^t(\tau)} \leqslant c\,\|p\|_{L^t(\omega)}, \quad p \in \mathcal{P}.$$

In [24] the author characterizes all Carleson measures on a normal domain (see Theorem 0.3). We call a function x in $P^2(\omega)$ an *outer* function if $A(G)x$ is dense in $P^2(\omega)$, where

$$A(G) = \{f : f \in C(\overline{G}) \quad \text{and} \quad f \ \text{ is analytic on } \ G\}.$$

Recall that a function g in $H^\infty(\mathbf{D})$ is a *weak-star generator* provided polynomials in g are weak-star dense in $H^\infty(\mathbf{D})$, where $\mathbf{D}$ is the open unit disc (consult D. Sarason [28], [29], [30]). We say a simply connected domain U is *perfectly connected* if U is the image of a weak-star generator of $H^\infty(\mathbf{D})$.

Now our equivalence theorems can be stated as follows:

THEOREM 1. *Let S be a subnormal operator. The operators S and S_ω are similar if and only if there exists a measure μ on $\overline{G}$ such that $S_\mu \cong S$ and μ has the following properties:*

(i) a.b.p.e. $P^2(\mu) \subseteq G$.

(ii) $\mu|\partial G \ll \omega$ *and* $\log \frac{(\mathrm{d}\mu|\partial G)}{\mathrm{d}\omega} \in L^1(\omega)$.

(iii) *If $x \in P^2(\omega)$ is an outer function and $|x|^2 = \frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega}$, then $|\widehat{x}|^{-2}\mu|G$ is a Carleson measure on G. (Here, $\widehat{x}$ is the analytic extension of x to* a.b.p.e. $P^2(\omega) = G$*, which is defined in Section 2.)*

THEOREM 3. *In order that S_μ be unitarily equivalent to S_ω for each positive measure μ with the properties that $[\mu] = [\omega]$ and $\int \log(\frac{\mathrm{d}\mu}{\mathrm{d}\omega})\,\mathrm{d}\omega > -\infty$, it is necessary and sufficient that G is a perfectly connected domain.*

THEOREM 5. *Let S be a subnormal operator. The operators S and S_ω are quasisimilar if and only if there exists a measure μ on $\overline{G}$ such that $S_\mu \cong S$ and μ has the following properties:*

(i) a.b.p.e. $P^2(\mu) \subseteq G$.

(ii) $\mu|\partial G \ll \omega$ *and* $\log(\frac{\mathrm{d}\mu|\partial G}{\omega}) \in L^1(\omega)$.

Theorem 1 and Theorem 5 were first proved by Clary in the case that G is the unit disc. Condition (i) is necessary in general. Theorem 3 implies that (i) can be omitted in the other two theorems if and only if G is a perfectly connected domain (for the proof, see the remark for Theorem 3 in Section 4).

A classical result of the Hardy space theory says that a function f in $P^2(m)$ (here, m is the normalized Lebesgue measure on the unit circle $\partial\mathbf{D}$) is outer if and only if $\mathcal{P}f$ is dense in $P^2(m)$. This is not true, in general. In the end of this paper we characterize all those domains on which our definition of f in $P^2(\omega)$ being an outer function is equivalent to the condition that $\mathcal{P}f$ is dense in $P^2(\omega)$.

2. PRELIMINARIES

ANALYTIC BOUNDED POINT EVALUATIONS. Let μ be a finite positive measure with compact support in the complex plane $\mathbf{C}$. For t in $[1,\infty)$, we let $P^t(\mu)$ be the closure of $\mathcal{P}$ in $L^t(\mu)$. A point w in $\mathbf{C}$ is a *bounded point evaluation* (b.p.e.) for $P^t(\mu)$ if there exists a positive constant c such that

$$|p(w)| \leqslant c\,\|p\|, \quad p \in \mathcal{P}.$$

In this case it is clear that $p \to p(w)$ extends to a bounded linear functional on $P^t(\mu)$. By the Riesz Representation Theorem there exists a function k_w in $L^q(\mu)$ such that

$$p(w) = \int pk_w \, d\mu.$$

Let $\hat{f}(w) = \int fk_w \, d\mu$. A point w is called an *analytic bounded point evaluation* (a.b.p.e.) if there exists a neighborhood U of w such that each point in U is a b.p.e. and $\hat{f}$ is analytic in U for each $f \in P^t(\omega)$.

In 1990 James E. Thomson proved the following remarkable theorem [33].

THOMSON'S THEOREM: *Let μ be a finite measure with compact support and let $t \in [1,\infty)$. Then there exists a Borel partition $\{\Delta_i\}_0^\infty$ of the support of μ such that*

$$P^t(\mu) = L^t(\mu|\Delta_0) \oplus \bigoplus_1^\infty P^t(\mu|\Delta_i)$$

and for each $i \geqslant 1$, $P^t(\mu|\Delta_i)$ contains no nontrivial characteristic functions. If $i \geqslant 1$ and W_i is the set of a.b.p.e.*'s for $P^t(\mu|\Delta_i)$, then W_i is a simply connected domain and $\Delta_i \subseteq \overline{W_i}$. Moreover, for $i \geqslant 1$, the evaluation map E: $f \to \hat{f}$ is one-to-one on $P^t(\mu_i)$. Finally, for $i \geqslant 1$, the Banach algebras $P^t(\mu_i) \cap L^\infty(\mu_i)$ and $H^\infty(W_i)$ are algebraically and isometrically isomorphic and weak-star homeomorphic via E.*

The space $P^t(\mu)$ is called *pure* if $\Delta_0 = 0$.

LEMMA 0.1. *Let μ be a compactly supported measure and let $U =$* a.b.p.e. $P^t(\mu)$. *If $P^t(\mu)$ is pure, then $A(U) \subset P^t(\mu)$.*

REMARK. This result was first proved in [20] for $t = 2$. J. Thomson had a proof for all $t \in [1,\infty)$, which is essentially the same as that of Lemma 5.5 of [33]. The author extended this result to a space $R^t(K,\mu)$ with a.b.p.e. $R^t(K,\mu)$ being finitely connected in [25].

HARMONIC MEASURE. A domain U is called a *Dirichlet domain* if the Dirichlet problem is solvable for every continuous function on ∂U. If U is a Dirichlet domain and if $u \in C(\partial U)$, let $\hat{u}$ be the unique function that is continuous on $\overline{U}$ and harmonic in U. Fix a point $a \in U$. The mapping $u \to \hat{u}(a)$ defines a positive functional on $C(\partial U)$ with norm one. Thus, there is a unique probability measure ω_a on ∂U such that

$$\hat{u}(a) = \int_{\partial G} \hat{u} \, d\omega_a, \quad u \in C(\partial U).$$

The measure ω_a is called the *harmonic measure* of U evaluated at a (consult [7] or [36]). When U is a simply connected domain, $\omega_a = m \circ \varphi^{-1}$, where φ is a conformal map of $\mathbb{D}$ onto U which sends 0 to a (here, we also use φ to denote its boundary value function on $\partial\mathbb{D}$). A simple application of Harnack's inequality shows that the harmonic measures of two points of U are boundedly equivalent.

NICELY CONNECTED DOMAIN. A simply connected *domain* U is said to be *nicely connected* if there exists a conformal map φ of $\mathbb{D}$ onto U so that it is univalent almost everywhere on $\partial\mathbb{D}$ with respect to m. In this case there is a Borel subset E of $\partial\mathbb{D}$ with $m(E) = 1$ such that the conformal map performs a one-to-one correspondence between the Borel subsets of E and $\varphi(E)$. The following theorem, which can be found in [9], characterizes nicely connected domains and is used repeatedly in this paper.

THEOREM 0.1. [Davie-Gamelin-Garnett]. *Let U be a simply connected domain in the complex plane. The following are equivalent:*

(i) *Every bounded analytic function on U is the pointwise limit on U of a bounded sequence in $A(U)$.*

(ii) *U is a nicely connected domain.*

(iii) *$A(U)$ is a Dirichlet algebra on ∂U.*

The following fact is repeatedly used in this paper also.

OBSERVATION. Suppose U is a nicely connected domain. Then

$$\int f \, d\omega_a = \int f \circ \varphi \, dm \qquad f \in L^1(\omega)$$

and

$$\int g \circ \varphi^{-1} \, d\omega_a = \int g \, dm \qquad g \in L^1(m)$$

where φ is the conformal map of $\mathbb{D}$ onto U such that $\omega_a = m \circ \varphi^{-1}$.

HARDY SPACES $H^t(U)$. Let U be a bounded domain and let $t \in [1, \infty)$. The Hardy space $H^t(U)$ is defined to be the set of all analytic functions f on U such that $|f|^t$ has a harmonic majorant on U. The norm of a function f in $H^t(U)$ can be defined as follows: Fix a point z_0 in U and put $\|f\| = u(z_0)^{1/t}$, where u is the least harmonic majorant of $|f|^t$ (see [11]). Harnack's inequality guarantees that the norms of $H^t(U)$ induced by two points of U are equivalent. $H^\infty(U)$ is the Banach space of bounded analytic functions on U with the sup norm.

