J. OPERATOR THEORY
32(1994), 209–241

ANGLES BETWEEN TWO SUBFACTORS

TAKASHI SANO and YASUO WATATANI

Communicated by William B. Arveson

ABSTRACT. We introduce angles between two subfactors of a type II_1 factor to study their relative position. We calculate angles for certain pairs. We also determine when fixed point algebras form a commuting square.

KEYWORDS: *Angle, subfactor, Jones index, commuting square, co-commuting square.*

AMS SUBJECT CLASSIFICATION: Primary 46L37; Secondary 46L10.

1. INTRODUCTION

We introduce the notion of angles (and opposite angles) between two subfactors by using the relevant Jones projections, and relative positions between two subfactors are studied. The existence of gaps in angles is shown for a certain pair of subfactors. We also characterize pairs of subfactors for which (sets of) angles and opposite angles reduce to $\{\frac{\pi}{2}\}$.

After recalling standard facts on angle operators in Section 2, we define (the set of) angles between two subalgebras as the spectrum of the angle operator of the corresponding Jones projections in Section 3. A quadrilateral (L, M, N, K) of type II_1 factors, where L is generated by M, N and $K = M \cap N$, forms a commuting square if and only if $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$. Therefore, our angle measures how far the quadruplet is from being the commuting square. We establish basic properties to make angles computable. We show that $\mathrm{Ang}_L(M, N)$ is a finite set when the Jones index $[L : K]$ is finite. We also define opposite angles by looking at the basic extensions. In Section 4, we show a kind of duality of angles between a quadrilateral and its second basic extensions. Let P be a type II_1 factor with an outer action of a finite group G with two subgroups A, B. Angles for the quadruplet

(P, P^A, P^B, P^G) are analyzed in Section 5. In Section 6, we investigate angles for a quadrilateral (L, M, N, K) of type II_1 factors satisfying $[L : M] = [L : N] = 2$, $[L : K] < \infty$, and $L \cap K' = \mathbf{C}$. We show that angles in this case have some gaps. (such a quadrilateral is described by the dihedral group.) In Section 7, we deal with a quadrilateral (L, M, N, K) with $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$. We obtain several characterizations for opposite angles to be trivial $(= \{\frac{\pi}{2}\})$. We also find a pair of sequences of projections satisfying the Jones relations.

We would like to thank H. Araki, H. Kosaki, and V. Jones for several discussions and comments.

2. ANGLES BETWEEN TWO SUBSPACES

Relative positions of two subspaces $\mathcal{M}$ and $\mathcal{N}$ in a Hilbert space $\mathcal{H}$ have been investigated by several authors with interesting applications ([1], [2], [5], [7], [8], [9], [12], [16], [18], [33]). Refer to the introduction of [29].

In general, $\mathcal{H}$ is the direct sum of four subspaces $\mathcal{M} \cap \mathcal{N}$, $\mathcal{M} \cap \mathcal{N}^\perp$, $\mathcal{M}^\perp \cap \mathcal{N}$, $\mathcal{M}^\perp \cap \mathcal{N}^\perp$, and the rest. The subspaces $\mathcal{M}$ and $\mathcal{N}$ (or the corresponding projections p and q) are in *generic position* ("*position p*" in [9]) if all of the above four subspaces are trivial. In this case one can find a Hilbert space $\mathcal{K}$ and positive contractions s and c on $\mathcal{K}$ such that $s^2 + c^2 = 1$, $\mathrm{Ker}\,(s) = \mathrm{Ker}\,(c) = 0$, and p, q are unitarily equivalent to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

respectively (for instance see [12]). In general, the following projections p_0 and q_0 are in generic position: $p_0 = p - p \wedge q - p \wedge q^\perp$, $q_0 = q - p \wedge q - p^\perp \wedge q$. Hence, p and q look like

$$p = I_{(p \wedge q)\mathcal{H}} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus I_{(p \wedge q^\perp)\mathcal{H}} \oplus 0 \oplus 0,$$

$$q = I_{(p \wedge q)\mathcal{H}} \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus I_{(p^\perp \wedge q)\mathcal{H}} \oplus 0.$$

Notice that c can be also defined as the restriction of

$$\sqrt{pqp - p \wedge q} = 0 \oplus \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus 0 \oplus 0$$

to $p_0\mathcal{H} = \mathcal{M} \ominus \{(\mathcal{M} \cap \mathcal{N}) \oplus (\mathcal{M} \cap \mathcal{N}^\perp)\}$ (and $s = \sqrt{1 - c^2}$). By the functional calculus, there exists a unique positive operator Θ $(0 \leqslant \Theta \leqslant \frac{\pi}{2})$ on $p_0\mathcal{H}$ such that

$$c = \cos\Theta \quad \text{and} \quad s = \sin\Theta.$$

The spectrum $\mathrm{Sp}\,\Theta$ of Θ is contained in $[0, \frac{\pi}{2}]$, but 0 and $\frac{\pi}{2}$ are not eigenvalues.

DEFINITION 2.1. The above operator Θ is called the *angle operator* between p and q, and we define the set of angles $\mathrm{Ang}(p,q)$ between p and q by

$$\mathrm{Ang}(p,q)=\begin{cases}\mathrm{Sp}\,\Theta, & \text{if } pq\neq qp\\ \{\frac{\pi}{2}\}, & \text{if } pq=qp\end{cases}.$$

REMARK. H. Araki ([1], [2]) studied a slightly different angle operator. Connes ([5]) used the following operators defined by

$$s(p,q)=|p-q|,\ c(p,q)=|p\vee q-p-q|=s(p\vee q-p,q).$$

C. Davis ([8]) also introduced the *closeness operator* $pqp+p^{\perp}q^{\perp}p^{\perp}$ and the *separation operator* $pq^{\perp}p+p^{\perp}qp^{\perp}$ to investigate two subspaces.

We can describe our angles in terms of these operators (cf. Corollary 3.1):

LEMMA 2.1. *Let p and q be projections with $pq\neq qp$, then we have the following:*

(i) $\mathrm{Ang}(p,q)\setminus\{0,\frac{\pi}{2}\}=\mathrm{Sp}\{\arcsin s(p,q)\}\setminus\{0,\frac{\pi}{2}\}$,
(ii) $\mathrm{Ang}(p,q)\setminus\{0,\frac{\pi}{2}\}=\mathrm{Sp}\{\arccos c(p,q)\}\setminus\{0,\frac{\pi}{2}\}$,
(iii) $\mathrm{Ang}(p,q)\setminus\{\frac{\pi}{2}\}=\mathrm{Sp}\{\arccos\sqrt{pqp-p\wedge q}\}\setminus\{\frac{\pi}{2}\}$,
(iv) $\mathrm{Ang}(p,q)\setminus\{\frac{\pi}{2}\}=\mathrm{Sp}\{\arccos(c(p,q)-p\wedge q)\}\setminus\{\frac{\pi}{2}\}$.

Proof. These equalities easily follow from the explicit expressions of $s(p,q)$, $c(p,q)$, etc. For instance,

$$s(p,q)=0\oplus\begin{pmatrix}s&0\\0&s\end{pmatrix}\oplus I\oplus I\oplus 0.\quad\blacksquare$$

The following lemmas are easy to check and their proofs are left to the readers.

LEMMA 2.2. *Let $\mathcal{M}_i$ and $\mathcal{N}_i$ be subspaces of a Hilbert space $\mathcal{H}_i$ $(i=1,2)$. Then the set of angles between $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ and $\mathcal{N}=\mathcal{N}_1\oplus\mathcal{N}_2$ is as follows:*

(i) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1)=\{\frac{\pi}{2}\}=\mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N})=\left\{\frac{\pi}{2}\right\},$$

(ii) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1)=\{\frac{\pi}{2}\}\neq\mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N})=\mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2),$$

(iii) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1)\neq\{\frac{\pi}{2}\}=\mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N})=\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1),$$

(iv) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) \neq \{\frac{\pi}{2}\} \neq \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N}) = \mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) \cup \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2).$$

LEMMA 2.3. *Let $\mathcal{H}$ be a Hilbert space. Consider two distinct one-dimensional subspaces $\mathcal{M}$ and $\mathcal{N}$ spanned by non-zero elements e and f respectively. Then the set of angles between $\mathcal{M}$ and $\mathcal{N}$ satisfies*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N}) = \left\{\arccos \frac{|(e,f)|}{\|e\|\cdot\|f\|}\right\}.$$

At the end of this section, we give a formula of angles for a tensor product.

PROPOSITION 2.1. *Let $\mathcal{H}_i$ be a Hilbert space and $\mathcal{M}_i, \mathcal{N}_i$ be non-trivial subspaces of $\mathcal{H}_i$ with the orthogonal projection p_i, q_i respectively ($i = 1,2$). Suppose that neither $\mathcal{M}_1 \cap \mathcal{M}_2$ nor $\mathcal{N}_1 \cap \mathcal{N}_2$ is 0. Then the set of angles between $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2$ is as follows:*

(i) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) = \{\frac{\pi}{2}\} = \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N}) = \left\{\frac{\pi}{2}\right\},$$

(ii) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) = \{\frac{\pi}{2}\} \neq \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N}) = \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2),$$

(iii) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) \neq \{\frac{\pi}{2}\} = \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\mathrm{Ang}(\mathcal{M},\mathcal{N}) = \mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1),$$

(iv) *If* $\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) \neq \{\frac{\pi}{2}\} \neq \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2)$, *then*

$$\begin{aligned}\mathrm{Ang}(\mathcal{M},\mathcal{N}) = \arccos\{&\cos(\mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1)) \cdot \cos(\mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2))\} \\ &\cup \mathrm{Ang}(\mathcal{M}_1,\mathcal{N}_1) \cup \mathrm{Ang}(\mathcal{M}_2,\mathcal{N}_2).\end{aligned}$$

Proof. The orthogonal projections p_i and q_i are represented by

$$p_i = 1 \oplus p_i^0 \oplus 1 \oplus 0 \oplus 0, \quad q_i = 1 \oplus q_i^0 \oplus 0 \oplus 1 \oplus 0$$

on $\mathcal{H}_i = (\mathcal{M}_i \cap \mathcal{N}_i) \oplus \mathcal{L}_i \oplus (\mathcal{M}_i \cap \mathcal{N}_i^\perp) \oplus (\mathcal{M}_i^\perp \cap \mathcal{N}_i) \oplus (\mathcal{M}_i^\perp \cap \mathcal{N}_i^\perp)$ ($i = 1,2$). We consider the "generic" subspace $\mathcal{L}$ of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\begin{aligned}\mathcal{L} = &(\mathcal{M}_1 \cap \mathcal{N}_1) \otimes \mathcal{L}_2 \oplus (\mathcal{M}_1 \cap \mathcal{N}_1^\perp) \otimes \mathcal{L}_2 \oplus (\mathcal{M}_1^\perp \cap \mathcal{N}_1) \otimes \mathcal{L}_2 \\ &\oplus (\mathcal{M}_1^\perp \cap \mathcal{N}_1^\perp) \otimes \mathcal{L}_2 \oplus \mathcal{L}_1 \otimes \mathcal{L}_2 \oplus \mathcal{L}_1 \otimes (\mathcal{M}_2 \cap \mathcal{N}_2) \\ &\oplus \mathcal{L}_1 \otimes (\mathcal{M}_2 \cap \mathcal{N}_2^\perp) \oplus \mathcal{L}_1 \otimes (\mathcal{M}_2^\perp \cap \mathcal{N}_2) \oplus \mathcal{L}_1 \otimes (\mathcal{M}_2^\perp \cap \mathcal{N}_2^\perp).\end{aligned}$$

The restrictions of $p_1 \otimes p_2$ and $q_1 \otimes q_2$ to the subspace $\mathcal{L}$ have the following forms:

$$\begin{aligned} p_1 \otimes p_2|_{\mathcal{L}} &= \mathbf{1}_{(\mathcal{M}_1 \cap \mathcal{N}_1)} \otimes p_2^0 \oplus \mathbf{1}_{(\mathcal{M}_1 \cap \mathcal{N}_1^\perp)} \otimes p_2^0 \oplus p_1^0 \otimes p_2^0 \\ &\quad \oplus p_1^0 \otimes \mathbf{1}_{(\mathcal{M}_2 \cap \mathcal{N}_2)} \oplus p_1^0 \otimes \mathbf{1}_{(\mathcal{M}_2 \cap \mathcal{N}_2^\perp)}, \\ q_1 \otimes q_2|_{\mathcal{L}} &= \mathbf{1}_{(\mathcal{M}_1 \cap \mathcal{N}_1)} \otimes q_2^0 \oplus \mathbf{1}_{(\mathcal{M}_1^\perp \cap \mathcal{N}_1)} \otimes q_2^0 \oplus q_1^0 \otimes q_2^0 \\ &\quad \oplus q_1^0 \otimes \mathbf{1}_{(\mathcal{M}_2 \cap \mathcal{N}_2)} \oplus q_1^0 \otimes \mathbf{1}_{(\mathcal{M}_2^\perp \cap \mathcal{N}_2)}. \end{aligned}$$

The assumption of case (iv) implies that $\mathcal{L}_1 \neq 0 \neq \mathcal{L}_2$. Hence, we get that

$$\mathrm{Ang}\,(p_1 \otimes p_2, q_1 \otimes q_2) = \mathrm{Ang}\,(p_2^0, q_2^0) \cup \mathrm{Ang}\,(p_1^0, q_1^0) \cup \mathrm{Ang}\,(p_1^0 \otimes p_2^0, q_1^0 \otimes q_2^0).$$

