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PURE SUB-JORDAN OPERATORS AND SIMULTANEOUS APPROXIMATION BY A POLYNOMIAL AND ITS DERIVATIVE

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ABSTRACT. An operator T on a Hilbert space H is said to be *Jordan* (of order 2) if $T = M + N$ where $M^*M = MM^*, MN = NM$ and $N^2 = 0$, and to be *sub-Jordan* if T has an extension to a Jordan operator J on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$. If the sub-Jordan operator T is such that its minimal Jordan extension J has spectrum in the unit circle $\mathbf{T}$ we say that T is $\mathbf{T}$-sub-Jordan. We present a functional model and solve an inverse spectral problem for this class of operators.

KEYWORDS: *Minimal Jordan extension, spectral measure, local resolvent, closed operator, annihilator.*

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1. INTRODUCTION

To state the main operator theory result of the paper, we first need a few definitions. A (bounded, linear) operator T on a Hilbert space H is said to be *Jordan* (of order 2) if $T = M + N$ where M is normal, M and N commute and N is nilpotent of order 2 ($M^*M = MM^*, MN = NM$ and $N^2 = 0$). If instead there is a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$, and a Jordan operator $J = M + N$ on $\mathcal{K}$ such that $\mathcal{H}$ is invariant for J and $J|\mathcal{H} = T$, we say that T is *sub-Jordan*. If T is sub-Jordan on $\mathcal{H}$ and there is no nonzero invariant subspace $\mathcal{M} \subset \mathcal{H}$ for T such that $T|\mathcal{M}$ is Jordan, we say that T is *pure sub-Jordan*. If T is a sub-Jordan operator such that the minimal Jordan extension has spectrum equal to a subset of the unit circle $\mathbf{T}$, we say that T is a $\mathbf{T}$-*sub-Jordan*. Finally, we say that a compact subset K of the unit circle $\mathbf{T}$ is regularly closed with respect to $\mathbf{T}$ if K is the closure of its $\mathbf{T}$-interior; intuitively, such a subset K is locally thick with respect to the unit circle. We can now state one of the main operator theory results in the paper.

THEOREM A. (See Theorem 4.1) *A compact subset K of the complex plane is the spectrum of a pure $\mathbf{T}$-sub-Jordan operator if and only if either K is the closed unit disk or K is a regularly closed (with respect to $\mathbf{T}$) subset of the unit circle $\mathbf{T}$.*

The proof of Theorem A relies on a functional model $P_2^2(\mu,\nu,\theta)$ for a sub-Jordan operator with cyclic vector. Here μ and ν are compactly supported measures in the plane, θ is a complex-valued function in $L^2(\nu)$, $P_2^2(\mu,\nu,\theta)$ is the closure of the manifold $\{p \oplus (\theta p + p') : p \text{ a polynomial}\}$ in $L^2(\mu) \oplus L^2(\nu)$ and the associated sub-Jordan operator $T(\mu,\nu,\theta)$ is the restriction of multiplication by the matrix function

$$\begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$$

to $P_2^2(\mu,\nu,\theta)$. This then is a canonical generalization of Bram's model for cyclic subnormal operators (see [19]).

Properties of μ,ν,θ for the case where $T(\mu,\nu,\theta)$ is pure are derived; for the case where μ and ν have support on the unit circle, it is possible to gather sufficiently more explicit information to arrive at Theorem A.

We also give an explicit characterization of the space $P_2^2(\mu,\nu,\theta)$ for a few computable special cases where μ and ν have support on $\mathbf{T}$; we note that such a characterization for the case where $\nu = 0$ is an easy consequence of a theorem of Szegö and Kolmogoroff-Krein (see [33], p. 49); here the main tool is a characterization of the annihilator of $P^2(\mu,\nu,\theta)$ in $L^2(\mu) \oplus L^2(\nu)$. We also conjecture a function algebraic characterization of the spectrum of a pure sub-Jordan operator analogous to that of Clancey-Putnam (see [17]) for the case of subnormal operators and verify the validity of the conjecture for a number of computable special situations. This analysis involves a review on what is known about simultaneous uniform approximation by a rational function and its derivative over some compact subset of the plane.

For the case of sub-Jordan operators with real spectrum (*real* sub-Jordan operators), more definitive corresponding operator theory and function theory results were obtained in [6] and [7]; indeed this paper is simply an adaptation (to the extent possible) of the techniques there to the complex case. Real sub-Jordan operators also have an intrinsic algebraic characterization (namely, the operator function $Q_T(s) = e^{-isT^*}e^{isT}$ is a second degree polynomial in the variable s)

and have a strong connection with conjugate point theory for Sturm-Liouville operators (see [2], [8], [29], [30], [31]). The circle analogue of this latter theory has been worked out by Agler (see [4]); the relevant class of operators turns out to be 2-isometries (operators T for which $I - 2T^*T + T^{*2}T^2 = 0$) rather than $\mathbf{T}$-sub-Jordan operators as one might initially expect by analogy with the real case. The extension theory for 2-isometries has been developed by Agler and McCullough (see [4], [43]). A Dirichlet space function model for 2-isometries was found by Richter (see [46]); a comparison of models suggests that a pure 2-isometry is similar to a pure $\mathbf{T}$-sub-Jordan.

To our knowledge an algebraic characterization of $\mathbf{T}$-sub-Jordan operators has not appeared in the literature; for the case of $\mathbf{T}$-Jordan operators, as well as the general class of Jordan operators of arbitrary order k, results appear in [15] and [23]. Also, we do not know of a connection between the class of $\mathbf{T}$-sub-Jordan or general sub-Jordan operators and another area of analysis such as Sturm-Liouville conjugate point theory; our motivation is to try to develop an extension of the rich interplay between operator and function theory that has been worked out in the theory of subnormal operators (see [19]).

We do not deal with the question of invariant subspaces for the class of sub-Jordan operators except for the following observations. If T is in fact subnormal, nontrivial invariant subspaces are known to exist as a consequence of S. Brown's theorem [14] (see also [19] and [42]); now the result of Thomson [48] on analytic bounded point evaluations gives an even stronger result. Subjordan operators are in particular subdecomposable; hence nontrivial invariant subspaces are known to exist if the spectrum of T has interior in the plane as a result of a theorem of Albrecht and Chevreau [5] (see also [42], Theorem IV.2.3), or if $R(\sigma(T)) \neq C(\sigma(T))$ (see [21]); here $C(K)$ is the algebra of continuous complex-valued functions on K and $R(K)$ is the uniform closure in $C(K)$ of rational functions with poles off K.