In the case that U is a nicely connected domain, we can embed $H^t(U)$ in $L^t(\omega)$. In fact, let φ be a conformal map φ of $\mathbf{D}$ onto U. Then it effects a point isomorphism between the measure spaces $(\partial\mathbf{D}, m)$ and $(\partial G, \omega)$. For $f \in H^\infty(\mathbf{D})$, we let $\tilde{f}$ be the boundary value function on $\partial\mathbf{D}$. Now let $h \in H^t(U)$. Then, $\tilde{h}$, the boundary value function of h, is defined to be $\tilde{h} = (\widetilde{h \circ \varphi}) \circ \tilde{\varphi}^{-1}$. The map $h \to \tilde{h}$ induces an isometric isomorphism of $H^t(U)$ onto its image in $L^t(\omega)$. In this paper, we shall not distinguish between h and $\tilde{h}$.

For a given domain U, a *point* $a \in \partial G$ is called *removable* for $H^t(U)$ if each f in $H^t(U)$ can extended analyticaly to a neighborhood of a. The next theorem is proved in [23].

THEOREM 0.2. [J. Qiu]. *Let $t \in [1, \infty)$ and let U be a bounded domain with harmonic measure ω such that no point of ∂U is removable for $H^t(U)$.. The following are equivalent:*

(i) *$\mathcal{P}$ is dense in $H^t(U)$.*

(ii) a.b.p.e. $P^t(\omega) = U$.

(iii) *U is a nicely connected domain and if ψ is a Riemann map of U onto $\mathbf{D}$, then its boundary value function belongs to $P^t(\omega)$.*

CARLESON MEASURES AND NORMAL DOMAINS. A well-known theorem of L. Carleson ([11], p.156) says that a measure τ on the unit disc $\mathbf{D}$ is a Carleson measure if and only if there exists a positive constant c such that

$$\tau(C_h) \leqslant ch,$$

for each Carleson square

$$C_h = \{z = re^{it} : 1 - h \leqslant r < 1;\ t_0 \leqslant t \leqslant t_0 + h\} \text{ on } \mathbf{D}.$$

The following theorem is the main result of [24], which characterizes normal domains and also describes all Carleson measures on a given normal domain.

THEOREM 0.3. [J. Qiu]. *Let U be a simply connected domain with harmonic measure ω. The following are equivalent:*

(i) *Every Carleson measure τ on U has the form $\tau = \eta \circ \alpha^{-1}$ for a Carleson measure η on $\mathbf{D}$ and a conformal map α of $\mathbf{D}$ onto U.*

(ii) *$\mathcal{P}$ is dense in $H^t(U)$ for all $t \in [1, \infty)$.*

(iii) *U is a normal domain.*

THE SWEEP OF A MEASURE. Let U be a bounded Dirichlet domain and μ be a finite positive measure on $\overline{U}$. For $u \in C(\partial U)$, let $\widehat{u}$ be the unique function that is continuous on $\overline{U}$ and harmonic in U. The map $u \to \int_{\overline{U}} \widehat{u}\, d\mu$ defines a positive functional on $C(\partial U)$ with norm one. Using the Riesz representation theorem, there is a unique probability measure $\widehat{\mu}$ so that

$$\int_{\overline{U}} \widehat{u}\, d\mu = \int_{\partial U} u\, d\widehat{\mu}, \quad u \in C(\partial U).$$

The measure $\widehat{\mu}$ is called the *sweep* of μ. The proof of the next lemma can be found in [7].

LEMMA 0.2. *If μ is a measure on $\overline{U}$, then $\widehat{\mu} = \mu|\partial U + \widehat{\mu|U}$.*

3. OUTER AND INNER FUNCTIONS

LEMMA 1. *Let f be a positive function in $L^1(\omega)$. In order that there exist an outer function g in $P^2(\omega)$ such that $|g|^2 = f$ a.e. $[\omega]$, it is necessary and sufficient that $\log f$ is in $L^1(\omega)$.*

Proof. Suppose $\log f \in L^1(\omega)$. We put $\omega = \omega_a$ for some fixed point a in G. Let φ be the conformal map of $\mathbb{D}$ onto G with $\varphi(0) = a$ and let $\psi = \varphi^{-1}$ (so $\omega = m \circ \varphi^{-1}$). Then both $f \circ \varphi$ and $\log(f \circ \varphi)$ are in $L^1(m)$. By the classical Hardy space theory on the disc, there exists an outer function x in $P^2(m)$ such that $|x|^2 = f \circ \varphi$. Let $g = x \circ \psi$. Then

$$|g|^2 = |x \circ \psi|^2 = f \text{ a.e. } [\omega] \text{ on } \partial G.$$

Choose a sequence $\{p_n\} \subset \mathcal{P}$ such that

$$\int |p_n - x|^2\, dm \to 0.$$

It follows that

$$\int |p_n \circ \psi - g|^2\, d\omega \to 0.$$

Since $\psi \in P^2(\omega) \cap L^\infty(\omega)$ (Theorem 0.2), we conclude that $g \in P^2(\omega)$.

Since G is a normal domain, G is nicely connected and thus $A(G)$ is a Dirichlet algebra on ∂G (Theorem 0.1). So $P^2(\omega) = H^2(G) =$ the closure of $H^\infty(G)$ in $H^2(G) =$ the closure of $A(G)$ in $H^2(G)$. Therefore, $A(G)$ is dense in $P^2(\omega)$. Using an abstract version of Szegö's theorem ([16], p.103), we conclude that $A(G)g$ is dense in $P^2(\omega)$. Hence g is an outer function.

Conversely, assume that $f = |g|^2$ for some (outer) function g in $P^2(\omega) = H^2(G)$. Then $g \circ \varphi \in H^2(\mathbb{D})$ and so $g \circ \varphi \in P^2(m)$. By a result of the classical Hardy space theory, $\log |g \circ \varphi| \in L^1(m)$. Hence $\log f \in L^1(\omega)$ and so we are done. ∎

We call a function f in $P^2(\omega)$ an *inner function* if $|f| = 1$ a.e. $[\omega]$. A consequence of the proof of Lemma 1 is the following inner and outer factorization result.

COROLLARY 1. *If $g \in P^2(\omega)$, then there are an outer function h and an inner function f such that $g = fh$.*

Proof. Suppose that $g \in P^2(\omega)$. Let φ be a conformal map of $\mathbf{D}$ onto G. As in the proof of Lemma 1, we have that $g \circ \varphi \in P^2(m)$. It follows by Szegö's theorem that

$$\int \log|g|\,d\omega = \int \log|g \circ \varphi|\,dm > \infty.$$

Using a classical Hardy space result, we can find an outer function f_1 in $P^2(m)$ and an inner function h_1 in $P^2(m)$ such that $g \circ \varphi = f_1 h_1$. Set

$$f = f_1 \circ \varphi^{-1} \quad \text{and} \quad h = h_1 \circ \varphi^{-1}.$$

From the proof of Lemma 1, we see that f is outer and h is inner. ∎

It is well-known that if G is the unit disc $\mathbf{D}$, then $\mathcal{P}f$ is dense in $P^2(m)$ for each outer function f. But this is not true in general.

QUESTION 1. For what normal domains G do we have $\mathcal{P}f$ is dense in $P^2(\omega)$ for every outer function $f \in P^2(\omega)$?

The answer to this question is given in Theorem 6.

4. SIMILARITY AND UNITARY EQUIVALENCE

We begin this section with several lemmas. The first two lemmas are well-known and we include them here for the reader's convenience.

LEMMA 2. *Let μ and ν be two compactly supported positive measures. If there exists a bounded operator A from $P^2(\mu)$ to $P^2(\nu)$ such that A has dense range and $S_\nu A = AS_\mu$, then* b.p.e. $P^2(\nu) \subseteq$ b.p.e. $P^2(\mu)$.

Proof. Suppose that $w \in$ b.p.e. $P^2(\nu)$. Proposition 9.2 in [7] implies a point $w \in$ b.p.e. $P^2(\nu)$ if and only if $\operatorname{Ran}[(S_\nu - w)]$ is not dense in $P^2(\nu)$. Thus, $\operatorname{Ran}[(A)(S_\mu - w)]$ is not dense in $P^2(\nu)$. Since A has dense range, it follows that $\operatorname{Ran}(S_\mu - w)$ is not dense in $P^2(\mu)$, which in turn implies that $w \in$ b.p.e. $P^2(\mu)$. The proof is complete. ∎

LEMMA 3. *Let μ and ν be two compactly supported positive measures. If S_μ and S_ν are quasisimilar, then* b.p.e. $P^2(\nu) =$ b.p.e. $P^2(\mu)$.

Proof. This follows directly from the definition of the quasisimilarity and the previous lemma. ∎

Let μ and ν be two measures. We use the symbol $\mu \ll \nu$ to state that μ is absolutely continuous with respect to ν.

LEMMA 4. *Let μ be a finite compactly supported measure. If S_μ and S_ω are quasisimilar, then the following hold:*

(i) $\mathrm{supp}(\mu) \subset \overline{G}$.