The projections $p_1^0 \otimes p_2^0$ and $q_1^0 \otimes q_2^0$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos^2\Theta & \cos\Theta\sin\Theta \\ \cos\Theta\sin\Theta & \sin^2\Theta \end{pmatrix} \otimes \begin{pmatrix} \cos^2\tilde{\Theta} & \cos\tilde{\Theta}\sin\tilde{\Theta} \\ \cos\tilde{\Theta}\sin\tilde{\Theta} & \sin^2\tilde{\Theta} \end{pmatrix},$$

where Θ (resp. $\tilde{\Theta}$) is the angle operator between p_1 and q_1 (resp. p_2 and q_2). Since

$$\sqrt{(p_1^0 \otimes p_2^0)(q_1^0 \otimes q_2^0)(p_1^0 \otimes p_2^0)|_{(p_1^0 \otimes p_2^0)(\mathcal{L}_1 \otimes \mathcal{L}_2)}} = \cos\Theta \otimes \cos\tilde{\Theta},$$

we get $\mathrm{Sp}\left(\sqrt{(p_1^0 \otimes p_2^0)(q_1^0 \otimes q_2^0)(p_1^0 \otimes p_2^0)|_{(p_1^0 \otimes p_2^0)(\mathcal{L}_1 \otimes \mathcal{L}_2)}}\right) = \mathrm{Sp}\,(\cos\Theta) \cdot \mathrm{Sp}\,(\cos\tilde{\Theta})$. Therefore, we get the conclusion of (iv). Other statements can be proved more easily. ∎

3. ANGLES BETWEEN TWO SUBFACTORS

Let L be a finite von Neumann algebra (with a normalized trace tr) with von Neumann subalgebras M, N. The trace tr determines the normal faithful conditional expectations $E_M = E_M^L : L \to M$ and $E_N = E_N^L : L \to N$ ([34]). They extend to the orthogonal projections $e_M = e_M^L$ and $e_N = e_N^L$ on the GNS representation space $L^2(L, \mathrm{tr})$. Let $\eta : L \hookrightarrow L^2(L, \mathrm{tr})$ be the canonical injection. The von Neumann algebra $\langle L, e_M \rangle$ generated by L and e_M is called the *basic construction* ([14]).

DEFINITION 3.1. The set of angles $\mathrm{Ang}_L(M, N)$ between two subalgebras M and N of L is defined by

$$\mathrm{Ang}_L(M, N) = \mathrm{Ang}\,(e_M, e_N).$$

For many examples angles will be calculated, and sometimes a set of angles reduces to a singleton. (See Corollary 3.1.(ii), Lemma 5.3, etc.)

REMARK. Minimal and maximal angles seem to be important. In fact, we have $\cos(\min\{\mathrm{Ang}_L(M,N)\}) = \sup\{|\langle\xi,\eta\rangle|;\ \xi \in (M \ominus (M\cap N))_1, \eta \in (N \ominus (M \cap N))_1\}$ ([5], pp. 391), and $\cos \mathrm{Ang}_L(M,N)$ can be considered as the canonical partial correlation coefficients in statistics (cf. [29]).

REMARK. Several authors study the following inclusions:

$$\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$$

([11], [21], [22], [24], [25], [26], [37]). They form a commuting square if and only if $\mathrm{Ang}_L(M,N) = \{\frac{\pi}{2}\}$ and $K = M \cap N$.

REMARK. Although $\mathrm{Ang}_L(M,N)$ depends on the choice of a trace on L, we do not worry about this because we will mainly consider the factor case.

It is straightforward to check:

LEMMA 3.1. *Let L be a finite von Neumann algebra with a trace* tr *and M,N be von Neumann subalgebras of L. Then we have the following:*

(i) $\mathrm{Ang}_L(M,N) = \mathrm{Ang}_L(N,M)$,

(ii) *For a finite von Neumann algebra $\tilde{L}(\supseteq L)$ with a trace $\tilde{\mathrm{tr}}$ such that* $\tilde{\mathrm{tr}}|_L = \mathrm{tr}$,

$$\mathrm{Ang}_{\tilde{L}}(M,N) = \mathrm{Ang}_L(M,N).$$

In fact, (ii) follows from $e_M^{\tilde{L}} = e_M^L \oplus 0$, and $e_N^{\tilde{L}} = e_N^L \oplus 0$ on $L^2(\tilde{L},\tilde{\mathrm{tr}}) = L^2(L,\mathrm{tr}) \oplus L^2(L,\mathrm{tr})^{\perp}$.

This lemma justifies the notation $\mathrm{Ang}(M,N)$. By Proposition 2.1 we have

PROPOSITION 3.1. *Let L_i be a finite von Neumann algebra with a trace tr_i and M_i, N_i be von Neumann subalgebras $(i = 1,2)$. The set of angles between $M = M_1 \otimes M_2$ and $N = N_1 \otimes N_2$ is as follows:*

(i) *If* $\mathrm{Ang}(M_1,N_1) = \{\frac{\pi}{2}\} = \mathrm{Ang}(M_2,N_2)$, *then*

$$\mathrm{Ang}(M,N) = \left\{\frac{\pi}{2}\right\},$$

(ii) *If* $\mathrm{Ang}(M_1,N_1) = \{\frac{\pi}{2}\} \neq \mathrm{Ang}(M_2,N_2)$, *then*

$$\mathrm{Ang}(M,N) = \mathrm{Ang}(M_2,N_2),$$

(iii) *If* $\mathrm{Ang}(M_1,N_1) \neq \{\frac{\pi}{2}\} = \mathrm{Ang}(M_2,N_2)$, *then*

$$\mathrm{Ang}(M,N) = \mathrm{Ang}(M_1,N_1),$$

(iv) *If* $\mathrm{Ang}(M_1, N_1) \neq \{\frac{\pi}{2}\} \neq \mathrm{Ang}(M_2, N_2)$, *then*

$$\mathrm{Ang}(M, N) = \arccos\{\cos(\mathrm{Ang}(M_1, N_1)) \cdot \cos(\mathrm{Ang}(M_2, N_2))\} \cup \mathrm{Ang}(M_1, N_1) \cup \mathrm{Ang}(M_2, N_2).$$

The well-known description of the Jones projection for an inclusion of crossed product algebras shows:

LEMMA 3.2. *Let L be a finite von Neumann algebra with a trace* tr, *and M, N be von Neumann subalgebras of L. Suppose that a trace-preserving outer action α of a countable discrete group G leaves M and N invariant. Then we have*

$$\mathrm{Ang}_{L\rtimes_\alpha G}(M\rtimes_\alpha G, N\rtimes_\alpha G) = \mathrm{Ang}_L(M, N),$$

where angles in the left-side are relative to the trace on $L\rtimes_\alpha G$ canonically determined by tr.

In this paper we will study the case that L, M, and N are factors of type II_1. It is easy to image that angles vary continuously and take any values in $]0, \frac{\pi}{2}]$. (See Example 5.6). But if we make some restrictions, a different stage seems to appear. When we restrict our attention to the fundamental case, that is, $L = M \vee N$ and $M \cap N$ are factors, and moreover if we assume that $[L : M] = [L : N] = 2$, and the relative commutant $(M \cap N)' \cap L = \mathbf{C}$, we will show that the possible values of angles have gaps. (In detail, see Section 6). This is analogous to the fact that Jones index have gaps.

DEFINITION 3.2. Let L, M, N, and K be von Neumann algebras. (L, M, N, K) is called *quadrilateral* if L is generated by M, N and $K = M \cap N$. Two quadrilaterals (L_1, M_1, N_1, K_1) and (L_2, M_2, N_2, K_2) are *isomorphic* if there exists a $*$-isomorphism φ from L_1 onto L_2 such that $\varphi(M_1) = M_2$, $\varphi(N_1) = N_2$, and $\varphi(K_1) = K_2$.

LEMMA 3.3. *If (L, M, N, K) is a quadrilateral of finite von Neumann algebras (with a trace* tr*), then $(\langle L, e_K\rangle, \langle L, e_M\rangle, \langle L, e_N\rangle, L)$ is also a quadrilateral.*

Proof. It is well-known that $\langle L, e_K\rangle = J_L K' J_L$, $\langle L, e_M\rangle = J_L M' J_L$, and $\langle L, e_N\rangle = J_L N' J_L$ on $L^2(L, \mathrm{tr})$ ([14]). Since $K = M \cap N$ and L is generated by M and N, $L' = M' \cap N'$ and K' is generated by M' and N' thanks to the double commutant theorem. ∎

From now on, we will concentrate on the case that involved algebras are factors of type II_1. If $[L : K]$ is finite, then we can define opposite angles between M and N as "vertical angles" of angles between $\langle L, e_M\rangle$ and $\langle L, e_N\rangle$.

DEFINITION 3.3. Let (L, M, N, K) be a quadrilateral of factors of type II_1 satisfying $[L : K] < \infty$. The *set of opposite angles* $\text{Op-ang}_L(M, N)$ is defined as $\text{Ang}_{\langle L, e_K\rangle}(\langle L, e_M\rangle, \langle L, e_N\rangle)$.

DEFINITION 3.4. Let

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$$

be a quadruple of factors of type II_1 satisfying $[L : K] < \infty$. We call it a *co--commuting square* if their commutants (on $L^2(L)$)

$$\begin{array}{ccc} M' & \subset & K' \\ \cup & & \cup \\ L' & \subset & N' \end{array}$$

form a commuting square.

REMARK. The above definitions does not depend on representation spaces. (See Proposition 4.1). For a quadruple

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup, \\ K & \subset & N \end{array}$$

the co-commuting condition is equivalent to that $\text{Op-ang}_L(M, N) = \{\frac{\pi}{2}\}$ and $L = M \vee N$.

PROPOSITION 3.2. *Let (L, M, N, K) be a quadrilateral of factors of type II_1. If $[L : K]$ is finite, then $\text{Ang}_L(M, N)$ is a finite set. (Actually $\#\text{Ang}_L(M, N) \leqslant [L : K]^2$.)*

Proof. Since $e_M k\eta(x) = e_M\eta(kx) = \eta(E_M(kx)) = \eta(kE_M(x)) = ke_M\eta(x)$ $(k \in K,\ x \in L)$, $e_M \in K' \cap \langle L, e_K\rangle$. Similarly $e_N \in K' \cap \langle L, e_K\rangle$. Hence, the von Neumann algebra generated by e_M and e_N is contained in $K' \cap \langle L, e_K\rangle$ so that the result follows from that $\dim\{K' \cap \langle L, e_K\rangle\} \leqslant [\langle L, e_K\rangle : K] = [L : K]^2$ ([14], Corollary 2.2.3). ∎

REMARK. If (L, M, N, K) is not a commuting square, with a little more effort one obtains

$$\#\text{Ang}_L(M, N) \leqslant \frac{[L : K]^2 - 2}{4}.$$

REMARK. Proposition 3.2 shows that if $[L : K]$ is finite, angles between two subfactors behave like those for two subspaces of a finite-dimensional Hilbert space.

From the construction, we know that $\langle L, e_K\rangle$ contains $\langle L, e_M\rangle$ and $\langle L, e_N\rangle$. In terms of a Pimsner-Popa basis ([22], [35]), the projections e_M and e_N are represented explicitly in $\langle L, e_K\rangle$.

LEMMA 3.4. *Let $L \supset M \supset K$ be factors of type II_1 with $[L:K]$ finite. Let $\{u_1, u_2, \ldots, u_m\}$ be a Pimsner-Popa basis of M over K. Then we have*

$$e_M^L = \sum_{i=1}^{m} u_i e_K^L u_i^*.$$

Proof. Since $E_M^L(x) \in M$ for $x \in L$, the conclusion follows from that $E_M^L(x) = \sum_i u_i E_K^M(u_i^* E_M^L(x)) = \sum_i u_i E_K^M(E_M^L(u_i^* x)) = \sum_i u_i E_K^L(u_i^* x)$. ∎

The following subtle fact is due to C. F. Skau ([28]):

LEMMA 3.5. *For a quadrilateral (L, M, N, K) of factors of type II_1, we have $e_K = e_M \wedge e_N$.*

By Lemma 2.1 and Lemma 3.5, $\mathrm{Ang}_L(M, N)$ can be represented simply as follows:

COROLLARY 3.1. *Let (L, M, N, K) be a quadrilateral of factors of type II_1. Suppose that $[L:K]$ is finite.*

(i) *If the diagram*

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$$

is not a commuting square, then $\mathrm{Ang}_L(M, N)$ contains neither 0 nor $\frac{\pi}{2}$. And we have

$$\mathrm{Ang}_L(M, N) = \mathrm{Sp}\,(\arccos\sqrt{e_M e_N e_M - e_K}) \setminus \left\{\frac{\pi}{2}\right\} = \mathrm{Sp}\,(\arccos|e_M - e_N|) \setminus \left\{0, \frac{\pi}{2}\right\}.$$

(ii) *Let $s = e_M e_N e_M - e_K$. If $s = 0$, then $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$. If $s^2 = \alpha s \neq 0$ for some scalar α, then $\mathrm{Ang}_L(M, N)$ consists of one point and $\mathrm{Ang}_L(M, N) = \{\arccos\sqrt{\alpha}\}$.*

Proof. (i) Assume that 0 (or $\frac{\pi}{2}$) $\in \mathrm{Ang}_L(M, N)$. Since $\#\mathrm{Ang}_L(M, N)$ is finite, 0 (or $\frac{\pi}{2}$) is an eigenvalue of the angle operator. But this contradicts the property of the spectrum of the angle operator. These equalities follow from Lemma 2.1.(i), (iii), and Lemma 3.5.