The paper is organized as follows. In Section 2 we give several equivalent formulations of the purity property for sub-Jordan operators. In Section 3 we set down our model for a sub-Jordan operator with cyclic vector, with special attention paid to the case of a pure sub-Jordan operator. In Section 4 we specialize the model to the case of pure $\mathbf{T}$-sub-Jordan operators and prove Theorem A. Section 5 formulates and analyzes a conjectured function algebraic characterization of the spectrum of a pure sub-Jordan operator analogous to the result of Clancey-Putnam ([17]) for the subnormal case. Finally, Section 6 gives more detailed information on the space $P_2^2(\mu, \nu, \theta)$ for the case where μ and ν have support in $\mathbf{T}$, and can be viewed as a start toward a Szegö-Kolmogoroff-Krein theorem with derivatives.

Finally, we mention that some of the results of the paper are in the second author's 1989 dissertation written under the direction of the first author.

2. A FORMULATION OF PURITY

The primary aim of this section is to define a notion of pure sub-Jordan operators which generalizes simultaneously the notions of purity for the case of subnormal operators (see [19]) and for real sub-Jordan operators given in [6]. To serve these dual purposes we actually give three notions which we later prove equivalent.

DEFINITION 2.1. Let T be a sub-Jordan operator on $\mathcal{H}$ with complex Jordan extension $J = M + N$ on $\mathcal{K}$. Then we say

(2.1.1) T is *pure* if there is no nonzero subspace $\mathcal{H}_0 \subset \mathcal{H}$ which is invariant for T such that $T|\mathcal{H}_0$ is Jordan.

(2.1.2) T is *pure* if there is no nonzero subspace $\mathcal{H}_0 \subset \mathcal{H}$ which is invariant for M, M^* and N.

(2.1.3) T is *pure* if there is no nonzero subspace $\mathcal{H}_0 \subset \mathcal{H}$ which is invariant for T such that $T|\mathcal{H}_0$ is normal.

We should note that in the real case (i.e. where $M = M^*$), all these notions coincide. To be exact, in (2.1.1), if T is real sub-Jordan, we have Definition 3.2 of [6], while Proposition 3.3 of [6] (replacing normal with self-adjoint) gives (2.1.3). Finally, again taking $M = M^*$, Proposition 3.6 of [6] is exactly (2.1.2).

Moreover these three notions generalize the idea of purity for subnormal operators. A pure subnormal operator is the restriction of a normal operator to an invariant subspace with the property that it has no nonzero reducing subspace on which it is normal (see [19], Definition 2.2, p. 127). Clearly, a subnormal operator A can be thought of as sub-Jordan; that is, the restriction of a Jordan operator with nilpotent part $N = 0$. If in addition A is a pure subnormal operator, then by (2.1.2) A can also be regarded as pure sub-Jordan. Conversely if A is a pure sub-Jordan operator in the sense of (2.1.3) with a normal extension, then A is a pure subnormal operator.

We now show the equivalence of (2.1.1), (2.1.2), (2.1.3), beginning with the first and third.

PROPOSITION 2.2. *Let T be a complex sub-Jordan operator on $\mathcal{K}$. Then T has a nonzero invariant subspace on which it is Jordan if and only if T has a nonzero invariant subspace on which it is normal.*

Proof. If $T\mathcal{H} \subset \mathcal{H}$ with $\mathcal{H}$ nonzero, $\mathcal{H} \neq \mathcal{K}$, and $T|\mathcal{H}$ is normal, then $T|\mathcal{H}$ is Jordan. Conversely, suppose T is a sub-Jordan operator on $\mathcal{K}$ and $\mathcal{H}$ is a nonzero invariant subspace of $\mathcal{K}$ so that $T|\mathcal{H} = M_0 + N_0$ is Jordan where M_0 is normal, M_0 commutes with N_0 and $N_0^2 = 0$. Without loss of generality, we assume $N_0 \neq 0$. Let $\mathcal{H}_0 = N_0\mathcal{H} \neq (0)$. Considering $h \in \mathcal{H}$, we see $M_0N_0h = N_0M_0h \in \mathcal{H}_0$. Furthermore, by the Fuglede-Putnam Theorem ([19], Theorem 5.4), $M_0^*N_0h = N_0M_0^*h \in \mathcal{H}_0$.

Thus $\mathcal{H}_0$ is reducing for M_0. Moreover

$$T|\mathcal{H}_0 = (M_0 + N_0)|\mathcal{H}_0 = M_0|\mathcal{H}_0.$$

Thus $\mathcal{H}_0$ is a nontrivial proper invariant subspace on which T is normal. ∎

In order to exhibit the equivalence of (2.1.1) and (2.1.2) we argue via spectral subspaces. Let $J = M + N$ be a Jordan operator on $\mathcal{K}$. We define a spectral measure $E_J(\cdot)$ for J to be simply the spectral measure $E_M(\cdot)$ for the normal operator M. The idea used here is a consequence of Proposition 3.7 of [6], that is, $\sigma(J) = \sigma(M)$. The following result gives that, with respect to $E_J(\cdot)$, J is a spectral operator in the sense of Dunford and Schwartz (see [20], pp. 1930–1931).

PROPOSITION 2.3. *For $J = M + N$, a Jordan operator on $\mathcal{K}$, let $E_J(\cdot)$ be the spectral measure for the normal operator M. Then for all Borel subsets δ of the complex plane $\mathbb{C}$*

$$E_J(\delta)J = JE_J(\delta), \tag{2.3.1}$$

and

$$\sigma(J|E_J(\delta)\mathcal{K}) \subset \bar{\delta}. \tag{2.3.2}$$

Furthermore,

(2.3.3) *for all $h, k \in \mathcal{K}$, the scalar valued measure $< E_J(\cdot)h, k >$ is countably additive.*

Proof. First of all (2.3.3) follows directly from the definition of a spectral measure for a normal operator (see [19], Section 2 and Section 3).

Since N commutes with M and M^* by the Fuglede-Putnam Theorem (see [19]), we have, by the spectral theorem for normal operators ([19], Theorem 3.1, p. 67) that both M and N commute with $E_J(\delta)$ for each Borel subset of the plane. Thus (2.3.1) follows.