(ii) a.b.p.e. $P^2(\mu) =$ a.b.p.e. $P^2(\omega)$.

(iii) $\mu|\partial G \ll \omega$ *and* $\log(\frac{d\mu|\partial G}{d\omega})$ *is in* $L^1(\omega)$.

To prove Lemma 4 we need the next result which was obtained by Olin and Yang (with a different proof) for an arbitrary simply connected domain in [20]. A generalization of their result can be found in [25].

LEMMA 5. *Let U be a bounded nicely connected domain with harmonic measure ω. If μ is a finite positive measure with compact support such that* a.b.p.e. $P^t(\mu) = U$ *and $P^t(\mu)$ is pure, then $\mu|\partial U \ll \omega$.*

First, we need the following version of the Abstract F. and M. Riesz Theorem (consult [14], p.158 or [8]).

THE ABSTRACT F. AND M. RIESZ THEOREM FOR THE ALGEBRA $A(U)$: *Let $\eta \perp A(U)$. Then η can be expressed as a series $\eta = \sum_{j\geqslant 0} \eta_j$, where each $\eta_j \perp A(U)$, the η_j's are pairwise mutually singular, η_0 is singular to all representing measures (of $A(U)$) for all points of $\overline{U}$, and for $j \geqslant 1$, η_j is absolutely continuous with respect to a representing measure (of $A(U)$) for some point of $\overline{U}$.*

Proof of Lemma 5. First, since U is nicely connected, it follows by Theorem 0.1 that $A(U)$ is a Dirichlet algebra on ∂U, and hence every point in ∂U is a peak point for $A(U)$. Second, the nontrivial Gleason part contains U ([14], Section 15) and every trivial Gleason part consists of a single point since $A(U)$ is a Dirichlet algebra (consult Section 15 of [14] for an explanation of our terminology and results). Consequently, U is the only nontrivial Gleason part of $A(U)$.

Now let $\eta \perp A(U)$. We claim that $\eta \ll \nu$ with respect to a representing measure ν for some point of U. By the Abstract F. and M. Riesz theorem, we have $\eta = \sum_{j\geqslant 0} \eta_j$, where each η_j is as in the theorem above. Using Wilkin's Theorem ([14], p.162), we have $\eta_0 = 0$. Let a be a peak point and let $f \in A(U)$ be a peak function for a. Clearly the sequence $f^n(z)$ boundedly pointwise converges to $\chi_{\{a\}}$, the characteristic function of $\{a\}$. Thus,

$$0 = \int \lim_{n\to\infty} f^n \, d\eta = \eta(\{a\}).$$

This implies that $\eta \ll \nu$ for a representing measure ν of $A(U)$, for some $a \in U$.

Since $A(U)$ is a Dirichlet algebra on ∂U, each point z in U has a unique representing measure whose support is contained in ∂U ([14], Lemma 31.1). Now if we let $\widehat{\nu}$ be the sweep of ν on ∂U, we then have

$$\int g(z)\, d\widehat{\nu} = \int \widehat{g(z)}\, d\nu = g(a) = \int g \, d\omega_a, \quad g \in A(U).$$

By uniqueness, we conclude $\widehat{\nu} = \omega_a$. Hence

$$\eta\big|\partial U \ll \nu\big|\partial U \ll \widehat{\nu}\big|\partial U = \omega_a.$$

Now suppose that $g \in L^q(\mu)$ such that

$$\int fg \, d\mu = 0, \quad f \in P^t(\mu),$$

where $1/q + 1/t = 1$. Since $A(U) \subseteq P^t(\mu)$, it follows that

$$\int fg \, d\mu = 0, \quad f \in A(U).$$

That is, $g\mu \perp A(U)$ and thus $g\mu\big|\partial U \ll \omega_a \ll \omega$. This implies that $(g\mu)_s = 0$, where $(g\mu)_s$ is the singular part of the Lebesgue decomposition of $g\mu$ with respect to ω. Consequently, $g \perp \chi_\Delta$, where Δ is the carrier of μ_s and μ_s is the singular part of the Lebesgue decomposition of μ with respect to ω. Now an application of the Hahn-Banach theorem yields $\chi_\Delta \in P^t(\mu)$. Since $P^t(\mu)$ is pure, it follows that $\chi_\Delta = 0$ a.e. $[\mu]$ and hence $\mu_s = 0$. ∎

The proof of Lemma 4. Assume that S_μ and S_ω are quasisimilar. Since $P^2(\omega)$ is pure, it follows that $P^2(\mu)$ is pure ([7], p.223). Theorem 4.11 of [33] together with Lemma 3 imply that

$$\begin{aligned} \text{a.b.p.e.}\, P^2(\mu) &= \text{b.p.e.}\, P^2(\mu) \\ &= \text{b.p.e.}\, P^2(\omega) \\ &= \text{a.b.p.e.}\, P^2(\omega) \\ &= G. \end{aligned}$$

Using Lemma 5, we see that

$$\mu|\partial G \ll \omega.$$

The fact that $\operatorname{supp}(\mu) \subseteq \overline{G}$ follows from Theorem 4.10 of [33].

Since S_μ and S_ω are quasisimilar, we can find an operator A: $P^2(\mu) \to P^2(\omega)$ such that $AS_\mu = S_\omega A$. For simplicity we may assume $\|A\| = 1$. Let φ be a conformal map of $\mathbf{D}$ onto G with $\varphi(0) = a$ and let ψ be its inverse function. We may assume, without loss of generality, that $\omega = m \circ \varphi^{-1}$. Set $u = A(1)$. Notice that $A(G)$ is contained in both $P^2(\omega)$ and $P^2(\mu)$ (Lemma 0.1). Now one can easily check that

$$A(f) = uf, \quad f \in A(G).$$

Since $|\psi| = 1$ on ∂G a.e. $[\omega]$, it follows that for all $n \geqslant 1$ and for each $f \in A(G)$

$$\begin{aligned}\int_{\partial G} |f|^2\, |u|^2\, \mathrm{d}\omega &= \int_{\partial G} |f|^2\, |u|^2\, |\psi|^{2n}\, \mathrm{d}\omega \\ &= \|A((\psi)^n f)\|^2 \\ &\leqslant \|(\psi)^n f\|^2 \\ &= \int_{\overline{G}} |\psi|^{2n}\, |f|^2\, \mathrm{d}\mu.\end{aligned}$$

If we let $n \to \infty$, then

$$\int_{\partial G} |f|^2\, |u|^2\, \mathrm{d}\omega \leqslant \int_{\partial G} |f|^2\, \mathrm{d}\mu, \quad f \in A(G). \tag{1}$$

Now we claim that

$$\{p \circ \varphi : p \in A(G)\} \quad \text{is dense in} \quad P^2(|h|^2 m) \quad \text{for each} \quad h \in P^2(m).$$

In fact, since $A(G)$ is bounded pointwise dense in $H^\infty(G)$ (Theorem 0.1), we see that $\{p \circ \varphi : p \in A(G)\}$ is bounded pointwise dense in $H^\infty(\mathbf{D})$. That is, each f in $H^\infty(\mathbf{D})$ is the pointwise limit of a bounded sequence of functions in $\{p \circ \varphi : p \in A(G)\}$ on $\mathbf{D}$. Now the Lebesgue dominated convergence theorem together with the density of $H^\infty(\mathbf{D})$ in $P^2(|h|^2 m)$ implies that $\{p \circ \varphi : p \in A(G)\}$ is dense $P^2(|h|^2 m)$.

Now using (1), we have

$$\int |f \circ \varphi|^2\, |u \circ \varphi|^2\, \mathrm{d}m \leqslant \int |f \circ \varphi|^2 \left(\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega}\right) \circ \varphi\, \mathrm{d}m, \quad f \in A(G).$$

Consequently,

$$\int |1-g|^2\, |u\circ\varphi|^2\, \mathrm{d}m \leqslant \int |1-g|^2\, (\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega})\circ\varphi\, \mathrm{d}m, \quad g\in P_0^2(m),$$

where $P_0^2(m)$ is the closure of $\{p : p\in\mathcal{P}$ and $p(0)=0\}$ in $L^2(m)$. Notice that the function $u\circ\varphi$ is in $P^2(m)$. So by Szegö's theorem (see [16], p.49)

$$\inf_{g\in P_0^2(m)} \int |1-g|^2\, (\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega}\circ\varphi)\, \mathrm{d}m > 0.$$

Also, using Szegö's theorem we get

$$\int \log(\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega}\circ\varphi)\, \mathrm{d}m > -\infty,$$

i.e.,

$$\int \log(\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega})\, \mathrm{d}\omega > -\infty.$$

Hence $\log(\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega})$ is in $L^1(\omega)$. The proof is complete. ∎

LEMMA 6. *Suppose that A is an invertible operator from $P^2(\mu)$ to $P^2(\omega)$ such that $AS_\mu = S_\omega A$. Let $u = A(1)$ and let $\alpha = \mu|\partial G$. Then there exists an outer function x in $P^2(\omega)$ such that $|x|^2 = \dfrac{\mathrm{d}\alpha}{\mathrm{d}\omega}$. Moreover, there exists an invertible function $h\in P^2(\omega)\cap L^\infty(\omega)$ such that $u = hx$.*