(ii) The diagram is a commuting square (i.e., $e_M e_N = e_K$) if and only if $s = e_M e_N e_M - e_K = 0$, because $(e_M e_N - e_K)(e_M e_N - e_K)^* = s$. Suppose that the diagram is not a commuting square and that $s^2 = \alpha s \neq 0$ for some scalar α. Thanks to Lemma 3.5, s has the following expression:

$$s = e_M e_N e_M - e_K = 0 \oplus \begin{pmatrix} \cos^2\Theta & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \oplus 0 \oplus 0.$$

Hence, the spectrum $\mathrm{Sp}(s) = \{0\} \cup \mathrm{Sp}(\cos^2\Theta)$. Because $\alpha \neq 0$ and 0 is not an eigenvalue of $\cos\Theta$, $\{\alpha\} = \mathrm{Sp}(\cos^2\Theta)$. Therefore,

$$\mathrm{Ang}_L(M,N) = \mathrm{Sp}\,\Theta = \{\arccos\sqrt{\alpha}\}. \quad \blacksquare$$

Besides angles, there are other invariants (for subspaces) introduced by C. Davis ([8], Theorem 5.1). In our setting they are $\mathrm{tr}(e_K)$, $\mathrm{tr}(e_M)$, $\mathrm{tr}(e_N)$, $\mathrm{tr}(e_M \wedge e_N^\perp)$, $\mathrm{tr}(e_M^\perp \wedge e_N)$, $\mathrm{tr}(e_M^\perp \wedge e_N^\perp)$, and the values of traces of the spectral projections of the angle operator. (These are invariants for a quadrilateral (L, M, N, K) of type II_1 factors.)

LEMMA 3.6. *Let (L, M, N, K) be a quadrilateral of factors of type II_1 satisfying $[L:K] < \infty$. By setting $r = e_M - e_M \wedge e_N - e_M \wedge e_N^\perp$, we have:*

(i) $[L:M]^{-1} = [L:K]^{-1} + \mathrm{tr}(r) + \mathrm{tr}(e_M \wedge e_N^\perp)$.

(ii) $[L:N]^{-1} = [L:K]^{-1} + \mathrm{tr}(r) + \mathrm{tr}(e_M^\perp \wedge e_N)$.

(iii) *If $\mathrm{Ang}_L(M,N)$ consists of one point $\theta \neq \frac{\pi}{2}$, then*

$$\mathrm{Ang}_L(M,N) = \theta = \left\{\arccos\left(\frac{\mathrm{tr}(e_M e_N) - \mathrm{tr}(e_K)}{\mathrm{tr}(e_M) - \mathrm{tr}(e_K) - \mathrm{tr}(e_M \wedge e_N^\perp)}\right)^{\frac{1}{2}}\right\}.$$

(iv) $[L:M] = [L:N]$ *if and only if* $\mathrm{tr}(e_M \wedge e_N^\perp) = \mathrm{tr}(e_M^\perp \wedge e_N)$.

Proof. (i) is clear from Lemma 3.5.

(ii) follows from the fact that r is equivalent to $e_N - e_M \wedge e_N - e_M^\perp \wedge e_N$ in the Murray-von Neumann sense.

By Corollary 3.1.(ii), $\mathrm{tr}(e_M e_N) - \mathrm{tr}(e_K) = \cos^2\theta \cdot \mathrm{tr}(r)$ so that we obtain (iii).

(iv) is an immediate consequence of (i) and (ii). $\blacksquare$

4. DUALITY FOR ANGLES

In this section, we give a kind of duality for angles between a quadrilateral (L, M, N, K) and that of the second basic construction. Simultaneously, we also show that the set of angles $\mathrm{Ang}_{K'}(M', N')$ between commutants does not depend on the Hilbert spaces on which L acts. More precisely,

PROPOSITION 4.1. *Let (L_1, M_1, N_1, K_1) and (L_2, M_2, N_2, K_2) be two quadrilaterals of factors of type II_1 on $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfying that $[L_1 : K_1]$ and $[L_2 : K_2]$ are finite. Assume that K_1' and K_2' are finite. If two quadrilaterals are isomorphic, then we have*

$$\mathrm{Ang}_{K_1'}(M_1', N_1') = \mathrm{Ang}_{K_2'}(M_2', N_2').$$

Proof. Since angles are invariant under a spatial isomorphism and an amplification thanks to Proposition 3.1, the result follows from the next lemma. $\blacksquare$

LEMMA 4.1. *If (L, M, N, K) is a quadrilateral of factors of type II_1 with $[L : K]$ finite and e is a non-zero projection in K, then*

$$\mathrm{Ang}_L(M, N) = \mathrm{Ang}_{L_e}(M_e, N_e).$$

Proof. Since there exist an integer $n \in \mathbf{N}$ and a projection $q \in K_e \otimes B(\mathbf{C}^n)$ such that

$$L \cong (L_e \otimes B(\mathbf{C}^n))_q,$$

it suffices to show

$$\mathrm{Ang}_L(M, N) \supseteq \mathrm{Ang}_{L_e}(M_e, N_e) \quad (e \in K).$$

If the quadrilateral (L, M, N, K) is a commuting square, the conclusion is obvious. In another case, by Corollary 3.1, the inclusion is equivalent to

$$\mathrm{Sp}\,(e^{L_e}_{M_e} e^{L_e}_{N_e} e^{L_e}_{M_e} - e^{L_e}_{K_e}) \subseteq \mathrm{Sp}\,(e^L_M e^L_N e^L_M - e^L_K).$$

This follows from

$$e^{L_e}_{M_e} e^{L_e}_{N_e} e^{L_e}_{M_e} - e^{L_e}_{K_e} = e(e^L_M e^L_N e^L_M - e^L_K)e|_{L^2(L_e)}. \quad \blacksquare$$

Let (L, M, N, K) be a quadrilateral of factors of type II_1 with $[L : K]$ finite. We have the basic constructions $L_1 = \langle L, e^L_K\rangle$, $M_1 = \langle L, e^L_M\rangle$, $N_1 = \langle L, e^L_N\rangle$. Put $K_1 = L$. Iterating the basic constructions, we also get $L_2 = \langle L_1, e^{L_1}_L\rangle$, $M_2 = \langle L_1, e^{L_1}_{M_1}\rangle$, $N_2 = \langle L_1, e^{L_1}_{N_1}\rangle$ and $K_2 = L_1$. We find a kind of duality of angles for two quadrilaterals (L, M, N, K) and (L_2, M_2, N_2, K_2).

PROPOSITION 4.2. *In the above situation, we have*

$$\mathrm{Ang}_{L_2}(M_2, N_2) = \mathrm{Ang}_L(M, N).$$

Proof. Let $J = J_L$ and $J_1 = J_{L_1}$ be the canonical involutions on $L^2(L)$ and $L^2(L_1)$. Two quadrilaterals $(JL_1J, JM_1J, JN_1J, JK_1J)$ and $(J_1L_1J_1, J_1M_1J_1, J_1N_1J_1, J_1K_1J_1)$ are isomorphic, and their commutants are (L, M, N, K) and (L_2, M_2, N_2, K_2) so that we get the conclusion by Proposition 4.1. $\blacksquare$

COROLLARY 4.1. *Let (L, M, N, K) be a quadrilateral of factors of type II_1 with $[L : K]$ finite:*

(i) $\mathrm{Ang}_{K'}(M', N') = \text{Op-ang}_L(M, N)$,
(ii) $\mathrm{Ang}_L(M, N) = \text{Op-ang}_{K'}(M', N')$.

Proof. Since $\text{Op-ang}_L(M, N) = \mathrm{Ang}_{L_1}(M_1, N_1) = \mathrm{Ang}_{JL_1J}(JM_1J, JN_1J)$, where J is the canonical involution on $L^2(L)$, and the commutant of (L, M, N, K) on $L^2(L)$ is $(JL_1J, JM_1J, JN_1J, JLJ)$. Hence, (i) follows from Proposition 4.1. Clearly (i) implies (ii). $\blacksquare$

5. EXAMPLES

Let P be a factor of type II_1 with an outer action α of a finite group G with subgroups A, B. In this case two quadruplets $(P \rtimes G, P \rtimes A, P \rtimes B, P)$ and (P, P^A, P^B, P^G) are naturally considered. While the former has trivial angles, the latter gives nontrivial angles. In fact, there appear several angles. In order to explain this, we prepare several lemmas.

Let H be a subgroup of G. Set $L = P \rtimes G$ and $M = P \rtimes H$. The action ν of G on $\ell^\infty(G/H)$ is defined by

$$\nu_g(f)(sH) = f(g^{-1}sH) \quad \text{for } f \in \ell^\infty(G/H),\ s \in G.$$

Let $\chi_{sH} \in \ell^\infty(G/H)$ be the characteristic function on sH. Consider the action μ of G on $P \otimes \ell^\infty(G/H)$ defined by

$$\mu_g(d \otimes f) = \alpha_g(d) \otimes \nu_g(f) \quad \text{for } g \in G,\ d \in P,\ f \in \ell^\infty(G/H).$$

LEMMA 5.1. *There exists a $*$-isomorphism*

$$\varphi : \langle L, e_M^L \rangle \to (P \otimes \ell^\infty(G/H)) \rtimes_\mu G$$

such that $\varphi((x\lambda_s)e_M^L(y\lambda_t)) = (x\alpha_s(y) \otimes \chi_{sH})\lambda_{st}$ for $x, y \in P$ and $s, t \in G$, where the λ_g's are the implementing unitaries of the crossed products.

Proof. Thanks to the fact that $\langle L, e_M^L \rangle \cong L \otimes_M L$ by V. Jones ([15]), φ is well defined. It is easy to see that φ is a $*$-homomorphism. Let us consider the following linear mapping:

$$\psi : (P \otimes \ell^\infty(G/H)) \rtimes_\mu G \to \langle L, e_M^L \rangle$$

defined by

$$\psi((x \otimes \chi_{sH})\lambda_g) = (x\lambda_s)(e_M^L(\lambda_{s^{-1}g}) \quad (x \in P).$$

Then we can easily check that ψ is precisely the inverse of φ. Hence φ turns out to be an isomorphism between $\langle L, e_M^L \rangle$ and $(P \otimes \ell^\infty(G/H)) \rtimes_m uG$. ∎

Let A and C be subgroups of G with $A \supseteq C$. We set $L = P \rtimes G$, $M = P \rtimes A$, and $K = P \rtimes C$. By Lemma 5.1, there exist $*$-isomorphisms

$$\varphi_1 : \langle L, e_M^L \rangle \to (P \otimes \ell^\infty(G/A)) \rtimes G,$$

and

$$\varphi_2 : \langle L, e_K^L \rangle \to (P \otimes \ell^\infty(G/C)) \rtimes G.$$

We also have natural inclusions:

$$I : \langle L, e_M^L \rangle \to \langle L, e_K^L \rangle,$$

and

$$J : (P \otimes \ell^\infty(G/A)) \rtimes G \to (P \otimes \ell^\infty(G/C)) \rtimes G.$$

In fact, let $\{u_1, \ldots, u_m\}$ be a Pimsner-Popa basis of M over K. Then by Lemma 3.4, I is given by

$$I(xe_M^L y) = \sum_i x u_i e_K^L u_i^* y \quad \text{for } x, y \in L.$$

Also, J is induced by the natural quotient map $G/C \to G/A$. More precisely, let $A = \bigcup_{j=1}^{m} t_j C$ be the coset decomposition with $t_1 = e$, then $\{\lambda_{t_j}; j = 1, \ldots, m\}$ is a Pimsner-Popa basis of M over K. Hence,

$$I(ze_M^L w) = \sum_j z \lambda_{t_j} e_K^L \lambda_{t_j}^* w \quad \text{for } z, w \in L.$$

J is also written explicitely as

$$J((p \otimes \chi_{sA})\lambda_g) = \sum_j (p \otimes \chi_{st_jC})\lambda_g \quad \text{for } p \in P, \ s, g \in G.$$

We then have the following relation among these maps whose proof is left to the reader as an exercise:

LEMMA 5.2. *The following diagram*

$$\begin{array}{ccc} \langle L, e_K^L \rangle & \xrightarrow{\varphi_2} & (P \otimes \ell^\infty(G/C)) \rtimes G \\ \uparrow I & & \uparrow J \\ \langle L, e_M^L \rangle & \xrightarrow{\varphi_1} & (P \otimes \ell^\infty(G/A)) \rtimes G \end{array}$$

is commutative.

The following proposition gives a possibility of explicit computation of the angles.