Now let δ be a Borel subset of $\mathbb{C}$. Since $E_J(\cdot)$ is the spectral projection for M we know $\sigma(M|E_J(\delta)\mathcal{K}) \subset \bar{\delta}$. Furthermore, by the above, the subspace $E_J(\delta)\mathcal{K}$ is invariant for both M and N. Let $M_\delta = M|E_J(\delta)\mathcal{K}$ and $N_\delta = N|E_J(\delta)\mathcal{K}$. Then $M_\delta N_\delta = N_\delta M_\delta$ and $(N_\delta)^2 = 0$. So by Proposition 3.7 of [6]

$$\begin{aligned}\sigma(J|E_J(\delta)\mathcal{K}) &= \sigma((M+N)|E_J(\delta)\mathcal{K}) \\ &= \sigma(M_\delta + N_\delta) = \sigma(M_\delta) \\ &= \sigma(M|E_J(\delta)\mathcal{K}) \subset \bar{\delta},\end{aligned}$$

which proves (2.3.2). ∎

Our argument giving the equivalence of (2.1.1) and (2.1.2) relies on a procedure for recovering spectral subspaces $E_J(\delta)\mathcal{K}$ for Borel subsets of the plane directly from J which is well known in the theory of decomposable operators (see [20] and [18]). We define for $k \in \mathcal{K}$, the set

$$\begin{aligned}\rho_J(k) = \{\lambda_0 \in \mathbb{C} :\ &\text{there exists a function } \lambda \to k(\lambda) \in \mathcal{K}, \\ &\text{defined and analytic on a neighborhood of } \lambda_0, \text{ so that } (\lambda I - J)k(\lambda) \equiv k\}.\end{aligned}$$

By [20], Theorem 2, p. 1933, J has the single valued extension property. Thus $k(\lambda)$ is unique and is referred to as the local resolvent of k. Furthermore, by definition, $\rho_J(k)$ is open and contains the resolvent set $\rho(J)$ of J for all k. Let $\sigma_J(k) = \mathbb{C} \setminus \rho_J(k)$. Then $\sigma_J(k)$ is compact and nonempty if $k \neq 0$. Given a Borel set $\delta \subset \mathbb{C}$, a spectral maximal subspace for J is one of the form

$$\mathcal{K}_J(\delta) = \{k \in \mathcal{K} : \sigma_J(k) \subset \delta\}.$$

By [20], Theorem 4, p. 1934,

$$\mathcal{K}_J(\delta) = E_J(\delta)\mathcal{K}.$$

The proof of the following theorem, which gives that (2.1.1) and (2.1.2) are equivalent, follows from the two succeeding lemmas.

THEOREM 2.4. *Let $J = M + N$ be a Jordan operator on a Hilbert space $\mathcal{K}$. Suppose $\mathcal{H} \subset \mathcal{K}$ is invariant for J. Then $J_0 = J|H$ is itself Jordan if and only if $\mathcal{H}$ is invariant for M, M^*, and N. Furthermore, if $J_0 = M_0 + N_0$ where M_0 is normal, M_0 commutes with N_0 and $N_0^2 = 0$, then $M_0 = M|\mathcal{H}$ and $N_0 = N|H$.*

LEMMA 2.5. *With J and J_0 defined and Jordan on $\mathcal{K}$ and $\mathcal{H}$ respectively as in Theorem 2.5, for every Borel set $\delta \subset \mathbb{C}$, the containment*

$$E_{J_0}(\delta)\mathcal{H} \subset E_J(\delta)\mathcal{H}$$

is valid, where $E_{J_0}(\cdot)$ and $E_J(\cdot)$ are the spectral measures for J_0 and J respectively.

Proof. (Note that spectral measures are unique via [20], Corollary 9, p. 1935; thus the use of the definite article in the lemma is justified). By the previously cited result from [20], we need only show

$$\mathcal{K}_{J_0}(\delta) \subset \mathcal{K}_J(\delta).$$

Let $h \in \mathcal{H}$ and $\lambda_0 \in \mathbb{C} \setminus \delta$. Then there exists a vector-valued function $\lambda \to h(\lambda)$, defined and analytic on a neighborhood of λ_0, so that $(\lambda I - J_0)h(\lambda) \equiv h$. But since $\mathcal{H} \subset \mathcal{K}$ this says that there is s $\mathcal{K}$-vector-valued function $\lambda \to h(\lambda)$, defined and analytic on a neighorhood of λ_0, so that $(\lambda I - J)h(\lambda) \equiv h$. That is, for $h \in \mathcal{H}$, $\sigma_{J_0}(h) \subset \delta$ implies $\sigma_J(h) \subset \delta$. Hence $\mathcal{K}_{J_0}(\delta) \subset \mathcal{K}_J(\delta)$. ∎

LEMMA 2.6. *For all $h \in \mathcal{H}$ and for all Borel sets δ in the plane,*

$$E_{J_0}(\delta)h = E_J(\delta)h.$$

Proof. By the definition of the spectral projections for J and J_0, we have, for a Borel set $\delta \subset \mathbb{C}$,

$$\mathcal{H} = E_{J_0}(\delta)\mathcal{H} \oplus E_{J_0}(\mathbb{C} \setminus \delta)\mathcal{H}$$

and

$$\mathcal{K} = E_J(\delta)\mathcal{K} \oplus E_J(\mathbb{C} \setminus \delta)\mathcal{K}.$$

Also from Lemma 2.6, $E_{J_0}(\delta)\mathcal{H} \subset E_J(\delta)\mathcal{H}$ and $E_{J_0}(\mathbb{C} \setminus \delta)\mathcal{H} \subset E_J(\mathbb{C} \setminus \delta)\mathcal{H}$. So if $h \in \mathcal{H}$, we have equality of the two orthogonal decompositions

$$E_{J_0}(\delta)h + E_{J_0}(\mathbb{C} \setminus \delta)h = E_J(\delta)h + E_J(\mathbb{C} \setminus \delta)h$$

which implies $E_{J_0}(\delta)h = E_J(\delta)h$. ∎

Proof of Theorem 2.4. From Lemma 2.6, $\mathcal{H}$ is invariant under $E_J(\delta)$ for all Borel sets $\delta \subset \mathbf{C}$. Since $E_J(\cdot) = E_M(\cdot)$, we have $\mathcal{H}$ is invariant under all the spectral projections for M. In particular, $\mathcal{H}$ is invariant for $M = \int \lambda \, dE(\lambda)$, and, by the Fuglede-Putnam theorem, also for M^*. Since $\mathcal{H}$ is invariant also for $J = M + N$, it follows immediately that $\mathcal{H}$ is invariant for N. This gives the implication in one direction. On the other hand, if we assume $\mathcal{H}$ is invariant for M, M^*, and N, it is easily seen that $J_0 = J|\mathcal{H} = M|\mathcal{H} + N|\mathcal{H}$ is Jordan. ∎

3. A MODEL FOR SUBJORDAN OPERATORS WITH CYCLIC VECTOR

We proceed with the construction of a model for a cyclic complex sub-Jordan operator T on $\mathcal{H}$ similar to that of the real case presented in [6], Section 3. The basic ideas used there are mimicked and some proofs follow exactly as presented in that article.