Proof. Since $AS_\mu = S_\omega A$, it is easy to verify that (as in the proof of Lemma 4)

$$\|A^{-1}\|^{-1}\, \|f\|_{L^2(\mu)} \leqslant \|uf\|_{L^2(\omega)} \leqslant \|A\|\, \|f\|_{L^2(\mu)}, \quad f\in A(G).$$

Replace f by $\psi^n f$. ψ is a conformal map of G onto $\mathbf{D}$ and is in $P^2(\omega)$ since G is a normal domain. Let $n\to\infty$, we obtain

$$\|A^{-1}\|^{-1}\, \|f\|_{L^2(\alpha)} \leqslant \|u f\|_{L^2(\omega)} \leqslant \|A\|\, \|f\|_{L^2(\alpha)}, \quad f\in A(G). \tag{2}$$

By Lemma 4 we see that $\log(\frac{\mathrm{d}\mu|\partial G}{\mathrm{d}\omega})\in L^1(\omega)$. It follows by Lemma 1 that there exists an outer function x such that $|x|^2 = \frac{\mathrm{d}\alpha}{\mathrm{d}\omega}$. Define an operator B on the manifold $\{fx : f\in A(G)\}$ via

$$B(xf) = uf \quad \text{for each} \quad f\in A(G).$$

Then for each $f \in A(G)$

$$\begin{aligned}\|uf\|_\omega^2 &= \int |uf|^2 \,d\omega \\ &= \int |A(f)|^2 \,d\omega \\ &\leqslant \|A\|^2 \int |f|^2 \,d\alpha \\ &= \|A\|^2 \int |f|^2 \frac{d\alpha}{d\omega} \,d\omega \\ &= \|A\|^2 \|fx\|_\omega^2.\end{aligned}$$

Hence B can be extended boundedly to $P^2(\omega)$. We use B to denote this extension too. Notice that the operator B commutes with S_ω. By Yoshino's theorem ([7], p.147), there is a function h in $P^2(\omega) \cap L^\infty(\omega)$ such that $B = M_h$, the multiplication operator induced by h on $P^2(\omega)$. Hence $u = hx$. Finally, (2) clearly indicates that M_h is bounded below, and hence h is invertible in $L^\infty(\omega)$. The proof is complete. ∎

LEMMA 7. *Let A be an invertible operator from $P^2(\mu)$ to $P^2(\omega)$ such that $AS_\mu = S_\omega A$. Let $u = A(1)$ and let $v = A^{-1}(1)$. Then*

$$v = A^{-1}(1) = \frac{1}{A(1)} = \frac{1}{u} \quad \text{a.e.} \quad [\mu].$$

Proof. Choose a sequence $\{p_n\} \subset \mathcal{P}$ such that

$$p_n \to A^{-1}(1) \quad \text{in} \quad P^2(\mu).$$

By passing to a subsequence if necessary, we see that

$$p_n \to A^{-1}(1) \quad \text{a.e.} \quad [\mu].$$

By the continuity of A, we get that

$$up_n \to 1 \quad \text{in} \quad P^2(\omega).$$

Thus, there exists a subsequence $\{p_{n_i}\}$ such that

$$up_{n_i} \to 1 \quad \text{a.e.} \quad [\omega].$$

But, a.b.p.e. $P^2(\mu) = G$ by Lemma 4. It follows that p_{n_i} converges to $\widehat{v}$, the analytic extension of v on G, uniformly on compact subsets of G. Since $v = \widehat{v}$ on G, it follows that $uv = 1$ a.e. $[\mu]$. Hence $v = \frac{1}{u}$ a.e. $[\mu]$. So we are done. ∎

LEMMA 8. *If τ is a Carleson measure on G, then $S_{\omega+\tau} \simeq S_\omega$.*

Proof. Suppose τ is a Carleson measure on G. Then there exists a positive constant c such that

$$\|p\|_\tau \leqslant c\,\|p\|_\omega, \quad p \in \mathcal{P}.$$

If we define the operator $I : P^2(\omega) \to P^2(\omega+\tau)$ densely [1] via

$$I(p) = p \quad \text{for each} \quad p \in \mathcal{P},$$

then it is easy to verify that I is an invertible operator. Clearly, I has the property that $IS_\omega = S_{\omega+\tau} I$. Hence $S_{\omega+\tau} \simeq S_\omega$. ∎

Recall that if $x \in P^2(\omega)$, then $\widehat{x}$ denotes the analytic extension of x to a.b.p.e. $P^2(\omega) = G$. The next theorem is the first main result of this paper.

THEOREM 1. *Let S be a subnormal operator. The following are equivalent:*

(i) $S \simeq S_\omega$.

(ii) *There is a Carleson measure τ on G such that $S \cong S_{\omega+\tau}$.*

(iii) *There exists a finite measure μ on $\overline{G}$ such that $S_\mu \cong S$ and μ has the following properties*

(a) a.b.p.e. $P^2(\mu) \subseteq G$.

(b) $\alpha = \mu|\partial G \ll \omega$ *and* $\log(\frac{d\alpha}{d\omega}) \in L^1(\omega)$.

(c) *If x is an outer function in $P^2(\omega)$ such that $|x|^2 = \frac{d\alpha}{d\omega}$, then $|\widehat{x}|^{-2}\mu|G$ is a Carleson measure on G.*

REMARKS.

I) Condition (a) in (iii) can be replaced by a.b.p.e. $P^2(\mu) = G$ since $S_\mu \simeq S_\omega$ implies that

$$\text{a.b.p.e. } P^2(\mu) = \text{a.b.p.e. } P^2(\omega) = G.$$

II) Condition (ii) says that, up to unitary equivalence, the similarity class of S_ω is:

$$\{S_{\omega+\tau} : \tau \text{ is a Carleson measure on } G\}.$$

III) If G is the unit disc $\mathbb{D}$, then (a) is satisfied for all μ with $\operatorname{supp}(\mu) \subseteq \overline{\mathbb{D}}$ and thus it can be removed. However, (a) can not be dropped in general (see Example 1 below). A natural question now arises:

---

[1] Here, 'densely' means the operator is defined on a manifold that is dense in the space. We use this terminology repeatly in the rest of this paper.

QUESTION 2. What normal domains have the property that Theorem 1 is valid with condition (iii).(a) omitted?

The answer is the perfectly connected domains, see the remark after Theorem 3.

The proof of Theorem 1. (i) $\Longrightarrow$ (ii). Assume $S \simeq S_\omega$. By Bram and Singer's theorem ([7], p.147) there is a measure μ such that $S_\mu \cong S$. So $S_\mu \simeq S_\omega$ also. It follows by Lemma 4 that

$$\operatorname{supp}(\mu) \subset \overline{G}, \quad \text{a.b.p.e.}\, P^2(\mu) \subseteq G, \quad \mu|\partial G \ll \omega, \quad \text{and} \quad \log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega).$$

Let T be an invertible operator from $P^2(\mu)$ to $P^2(\omega)$ such that $TS_\mu = S_\omega T$. Choose the functions x, h, and u as in Lemma 6. Using Lemma 7, we have

$$\frac{1}{x} = \frac{h}{u} = hT^{-1}(1) \quad \text{on} \quad \partial G \quad \text{a.e.} \quad [\mu]$$

and

$$\frac{1}{\widehat{x}} = \frac{\widehat{h}}{\widehat{u}} = \widehat{h}\widehat{T^{-1}(1)} \quad \text{on} \quad G.$$

Therefore, there exist x_1 in $P^2(\mu)$ and h_1 in $P^2(\mu) \cap L^\infty(\mu)$ such that $x = x_1$ and $h = h_1$ a.e. $[\mu]$ and

$$\frac{1}{x_1} = h_1 v = h_1 T^{-1}(1) \quad \text{a.e.} \quad [\mu].$$

Set $\eta = |x_1|^{-2}\mu$. Then for each $p \in \mathcal{P}$

$$\begin{aligned} \|p\|_\eta &= \left(\int \left|\frac{p}{x_1}\right|^2 d\mu\right)^{\frac{1}{2}} \\ &= \left(\int \left|\frac{ph_1}{u}\right|^2 d\mu\right)^{\frac{1}{2}} \\ &= \|h_1 T^{-1}(p)\|_\mu \\ &\leqslant \|h_1\|_\infty \|T^{-1}\| \, \|p\|_\omega. \end{aligned}$$

This implies that a.b.p.e. $P^2(\eta) \subseteq G$. It follows by Lemma 0.1 that

$$A(G) \subseteq P^2(\eta).$$

Now define an operator $F : P^2(\eta) \to P^2(\mu)$ densely via

$$F(f) = (\frac{1}{x_1})f \quad \text{for every} \quad f \in A(G).$$

One verifies that F is an isometry. Also, we have

$$S_\eta F = FS_\mu.$$

We want to show that F is onto. In fact, since

$$F[A(G)] = h_1 T^{-1}[A(G)]$$

and since h_1 is invertible in $P^2(\mu) \cap P^\infty(\mu)$, it follows that F has dense range; and thus F is an isometric isomorphism. Hence $S_\eta \cong S_\mu$. Now set

$$\tau = |x_1|^{-2}\mu|G.$$

Then

$$\eta = \omega + \tau$$

and

$$S \cong S_\mu \cong S_\eta = S_{\omega+\tau}.$$

Clearly

$$\|p\|_\tau \leqslant \|p\|_\eta \leqslant \|h\|_\infty \|T^{-1}\| \, \|p\|_\omega.$$

Hence, τ is a Carleson measure on G.