PROPOSITION 5.1. *Let P be a factor of type II_1 with an outer action α of a finite group G generated by subgroups A and B. Set $L = P \rtimes_\alpha G$, $M = P \rtimes_\alpha A$, $N = P \rtimes_\alpha B$, and $K = P \rtimes_\alpha C$ ($C := A \cap B$). Then (L, M, N, K) is a quadrilateral and we have*

$$\mathrm{Ang}_L(M, N) = \left\{\frac{\pi}{2}\right\},$$

$$\text{Op-ang}_L(M, N) = \mathrm{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A), \ell^\infty(G/B)).$$

Proof. The only nontrivial assertion is the last one. By Lemmas 5.1, 5.2, 3.2, and Proposition 3.1, we have

$$\begin{aligned}\text{Op-ang}_L(M,N) &= \text{Ang}_{\langle L,e_K\rangle}(\langle L,e_M\rangle,\langle L,e_N\rangle)\\ &= \text{Ang}_{(P\otimes\ell^\infty(G/C))\rtimes G}((P\otimes\ell^\infty(G/A))\rtimes G,(P\otimes\ell^\infty(G/B))\rtimes G)\\ &= \text{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A),\ell^\infty(G/B)).\ \blacksquare\end{aligned}$$

By taking the commutants, we also have the following:

PROPOSITION 5.2. *Let P be a factor of type II_1 with an outer action α of a finite group G generated by subgroups A and B. Set $L = P^C$, $M = P^A$, $N = P^B$, and $K = P^G$, where $C = A\cap B$. Then (L,M,N,K) is a quadrilateral and we have*

$$\text{Ang}_L(M,N) = \text{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A),\ell^\infty(G/B)),$$

$$\text{Op-ang}_L(M,N) = \left\{\frac{\pi}{2}\right\}.$$

Proof. By [20], we have the isomorphism $(P^G)'\cong P'\rtimes G$, $(P^A)'\cong P'\rtimes A$, $(P^B)'\cong P'\rtimes B$, and $(P^C)'\cong P'\rtimes C$, where the commutants are taken on $L^2(P,\text{tr})$. Then the conclusions follows from Corollary 4.1 and Proposition 5.1. ∎

Now we compute $\text{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A),\ell^\infty(G/B))$. First we study a sufficient condition that $\text{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A),\ell^\infty(G/B))$ reduces to a singleton.

LEMMA 5.3. *Let G be a finite group and A,B be subgroups with $C = A\cap B = \{e\}$. Assume that $|A\setminus G/A| = 2$, $A\neq\{e\}\neq B$. Then $G = ABA$, $\text{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A),\ell^\infty(G/B))$ reduces to a singleton, and we have*

$$\text{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A),\ell^\infty(G/B)) = \left\{\arccos\left(\frac{|G|-|A||B|}{|B|(|G|-|A|)}\right)^{\frac12}\right\}.$$

Proof. Let $G = AeA\cup Ab_0A$ be the double coset decomposition for some $b_0\in B\setminus\{e\}$. Clearly $G = ABA$. Set $L = \ell^\infty(G)$, $M = \{f\in\ell^\infty(G); f(xa) = f(x), a\in A\}\cong\ell^\infty(G/A)$, $N = \{f\in\ell^\infty(G); f(xb) = f(x), b\in B\}\cong\ell^\infty(G/B)$, and $K = \mathbf{C}$. We identify $L^2(L,\text{tr})$ with $\mathbf{C}^{|G|}$. Then the projections e_M^L, e_N^L, and e_K^L are given by $|G|\times|G|$-matrices; $e_M^L = (e_M^L(g,h))_{g,h\in G}$, $e_N^L = (e_N^L(g,h))_{g,h\in G}$, $e_K^L = (e_K^L(g,h))_{g,h\in G}$ such that $e_K^L = \frac{1}{|G|}$, $e_M^L(g,h) = \frac{1}{|A|}\delta_{gA,hA} = \dfrac{|gA\cap hA|}{|A|^2}$, and $e_N^L(g,h) = \frac{1}{|B|}\delta_{gB,hB} = \dfrac{|gB\cap hB|}{|B|^2}$. Put $s = e_M^Le_N^Le_M^L - e_K^L$ and $\alpha = \dfrac{|G|-|A||B|}{|B|(|G|-|A|)}$. If $s = 0$, then $\text{Ang}_L(M,N) = \{\frac{\pi}{2}\}$ by Corollary 3.1. Then by Proposition 7.1, which will be shown later, $|G| = |A||B|$ so that $\alpha = 0$. Hence, the conclusion holds in this case. Therefore, we may assume that $s \neq 0$. By Corollary 3.1.(ii), it is sufficient to show that $s^2 = \alpha s$. Since $(e_M^L e_N^L)(x,y) = \dfrac{|xA \cap yB|}{|A||B|}$ $(x, y \in G)$, we have

$$
\begin{aligned}
s(x,y) &= (e_M^L e_N^L e_M^L)(x,y) - e_K^L(x,y) = \sum_{z\in G}(e_M^L e_N^L)(x,z)e_M^L(z,y) - e_K^L(x,y) \\
&= \sum_{z\in yA} \frac{|xA\cap zB|}{|A||B|}\frac{1}{|A|} - \frac{1}{|G|} = \sum_{a\in A}\frac{|xA\cap yaB|}{|A|^2|B|} - \frac{1}{|G|} \\
&= \sum_{a_1,a_2\in A}\frac{|\{y^{-1}x\}\cap a_1Ba_2|}{|A|^2|B|} - \frac{1}{|G|} = \sum_{a_1,a_2\in A, b\in B}\frac{|\{y^{-1}x\}\cap\{a_1ba_2\}}{|A|^2|B|} - \frac{1}{|G|}.
\end{aligned}
$$

To continue further we need to look at the map

$$\varphi : A\times B\times A \to G \quad \text{such that} \quad \varphi(a_1,b,a_2) = a_1ba_2.$$

Then for $x \in G$, $a_1, a_2 \in A$, $|\varphi^{-1}(a_1xa_2)| = |\varphi^{-1}(x)|$, and $|\varphi^{-1}(x^{-1})| = |\varphi^{-1}(x)|$. Since $\varphi^{-1}(e) = \{(a, e, a^{-1}) \in A\times B\times A; a \in A\}$,

$$|\varphi^{-1}(a)| = |\varphi^{-1}(e)| = |A| \qquad (a\in A).$$

If $x \in Ab_0A$, then

$$|\varphi^{-1}(x)| = |\varphi^{-1}(b_0)| = \frac{|A|^2|B| - |A|^2}{|G| - |A|} =: \beta.$$

Therefore, we have

$$s(x,y) = \frac{|\varphi^{-1}(y^{-1}x)|}{|A|^2|B|} - \frac{1}{|G|} = \frac{|\varphi^{-1}(x^{-1}y)|}{|A|^2|B|} - \frac{1}{|G|}.$$

Moreover, we get that

$$
\begin{aligned}
(s^2)(x,y) &= \sum_{z\in G} s(x,z)s(z,y) = \sum_{z\in G}\left(\frac{|\varphi^{-1}(x^{-1}z)|}{|A|^2|B|} - \frac{1}{|G|}\right)\left(\frac{\varphi^{-1}(z^{-1}y)|}{|A|^2|B|} - \frac{1}{|G|}\right) \\
&= \left(\sum_{z\in G}\frac{|\varphi^{-1}(x^{-1}z)|\,|\varphi^{-1}(z^{-1}y)|}{|A|^4|B|^2}\right) - \frac{2}{|A|^2|B|\,|G|}\left(\sum_{z\in G}|\varphi^{-1}(z)|\right) + \frac{1}{|G|} \\
&= \left(\sum_{z\in G}\frac{|\varphi^{-1}(x^{-1}z)|\,|\varphi^{-1}(z^{-1}y)|}{|A|^4|B|^2}\right) - \frac{2|A|^2|B|}{|A|^2|B|\,|G|} + \frac{1}{|G|} \\
&= \left(\sum_{z\in G}\frac{|\varphi^{-1}(x^{-1}z)|\,|\varphi^{-1}(z^{-1}y)|}{|A|^4|B|^2}\right) - \frac{1}{|G|}.
\end{aligned}
$$

Now we need to consider two cases.

Case 1: $x^{-1}y \in A$. We get that

$$s(x,y) = \frac{|\varphi^{-1}(y^{-1}x)|}{|A|^2|B|} - \frac{1}{|G|} = \frac{|A|}{|A|^2|B|} - \frac{1}{|G|} = \frac{|G| - |A|\,|B|}{|A|\,|B|\,|G|}.$$

We also have that

$$s^2(x,y) = \left(\sum_{h\in G} \frac{|\varphi^{-1}(x^{-1}yh)|\,|\varphi^{-1}(h^{-1})|}{|A|^4|B|^2}\right) - \frac{1}{|G|} = \left(\sum_{h\in G} \frac{|\varphi^{-1}(h)|^2}{|A|^4|B|^2}\right) - \frac{1}{|G|}$$
$$= \frac{|A|\,|A|^2 + (|G| - |A|)\cdot\beta^2}{|A|^4|B|^2} - \frac{1}{|G|} = \frac{(|G| - |A|\,|B|)^2}{|A|\,|B|^2(|G| - |A|)|G|}.$$

Hence, $s^2(x,y) = \alpha s(x,y)$.

Case 2: $x^{-1}y \notin A$. We get that

$$s(x,y) = \frac{|\varphi^{-1}(x^{-1}y)|}{|A|^2|B|} - \frac{1}{|G|} = \frac{\beta}{|A|^2|B|} - \frac{1}{|G|} = \frac{|G| - |A|\,|B|}{|G|\,|B|(|A| - |G|)}.$$

Put $u = x^{-1}y \notin A$, then we have that

$$s^2(x,y) = \left(\sum_{z\in G} \frac{|\varphi^{-1}(x^{-1}yz)|\,|\varphi^{-1}(z^{-1})|}{|A|^4|B|^2}\right) - \frac{1}{|G|}$$
$$= \left(\sum_{z\in AeA} \frac{|\varphi^{-1}(uz)|\,|\varphi^{-1}(z)|}{|A|^4|B|^2}\right) + \left(\sum_{z\in Ab_0A} \frac{|\varphi^{-1}(uz)|\,|\varphi^{-1}(z)|}{|A|^4|B|^2}\right) - \frac{1}{|G|}$$
$$= \frac{|A|\cdot\beta\cdot|A|}{|A|^4|B|^2} - \frac{1}{|G|} + \frac{|A|\cdot\beta\cdot|A| + (|G| - 2|A|)\cdot\beta^2}{|A|^4|B|^2}$$
$$= \frac{-(|G| - |A|\,|B|)^2}{|B|^2(|G| - |A|)^2|G|},$$

because $\{uz; z\in Ab_0A\} = A \cup G_0$ (disjoint union) for some $G_0 \subset Ab_0A$. Hence, we also have that $s^2(x,y) = \alpha s(x,y)$. This completes the proof. ∎

H. Kosaki showed us the following powerful method to compute angles. By Lemma 3.1.(ii), we have

$$\mathrm{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A), \ell^\infty(G/B)) = \mathrm{Ang}_{\ell^\infty(G)}(\ell^\infty(G/A), \ell^\infty(G/B)).$$

The orthogonal projection $e_{\ell^\infty(G/A)}$ is $\frac{1}{|A|}\sum_{a\in A}\rho(a)$, where $(\rho(g)f)(h) = f(hg)$ for $h\in G$, $f\in \ell^2(G)$. Since $\rho = \sum_{\pi\in\hat{G}}^{\oplus}(\dim\mathcal{H}_\pi)\pi$, we have

$$e_{\ell^\infty(G/A)} = \sum\nolimits^{\oplus}(\dim\mathcal{H}_\pi)\left(\frac{1}{|A|}\sum_{a\in A}\pi(a)\right),$$
$$e_{\ell^\infty(G/B)} = \sum\nolimits^{\oplus}(\dim\mathcal{H}_\pi)\left(\frac{1}{|B|}\sum_{b\in B}\pi(b)\right).$$

Let $\{\mathcal{H}_\pi\}_A = \{\xi \in \mathcal{H}_\pi; \pi(a)\xi = \xi, a \in A\}$, $\{\mathcal{H}_\pi\}_B = \{\xi \in \mathcal{H}_\pi; \pi(b)\xi = \xi, b \in B\}$ be the ranges of the orthogonal projections $\frac{1}{|A|}\sum_{a\in A}\pi(a)$ and $\frac{1}{|B|}\sum_{b\in B}\pi(b)$. Hence, in order to compute $\mathrm{Ang}_{\ell^\infty(G/C)}(\ell^\infty(G/A), \ell^\infty(G/B))$, it suffices to compute angles between two subspaces $\{\mathcal{H}_\pi\}_A$ and $\{\mathcal{H}_\pi\}_B$ in each irreducible representation space $\mathcal{H}_\pi$. Therefore, combining with Proposition 5.2, we have shown that:

PROPOSITION 5.3. *Under the same assumption as Proposition 5.2, we have*

$$\mathrm{Ang}_L(M,N) = \bigcup_{\pi\in G} \mathrm{Ang}\,(\{\mathcal{H}_\pi\}_A, \{\mathcal{H}_\pi\}_B) \setminus \left\{\frac{\pi}{2}\right\},$$

if there exists an irreducible representation π_0 *such that*

$$\mathrm{Ang}\,(\{\mathcal{H}_{\pi_0}\}_A, \{\mathcal{H}_{\pi_0}\}_B) \neq \left\{\frac{\pi}{2}\right\}.$$

Otherwise

$$\mathrm{Ang}_L(M,N) = \left\{\frac{\pi}{2}\right\}.$$

EXAMPLE 5.1. Let P be a factor of type II_1 with an outer action of an abelian finite group G generated by two subgroups A, B. Put $C = A \cap B$. The orthogonal projections $e_{\ell^\infty(G/A)}$ and $e_{\ell^\infty(G/B)}$ clearly commute. Hence, for the quadrilateral (P^C, P^A, P^B, P^G), we have

$$\mathrm{Ang}_{P^C}(P^A, P^B) = \text{Op-ang}_{P^C}(P^A, P^B) = \left\{\frac{\pi}{2}\right\}.$$

For example $G = Z_2 \oplus Z_2$, $A = Z_2 \oplus 0$ and $B = 0 \oplus Z_2$.