There the authors characterized $\mathcal{H}$ as the closure of the graph of a closable differential operator D if T is pure. Here the purity of T does not imply the closability of D in general and hence specific results using the structure of the closure of the graph cannot be utilized as before. Though less explicit than the model for the real case in [6], an analysis of purity is still possible.

DEFINITION 3.1. Let T be a complex sub-Jordan operator on $\mathcal{H}$ with complex Jordan extension $J = M + N$ on $\mathcal{K}$ where M is normal, $N^2 = 0$, and $MN = NM$. We say J is a *minimal Jordan extension of* T if $\mathcal{K}$ is the smallest space containing $\mathcal{H}$ for which M, M^*, and N are invariant.

THEOREM 3.2. *Let T be a sub-Jordan operator on $\mathcal{H}$ with cyclic vector and minimal Jordan extension J on $\mathcal{K}$. Then there are finite positive measures μ and ν compactly supported in the complex plane with ν absolutely continuous with respect to μ, and a function $\theta \in L^2(\nu)$ so that, if we set*

$$\mathcal{K} = L^2(\mu) \oplus L^2(\nu) \tag{3.2.1}$$

then

$$\begin{array}{c}\textit{J is unitarily equivalent to multiplication} \\ \textit{by the matrix function } \begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix} \textit{ on } \mathcal{K},\end{array} \tag{3.2.2}$$

and

(3.2.3) *T is unitarily equivalent to J restricted to the closure of the graph of $D = \theta + \frac{\mathrm{d}}{\mathrm{d}z}$ defined on all polynomials; moreover the vector $\begin{bmatrix} 1 \\ \theta \end{bmatrix}$ is cyclic for this restricted operator.*

Conversely, an operator T defined by (3.2.3) *is a cyclic sub-Jordan operator with minimal Jordan extension J defined by* (3.2.2) *on $\mathcal{K}$ as defined by* (3.2.1) *provided that*

(3.2.4) *the inclusion map $1 : L^2(\mu) \to L^2(\nu)$ is continuous.*

Proof. Let $J = M + N$ where M is normal, $N^2 = 0$, and $MN = NM$. Let $\mathcal{K}_1 = (\operatorname{Ran} N)^\perp$ and $\mathcal{K}_2 = \operatorname{Ran} N$. Letting $M_i = M|\mathcal{K}_i$, we see by Fuglede-Putnam that $\mathcal{K}_i$ is reducing for M_i, hence M_i is normal for $i = 1, 2$. Furthermore, for some operator $\Gamma : \mathcal{K}_1 \to \mathcal{K}_2$ with dense range we can write

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Note also, since $MN = NM$, we have $\Gamma M_1 = M_2 \Gamma$.

By [19], Theorem 9.1, p. 99, there exist two sequences of measures $\{\mu_i\}_{i=1}^\infty$ and $\{\nu_j\}_{i=1}^\infty$ satisfying $\mu_{i+1} \ll \mu_i$ and $\nu_{j+1} \ll \nu_j$ with M_1 represented as multiplcation by z on $\mathcal{K}_1 = \oplus \sum\limits_{i=1}^{\infty} L^2(\mu_i)$ and M_2 as multiplication by z on $\mathcal{K}_2 = \oplus \sum\limits_{j=1}^{\infty} L^2(\nu_j)$. With respect to these decompositions of $\mathcal{K}_1$ and $\mathcal{K}_2$, Γ can be viewed as the matrix operator $\Gamma = [\Gamma_{ij}]$ where $\Gamma_{ij} : L^2(\mu_j) \to L^2(\mu_i)$. If we designate, for $i, j = 1, 2, \ldots, M_{1i} = M_1|L^2(\mu_i)$ and $M_{2j} = M_2|L^2(\mu_j)$, it follows that $M_{2i}\Gamma_{ij} = \Gamma_{ij} M_{1j}$. Therefore by Abrahamse [1], for each i, j we get the existence of a function $\Gamma_{ij}(z)$ so that for all $h \in L^2(\mu_j)$,

$$(\Gamma_{ij} h)(z) = \Gamma_{ij}(z) h(z).$$

Let $\Gamma(z)$ be the matrix valued function with ij-coordinate $\Gamma_{ij}(z)$. Then for each $h(z) = h_1(z) \oplus h_2(z) \oplus \cdots \in \oplus \sum\limits_{j=1}^{\infty} L^2(\mu_j)$

$$(\Gamma h)(z) = \oplus \sum_{j=1}^{\infty} \Gamma_{ij}(z) h_j(z).$$

We argue that T being cyclic and J being a minimal Jordan extension of T implies that without loss of generality there are compactly supported measures μ and ν so that $\mathcal{K}_1 = L^2(\mu)$ and $\mathcal{K}_2 = L^2(\nu)$. Let $\zeta_i = \zeta_{i1} \oplus \zeta_{i2} \oplus \cdots \in \mathcal{K}_i$ for $i = 1, 2$ be so that $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$ is a cyclic vector for T. Then

$$\begin{aligned} H &= \{p(T)\zeta : p \text{ a polynomial}\}^- \\ &= \left\{ \begin{bmatrix} p(z)\zeta_1(z) \\ p'(z)\Gamma(z)\zeta_1(z) + p(z)\zeta_2(z) \end{bmatrix} : p \text{ a polynomial} \right\}^- . \end{aligned}$$

Now, if we let

$$\begin{aligned} \widehat{\mathcal{K}}_1 &= \left\{p(z,\bar{z})\zeta_1(z) \ : \ p \text{ a polynomial in two arguments}\right\}^-, \\ \widehat{\mathcal{K}}_2 &= \left[\operatorname{span}\left\{p(z,\bar{z})\Gamma(z)\zeta_1(z), p(z,\bar{z})\zeta_2(z) : p \text{ a polynomial in two arguments}\right\}\right]^-, \end{aligned}$$

and $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}_1 \oplus \widehat{\mathcal{K}}_2$, we see $\mathcal{H} \subset \widehat{\mathcal{K}}$. Moreover, if $\begin{bmatrix} h \\ k \end{bmatrix} \in \widehat{\mathcal{K}}$, then

$$\begin{aligned} N \begin{bmatrix} h \\ k \end{bmatrix} &= \begin{bmatrix} 0 \\ \Gamma(z)h(z) \end{bmatrix} \in \widehat{\mathcal{K}}, \\ M \begin{bmatrix} h \\ k \end{bmatrix} &= \begin{bmatrix} zh(z) \\ zk(z) \end{bmatrix} \in \widehat{\mathcal{K}}, \end{aligned}$$

and

$$M^* \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} \bar{z}h(z) \\ \bar{z}k(z) \end{bmatrix} \in \widehat{\mathcal{K}}.$$

Thus $\widehat{\mathcal{K}}$ is invariant for M, M^* and N. So by minimality of J, $\mathcal{K} \subset \widehat{\mathcal{K}}$. In particular $\mathcal{K}_1 \subset \widehat{\mathcal{K}}_1$. We conclude M_1 is star-cyclic. By Theorem 4.3, p. 14 of [19], there is a measure μ compactly supported in $\mathbf{C}$ so that M_1 is unitarily equivalent to multiplication by z on $L^2(\mu)$.