(ii) $\Longrightarrow$ (iii). It is clear.

(iii) $\Longrightarrow$ (i). Let μ be a measure as in (iii). Set

$$\eta = \omega + |\widehat{x}|^{-2}\mu|G.$$

Since $|\widehat{x}|^{-2}\mu|G$ is a Carleson measure on G, it follows by Lemma 8 that $S_\eta \simeq S_\omega$. If we repeat the process of the proof of (i) $\Longrightarrow$ (ii), we see that $S_\eta \cong S_\mu$. Thus, $S \cong S_\mu \simeq S_\omega$. So the proof is complete. ∎

EXAMPLE 1. Let $G = \mathbf{D} \setminus \{z : |z - 1/2| \leqslant 1/2\}$. Then $\mathcal{P}$ is not dense in $L^2_a(G)$, the Bergman space on G (for the existence of such a domain, consult [4]). Indeed a.b.p.e. $P^2(A) = \mathbf{D}$, where A is area measure on G. Set

$$\mu = A + \omega.$$

Then a.b.p.e. $P^2(\mu) = \mathbf{D}$. Let $\widehat{\mu}$ be the sweep of μ on ∂G. It follows from Lemma 0.2 that $\widehat{\mu} \ll \omega$. Clearly, we have

$$\log\left(\frac{d\widehat{\mu}}{d\omega}\right) \in L^1(d\omega)$$

and $|\widehat{x}|^{-2}\widehat{\mu}|G = 0$ is a Carleson measure on G (here, $|x|^2 = \frac{d\widehat{\mu}}{d\omega}$ as in Theorem 1). On the other hand, $\mathcal{P}$ is dense in $H^2(G)$ (Akeroyd [2]). It follows by Theorem 0.2 that $G = $ a.b.p.e. $P^2(\omega)$. Hence a.b.p.e. $P^2(\widehat{\mu}) \neq$ a.b.p.e. $P^2(\widehat{\mu})$. Therefore, $S_{\widehat{\mu}}$ and S_ω are even not quasisimilar. ∎

NOTE. In this example, area measure is not a Carleson measure on G since a.b.p.e. $P^2(A)$ strictly contains G.

The next theorem describes the unitary equivalence class of S_ω.

THEOREM 2. *Let H be a Hilbert space and let B be a bounded operator on H. Then B and S_ω are unitarily equivalent if and only if B is a cyclic subnormal operator and there exists a measure μ on $\overline{G}$ such that $B \cong S_\mu$ and μ has the properties:*

(i) a.b.p.e. $P^2(\mu) \subseteq G$.
(ii) $[\mu] = [\omega]$.
(iii) $\int \log(\frac{d\mu}{d\omega})\, d\omega > -\infty$.

REMARK. First, note that Example 1 also shows that (i) in Theorem 2 is necessary. In Theorem 3 we characterize all those domains G for which (i) can be removed.

Proof. Suppose that $B \cong S_\omega$. Since S_ω is cyclic, it follows that B is cyclic. Since S_ω is subnormal, one verifies that B is subnormal too (for example, one may apply (f) of Theorem 1.9 of [7], p.118). ω is a measure having properties (i), (ii) and (iii).

Conversely, assume that $B \cong S_\mu$ for some μ that has the following properties

$$\text{a.b.p.e. } P^2(\mu) \subset G, \quad [\mu] = [\omega] \quad \text{and} \quad \int \log(\frac{d\mu}{d\omega})\, d\omega > -\infty.$$

Using Lemma 1, we can find an outer function x in $P^2(\omega)$ so that $|x|^2 = \frac{d\mu}{d\omega}$. Since

$$\text{a.b.p.e. } P^2(\mu) \subseteq \text{a.b.p.e. } P^2(\omega) = G,$$

it follows that

$$A(G) \subseteq P^2(\mu).$$

Now if we define an operator A from $P^2(\mu)$ to $P^2(\omega)$ densely via

$$A(p) = xp \quad \text{for each} \quad p \in A(G),$$

then A clearly is an isometric isomorphism. Also, A has the properties that $AS_\mu = S_\omega A$. Thus $S_\mu \cong S_\omega$. This proves the theorem. ∎

The next theorem not only answers Question 2 but also gives a characterization of perfectly connected domains.

THEOREM 3. *In order that S_μ and S_ω be unitarily equivalent for each positive measure μ with the properties that $[\mu] = [\omega]$ and $\int \log(\frac{d\mu}{d\omega})\, d\omega > -\infty$, it is necessary and sufficient that G is a perfectly connected domain.*

REMARK. Using Theorem 3 we now show that the answer for Question 2 is the perfect connected domains. Assume that Theorem 1 with hypothesis (iii).(a) omitted is valid for a normal domain G. This means that S_μ and S_ω the similar for every measure μ that satisfies conditions (b) and (c) of Theorem 1. Since 0 measure is a Carleson measure on G, we derive that S_μ are S_ω similar (so a.b.p.e. $P^2(\mu)$ = a.b.p.e. $P^2(\omega)$) for all those μ satisfying (b) and (c) in Theorem 1 and having the property that $\mu|G = 0$. (Therefore, we see that if μ satisfies (b) and (c) and if $\mu|G = 0$, then μ has properties (i), (ii) and (iii) in Theorem 2.) By Theorem 2 we conclude that S_μ and S_ω are unitarily equivalent for all those μ with the properties that $[\mu] = [\omega]$ and $\log \frac{d\mu}{d\omega} \in L^1(\omega)$. Hence, it follows from Theorem 3 that G is a perfectly connected domain.

For the proof of Theorem 3, we need to recall Sarason's weak-star density theorem for polynomials. For a compact set K, let $R(K)$ denote the uniform closure in $C(K)$ of the set of the rational functions with poles off K. Recall that $R(K)$ is a Dirichlet algebra if $\{\mathrm{Re}\, f : f \in R(K)\}$ is dense in $C_{\mathbb{R}}(\partial K)$, the real continuous function algebra on ∂K. Sarason's theorem for weak-star density of polynomials now can be stated (see [31] or [8], p.301).

SARASON'S THEOREM. *For a compactly supported positive measure μ on the complex plane, there is a compact set K and measures μ_a and μ_s having the following properties:*

(i) $\mu = \mu_a + \mu_s$, $\mu_a \perp \mu_s$ *and* $P^\infty(\mu) = L^\infty(\mu_s) \oplus P^\infty(\mu_a)$.

(ii) K *contains the support of* μ_a, $R(K) \subseteq P^\infty(\mu_a)$ *and* $R(K)$ *is a Dirichlet algebra.*

(iii) *There is an isometric isomorphism* α *from* $H^\infty(\mathrm{int}\,(K))$ *to* $P^\infty(\mu_a)$ *such that* α *is also a weak-star homeomorphism and* $\alpha(f) = f$ *for every* $f \in R(K)$.

REMARK. The set K is known as the Sarason hull of μ.

For a compactly supported measure μ, let b.p.e. $P^\infty(\mu)$ denote the set of all points λ such that the linear map $p \to p(\lambda)$, $p \in \mathcal{P}$, can be extended to a weak-star continuous linear functional on $P^\infty(\mu)$.

The following lemma can be found in [8], p.306.

LEMMA 9. *If μ is a finite measure with compact support, then* b.p.e. $P^\infty(\mu_a) = \text{int}\,(K)$.

Proof of Theorem 3. Suppose that G is a perfectly connected domain. Then there is a conformal map φ of $\mathbf{D}$ onto G such that $\{p\circ\varphi : p \in \mathcal{P}\}$ is weak-star dense in $H^\infty(\mathbf{D})$; this clearly is equivalent to saying $\mathcal{P}$ is weak-star dense in $H^\infty(G)$ and the latter is equivalent to having

$$P^\infty(\omega) = H^\infty(G).$$

Let μ be a measure such that

$$[\mu] = [\omega] \quad \text{and} \quad \int \log(\frac{d\mu}{d\omega})\, d\omega > -\infty.$$

It follows that

$$\begin{aligned} A(G) &\subseteq H^\infty(G) \\ &= P^\infty(\omega) \\ &= P^\infty(\mu) \\ &\subseteq P^2(\mu). \end{aligned}$$

By Lemma 1 there is an outer function x in $P^2(\omega)$ such that $|x^2| = \frac{d\mu}{d\omega}$ and thus there is an isometric isomorphism A: $P^2(\mu) \to P^2(\omega)$, defined by extending the following isometrical map

$$A(a) = ax \quad \text{for each} \quad a \in A(G).$$

Clearly the operator A has the property that $AS_\mu = S_\omega A$. Hence $S_\mu \cong S_\omega$.