EXAMPLE 5.2. Let $G = S_3$ be the symmetric group on $\{1,2,3\}$ generated by two subgroups A, B defined as the symmetric groups on $\{1,2\}$ and $\{2,3\}$ respectively. Then $A \cap B = \{e\}$ and $|A \setminus G/A| = 2$. Let P be a factor of type II_1 with an outer action of G. Then for the quadrilateral (P, P^A, P^B, P^G), we have

$$\mathrm{Ang}_P(P^A, P^B) = \left\{\frac{\pi}{3}\right\}, \quad \text{Op-ang}_P(P^A, P^B) = \left\{\frac{\pi}{2}\right\}.$$

EXAMPLE 5.3. Let $G = S_4$ be the symmetric group on $\{1,2,3,4\}$ with two subgroups A, B defined as the symmetric groups on $\{1,2,3\}$ and $\{3,4\}$ respectively. Then $A \cap B = \{e\}$, G is generated by A and B, and $|A \backslash G/A| = 2$. Let P be a factor of type II_1 with an outer action of G. For the quadrilateral (P, P^A, P^B, P^G), we have

$$\mathrm{Ang}_P(P^A, P^B) = \left\{\arccos\sqrt{\frac{1}{3}}\right\}, \quad \text{Op-ang}_P(P^A, P^B) = \left\{\frac{\pi}{2}\right\}.$$

EXAMPLE 5.4. Let $G = D_n$ be the dihedral group of order $2n$ for $n \geqslant 3$ (cf. [31], [36]). Then D_n has the following presentation:

$$D_n = \langle x, y; x^n = y^2 = (xy)^2 = e\rangle.$$

Let H, A, and B be the subgroups of D_n generated by x, y, and $z = xy$ respectively. Then we have that $H \cong Z_n$, $A \cong B \cong Z_2$, and $D_n \cong H \rtimes A$. Let P be a factor of type II_1 with an outer action of G. For the quadrilateral (P, P^A, P^B, P^G), we have

$$\operatorname{Ang}_P(P^A, P^B) = \left\{\frac{k}{n}\pi; k = 1, \dots, \left[\frac{n-1}{2}\right]\right\}, \quad \text{Op-ang}_P(P^A, P^B) = \left\{\frac{\pi}{2}\right\},$$

where $[x]$ is the greatest integer not exceeding x.

Proof. The irreducible unitary representations of the dihedral group D_n are well-known: there exist $[\frac{n-1}{2}]$ irreducible unitary representations w_k of degree 2 such that for $1 \leqslant k \leqslant [\frac{n-1}{2}]$

$$w_k(x) = \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^{-k} \end{pmatrix}, \quad w_k(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\alpha = e^{\frac{2\pi i}{n}}).$$

And other irreducible representations are degree 1. For each w_k, $w_k(xy)$ and $w_k(y)$ have ${}^t(1, \alpha^{-k})$ and ${}^t(1, 1)$ as an eigenvector respectively. Now we can calculate angles in each representation space $\mathcal{H}_k$ by Lemma 2.3: we obtain that the angle in $\mathcal{H}_k$ is $\theta_k = \frac{k}{n}\pi$ for $1 \leqslant k \leqslant [\frac{n-1}{2}]$. Hence, we get the conclusion thanks to Proposition 5.3. ∎

We give an example of two subfactors with non-trivial angles and non-trivial opposite angles by making use of the formula of a tensor product for angles, that is, Proposition 3.1. The following example is suggested by V. Jones.

EXAMPLE 5.5. Let $G_1 = S_3$ be the symmetric group on $\{1, 2, 3\}$. Let A_1 and B_1 be the symmetric group on $\{1, 2\}$ and $\{2, 3\}$ respectively. Let K_1 be a factor of type II_1 with an outer action of G_1. Put $L_1 = K_1 \rtimes G_1$, $M_1 = K_1 \rtimes A_1$, $N_1 = K_1 \rtimes B_1$. Let $G_2 = S_4$ be the symmetric group on $\{1, 2, 3, 4\}$. Let A_2 and B_2 be the symmetric group on $\{1, 2, 3\}$ and $\{3, 4\}$ respectively. Let L_2 be a factor of type II_1 with an outer action of G_2. Put $M_2 = L_2^{A_2}$, $N_2 = L_2^{B_2}$, $K_2 = L_2^{G_2}$. For the quadrilateral $(L, M, N, K) = (L_1 \otimes L_2, M_1 \otimes M_2, N_1 \otimes N_2, K_1 \otimes K_2)$, we have

$$\operatorname{Ang}_L(M, N) = \left\{\arccos\sqrt{\frac{1}{3}}\right\}, \quad \text{Op-ang}_L(M, N) = \left\{\frac{\pi}{3}\right\}$$

thanks to Proposition 3.1, Examples 5.1, 5.2.

EXAMPLE 5.6. First we recall the CAR-algebra ([27]). Let $\mathcal{H}$ be a separable Hilbert space over $\mathbf{C}$ with inner product $\langle\cdot,\cdot\rangle$. Consider a linear mapping a from $\mathcal{H}$ to $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ such that the following anticommutation relations hold: for $f, g \in \mathcal{H}$

$$a(f)a(g) + a(g)a(f) = 0,$$

$$a(f)a(g)^* + a(g)^*a(f) = \langle f, g\rangle.$$

The CAR-algebra $\mathcal{A}(\mathcal{H})$ is a UHF algebra with the unique normalized trace τ. By the GNS construction $\{\pi_\tau, \mathcal{H}_\tau = L^2(\mathcal{A}(\mathcal{H})), \xi_\tau\}$, we obtain the von Neumann algebra generated by $\pi_\tau(\mathcal{A}(\mathcal{H}))$ and we denote this by $\mathcal{A}(\mathcal{H})''$.

Now we give an example via the CAR-algebra and subalgebra. We would like to express our thanks to H. Araki for his suggestion of this example.

First let us consider a two-dimensional subspace $\mathcal{H}$ with an orthonormal basis e_1 and e_2. For the two one-dimensional subspaces $\mathcal{H}_1, \mathcal{H}_2$ spanned by e_1 and $f_1 = (\cos\theta)e_1 + (\sin\theta)e_2$, $0 < \theta < \frac{\pi}{2}$, then the set of angles between $\mathcal{A}(\mathcal{H}_1)$ and $\mathcal{A}(\mathcal{H}_2)$ is given by

$$\mathrm{Ang}\,(\mathcal{A}(\mathcal{H}_1), \mathcal{A}(\mathcal{H}_2)) = \{\theta, \arccos(\cos^2\theta)\}.$$

Proof. Remark the following orthogonal decomposition:

$$\begin{aligned} L^2(\mathcal{A}(\mathcal{H})) = {} & [a(e_1), a(f_1)] \oplus [a(e_1)^*, a(f_1)^*] \oplus \\ & \oplus [a(e_1)a(e_1)^*, a(e_1)^*a(e_1), a(f_1)a(f_1)^*, a(f_1)^*a(f_1)] \oplus H_0, \end{aligned}$$

where $[x_i]$ denote a linear subspace spanned by x_i and H_0 is the remainder. By Lemma 2.2,

$$\begin{aligned} \mathrm{Ang}\,(\mathcal{A}(\mathcal{H}_1), \mathcal{A}(\mathcal{H}_2)) = {} & \mathrm{Ang}\,([a(e_1)], [a(f_1)]) \cup \mathrm{Ang}\,([a(e_1)^*], [a(f_1)^*]) \\ & \cup \mathrm{Ang}\,([a(e_1)a(e_1)^*, a(e_1)^*a(e_1)], [a(f_1)a(f_1)^*, a(f_1)^*a(f_1)]). \end{aligned}$$

We compute

$$\mathrm{Ang}\,([a(e_1)], [a(f_1)]) = \mathrm{Ang}\,([a(e_1)^*], [a(f_1)^*]) = \{\theta\}$$

by Lemma 2.3. Thanks to the anticommutation relation, we can express

$$\mathcal{K}_1 = [a(e_1)a(e_1)^*, a(e_1)^*a(e_1)] = [1, u = a(e_1)a(e_1)^* - a(e_1)^*a(e_1)],$$
$$\mathcal{K}_2 = [a(f_1)a(f_1)^*, a(f_1)^*a(f_1)] = [1, v = a(f_1)a(f_1)^* - a(f_1)^*a(f_1)].$$

Because $\langle 1, u\rangle = \langle 1, v\rangle = 0$, it follows from Lemma 2.3 that

$$\mathrm{Ang}(\mathcal{K}_1, \mathcal{K}_2) = \mathrm{Ang}([u], [v]) = \{\arccos(\cos^2\theta)\}. \quad \blacksquare$$

We also get the following result: Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}$. For a fixed $n \in \mathbf{N}$ and parameters θ_k $(0 < \theta_k < \frac{\pi}{2}$, $k = 1, \ldots, n)$, we consider two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ spanned by $\{e_{2k-1}, e_s; 1 \leqslant k \leqslant n, 2n+1 \leqslant s\}$ and $\{f_k = (\cos\theta_k)e_{2k-1} + (\sin\theta_k)e_{2k}, e_s; 1 \leqslant k \leqslant n, 2n+1 \leqslant s\}$ respectively. Then the angles between $\mathcal{A}(\mathcal{H}_1)''$ and $\mathcal{A}(\mathcal{H}_2)''$ is as follows:

$$\mathrm{Ang}(\mathcal{A}(\mathcal{H}_1)'', \mathcal{A}(\mathcal{H}_2)'') = \left\{\arccos\Big\{\prod_k (\cos\theta_k)^{\varepsilon(k)}, \varepsilon(k) = 0, 1, 2\Big\}\right\} \setminus \{0\}.$$

Proof. This result follows from the previous one for the two-dimensional case, the following general fact, and Proposition 2.1. $\blacksquare$

Generally speaking, consider a separable Hilbert space $\mathcal{H}$ with an orthogonal decomposition $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose that each subspaces $\mathcal{H}_i$ contains two subspaces $\mathcal{M}_i, \mathcal{N}_i$. Put $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$. Thanks to the product property of trace τ, that is, $\tau(xy) = \tau(x)\tau(y)$ for $x \in \mathcal{A}(\mathcal{H}_1)$ and $y \in \mathcal{A}(\mathcal{H}_2)$ ([27]), we can identify $L^2(\mathcal{A}(\mathcal{H}))$ with $L^2(\mathcal{A}(\mathcal{H}_1)) \otimes L^2(\mathcal{A}(\mathcal{H}_2))$ and simultaneously $L^2(\mathcal{A}(\mathcal{M}))$ (respectively $L^2(\mathcal{A}(\mathcal{N}))$) with $L^2(\mathcal{A}(\mathcal{M}_1)) \otimes L^2(\mathcal{A}(\mathcal{M}_2))$ (respectively $L^2(\mathcal{A}(\mathcal{N}_1)) \otimes L^2(\mathcal{A}(\mathcal{N}_2))$) as Hilbert spaces in the following way:

$$\sum_i x_i \otimes y_i \longmapsto \sum_i x_i y_i \quad (x_i \in \mathcal{A}(\mathcal{H}_1), y_i \in \mathcal{A}(\mathcal{H}_2)).$$

Therefore, we obtain a formula of angles for CAR-algebras:

PROPOSITION 5.4. *In the above set-up, we have*

$$\mathrm{Ang}(\mathcal{A}(\mathcal{M}), \mathcal{A}(\mathcal{N})) = \mathrm{Ang}(L^2(\mathcal{A}(\mathcal{M}_1)) \otimes L^2(\mathcal{A}(\mathcal{M}_2)), L^2(\mathcal{A}(\mathcal{N}_1)) \otimes L^2(\mathcal{A}(\mathcal{N}_2)).$$

6. GAPS OF ANGLES

We will discuss the distribution of the possible values of angles by examining quadrilaterals whose upper sides have length 2. The following theorem shows the possible values of angles in this case have gaps.

THEOREM 6.1. *Let (L, M, N, K) be a quadrilateral of type II_1 factors. Suppose that $[L : M] = [L : N] = 2$, $[L : K]$ is finite, and $L \cap K' = \mathbf{C}$. Then either*

$$\operatorname{Ang}_L(M, N) = \left\{\frac{k}{n}\pi; k = 1, 2, \ldots, \left[\frac{n-1}{2}\right]\right\}$$

for some integer $n \geqslant 3$, or

$$\operatorname{Ang}_L(M, N) = \left\{\frac{\pi}{2}\right\}.$$

In addition, we have

$$\operatorname{Op-ang}_L(M, N) = \left\{\frac{\pi}{2}\right\}.$$

It suffices to show the next dual version (Corollary 4.1).