Furthermore, since $\mathcal{K}_2 = \operatorname{Ran} N$ and

$$N\zeta = \begin{bmatrix} 0 \\ \Gamma(z)\zeta_1(z) \end{bmatrix},$$

we see $\mathcal{K}_2$ is spanned (over the polynomials) by $\begin{bmatrix} 0 \\ \Gamma(z)\zeta_1(z) \end{bmatrix}$. Thus M_2 is star--cyclic and we have as above a measure ν compactly supported in $\mathbf{C}$ so that M_2 is unitarily equivalent to multiplication by z on $L^2(\nu)$. This gives (3.2.1).

Since $\Gamma : L^2(\mu) \to L^2(\nu)$ with $M_2\Gamma = \Gamma M_1$, it follows from [1] that there is a measurable function $\Gamma(z)$ satisfying, for some $c > 0, |\Gamma(z)| \leqslant c(\mathrm{d}\mu/\mathrm{d}\nu)^{\frac{1}{2}}$,

$\Gamma(z) = 0$ μ - almost everywhere on the set $\{z|(\mathrm{d}\mu/\mathrm{d}\nu)(z) > 0\}$, such that for every $h \in L^2(\mu)$, for ν - a.e. z,

$$(\Gamma h)(z) = \Gamma(z)h(z).$$

Furthermore, by arguments identical to those of Lemma 3.13 of [6], it follows that ν is absolutely continuous with respect to $\mu, \Gamma \in L^2(\nu)$ and $\Gamma \neq 0$ ν - a.e. This last fact gives that the transformation

$$V : L^2(\mu) \oplus L^2(\nu) \to L^2(\mu) \oplus L^2(|\Gamma|^2 \mathrm{d}\nu)$$

defined by

$$V \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h \\ \Gamma^{-1} k \end{bmatrix}$$

is unitary (see the proof of Lemma 3.14 in [6]). Therefore, replacing $\mathrm{d}\nu$ by $|\Gamma|^2$ and J by VJV^{-1} but retaining the prior notation, we have

$$J = \begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$$

on $L^2(\mu) \oplus L^2(\nu)$. Thus (3.2.2) holds.

Now consider the cyclic vector $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$ for T. Suppose there is a measurable set A contained in $\operatorname{supp} \mu$ (the support of μ) so that $\mu(A) > 0$ and $\zeta_1 = 0$ on A. Letting $\widehat{\mathcal{K}} = \chi_{A^c} L^2(\mu) \oplus L^2(\nu)$ where χ_X denotes the characteristic function of the set X, we see that $\mathcal{H} \subset \widehat{\mathcal{K}}$, $\widehat{\mathcal{K}}$ is a proper subspace of $\mathcal{K}$, and $\widehat{\mathcal{K}}$ is invariant for M, M^*, and N. This contradicts the minimality of the complex Jordan extension J on $\mathcal{K}$. So we may assume $\zeta_1 \neq 0$ μ - a.e.

Note that $\nu \ll \mu$ implies also $\zeta_1 \neq \nu$ - a.e. Consider the isometry

$$\begin{bmatrix} h \\ h \end{bmatrix} \to \begin{bmatrix} \zeta_1^{-1} h \\ \zeta_1^{-1} k \end{bmatrix}$$

from $L^2(\mu) \oplus L^2(\nu)$ into $L^2(|\zeta_1|^2 \mathrm{d}\mu) \oplus L^2(|\zeta_1|^2 \mathrm{d}\nu)$. Using this we may take as our cyclic vector for T the vector $\zeta = \begin{bmatrix} 1 \\ \theta \end{bmatrix}$ (where $\theta = \zeta_1^{-1}\zeta_2 \in L^2(|\zeta_1|^2 \mathrm{d}\nu)$ which we rename $L^2(\mathrm{d}\nu)$ just as we identify $L^2(|\zeta_1|^2 \mathrm{d}\mu)$ with $L^2(\mathrm{d}\mu)$).

Finally, noting that

$$T^n \begin{bmatrix} 1 \\ \theta \end{bmatrix} = \begin{bmatrix} z^n \\ \theta(z) z^n + n z^{n-1} \end{bmatrix},$$

we see that

$$\mathcal{H} = \left\{ \begin{bmatrix} p \\ Dp \end{bmatrix} : p \text{ a polynomial} \right\}^-$$

where $D = \theta + \mathrm{d}/\mathrm{d}z : L^2(\mu) \to L^2(\nu)$. This gives (3.2.3) and completes the proof in one direction.

Conversely if μ and ν are positive compactly supported measures with $\nu \ll \mu$, if $\theta \in L^2(\nu)$ so that (3.2.1), (3.2.2), (3.2.3) and (3.2.4) hold then clearly T is complex sub-Jordan on $\mathcal{H}$ with cyclic vector $\begin{bmatrix} 1 \\ \theta \end{bmatrix}$ and with complex Jordan extension J. We need only show J is minimal. Let $\widehat{\mathcal{K}}$ be a space so that $\mathcal{H} \subseteq \widehat{\mathcal{K}} \subseteq \mathcal{K}$ and which is invariant for M, M^*, and N. Then we have

$$N \begin{bmatrix} 1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \widehat{\mathcal{K}}.$$

Therefore if $p(\cdot,\cdot)$ is a polynomial in two arguments it follows that

$$p(M, M^*) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p(z,\bar{z}) & 0 \\ 0 & p(z,\bar{z}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ p(z,\bar{z}) \end{bmatrix} \in \widehat{\mathcal{K}}.$$

Since p is arbitary, we conclude that $\begin{bmatrix} 0 \\ L^2(\nu) \end{bmatrix} \subseteq \widehat{\mathcal{K}}$. In particular for all polynomials $p(\cdot)$ in one argument, $\begin{bmatrix} 0 \\ Dp \end{bmatrix} \in \widehat{\mathcal{K}}$. Thus

$$\begin{bmatrix} p(z) \\ (Dp)(z) \end{bmatrix} - \begin{bmatrix} 0 \\ (Dp)(z) \end{bmatrix} = \begin{bmatrix} p(z) \\ 0 \end{bmatrix} \in \widehat{\mathcal{K}}.$$