For the proof of the other direction, let us assume that $S_\mu \cong S_\omega$ for each positive μ that has the properties

$$[\mu] = [\omega] \quad \text{and} \quad \int \log(\frac{d\mu}{d\omega})\, d\omega > -\infty.$$

Let $f \in L^1(\omega)$. Clearly

$$(|f| + 1)\omega = \omega \quad \text{and} \quad \log(|f| + 1) \in L^1(\omega).$$

It follows, by our assumption, that $S_{(|f|+1)\omega} \cong S_\omega$. Thus,

$$\text{a.b.p.e.}\, P^2(|f|\omega) \subseteq \text{a.b.p.e.}\, P^2((|f| + 1)\omega) = \text{a.b.p.e.}\, P^2(\omega).$$

Now if η is a measure such that $[\eta] = [\omega]$, then by the Randon–Nikodym theorem there exists $h \in L^1(\omega)$ such that $h = \frac{d\eta}{d\omega}$. Using the previous argument, we get

$$\text{a.b.p.e.}\, P^2(|\eta|) \subseteq \text{a.b.p.e.}\, P^2(\omega).$$

Since ∂G has no removable point for $H^2(G)$ and since G is normal, it follows by Theorem 1 in [23] that

$$\text{a.b.p.e.}\, P^2(\omega) = G.$$

Thus, we have

$$\text{a.b.p.e.}\, P^2(|\eta|) \subset G \text{ for each } \eta \text{ with } [\eta] = [\omega].$$

We now claim:

$$\text{b.p.e.}\, P^\infty(\omega) = G.$$

Actually, using a result of T. Gamelin, J. McCarthy and J. Thomson (see Theorem 5.9 of [33], or [17]) we can find a finite measure ν such that $[\nu] = [\omega]$ and

$$P^\infty(\omega) = P^2(\nu) \cap L^\infty(\nu).$$

If we apply Sarason's theorem to $P^\infty(\omega)$ and Thomson's theorem to $P^2(\nu)$ respectively, then

$$H^\infty(\text{b.p.e.}\, P^\infty(\omega)) = H^\infty(\text{a.b.p.e.}\, P^2(\nu)).$$

Consequently,

$$\text{b.p.e.}\, P^\infty(\omega) = \text{a.b.p.e.}\, P^2(\nu).$$

On the other hand, clearly

$$G \subseteq \text{b.p.e.}\, P^\infty(\omega).$$

It follows from the last equality that

$$G \subseteq \text{a.b.p.e.}\, P^2(\nu).$$

Hence,

$$G = \text{a.b.p.e.}\, P^2(\nu) = \text{b.p.e.}\, P^\infty(\omega).$$

Now applying Sarason's theorem again, we conclude that $\mathcal{P}$ is weak-star dense in $H^\infty(G)$. Using the argument at the beginning of the proof, we see that G is a perfectly connected domain. ∎

It is natural to ask whether we can extend Theorem 1 and Theorem 2 to non-normal domains. The question should be phrased as follows (keep in mind that U is strictly contained in a.b.p.e. $P^2(\omega)$ when U is not a normal domain):

QUESTION 3. Let U be a non-normal simply connected domain with harmonic measure ω. Let μ be a positive measure with the following properties:

(i) a.b.p.e. $P^2(\mu) =$ a.b.p.e. $P^2(\omega)$.

(ii) $[\mu] = [\omega]$ and $\int \log(\frac{d\mu}{d\omega})\, d\omega > -\infty$.

In addition, let μ_0 and ω_0 be the restrictions of μ and ω on the boundary of a.b.p.e. $P^2(\omega)$ respectively and assume that $S_{\mu_0} \cong S_{\omega_0}$.

Are S_μ and S_ω similar?

NOTE. The measure μ stated in Question 3 satisfies all conditions in both Theorem 1 and Theorem 2. (But, U is not a normal domain!)

The following example shows that the question has a negative answer.

EXAMPLE 2. Let V be an open disc whose boundary contains the origin such that one of its diameters lies on the nonnegative real axis. Let J be a closed segment which joins 0 and a point inside the disc such that J forms an angle $\frac{\pi}{3}$ at 0 with the nonnegative imaginary axis. Let E be the closed domain enclosed by the triangle that is symmetric to the real axis and has J as one of its sides. Now set

$$U = V \setminus E.$$

Let ω be the harmonic measure of U and set

$$\omega_0 = \omega\big|\partial V \quad \text{and} \quad \omega_1 = \omega\big|\partial E.$$

Using Lemma 2.8 in [3], we see that ω and the measure $|z|^2 s$ are boundedly equivalent near 0, where s is the arclength measure on ∂U. Set

$$s_0 = s\big|\partial V \quad \text{and} \quad s_1 = s\big|\partial E.$$

Then $|z|^{-2}\omega_1$ and s_1 are boundedly equivalent. It is obvious that s_1 is a Carleson measure on V (see [11]). Thus, there is a positive constant c such that

$$\|p\|_{L^2(|z|^{-2}\omega_1)} \leqslant c\, \|p\|_{L^2(s_0)}, \quad p \in \mathcal{P}.$$

This implies that there exists another positive constant c_0 such that

$$\|p\|_{L^2(\omega_1)} \leqslant c_0\, \|p\|_{L^2(\omega_0)}, \quad p \in \mathcal{P}.$$

Now we define A: $P^2(\omega) \to P^2(\omega_0)$ densely via

$$A(p) = p \quad \text{for each} \quad p \in \mathcal{P}.$$

The last inequality implies that A is invertible. Apparently, $AS_\omega = S_{\omega_0}A$; hence S_ω and S_{ω_0} are similar. Since $\log(\frac{d\omega_0}{ds_0}) \in L^1(s_0)$, it follows by Theorem 3 that $S_{s_0} \cong S_{\omega_0}$ (s_0 is boundedly equivalent to the harmonic measure of V). Thus S_{s_0} and S_ω are similar.

Now we define a measure μ on ∂U by setting

$$\mu = s_0 + |z|^{-\frac{1}{2}}s_1.$$

The measure $|z|^{-\frac{1}{2}}s_1$ is not a Carleson measure since it does not satisfy the 'window' condition of the original definition of a Carleson measure [11], p.156. Therefore, S_μ and S_{s_0} are not similar. Combining this fact with the previous argument, we conclude that S_μ and S_ω are not similar. However, one can easily verify that we do have

(i) a.b.p.e. $P^2(\mu) =$ a.b.p.e. $P^2(\omega) = V$

(ii) $[\mu] = [\omega]$ and $\log\frac{d\mu}{d\omega} \in L^1(\omega)$.

(iii) $S_{\omega_0} \cong S_{\mu_0}$, where $\mu_0 = \mu|\partial V$.

NOTE. The above domain U is one of the simplest non-normal domains. So it seems that there is no hope for us to extend Theorem 1 and Theorem 2 to a larger class of domains. However, if we only consider an operator that has the form $S_{\omega+\tau}$ (where τ is a positive measure on U), then the author has shown that $S_{\omega+\tau} \simeq S_\omega$ if and only if τ is a Carleson measure on the domain U (see Theorem 6.3 of [26] or see [22]).

There is not much known concerning operators similar to shift operators associated with more general domains. We close this section with a theorem which deals with arbitrary domains.

THEOREM 4. *Let U be a non-normal bounded domain with harmonic measure ω. Let μ be a finite positive measure. Let μ_0 and ω_0 be the restrictions of the measures μ and ω to the boundary of* a.b.p.e. $P^2(\omega)$, *respectively. Then*

(i) *In order that $S_\mu \cong S_\omega$, it is necessary that $S_{\mu_0} \cong S_{\omega_0}$.*

(ii) *In order that $S_\mu \simeq S_\omega$, it is necessary that $S_{\mu_0} \simeq S_{\omega_0}$.*

Proof. The proofs of (i) and (ii) are almost identical, and so we are only going to prove (i). Suppose $S_\mu \cong S_\omega$. Let X be an isometric isomorphism from $P^2(\mu)$ to $P^2(\omega)$ such that $XS_\mu = S_\omega X$. If we let $v = X(1)$ and $u = X^{-1}(1)$, then for every $p \in \mathcal{P}$

$$||p||_\omega = ||up||_\mu$$

and

$$||p||_\mu = ||vp||_\omega.$$

Let g be a Riemann map from a.b.p.e. $P^2(\omega)$ to $\mathbb{D}$ and extend g to ∂[a.b.p.e. $P^2(\omega)$] by defining it to be its boundary value function (in nontangential limit sense) on ∂[a.b.p.e. $P^2(\omega)$] (this is possible since a.b.p.e. $P^2(\omega)$ is nicely connected by Thomson's theorem and Theorem 94 of [19]). Then g can be regarded as a function in both $P^2(\omega)$ and $P^2(\mu)$. Replacing p by pg^n in the above equalities (this can be done as follows: Let $\{q_k\}$ be a sequence in $\mathcal{P}$ that converges to pg^n in $P^2(\omega)$. By the same argument in the proof of Lemma 7 one shows the first equality above holds for pg^n. Others can be proved similarly.) respectively, and letting $n \to \infty$, we see that for each $p \in \mathcal{P}$

$$\|p\|_{\omega_0} = \|up\|_{\mu_0} \tag{3}$$

and

$$\|p\|_{\mu_0} = \|vp\|_{\omega_0}.$$

Now define A: $P^2(\mu_0) \to P^2(\omega_0)$ densely by

$$A(p) = vp \quad \text{for each} \quad p \in \mathcal{P}.$$

The operator A obviously is an isometry and it has the property that $AS_{\mu_0} = S_{\omega_0}A$. If we can show that A is onto, then we are done. To do this, let us pick a function $f \in P^2(\omega_0)$. Then (3) implies that $uf \in P^2(\mu_0)$ and thus uvf is in the range of A. But $uv = 1$ almost everywhere with respect to ω (Lemma 7), and so f is in the range of A. Therefore, A is onto. Hence, A is an isometric isomorphism. ∎