THEOREM 6.2. *Let (L, M, N, K) be a quadrilateral of type II_1 factors. Suppose that $[M : K] = [N : K] = 2$, $[L : K]$ is finite, and $L \cap K' = \mathbf{C}$. Then either*

$$\operatorname{Op-ang}_L(M, N) = \left\{\frac{k}{n}\pi; k = 1, 2, \ldots, \left[\frac{n-1}{2}\right]\right\}$$

for some integer $n \geqslant 3$, or

$$\operatorname{Op-ang}_L(M, N) = \left\{\frac{\pi}{2}\right\}.$$

In addition, we have

$$\operatorname{Ang}_L(M, N) = \left\{\frac{\pi}{2}\right\}.$$

Proof. By [10], there exist outer automorphisms β and $\gamma \in \operatorname{Aut}(K)$ such that $\beta^2 = \gamma^2 = \mathrm{id}_K$, $M \cong K \rtimes_\beta Z_2$ and $N \cong K \rtimes_\gamma Z_2$. Consider the quotient map $\varepsilon : \operatorname{Aut}(K) \to \operatorname{Out}(K)$. Let G be the subgroup of $\operatorname{Out}(K)$ generated by $\varepsilon(\beta)$ and $\varepsilon(\gamma)$. Let β and γ be implemented by $u \in M$ and $v \in N$; $\beta = \operatorname{ad} u$, $\gamma = \operatorname{ad} v$. For each element $g \in G$, we can choose a unitary operator $w_g \in L$ and an automorphism $\theta_g \in \operatorname{Aut}(K)$ such that $\theta_g = \operatorname{ad} w_g$, $\varepsilon(\theta_g) = g$, $w_e = e$, $\theta_e = \mathrm{id}$, $w_{\varepsilon(\beta)} = u$, $w_{\varepsilon(\gamma)} = v$, and w_g is a word consisting of u and v. For any distinct elements $g, h \in G$, $\varepsilon(\theta_h^{-1}\theta_g) = \varepsilon(\theta_h)^{-1}\varepsilon(\theta_g) = h^{-1}g \neq e$, hence, $\theta_h^{-1}\theta_g$ is outer. And the definition implies that $(\theta_h^{-1}\theta_g)(x)w_h^* w_g = w_h^* w_g x$ for $x \in K$ so that $(\theta_h^{-1}\theta_g)(x)E_K^L(w_h^* w_g) = E_K^L(w_h^* w_g)x$. Since $\theta_h^{-1}\theta_g$ is free ([3], [17]),

$E_K^L(w_h^* w_g) = 0$. Hence, $w_g L^2(K)$ and $w_h L^2(K)$ are mutually orthogonal for any distinct elements g and $h \in G$, because $\operatorname{tr}(y^* w_h^* w_g x) = \operatorname{tr}(y^* E_K^L(w_h^* w_g)x) = 0$ for $x, y \in K$. Therefore, the operator $p := \sum_{g \in G} w_g e_K^L w_g^*$ converges strongly in $\langle L, e_K^L \rangle$. Since $1 \geqslant \operatorname{tr}(p) = \sum_{g \in G} \operatorname{tr}(e_K^L) = \frac{|G|}{[L:K]}$, G is finite and $|G| \leqslant [L : K]$. Next we show that $w_g w_h \in K w_{gh}$ $(g, h \in G)$. In fact, since $\varepsilon(\theta_g \theta_h) = \varepsilon(\theta_g)\varepsilon(\theta_h) = gh = \varepsilon(\theta_{gh})$, there exists a unitary $w \in K$ such that $\theta_g \theta_h = (\operatorname{ad} w)\theta_{gh}$, which is equivalent to that $w_g w_h x w_h^* w_g^* = w w_{gh} x w_{gh}^* w^*$ $(x \in K)$. Hence, we get that $w_{gh}^* w^* w_g w_h \in K' \cap L \subseteq \mathbf{C}$. By ([13], Corollary 4.1.7 or [30]), we can find a unitary representation y of G on $L^2(L)$ such that $y \in K w_g$, that is, $y_g = h_g w_g$ for some unitary $h_g \in K$. Using this unitary representation, we introduce an outer action $\alpha : G \to \operatorname{Aut}(K)$ defined by $\alpha_g = \operatorname{ad} y_g$. In fact since $\alpha_g = \operatorname{ad} h_g \cdot \theta_g$, $\varepsilon(\alpha_g) = \varepsilon(\theta_g) = g$, hence, α is outer. Now we define the map φ which gives a $*$-homomorphism from $K \rtimes_\alpha G$ into L by

$$\varphi\Big(\sum_{g \in G} x_g \lambda_g\Big) = \sum_{g \in G} x_g y_g \quad (x_g \in K).$$

Since α is outer, φ is injective. Because L is generated by $M \cong K \rtimes_\beta Z_2$ and $N \cong K \rtimes_\gamma Z_2$, L is generated by K and $\{w_g\}_{g \in G}$. Since $y_g = h_g w_g$ $(h_g \in K)$, L is generated by K and $\{y_g\}_{g \in G}$. Hence, φ is surjective.

Let A and B be subgroups of G singly generated by $\varepsilon(\beta)$ and $\varepsilon(\gamma)$. Then φ induces a $*$-isomorphism between two quadrilaterals (L, M, N, K) and $(K \rtimes G, K \rtimes A, K \rtimes B, K)$. Since the finite group G is generated by two elements $x = \varepsilon(\beta)$ and $y = \varepsilon(\gamma)$ of order 2, G has the following presentation: $G = \langle x, y; x^2 = y^2 = (xy)^n = 1 \rangle$ for some integer n ([31], Theorem 6.8). Suppose that $n = 1$. Then $x = y$, so $M = N = K$. This contradicts that $[M : K] = [N : K] = 2$. Thus, the case $n = 1$ does not occur. Hence, the group G is isomorphic to the dihedral group D_n of order $2n$ for some integer $n \geqslant 3$, or $Z_2 \oplus Z_2$. These cases are already studied and angles and opposite angles are calculated in Examples 5.1, 5.4. Therefore, we conclude the proof. ∎

REMARK. Theorem 6.1 is a Goldman's type theorem for a quadrilateral whose upper sides are index 2. Such a quadrilateral is explicitly characterized by the dihedral group D_n. The following corollary is obtained from Theorem 6.1 and the complete classification of outer actions of finite groups on hyperfinite factors by V. Jones ([13]).

COROLLARY 6.1. *For each integer $n \geqslant 2$, there exists a unique quadrilateral (L, M, N, K) of hyperfinite II_1-factors such that $[L : M] = [L : N] = 2$, $[M : K] = [N : K] = n$, and $L \cap K' = C$.*

REMARK. The similar fact corresponding to Theorem 6.2 also holds.

7. OPPOSITE ANGLES, INDEX, AND WORD LENGTH

Let (L, M, N, K) be a quadrilateral of II_1-factors with $[L : K]$ finite. The equation

$$\mathrm{Ang}_L(M, N) = \text{Op-ang}_L(M, N)$$

may or may not hold. In this paper, by Example 5.5, we will only consider the special case that $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$. Under this assumption, we have $\text{Op-ang}_L(M, N) = \{\frac{\pi}{2}\}$, if and only if the quadrilateral (L, M, N, K) is a parallelogram, if and only if L is a linear span of words of length 2 consisting of elements in M and N. Denote $(\langle L, e_K^L\rangle, \langle L, e_M^L\rangle, \langle L, e_N^L\rangle, L)$ by (L_1, M_1, N_1, L).

LEMMA 7.1. *The conditional expectation* $E_{M_1}^{L_1} : L_1 = \langle L, e_K^L\rangle \to M_1 = \langle L, e_M^L\rangle$ *satisfies*

$$E_{M_1}^{L_1}(xe_K^L y) = \frac{[L : M]}{[L : K]} xe_M^L y \quad (x, y \in L).$$

Proof. It suffices to check that $E_{M_1}^{L_1}(e_K^L) = \frac{[L:M]}{[L:K]} e_M^L$, that is, $\mathrm{tr}\,(e_K^L xe_M^L y) = \frac{[L:M]}{[L:K]}\mathrm{tr}\,(e_M^L xe_M^L y)$ $(x, y \in L)$. In fact,

$$\begin{aligned}\frac{[L : M]}{[L : K]}\mathrm{tr}\,(e_M^L xe_M^L y) &= \frac{[L : M]}{[L : K]}\mathrm{tr}\,(E_M^L(x)e_M^L y) = \frac{[L : M]}{[L : K]}\frac{1}{[L : M]}\mathrm{tr}\,(E_M^L(x)y)\\ &= \frac{1}{[L : K]}\mathrm{tr}\,(E_M^L(x)E_M^L(y)),\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\,(e_K^L xe_M^L y) &= \mathrm{tr}\,(e_M^L e_K^L xe_M^L y) = \mathrm{tr}\,(e_K^L xe_M^L ye_M^L) = \mathrm{tr}\, e_K^L xE_M^L(y))\\ &= \frac{1}{[L : K]}\mathrm{tr}\,(xE_M^L(y)) = \frac{1}{[L : K]}\mathrm{tr}\,(E_M^L(x)E_M^L(y)). \quad \blacksquare\end{aligned}$$

LEMMA 7.2. *The following are equivalent:*

(i) $\text{Op-ang}_L(M, N) = \{\frac{\pi}{2}\}$.

(ii) $E_{M_1}^{L_1}(e_N^L) = \frac{1}{[L:N]}$.

(iii) $E_{N_1}^{L_1}(e_M^L) = \frac{1}{[L:M]}$.

(ii)′ $E_{M_1}^{L_1}(e_N^L) \in \mathbf{C}$.

(iii)′ $E_{N_1}^{L_1}(e_M^L) \in \mathbf{C}$.

Proof. From the assumption (i), $E^{L_1}_{M_1}E^{L_1}_{N_1} = E^{L_1}_{N_1}E^{L_1}_{M_1} = E^{L_1}_{L}$. (ii) follows from Lemma 7.1, since

$$E^{L_1}_{M_1}(e^L_N) = E^{L_1}_{M_1}\left(\frac{[L:K]}{[L:N]}E^{L_1}_{N_1}(e^L_K)\right) = \frac{[L:K]}{[L:N]}E^{L_1}_{L}(e^L_K)$$
$$= \frac{[L:K]}{[L:N]}\frac{1}{[L:K]} = \frac{1}{[L:N]}.$$

Conversely (ii) implies that

$$E^{L_1}_{M_1}E^{L_1}_{N_1}(xe^L_K y) = E^{L_1}_{M_1}(xE^{L_1}_{N_1}(e^L_K)y) = E^{L_1}_{M_1}\left(x\frac{[L:N]}{[L:K]}e^L_N y\right)$$
$$= \frac{[L:N]}{[L:K]}xE^{L_1}_{M_1}(e^L_N)y = \frac{[L:N]}{[L:K]}x\frac{1}{[L:N]}y = E^{L_1}_{L}(xe^L_K y)$$

so that we get (i). Because $\operatorname{tr}(E^{L_1}_{M_1}(e^L_N)) = \frac{1}{[L:N]}$, (ii) and (ii)′ are equivalent. Other equivalences are proved similarly. ∎

LEMMA 7.3. *The following are equivalent:*

(i) (L, M, N, K) *is a parallelogram, that is* $[L:M] = [N:K]$ *and* $[L:N] = [M:K]$.

(ii) $[L:M] = [N:K]$.

(iii) $[L:N] = [M:K]$.

Proof. They immediately follow from $[L:K] = [L:M][M:K] = [L:N][N:K]$. ∎

LEMMA 7.4. *If* $\operatorname{Ang}_L(M,N) = \{\frac{\pi}{2}\}$, (L, M, N, K) *is a parallelogram, and* $L = \overline{M\cdot N}^{\sigma\text{-strong}}$, *then* $\text{Op-ang}_L(M,N) = \{\frac{\pi}{2}\}$.

Proof. By Lemma 7.2, it suffices to verify that $E^{L_1}_{M_1}(e^L_N) = \frac{1}{[L:N]}$, that is,

$$\operatorname{tr}(e^L_N(xe^L_M y)) = \frac{1}{[L:N]}\operatorname{tr}(xe^L_M y) \quad (x, y \in L).$$

From the assumption, we may put $x = n_1m_1$, $y = m_2n_2$ for $n_1, n_2 \in N$, $m_1, m_2 \in M$. Then

$$\operatorname{tr}(e^L_N(xe^L_M y)) = \operatorname{tr}(e^L_N n_1m_1e^L_M m_2n_2) = \operatorname{tr}(n_2n_1e^L_N e^L_M m_1m_2) = \operatorname{tr}(n_2n_1e^L_K m_1m_2)$$
$$= \frac{1}{[L:K]}\operatorname{tr}(n_2n_1m_1m_2) = \frac{1}{[L:K]}\operatorname{tr}(xy),$$

and

$$\frac{1}{[L:N]}\operatorname{tr}(xe^L_M y) = \frac{1}{[L:N]}\frac{1}{[L:M]}\operatorname{tr}(xy) = \frac{1}{[L:K]}\operatorname{tr}(xy). \quad ∎$$

LEMMA 7.5. *Let (L, M, N, K) be a quadrilateral of II_1-factors with $[L:K]$ finite. Let $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ are Pimsner-Popa bases of M and N over K. Then the operator $a = \sum_{i,j} u_i v_j e_K^L v_j^* u_i^*$ have the following properties:*

(i) $\mathrm{tr}\,(a) = \frac{[M:K][N:K]}{[L:K]}$.