Now let $q(\cdot)$ be a polynomial in one argument. Then by the above

$$q(M^*) \begin{bmatrix} p(z) \\ 0 \end{bmatrix} = \begin{bmatrix} q(\bar{z})p(z) \\ 0 \end{bmatrix} \in \widehat{\mathcal{K}}.$$

Consider the collection

$$\{q(\bar{z})p(z) : q, p \text{ polynomials}\}.$$

By the Stone-Weierstrass theorem in the complex case (see [47], Theorem 7.3.8, p. 155), this is a dense set in $L^2(\mu)$. Therefore, $\begin{bmatrix} L^2(\mu) \\ 0 \end{bmatrix} \subset \widehat{\mathcal{K}}$. We have shown $\mathcal{K} = \begin{bmatrix} L^2(\mu) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L^2(\nu) \end{bmatrix} \subset \widehat{\mathcal{K}}$. Thus J is minimal. ∎

For measures μ and ν compactly supported in the complex plane with $\nu \ll \mu$ such that $L^2(\mu)$ is a subset of $L^2(\nu)$ and for a function $\theta \in L^2(\nu)$, we denote by

$T(\mu,\nu,\theta)$ the restriction of the operator $J(\mu,\nu,\theta) \equiv \begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$ on $L^2(\mu) \oplus L^2(\nu)$ to the subspace

$$P_2^2(\mu,\nu,\theta) \equiv \left\{ \begin{bmatrix} p \\ Dp \end{bmatrix} : p \text{ a polynomial} \right\}^-$$

where $Dp = \theta p + p'$ and the closure is in $L^2(\mu) \oplus L^2(\nu)$. Theorem 3.2 states that any cyclic sub-Jordan operator is unitarily equivalent to $T(\mu,\nu,\theta)$ on $P_2^2(\mu,\nu,\theta)$ for some triple (μ,ν,θ). Also, it should be noted that by taking $\nu = 0$ we get Bram's characterization of cyclic subnormal operators (see [19]). The next two theorems describe exactly when $T(\mu,\nu,\theta)$ is actually Jordan or the other extreme, pure.

THEOREM 3.3. *$T(\mu,\nu,\theta)$ is Jordan if and only if $P_2^2(\mu,\nu,\theta) = L^2(\mu) \oplus L^2(\nu)$.*

Proof. Obviously, by the preceeding theorem, if $P_2^2(\mu,\nu,\theta) = L^2(\mu) \oplus L^2(\nu)$, then $T = T(\mu,\nu,\theta)$ is Jordan. So assume T is Jordan on $P_2^2(\mu,\nu,\theta)$. Then by Theorem 2.5, $\mathcal{H} = P_2^2(\mu,\nu,\theta)$ is invariant for M, M^*, and N. Thus by the same argument used in the last part of the proof of Theorem 3.2, replacing $\widehat{\mathcal{K}}$ with $\mathcal{H}$, we see $P_2^2(\mu,\nu,\theta) = L^2(\mu) \oplus L^2(\nu)$. ∎

THEOREM 3.4. *$T(\mu,\nu,\theta)$ is pure if and only if there is no nonzero positive measure α, compactly supported in the complex plane, for which $P_2^2(\mu,\nu,\theta)$ contains the subspace $L^2(\alpha) \oplus (0)$ or $(0) \oplus L^2(\alpha)$.*

Proof. By Theorem 2.4, $T = T(\mu,\nu,\theta)$ is not pure if and only if there is a nontrivial subspace $\mathcal{H}_0$ of $P_2^2(\mu,\nu,\theta)$ which is invariant for M, M^*, and N. So assume $\mathcal{H}_0$ is such a subspace and consider three cases:

(1) $\mathcal{H}_0 \not\subset \ker N$. $\widehat{\mathcal{H}} = N\mathcal{H}_0$. First note that $N\widehat{\mathcal{H}} \neq (0)$. Secondly,

$$M\widehat{\mathcal{H}} = MN\mathcal{H}_0 = NM\mathcal{H}_0 \subset N\mathcal{H}_0 = \widehat{\mathcal{H}}.$$

Finally, again by Fuglede-Putnam theorem, we have

$$M^*\widehat{\mathcal{H}} = M^*N\mathcal{H}_0 = NM^*\mathcal{H}_0 \subset N\mathcal{H} = \widehat{\mathcal{H}}.$$

Thus $\widehat{\mathcal{H}} = N\mathcal{H}_0 \subset (0) \oplus L^2(\nu)$ and $T|\widehat{\mathcal{H}}$ is normal. Thus there is a measure α so that $\widehat{\mathcal{H}} = 0 \oplus L^2(\alpha)$.

(2) $\mathcal{H}_0 \subset \operatorname{Ran} N = 0 \oplus L^2(\nu)$. Here as in the previous case, $T|\mathcal{H}_0$ is normal. Thus $\mathcal{H}_0 = 0 \oplus L^2(\alpha)$ for some measure α.

(3) $\mathcal{H}_0 \subset \operatorname{Ker} N$ but $\mathcal{H}_0 \not\subset \operatorname{Ran} N$. Then we consider the nonzero subspace

$$\widehat{\mathcal{H}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H}_0 \subset L^2(\mu) \oplus (0).$$

Let $\begin{bmatrix} h \\ 0 \end{bmatrix} \in \widehat{\mathcal{H}}$. Then there is a $k \in L^2(\nu)$ so that $\begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H}_0$. Now $M \begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H}_0$ implies $M \begin{bmatrix} h \\ 0 \end{bmatrix} \in \widehat{\mathcal{H}}$. Thus $\widehat{\mathcal{H}}$ is invariant for M. Similarly $\widehat{\mathcal{H}}$ is invariant for M^* since $\mathcal{H}_0$ is invariant for M^*. Finally, noting that for $\begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H}_0, N \begin{bmatrix} h \\ k \end{bmatrix} = 0$ implies $N \begin{bmatrix} h \\ 0 \end{bmatrix} = 0$ for $\begin{bmatrix} h \\ 0 \end{bmatrix} \in \widehat{\mathcal{H}}$. We conclude that $T|\widehat{\mathcal{H}}$ is normal and hence there is a measure α so that $\widehat{\mathcal{H}} = L^2(\alpha) \oplus 0$.

Conversely, suppose $P_2^2(\mu, \nu, \theta)$ contains a non-zero subspace of the form $\mathcal{H}_0 = L^2(\alpha) \oplus (0)$. If $\mathcal{H}_0 \subset \operatorname{Ker} N$ then we are in case (3), above, and it follows that $T|\mathcal{H}_0$ is normal. Thus by Proposition 2.4, T is not pure. On the other hand, if $\mathcal{H}_0 \not\subset \operatorname{Ker} N$, then $\widehat{\mathcal{H}} = N\mathcal{H}_0$ is nonzero and as seen in case (1), $T|\widehat{\mathcal{H}}$ is normal. Hence T is not pure.