5. QUASISIMILARITY THEOREM

Quasisimilarity of subnormal operators has been studied by a number of authors. W. S. Clary ([6], 1973) first characterized all subnormal operators quasisimilar to the unilateral shift. W. Hastings ([15], 1979) extended his result to operators that are finite direct sums of unilateral shifts. In 1990 J. McCarthy [18] extended Clary's result to rationally cyclic shift operator (where a hypo-Dirichlet algebra condition was imposed).

The next theorem is our primary result concerning quasisimilarity of subnormal operators.

THEOREM 5. *Suppose S is a subnormal operator. Then S and S_ω are quasisimilar if and only if there exists a finite measure μ on $\overline{G}$ such that $S_\mu \cong S$ and μ has the following two properties:*

(i) a.b.p.e. $P^2(\mu) \subseteq G$.

(ii) $\mu|\partial G \ll \omega$ *and* $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$.

REMARK. We would like to point out the differences between J. McCarthy's similarity result and Theorem 5. In [18] McCarthy shows that if $R(K)$ is a hypo-Dirichlet algebra, then $R_\mu \sim R_\omega$ if and only if $\mu|\partial K \ll \omega$ and $\log(\frac{d\mu|\partial K}{d\omega}) \in L^1(\omega)$ (here μ is a finite measure on K, ω is the harmonic measure of int(K), and R_μ is given by $R_\mu f = zf$ for each $f \in R^2(K,\omega)$, the closure of $R(K)$ in $L^2(\omega)$). McCarthy's result does not cover this theorem even when $\overline{G}^\circ = G$ and $R(\overline{G})$ is a Dirichlet algebra (for example, let G be a crescent domain). The reason is that the spaces $P^2(\mu)$ and $R^2(\overline{G},\mu)$ are different and thus the operators S_μ and R_μ are different, in general, no matter whether $R(\overline{G})$ is a Dirichlet algebra or not.

In summary, we generalize Clary's result in two different directions.

Proof. Suppose S and S_ω are quasisimilar. It is easy to verify that S is a cyclic operator because S_ω is cyclic. By Bram and Singer's theorem ([7], p.147), there is a measure μ such that $S_\mu \cong S$ and thus S_ω and S_μ are quasisimilar. Using Lemma 4, we have

$$\text{supp}(\mu) \subseteq \overline{G}, \quad \text{a.b.p.e.}\, P^2(\mu) = G, \quad \mu|\partial G \ll \omega, \quad \text{and} \quad \log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega).$$

For the proof of sufficiency, assume that μ is a measure with supp$(\mu) \subset \overline{G}$ such that (i) and (ii) listed in Theorem 5 are satisfied. We first want to show that the inclusion operator

$$I : P^2(\mu) \to P^2(\mu|\partial G),$$

given by

$$I(a) = a \quad \text{for all} \quad a \in P^2(\mu),$$

is injective. In fact, if one notices that a.b.p.e. $P^2(\mu) =$ a.b.p.e. $P^2(\mu|\partial G) = G$ (by Theorem 2), then it is routine to check that I must be injective.

Since $S_{\mu|\partial G} \cong S_\omega$, there is an isometric isomorphism $J : P^2(\mu|\partial G) \to P^2(\omega)$ such that

$$JS_{\mu|\partial G} = S_\omega J.$$

Now let $Y = JI$. Then Y is injective and has dense range. Clearly, we also have

$$YS_\mu = S_\omega Y.$$

To finish our proof we need to find another operator X from $P^2(\omega)$ to $P^2(\mu)$ that is injective, has dense range, and has the property

$$XS_\omega = S_\mu X.$$

Let φ be a conformal map of $\mathbf{D}$ onto G. Without loss of generality, we may assume that $\omega = m\circ\varphi^{-1}$. Extend φ to $\partial\mathbf{D}$ by defining it to be its boundary value function on $\partial\mathbf{D}$. Since G is nicely connected, $v = \mu\circ\varphi$ defines a measure on $\overline{\mathbf{D}}$. Since

$$\mu|\partial G \ll \omega \quad \text{and } \log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega),$$

it follows that

$$\nu|\partial D \ll m \quad \text{and} \quad \log(\frac{d\nu|\partial D}{dm}) \in L^1(m).$$

Let $\hat{\nu}$ be the sweep of ν. Since

$$\|a\|_{L^2(\nu)} \leqslant \|a\|_{L^2(\hat{\nu})} \quad \text{for each} \quad a \in A(\mathbf{D}),$$

it follows that the map A, given by $A(a) = a$ for each $a \in A(\mathbf{D})$, extends to be a bounded operator from $P^2(\hat{\nu})$ to $P^2(\nu)$. Using Lemma 0.2, we see that $\log(\frac{d\hat{\nu}}{dm}) \in L^1(m)$. Thus, $S_{\hat{\nu}} \cong S_m$ (Theorem 2). Choose an isometric isomorphism R from $P^2(m)$ to $P^2(\hat{\nu})$ such that $RS_m = S_{\hat{\nu}}R$ and set $C = AR$. Then C is an injective operator with dense range. Let $u = C(1)$. Then $A(\mathbf{D})u$ is dense in $P^2(\nu)$. Let ψ be the inverse of φ and let $v = u\circ\psi$. It is easy to verify that $v \in P^2(\mu)$. Now, if we can show that $A(G)v$ is dense in $P^2(\mu)$, then the operator X from $P^2(\omega)$ to $P^2(\mu)$, defined densely by

$$X(a) = va \quad \text{for each} \quad a \in A(G),$$

is the desired operator, which is an injective bounded operator with dense range and has the property that $S_\mu X = XS_\omega$. Hence $S_\mu \sim S_\omega$.

To show that $A(G)v$ is dense in $P^2(\mu)$, we pick a function $f \in P^2(\mu)$. Then $f\circ\varphi \in P^2(\nu)$, and hence it can be approximated by the functions in $A(\mathbf{D})u$ in the $P^2(\nu)$-norm. This is equivalent to saying that f is approximable by functions in $\{v(a\circ\psi) : a \in A(\mathbf{D})\}$ in the $P^2(\mu)$-norm. Now the Lebesgue dominated convergence theorem together with the fact that $A(G)$ is pointwise bounded dense in $H^\infty(G)$ implies that $A(G)v$ is dense in $P^2(\mu)$. ∎

NOTE. As in Theorem 1 and Theorem 2, condition (i) in Theorem 5 is necessary. The following corollary shows that condition (i) may be removed if and only if G is perfectly connected.

COROLLARY 2. *In order that S_μ and S_ω be quasisimilar for every finite measure μ with $\mu \ll \omega$ and $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$, it is necessary and sufficient that G is a perfectly connected domain.*

Proof. The sufficiency follows from the previous theorem. For the proof of necessity, let us assume that $S_\mu \sim S_\omega$ for every finite positive measure μ with $\log(\frac{d\mu|\partial G}{d\omega}) \in L^1(\omega)$. We have, in particular, $S_\mu \sim S_\omega$ for each μ with $[\mu] = [\omega]$ and $\log(\frac{d\mu}{d\omega}) \in L^1(\omega)$. So it follows by Theorem 2 that $S_\mu \cong S_\omega$ for every finite positive measure μ with $[\mu] = [\omega]$ and $\log(\frac{d\mu}{d\omega}) \in L^1(\omega)$ (since $S_\mu \sim S_\omega$ implies that a.b.p.e. $P^2(\mu) \subseteq$ a.b.p.e. $P^2(\omega)$). By Theorem 3, we conclude that G is perfectly connected. ∎

It is natural to ask whether Theorem 5 can be generalized to non-normal domains. One may ask the following question.

QUESTION 4. Let U be a non-normal simply connected domain with harmonic measure ω. Assume μ is a finite positive measure such that

(i) $P^2(\mu)$ contains no non-trivial characteristic functions.

(ii) a.b.p.e. $P^2(\omega) =$ a.b.p.e. $P^2(\mu)$.

(iii) $\log(\frac{d\mu|\partial G_0}{d\omega_0}) \in L^1(\omega_0)$, where μ_0 and ω_0 are the restrictions of μ and ω on the boundary of a.b.p.e. $P^2(\omega)$ respectively.