(ii) *If* $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$, *then a is a projection.*

Proof. (i)

$$\mathrm{tr}\,(a) = \mathrm{tr}\left(E_L^{L_1}\Big(\sum_{i,j} u_i v_j e_K^L v_j^* u_i^*\Big)\right) = \mathrm{tr}\Big(\sum_{i,j} u_i v_j E_L^{L_1}(e_K^L) v_j^* u_i^*\Big)$$
$$= \frac{1}{[L:K]}\mathrm{tr}\Big(\sum_{i,j} u_i v_j v_j^* u_i^*\Big) = \frac{[M:K][N:K]}{[L:K]}.$$

(ii) By Lemma 3.4, $e_N^L = \sum_j v_j e_K^L v_j^*$, hence, $a = \sum_i u_i e_N^L u_i^*$. a is clearly positive and idempotent. In fact,

$$a^2 = \sum_{i,j} u_j E_N^L(u_i^* u_j) e_N^L u_j^* = \sum_j \Big(\sum_i u_i E_K^M(u_i^* u_j)\Big) e_N^L u_j^* = \sum_j u_j e_N^L u_j^* = a,$$

by the commuting square condition. ∎

LEMMA 7.6. *If* $L = \overline{M \cdot N}^{\sigma\text{-strong}}$ *and* $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$, *then (L, M, N, K) is a parallelogram.*

Proof. We shall show that the operator a of the previous lemma is identical on $L^2(L)$. Since $L^2(L) = \overline{\eta(M \cdot N)}^{\|\ \|_2}$ by the assumption, it suffices to show that

$$a\eta(mn) = \eta(mn) \quad (m \in M,\ n \in N).$$

In fact, thanks to the commuting square condition,

$$a\eta(mn) = \sum_i u_i e_N^L u_i^* \eta(mn) = \eta\Big(\sum_i u_i E_N^L(u_i^* m) n\Big)$$
$$= \eta\Big(\sum_i u_i E_K^M(u_i^* m) n\Big) = \eta(mn).$$

Hence, $1 = \mathrm{tr}\,(a) = \frac{[M:K][N:K]}{[L:K]}$ by Lemma 7.5.(i). Combining this with the equation $[L:K] = [L:M][M:K]$, we get $[L:M] = [N:K]$. By Lemma 7.3, (L, M, N, K) is a parallelogram. ∎

LEMMA 7.7. *If* $\mathrm{Ang}_L(M,N) = \{\frac{\pi}{2}\}$, *then* $[L:M] \geqslant [N:K]$, $[L:N] \geqslant [M:K]$.

Proof. Since (L,M,N,K) is a commuting square, $E^L_M(x) = E^N_K(x)$ $(x \in N)$, hence, the restriction of the Pimsner-Popa inequality ([22], Theorem 2.2) implies

$$E^N_K(x) \geqslant \frac{1}{[L:M]}x, \quad x \in N_+.$$

Therefore, by ([22], Theorem 2.2), $[L:M] \geqslant [N:K]$. ∎

The following theorem shows that there exists a close relation among index, angles, and the degree of the non-commutativity of M and N measured by word length.

THEOREM 7.1. *For any quadrilateral* (L,M,N,K) *of* II_1*-factors with* $[L:K]$ *finite and* $\mathrm{Ang}_L(M,N) = \{\frac{\pi}{2}\}$, *the following are equivalent:*

(i) $\text{Op-ang}_L(M,N) = \{\frac{\pi}{2}\}$.

(ii) (L,M,N,K) *is a parallelogram (i.e.,* $[L:M] = [N:K]$*).*

(iii) $L = M \cdot N := \left\{ \sum_{\text{finite}} m_i n_i : m_i \in M,\ n_i \in N \right\}$.

(iii)$'$ $L = N \cdot M$.

(iv) $L = \overline{M \cdot N}^{\sigma\text{-strong}}$.

(iv)$'$ $L = \overline{N \cdot M}^{\sigma\text{-strong}}$.

Proof. (i)⇒(ii) Since $\mathrm{Ang}_L(M,N) = \{\frac{\pi}{2}\}$, $[L:M] \geqslant [N:K]$ by Lemma 7.7. Similarly $\text{Op-ang}_L(M,N) = \{\frac{\pi}{2}\}$ implies that $[N:K] = [L_1:N_1] \geqslant [M_1:L] = [L:M]$. Thus, $[L:M] = [N:K]$. Therefore, (ii) follows from Lemma 7.3.

(ii)⇒(iii) By Lemma 7.5.(i) and the assumption (ii), $\mathrm{tr}\,(a) = \frac{[M:K][N:K]}{[L:K]} = 1$. Hence, $1 = a = \sum_{i,j} u_i v_j e^L_K v_j^* u_i^*$, by Lemma 7.5.(ii). Therefore, $\eta(x) = a\eta(x) = \eta\Big(\sum_{i,j} u_i v_j E^L_K(v_j^* u_i^* x)\Big)$ $(x \in L)$, that is, $x = \sum_{i,j} u_i v_j E^L_K(v_j^* u_i^* x) \in M \cdot N$.

(iii)⇒(iv) is clear.

(iv)⇒(i) It follows from Lemma 7.6 that (L,M,N,K) is a parallelogram. Hence (i) follows from Lemma 7.4. ∎

REMARK. In Theorem 7.1, we only think of quadrilaterals, hence, we assume $L = M \vee N$ by definition. But, with a little care, we get that the above proof simultaneously gives the following, without the assumption $L = M \vee N$.

COROLLARY 7.1. *Let*

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$$

be a commuting square of factors of type II_1 satisfying $[L:K]<\infty$. Then the following are equivalent:

(i) *The quadruple*

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$$

is co-commuting (i.e., their commutants (on $L^2(L)$)

$$\begin{array}{ccc} M' & \subset & K' \\ \cup & & \cup \\ L' & \subset & N' \end{array}$$

form a commuting square).

(ii) *(L,M,N,K) is a parallelogram (i.e., $[L:M]=[N:K]$).*

(iii) $L = M\cdot N := \left\{ \sum_{\text{finite}} m_i n_i : m_i \in M,\ n_i \in N \right\}$.

(iii)′ $L = N\cdot M$.

(iv) $L = \overline{M\cdot N}^{\sigma\text{-strong}}$.

(iv)′ $L = \overline{N\cdot M}^{\sigma\text{-strong}}$.

REMARK. Theorem 7.1 is analogous to the following known fact in Group theory ([31], 3.13): Suppose that G, A and B are groups such that $G\supset A, B$ and $|G:A|$ is finite. Then the following are equivalent:

(i) $G = A\cdot B = \{a\cdot b : a\in A,\, b\in B\}$,

(ii) $|G:A| = |B : A\cap B|$.

The following proposition is a generalization of ([11], Proposition 4.2.8):

PROPOSITION 7.1. *Let P be a factor of type II_1 with an outer action α of a finite group G generated by subgroups A, B. Put $C = A\cap B$. Then for the quadrilateral $(L,M,N,K) = (P^C, P^A, P^B, P^G)$, the following are equivalent:*

(i) $\mathrm{Ang}_L(M,N) = \{\frac{\pi}{2}\}$,

(ii) *(L,M,N,K) is a parallelogram,*

(iii) $G = A\cdot B = \{ab : a\in A,\, b\in B\}$.

Proof. Applying Theorem 7.1 to the quadrilateral of commutants

$$(K', M', N', L') \cong (P'\rtimes G, P'\rtimes A, P'\rtimes B, P'\rtimes C)$$

on $L^2(P)$, (i) and (ii) are equivalent. And by the previous Remark, (iii) is equivalent to that (K', M', N', L') is a parallelogram, which is equivalent to the condition (ii). ∎

REMARK. There exists a finite group G such that G is not semi-direct product of A and B, but satisfies the condition (iii) $G = A \cdot B$ in Proposition 7.1. For instance, $G = S_4$, $A = S_3$ and $B = Z_4$ where G acts on $\{1,2,3,4\}$ as permutation and A on $\{1,2,3\}$ and B as shifts on $\{1,2,3,4\}$.

Under the same assumption of Proposition 7.1, we have the following:

COROLLARY 7.2. *If* $[G : A]$ *is relatively prime to* $[G : B]$, *then* (P^C, P^A, P^B, P^G) *is a commuting square.*

Proof. See ([31], 3.13). ∎

Theorem 7.1 treats the case that any element of L can be represented as a sum of the words of at most length 2. We shall consider the case that any element of L should be represented as a word of length one. But there exist no non-trivial cases as follows:

PROPOSITION 7.2. *Let* (L, M, N, K) *be a quadrilateral of* II_1*-factors with* $[L : K]$ *finite. The following are equivalent:*

(i) $L = \overline{M + N}^{\sigma\text{-strong}}$,

(ii) $L = M$ *or* $L = N$.

Proof. (ii) implies (i) clearly. Conversely, assume that the condition (i). Then $L^2(L) = \overline{L^2(M) + L^2(N)}^{\|\ \|_2}$, or $1 = e_M \vee e_N$. Since $e_M \vee e_N - e_M \sim e_N - e_M \vee e_N$, $1 = \mathrm{tr}(e_M \vee e_N) = \mathrm{tr}(e_M) + \mathrm{tr}(e_N) - \mathrm{tr}(e_M \wedge e_N)$. Thus by Lemma 3.5, $1 = \frac{1}{[L:M]} + \frac{1}{[L:N]} - \frac{1}{[L:K]}$. Assume that $L \neq M$ and $L \neq N$. In this case, $[L : M]$ and $[L : N] \geqslant 2$. Thus by the above equality, $1 \leqslant \frac{1}{2} + \frac{1}{2} - \frac{1}{[L:K]} < 1$. This leads to a contradiction. Hence $L = M$ or $L = N$. ∎

Theorem 7.1 provides us a non-trivial quadrilateral with $\mathrm{Ang}_L(M, N) \neq \{\frac{\pi}{2}\}$ such that the length of sides is not integers as follows:

EXAMPLE 7.1. Jones' two-sided sequence of projections has been studied by several authors ([6], [11], [21], [22] etc.). M. Choda [4] investigated index for subfactors generated by Jones' two-sided sequence of projections. Let $\{e_i : i = 0, \pm 1, \ldots\}$ be a family of projections with the properties

(a) $e_i e_{i\pm 1} e_i = \lambda e_i$, for some $\lambda \leqslant 1$,

(b) $e_i e_j = e_j e_i$, $|i - j| \geqslant 2$,

(c) the von Neumann algebra L generated by the family $\{e_i : i = 0, \pm 1, \ldots\}$ is a hyperfinite II_1-factor with the trace tr,

(d) $\mathrm{tr}(we_i) = \lambda \mathrm{tr}(w)$ if w is a word on 1 and e_j $(j \leqslant i - 1)$. Let M, N, K be von Neumann subalgebras generated by $\{e_i : i \in Z \setminus \{0\}\}$, $\{e_i : i \in Z \setminus \{1\}\}$, and $\{e_i : i \in Z \setminus \{0, 1\}\}$ respectively. She showed that if $\lambda = \frac{1}{4}\sec^2\frac{\pi}{m}$ for some integer

m ($\geqslant 3$), then the index $[L : M] = \frac{m}{4}\mathrm{cosec}^2\frac{\pi}{m}$. We will show that (L, M, N, K) is a quadrilateral and $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$. Let A and B be algebras generated by 1 and $\{e_i : i = -1, -2, -3, \ldots\}$, 1 and $\{e_i : i = 1, 2, 3, \ldots\}$ respectively. Then $A \cdot B = \left\{\sum_i a_i b_i : a_i \in A,\ b_i \in B\right\}$ is σ-weakly dense in M. Thus in order to show that $E_M^L(e_0) = \lambda$, it is enough to verify that $\mathrm{tr}\,(e_0 ab) = \lambda \mathrm{tr}\,(ab)$, $a \in A$, $b \in B$. By Markov trace property, ([37], 3.1), $\mathrm{tr}\,(e_0 ab) = \mathrm{tr}\,(e_0 a)\mathrm{tr}\,(b) = \lambda \mathrm{tr}\,(a)\mathrm{tr}\,(b) = \lambda \mathrm{tr}\,(ab)$. On the other hand, $\left\{\sum_i x_i e_0 y_i + z : x_i, y_i, z \in \mathrm{Alg}\{1, e_i : i \in Z \setminus \{0, 1\}\}\right\}$ is σ-weakly dense in N, where $\mathrm{Alg}\,X$ means an algebra generated by X. Since $E_M^L(x e_0 y) = x E_M^L(e_0) y = \lambda xy \in K$, for $x, y \in \mathrm{Alg}\{1, e_i : i \in Z \setminus \{0, 1\}\}$, we get $E_M^L(N) \subset K$. Hence, by ([11], Proposition 4.2.1), (L, M, N, K) is a commuting square, that is, $\mathrm{Ang}_L(M, N) = \{\frac{\pi}{2}\}$.