Finally, if $P_2^2(\mu, \nu, \theta)$ contains a nonzero subspace of the form $\mathcal{H}_0 = (0) \oplus L^2(\alpha)$, then $\mathcal{H}_0 \subset \operatorname{Ran} N$. Hence we are in case (2) and it follows that T is not pure. ∎

4. PURE $\mathbf{T}$-SUBJORDAN OPERATORS

In [6], the main operator theoretic result was that the spectrum of a real pure sub-Jordan is a regularly closed subset of $\mathbf{R}$. A regularly closed set is one which can be recovered by closing its interior. In this section we show an analogous result in the case that the minimal Jordan extension of a pure sub-Jordan operator T has its spectrum contained in the unit circle $\mathbf{T} = \{z : |z| = 1\}$; such sub-Jordan operators we shall refer to as $\mathbf{T}$-sub-Jordan. We state the theorem now. The proof is a consequence of the lemmas and propositions which succeed it.

THEOREM 4.1. *Let T be a pure $\mathbf{T}$-sub-Jordan operator. Then*

$$\sigma(T) = \mathbf{D}^- = \{z : |z| \leqslant 1\}$$

or

$$\sigma(T) \textit{ is a regularly closed subset of } \mathbf{T}.$$

PROPOSITION 4.2. *Suppose that T is a $\mathbf{T}$-sub-Jordan operator such that its minimal Jordan extension J has $\sigma(J) = \mathbf{T}$. Then either $\sigma(T) = \mathbf{T}$ or $\sigma(T) = \mathbf{D}^-$.*

Proof. By Lemma 6.7 of [8], $\sigma(J) \subset \sigma(T)$. By Theorem 3.8 of [3], $\sigma(T)\setminus\sigma(J)$ is either empty or a union of components of $\mathbf{C} \setminus \sigma(J)$. Since $\sigma(J) = \mathbf{T}$, it follows that either $\sigma(T) = \mathbf{T}$ or $\sigma(T) = \mathbf{D}^-$. ∎

As a result of the preceeding proposition it is left to show that Theorem 4.1 holds when $\sigma(J)$ is a proper subset of $\mathbf{T}$. If in addition we assume that T is cyclic, by Theorem 3.2 we may assume that T is given by its model $T = T(\mu, \nu, \theta)$ where μ and ν are measures with common support equal to a proper subset of $\mathbf{T}$ and $\theta \in L^2(\nu)$. Then the minimal normal extension J of T is equal to the operator $J = J(\mu, \nu, \theta)$ of multiplication by $\begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$ on $L^2(\mu) \oplus L^2(\nu)$. When $\nu = 0$ (and hence $\theta = 0$ also), we abbreviate $P^2_2(\mu, \nu, \theta)$ to $P^2(\mu)$.

LEMMA 4.3. *If $T = T(\mu, \nu, \theta)$ is a pure cyclic $\mathbf{T}$-sub-Jordan operator such that the spectrum of its minimal Jordan extension $J = J(\mu, \nu, \theta)$ is a proper subset of $\mathbf{T}$, then $P^2(\mu) = L^2(\mu)$ and $P^2(\nu) = L^2(\nu)$.*

Proof. By the construction and the fact that $\sigma(M) = \sigma(J)$ (see [6], Proposition 3.7), we note that $\sigma(J) = (\operatorname{supp}\mu) \cup (\operatorname{supp}\nu)$. But since $\nu \ll \mu$ it follows that $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$. Hence $\sigma(J) = \operatorname{supp}\mu$. Thus if $\sigma(J)$ is a proper subset of $\mathbf{T}$, we have $\mathbf{T} \setminus \operatorname{supp}\mu$ and $\mathbf{T} \setminus \operatorname{supp}\nu$ must contain a subarc of $\mathbf{T}$. This is a consequence of $\mathbf{T} \setminus \operatorname{supp}\mu \neq \emptyset$ being an open subset of $\mathbf{T}$. Therefore both $d\mu/|dz|$ and $d\nu/|dz|$ are zero on a subset of $\mathbf{T}$ of positive $|dz|$ measure. Thus both

$$\int \log\left(\frac{d\mu}{|dz|}\right)|dz| = -\infty$$

and

$$\int \log\left(\frac{d\nu}{|dz|}\right)|dz| = -\infty$$

hold. So by Szëgo's Theorem (see [33], pp. 49-50), $L^2(\mu) = P^2(\mu)$ and $L^2(\nu) = L^2(\nu)$. ∎

LEMMA 4.4. *If $T(\mu, \nu, \theta)$ is pure $\mathbf{T}$-sub-Jordan and $\sigma(J(\mu, \nu, \theta)) \neq \mathbf{T}$, then* $\operatorname{supp}\mu = \operatorname{supp}\nu$.

Proof. Since $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$, it suffices to show $(\operatorname{supp}\mu) \setminus (\operatorname{supp}\nu) = \emptyset$. Arguing contrapositively, let $A \subseteq (\operatorname{supp}\mu) \setminus (\operatorname{supp}\nu)$ with $\mu(A) > 0$. Using, again by Szëgo's theorem, that $L^2(\mu|A) = P^2(\mu|A)$ we get that χ_A is a non-zero element of $P^2(\mu|A)$. Thus there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ so that $p_n \to \chi_A$ as $n \to \infty$ in $L^2(\mu|A)$. Now for each n, $p_n \oplus Dp_n \in P_2^2(\mu,\nu,\theta)$ and $p_n \oplus Dp_n \to \chi_A \oplus 0$ in $L^2(\mu) \oplus L^2(\nu)$.

Thus $\chi_A \oplus 0 \in P_2^2(\mu,\nu,\theta)$. Therefore $L^2(\mu|A) \oplus 0 \subset P_2^2(\mu,\nu,\theta)$ and by Theorem 3.4 it follows that T is not pure. ∎

The following result plays no role in the proof of Theorem 4.1, but has significance as an analogue of the situation in the model for real pure sub-Jordan operators found in Section 3 of [6]. For that reason we include it here.