Are S_μ and S_ω quasisimilar?

The answer is negative.

EXAMPLE 3. Let V be an open crescent enclosed by two circles C_0 and C_1 such that C_0 is its outer boundary. Let D_0 and D_1 be the open discs enclosed by C_0 and C_1, respectively. Let J be a proper closed arc of C_1 which intersects C_0 and has its endpoints in D_0. Set

$$U = D_0 \setminus J$$

and let ω be the harmonic measure of U. We now claim:

$$\text{a.b.p.e.}\, P^2(\omega) = D_0.$$

For the proof of the claim, we first need to show that a.b.p.e. $P^2(\omega)$ is a nicely connected domain. This follows from Thomson's theorem and Theorem 94 of

Miller-Olin-Thomson [19]. Second, since ω is the harmonic measure of U, we have that

$$U \subseteq \text{a.b.p.e.}\, P^2(\omega).$$

Now, since D_0 is the smallest nicely connected domain containing U and since a.b.p.e. $P^2(\omega)$ must be contained in D_0, the conclusion follows.

On the other hand, $\mathcal{P}$ is dense in $H^2(V)$ ([2], Akeroyd). It follows by Theorem 0.2 that a.b.p.e. $P^2(\lambda) = V$, where λ is the harmonic measure of V. If we denote the restriction of λ on C_0 by λ_0 and denote arclength measure on C_0 by s, then Szegö's theorem implies that $\log(\frac{d\lambda_0}{ds})$ is not in $L^1(s)$. Now a well-known argument about harmonic measure shows that ω_0 and λ_0 are boundedly equivalent on ∂C_0. So we conclude that $\log(\frac{d\omega_0}{ds})$ is not in $L^1(s)$. Using Theorem 5, we see that S_ω and S_s are not quasisimilar (note that s is boundedly equivalent to the harmonic measure of D_0 and so Theorem 5 can be applied here). But, we obviously have

(i) $P^2(s)$ contains no non-trivial characteristic functions.

(ii) a.b.p.e. $P^2(s) = D_0 =$ a.b.p.e. $P^2(\omega)$.

(iii) $\log(\frac{ds}{d\omega_0}) \in L^1(\omega_0)$. ∎

Now we present the theorem that answers Question 1.

THEOREM 6. *In order that $\mathcal{P}g$ be dense in $P^2(\omega)$ for every outer function g in $P^2(\omega)$, it is necessary and sufficient that G is a perfectly connected domain.*

Proof. Suppose that G is a perfectly connected domain. Let g be an outer function in $P^2(\omega)$. The hypothesis implies that a.b.p.e. $P^2(|g|^2\omega) \subseteq G$ (since, $G =$ b.p.e. $P^\infty(|g|^2\omega)$). It follows from Lemma 0.1 that $A(G) \subseteq P^2(|g|^2\omega)$. Since g is outer, we conclude that $\mathcal{P}g$ is dense in $P^2(\omega)$.

Conversely, suppose that $\mathcal{P}g$ is dense in $P^2(\omega)$ for every outer function g in $P^2(\omega)$. Let μ be any measure such that

$$[\mu] = [\omega] \quad \text{and} \quad \log(\frac{d\mu}{d\omega}) \in L^1(\omega).$$

Using Lemma 1, we can find an outer function x such that $\frac{d\mu}{d\omega} = |x^2|$. If we define an operator A from $P^2(\mu)$ to $P^2(\omega)$ via

$$A(p) = xp \quad \text{for each} \quad p \in \mathcal{P},$$

then A is an isometric isomorphism with the property that $AS_\mu = S_\omega A$. Therefore S_μ and S_ω are unitarily equivalent. Now the conclusion follows from Theorem 3. ∎

Acknowledgements. This paper is a part of the author's Ph.D. Thesis written under the supervision of Robert F. Olin at Virginia Polytechnic Institute and State University. I wish to thank Professor Olin for his support during this investigation. I am indebted to Professor James Thomson for helpful conversations. I sincerely thank the referee for his very careful reviewing and for all his helpful suggestions and effort that had helped me to get this final version.

REFERENCES

1. L.V. AHLFORS, *Complex Analysis*, Third Ed., McGraw-Hill, New York 1979.
2. J. AKEROYD, Polynomial approximation in the mean with respect to harmonic measure on crescents.II, *Trans. Amer. Math. Soc.* **303**(1987), 193–199.
3. J. AKEROYD, D. KHAVINSON, H.S. SHAPIRO, Remarks concerning cyclic vectors in Hardy spaces and Bergman spaces, *Michigan Math. J.* **38**(1991), 191–205.
4. J.E. BRENNAN, Approximation in the mean by polynomials on non-Carathéodory domains, *Ark. Mat.* **15**(1977), 117–168.
5. L. CARLESON, Interpolations by bounded analytic functions and the Corona problem, *Ann. of Math. (2)* **76**(1962), 547–559.
6. W.S. CLARY, *Quasi-similarity and subnormal operators*, Ph.D. thesis, University of Michigan 1973.
7. J.B. CONWAY, *Subnormal Operators*, Pitman Publishing Inc., Research Notes in Math. **51**, Marshfield, Mass. 1982.
8. J.B. CONWAY, *The theory of subnormal operators*, Math. Surveys Monographs, Vol. **36**, Amer. Math. Soc. 1991.
9. A.M. DAVIE, Dirichlet algebras of analytic functions, *J. Funct. Anal.* **6**(1970), 348–356.
10. R.G. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York 1972.
11. P.L. DUREN, *Theory of H^p-Spaces*, Academic Press, New York 1970.
12. W.H.J. FUCHS, *Topics in the Theory of one Complex Variable*, D. Von Nostrand Co., Princeton, N. J. 1967.
13. J.B. GARNETT, *Bounded Analytic Functions*, Academic Press 1981.
14. T.W. GAMELIN, *Rational Approximation Theory*, unpublished UCLA Lecture Notes 1975.
15. W.W. HASTINGS, Subnormal operators quasisimilar to an isometry, *Trans. Amer. Math. Soc.* **256**(1979), 145–161.
16. K. HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J. 1962.
17. J.E. MCCARTHY, Analytic structures for subnormal operator, *Integral Equations Operator Theory* **13**(1990), 251–270.
18. J.E. MCCARTHY Quasisimilarity of rationally cyclic subnormal operators, *J. Operator Theory* **24**(1990), 105–116.
19. T. MILLER, R. OLIN, J. THOMSON, Subnormal operators and representations of algebra of bounded analytic functions and other uniform algebras, *Mem. Amer. Math. Soc.* **354**(1986).
20. R. OLIN, L. YANG, The commutant of multiplication by z on the closure of polynomials in $L^t(\mu)$, to appear.

21. R.F. OLIN, J.Z. QIU, Carleson measures on a multiply connected domain and rationally cyclic subnormal operators, to appear.
22. J.Z. QIU, Boundary values of analytic functions in the Banach spaces $P^t(\sigma)$ on crescents, to appear, *Illinois J. Math.*
23. J.Z. QIU, Density of polynomials, to appear, *Houston J. Math.*
24. J.Z. QIU, Carleson measures and polynomial approximation, to appear, *Proc. Amer. Math. Soc.*
25. J.Z. QIU, The structure of $R^t(K,\tau)\cap L^\infty(\tau)$ and rational approximation, to appear.
26. J.Z. QIU, *Polynomial approximation and Carleson measures on a general domain and equivalence classes of subnormal operators*, Ph.D. Thesis, Virginia Tech. 1993.
27. R.C. ROAN, Composition operators on H^p with dense range, *Indiana Univ. Math. J.* **27**(1978), 159–162.
28. D. SARASON, Invariant subspaces and unstarred operator algebras, *Pacific J. Math.* **17**(1966).
29. D. SARASON, Weak-star generators of H^∞, *Pacific J. Math.* **17**(1966), 519–528.
30. D. SARASON, On the order of a simply connected domain, *Michigan Math. J.* **15**(1968), 129–133.
31. D. SARASON, Weak-star density of polynomials, *J. Reine Angew. Math.*, **252**(1972), 1–15.
32. A.L. SHIELDS, Weighted shift operators and analytic function theory, in *Math. Surveys Monographs*, **13**: Topics in Operator Theory, ed. C. Pearcy, Amer. Math. Soc., Providence, R. I., 1974, 49–128.
33. J. THOMSON, Approximation in the mean by polynomials, *Ann. of Math. (2)* **133** (1991), 477–507.
34. T.T. TRENT, $H^2(\mu)$ spaces and bounded point evaluations, *Pacific J. Math.* **80**, 279–292.
35. T.T. TRENT, Carleson measure inequalities and kernel functions in $H_2(\mu)$. *J. Operator Theory* **11**(1984), 157–169.
36. M. TSUJI, *Potential theory in modern function theory*, Chelsea, New York 1975.

JAMES Z. QIU
Department of Mathematics
Virginia Tech
Blacksburg, VA 24061
U.S.A.

Received March 15, 1992; revised May 27, 1994.