If $\lambda = \frac{1}{4}\sec^2\frac{\pi}{m}$, then $[L : M] = [L : N] = \frac{m}{4}\mathrm{cosec}^2\frac{\pi}{m}$ and $[M : K] = [N : K] = 4\cos^2\frac{\pi}{m}$. Since $\frac{m}{4}\mathrm{cosec}^2\frac{\pi}{m} \neq 4\cos^2\frac{\pi}{m}$ as $m \geqslant 5$, $\text{Op-ang}_L(M, N) \neq \{\frac{\pi}{2}\}$. Hence, the basic constructions $(\langle L, e_K^L\rangle, \langle L, e_M^L\rangle, \langle L, e_N^L\rangle, L) = (L_1, M_1, N_1, L)$ is a quadrilateral with $\mathrm{Ang}_{L_1}(M_1, N_1) \neq \{\frac{\pi}{2}\}$. This quadrilateral is the desired one. (For $m = 4$, the quadrilateral (L, M, N, K) is a parallelogram. Thus, by Theorem 7.1, $\mathrm{Ang}_{L_1}(M_1, N_1) = \{\frac{\pi}{2}\}$.)

Finally we shall consider the iteration of basic constructions of quadrilaterals. Let (L, M, N, K) be a quadrilateral of type II_1 factors with $[L : K]$ finite. Define the increasing sequence (L_n, M_n, N_n, K_n), $n = 0, 1, 2, \ldots$ of quadrilaterals by the relations

$$L_0 = L, \quad M_0 = M, \quad N_0 = N, \quad K_0 = K,$$
$$L_1 = \langle L, e_K^L\rangle, \quad M_1 = \langle L, e_M^L\rangle, \quad N_1 = \langle L, e_N^L\rangle \quad K_1 = L_0,$$
$$L_{n+1} = \langle L_n, e_{K_n}^{L_n}\rangle, \quad M_{n+1} = \langle L_n, e_{M_n}^{L_n}\rangle, \quad N_{n+1} = \langle L_n, e_{N_n}^{L_n}\rangle, \quad K_{n+1} = L_n.$$

Put $\mu = [L : M]^{-1}$, $\nu = [L : N]^{-1}$, $\tau = [L : K]^{-1}$, and

$$p_{n+1} = e_{M_n}^{L_n}, \quad q_{n+1} = e_{N_n}^{L_n}, \quad r_{n+1} = e_{K_n}^{L_n},$$

for $n = 0, 1, 2, \ldots$. Then it is clear that $\{r_n : n = 1, 2, \ldots\}$ satisfies the Jones relation that $r_i r_j = r_j r_i$ ($|i-j| \geqslant 2$), $r_i r_{i\pm1} r_i = \tau r_i$. We also get that if $|i-j| \geqslant 2$, then $p_i p_j = p_j p_i$, $q_i q_j = q_j q_i$, $p_i q_j = q_j p_i$. But the other commutation relations seems to be complicated. Hence, we shall consider only the special case that $\mathrm{Ang}_L(M, N) = \text{Op-ang}_L(M, N) = \{\frac{\pi}{2}\}$. The Jones relations appear again if we take cross terms:

PROPOSITION 7.3. *Suppose that* $\mathrm{Ang}_L(M,N) = \text{Op-ang}_L(M,N) = \{\frac{\pi}{2}\}$. *Let*

$$e_n = \begin{cases} p_n, & n \text{ is odd} \\ q_n, & n \text{ is even} \end{cases}, \qquad f_n = \begin{cases} q_n, & n \text{ is odd} \\ p_n, & n \text{ is even} \end{cases}.$$

Then $\{e_n : n = 1,2,\ldots\}$ *and* $\{f_n : n = 1,2,\ldots\}$ *satisfies the following relations:*

(i) $e_ie_j = e_je_i$ $(|i-j| \geqslant 2)$,

(ii) $e_ie_{i\pm 1}e_i = \mu e_i$,

(iii) $f_if_j = f_jf_i$ $(|i-j| \geqslant 2)$,

(iv) $f_if_{i\pm 1}f_i = \nu f_i$,

(v) $e_mf_n = f_ne_m$, *for any* $n, m = 1, 2, \ldots$.

Proof. We have

$$[L_n : M_n] = \begin{cases} [L : M], & n \text{ is even} \\ [L : N], & n \text{ is odd} \end{cases}, \quad [L_n : N_n] = \begin{cases} [L : N], & n \text{ is even} \\ [L : M], & n \text{ is odd} \end{cases}.$$

We note that $p_n \in M_n \subset L_n$ and $q_n \in N_n \subset L_n$. Thus p_n commutes with $\{p_k : k = 1,2,\ldots,n-1\}$ and $\{q_k : k = 1,2,\ldots,n-2\}$. Similarly, q_n commutes with $\{p_k : k = 1,2,\ldots,n-2\}$ and $\{q_k : k = 1,2,\ldots,n-1\}$. By the assumption that $\mathrm{Ang}_L(M,N) = \text{Op-ang}_L(M,N) = \{\frac{\pi}{2}\}$, $p_nq_n = q_np_n$ for $n = 1,2,\ldots$. Thus, (i), (iii) and (v) are proved. For (ii) and (iv), we get

$$\begin{aligned} q_{n+1}p_nq_{n+1} &= E^{L_n}_{N_n}(p_n)q_{n+1} = [L_{n-1} : M_{n-1}]^{-1}q_{n+1} \quad \text{(by Lemma 7.2)} \\ &= \begin{cases} \mu q_{n+1}, & n \text{ is odd} \\ \nu q_{n+1}, & n \text{ is even} \end{cases}. \end{aligned}$$

Similarly, we have

$$p_{n+1}q_np_{n+1} = \begin{cases} \nu p_{n+1}, & n \text{ is odd} \\ \mu p_{n+1}, & n \text{ is even} \end{cases}.$$

Next we shall show that $p_1q_2p_1 = \mu p_1$, that is, $e^L_M e^{L_1}_{N_1} e^L_M = \mu e^L_M$. Let $\{u_1, \ldots, u_m\}$ be a Pimsner-Popa basis of M over K. Then $e^L_M = \sum_i u_ie^L_Ku_i^* \in M_1 \subset L_1$ by Lemma 3.4. We may assume that these operators act on the Hilbert space

$L^2(L_1, \mathrm{tr})$. For $\eta(xe_K^L y) \in L^2(L_1, \mathrm{tr})$ $(x, y \in L)$, we have

$$\begin{aligned}
e_M^L e_{N_1}^{L_1} e_M^L \eta(xe_K^L y) &= e_M^L e_{N_1}^{L_1} \eta(e_M^L x e_K^L y) \\
&= e_M^L e_{N_1}^{L_1} \eta\left(\left(\sum_i u_i e_K^L u_i^*\right) x e_K^L y\right) \\
&= e_M^L e_{N_1}^{L_1} \eta\left(\sum_i u_i E_K^L(u_i^* x) e_K^L y\right) \\
&= e_M^L \eta\left(E_{N_1}^{L_1}\left(\sum_i u_i E_K^L(u_i^* x) e_K^L y\right)\right) \\
&= e_M^L \eta\left(\sum_i \frac{[L:N]}{[L:K]} u_i E_K^L(u_i^* x) e_N^L y\right) \\
&= \frac{[L:N]}{[L:K]} e_M^L \eta(E_M^L(x) e_N^L y) = \frac{[L:N]}{[L:K]} \eta(e_M^L E_M^L(x) e_N^L y) \\
&= \frac{[M:K]}{[L:K]} \eta(e_M^L E_M^L(x) e_N^L y) = \mu \eta(e_M^L x e_M^L e_N^L y) \\
&= \mu \eta(e_M^L x e_K^L y) = \mu e_M^L \eta(x e_K^L y),
\end{aligned}$$

thanks to Lemma 7.1 and 3.4, Theorem 7.1, and $\mathrm{Ang}_L(M, N) = \left\{\frac{\pi}{2}\right\}$. This shows that $p_1 q_2 p_1 = \mu p_1$. Similar arguments complete the proof of (ii) and (iv). ∎

REFERENCES

1. H. ARAKI, A lattice of von Neumann algebras associated with the quantum theory of a free Bose field, *J. Math. Phys.* **4**(1963), 1343–1362.
2. H. ARAKI, On quasifree states of CAR and Bogoliubov automorphisms, *Publ. Res. Inst. Math. Sci.*, Kyoto University **6**(1970/71), 385–442.
3. H. CHODA, On freely acting automorphisms of operator algebras, *Kodai Math. Sem. Rep.* **26**(1974), 1–21.
4. M. CHODA, Index for factors generated by Jones' two sided sequence of projections, *Pacific J. Math.* **139**(1989), 1–16.
5. A. CONNES, Outer conjugacy classes of automorphisms of factors, *Ann. Sci. École Norm. Sup. (4)* **8**(1975), 383–419.
6. A. CONNES; D. EVANS, Embeddings of $U(1)$-current algebras in non-commutative algebras of classical statistical mechanics, *Commun. Math. Phys.* **121**(1989), 507–525.
7. C. DAVIS, Generators of the ring of bounded operators, *Proc. Amer. Math. Soc.* **6**(1955), 970–972.

8. C. DAVIS, Separation of two linear subspaces, *Acta Sci. Math. (Széged)* **19**(1958), 172–187.

9. D. DIXMIER, Position relative de deux variétés linéaires fermées dans un espace de Hilbert, *Rev. Sci.* **86**(1948), 387–399.

10. M. GOLDMAN, On subfactors of factors of type II_1, *Michigan Math. J.* **7**(1960), 167–172.

11. F. GOODMAN, P. DE LA HARPE AND V. JONES, *Coxeter graphs and towers of algebras*, Math. Sci. Res. Inst. Publ. vol. 14, Springer Verlag 1989.

12. P. HALMOS, Two subspaces, *Trans. Amer. Math. Soc.* **144**(1969), 381–389.

13. V. JONES, *Actions of finite groups on the hyperfinite type II_1 factor*, Mem. Amer. Math. Soc. no. **237**(1980).

14. V. JONES, Index for subfactors, *Invent. Math.* **72**(1983), 1–25.

15. V. JONES, Index for subrings of rings, *Contemp. Math.* Amer. Math. Soc. **43**(1985), 181–190.

16. C. JORDAN, Essai sur la géométrie à n dimensions, *Bull. Soc. Math. France* **3**(1875), 103–174.

17. R. KALLMAN, A generalization of free action, *Duke Math. J.* **36**(1969), 781–789.

18. T. KATO, *Perturbation theory for linear operators*, Springer Verlag, Berlin 1966.

19. F. MURRAY AND J. VON NEUMANN, On rings of operators, *Ann. of Math.* **37**(1936), 116–229.

20. M. NAKAMURA AND Z. TAKEDA, A Galois theory for finite factors, *Proc. Japan Acad. Ser. A Math. Sci.* **36**(1960), 258–260.

21. A. OCNEANU, *Quantized groups, string algebras and Galois theory for algebras*, in Operator Algebras and Applications, vol. 2, London Math. Soc. Lecture Note Ser. vol. 136, (1988), pp. 119–172.

22. M. PIMSNER; S. POPA, Entropy and index for subfactors, *Ann. Sci. École Norm. Sup. (4)* **19**(1986), 57–106.

23. S. POPA, Maximal injective subalgebras in factors associated with free groups, *Adv. Math.* **50**(1983), 27–48.

24. S. POPA, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras, *J. Operator Theory* **9**(1983), 253–268.

25. S. POPA, Relative dimension, towers of projections and commuting squares of subfactors, *Pacific J. Math.* **137**(1989), 95–122.

26. S. POPA, Classification of subfactors: the finite depth case, *Invent. Math.* **101**(1990), 19–43.

27. R. POWERS, *Representation of canonical anticommutation relation*, Ph. D. Dissertation, Princeton 1968.

28. C. SKAU, Finite subalgebras of a von Neumann algebra, *J. Funct. Anal.* **25**(1977), 211–235.

29. V. SUNDER, N-subspaces, *Canad. J. Math.* **40**(1988), 38–54.

30. C. SUTHERLAND, Cohomology and extensions of operator algebras I; II, *Publ. Res. Inst. Math. Sci.*, Kyoto University **16**(1980), 105–133; 135–174.

31. M. SUZUKI, *Group theory I*, Springer Verlag 1982.

32. M. TAKESAKI, Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.* **131**(1973), 249–310.

33. M. TAKESAKI, *Theory of operator algebras I*, Springer Verlag 1979.

34. H. UMEGAKI, Conditional expectation in an operator algebra I, *Tôhoku Math. J. (2)* **6**(1954), 177–181.

35. Y. WATATANI, *Index for C^*-subalgebras*, Mem. Amer. Math. Soc., no. **424**(1990).

36. M. WEINSTEIN, *Examples of groups*, Polygonal Publishing House 1977.

37. H. WENZL, Hecke algebras of type A_n and subfactors, *Invent. Math.* **92**(1988), 349–383.

TAKASHI SANO
Department of Mathematics
Faculty of Science
Kyushu University
Fukuoka, 812
JAPAN

Current address

Department of Mathematics
Faculty of Science
Yamagata University
Yamagata, 990
JAPAN

YASUO WATATANI
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo, 060
JAPAN

Current address

Graduate School of Mathematics
Kyushu University
Ropponmatsu, Fukuoka 810
JAPAN

Received June 4, 1993; revised September 10, 1993.