PROPOSITION 4.5. *If $T(\mu,\nu,\theta)$ is a pure $\mathbf{T}$-sub-Jordan operator and $\sigma(J(\mu,\nu,\theta)) \neq \mathbf{T}$, then $D = \theta p + p' : L^2(\mu) \to L^2(\nu)$ is closable.*

Proof. Suppose there is a nonzero $k \in L^2(\nu)$ so that $0 \oplus k \in P_2^2(\mu,\nu,\theta)$. We may assume $k = 0$ on $\mathbf{T} \setminus \operatorname{supp}\nu$ and get that the essential support K of k is a subset of $\operatorname{supp}\nu$. Set

$$k^{-1} = \begin{cases} \frac{1}{k} & k \neq 0 \\ 0 & k = 0 \end{cases}.$$

Then $k^{-1} \in L^2(|k|^2 d\nu)$. Since the support of ν is a proper subset of $\mathbf{T}$, $P^2(|k|^2 d\nu) = L^2(|k|^2\nu)$; hence there exists a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} p_n = k^{-1}$ in $L^2(|k|^2\nu)$. Then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\mathbf{T}} |p_n - k^{-1}|^2 |k|^2 \, d\nu \\ &= \lim_{n\to\infty} \int_K |p_n k - 1|^2 \, d\nu. \end{aligned}$$

Thus $p_n k \to \chi_K$ in $L^2(\nu)$. Using this and the fact that

$$p_n(J) \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ p_n k \end{bmatrix} \to \begin{bmatrix} 0 \\ \chi_K \end{bmatrix}$$

we get $0 \oplus \chi_K \in P_2^2(\mu,\nu,\theta)$. Hence $0 \oplus L^2(\nu|K) \subseteq P_2^2(\mu,\nu,\theta)$ which implies T is not pure by Theorem 3.4. ∎

In [6] a duality technique was used to characterize the space $P_2^2(\mu,\nu,\theta)$; the key point was an explicit characterization of the annihilator $P_2^2(\mu,\nu,\theta)^{\perp}$ of $P_2^2(\mu,\nu,\theta)$ inside $L^2(\mu) \oplus L^2(\nu)$. We next adapt these ideas to the case where μ

and ν have support on the unit circle $\mathbf{T}$. For μ a measure on the circle, we identify the dual space of $L^2(\mu)$ with $L^2(\mu)$ via the bilinear pairing

$$L_g(f) = \int_{\mathbf{T}} f(z)g(z)\,\mathrm{d}\mu(z), \quad f, g \in L^2(\mu).$$

We also identify the dual of the space $C(\mathbf{T})$ with the space $M(\mathbf{T})$ of finite Borel measures on $\mathbf{T}$ via the pairing

$$L_\mu(f) = \int_{\mathbf{T}} f(z)\,\mathrm{d}\mu(z), \quad f \in C(\mathbf{T}),\ \mu \in M(\mathbf{T}).$$

Under this pairing as a consequence of the F. and M. Riesz Theorem (see [27]) the annihilator of $A(\mathbf{T})$ (the uniform closure of the analytic polynomials in $C(\mathbf{T})$) can be identified with H_0^1, or more precisely, with measures of the form $g|\mathrm{d}z|$ where $g \in H_0^1$ and $|\mathrm{d}z|$ is arc length measure on $\mathbf{T}$. Here H_0^1 is the Hardy subspace of $L^1(|\mathrm{d}z|)$ consisting of functions f with vanishing nonpositive Fourier coefficients; such functions are the nontangential limit boundary value functions of functions analytic on $\mathbf{D}$ and vanishing at 0 (see [33] or [27]).

Throughout the rest of this section we assume that μ and ν are finite Borel measures with support on $\mathbf{T}$ and $\theta \in L^2(\nu)$.

PROPOSITION 4.6. *The function pair $f \oplus g \in L^2(\mu) \oplus L^2(\nu)$ is an element of $P_2^2(\mu,\nu,\theta)^\perp$ if and only if*

$$g\,\mathrm{d}\nu = gw|\mathrm{d}z| \quad \textit{where } w = \frac{\mathrm{d}\nu}{|\mathrm{d}z|} \tag{4.6.1}$$

$$\int_{\mathbf{T}} [f(z)\,\mathrm{d}\mu(z) + \theta(z)g(z)w(z)|\mathrm{d}z|] = 0 \tag{4.6.2}$$

and

$$\frac{\mathrm{d}z}{|\mathrm{d}z|}\int_z^{z_0} [f(\zeta)\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|] + g(z)w(z) \in H_0^1 \textit{ for all } z_0 \in \mathbf{T}. \tag{4.6.3}$$

(Here $\int_{z_1}^{z_0} h(\zeta)\,\mathrm{d}\alpha(\zeta)$ is the integral along the arc from z_1 to z_0 taken with positive orientation.)

Proof. If $f \oplus g \in P_2^2(\mu,\nu,\theta)^\perp$, then for all polynomials p,

$$\int_{\mathbf{T}} pf\,\mathrm{d}\mu + \int_{\mathbf{T}} (\theta p + p')g\,\mathrm{d}v = 0. \tag{4.6.4}$$

Let $z_0 \in \mathrm{T}$ and write

$$p(z) = p(z_0) + \int_{z_0}^{z} p'(\zeta)\,\mathrm{d}\zeta.$$

Substituting in (4.6.4) results in

$$\begin{aligned}
0 &= \int_{\mathbf{T}} \Big[p(z_0) + \int_{z_0}^{z} p'(\zeta)\,\mathrm{d}\zeta\Big] f(z)\,\mathrm{d}\mu(z) \\
&\quad + \int_{\mathbf{T}} \Big\{\theta(z)\Big[p(z_0) + \int_{z_0}^{z} p'(\zeta)\,\mathrm{d}\zeta\Big] + p'(z)\Big\} g(z)\,\mathrm{d}\nu(z) \\
&= \int_{\mathbf{T}} p(z_0)[f(z)\,\mathrm{d}\mu(z) + \theta(z)g(z)\,\mathrm{d}\nu(z)] \\
&\quad + \int_{\mathbf{T}} \Big[\int_{z_0}^{z} p'(\zeta)\frac{\mathrm{d}\zeta}{|\mathrm{d}\zeta|}|\mathrm{d}\zeta|\Big][f(z)\,\mathrm{d}\mu(z) + \theta(z)g(z)\,\mathrm{d}\nu(z)] \\
&\quad + \int_{\mathbf{T}} p'(z)g(z)\,\mathrm{d}\nu(z).
\end{aligned} \tag{4.6.5}$$

Using Fubini's Theorem we interchange the order of integration in the second integral of (4.6.5) to obtain

$$\begin{aligned}
0 &= p(z_0)\int_{\mathbf{T}} [f(z)\,\mathrm{d}\mu(z) + \theta(z)g(z)\,\mathrm{d}\nu(z)] \\
&\quad + \int_{\mathbf{T}} p'(z)\Big\{\frac{\mathrm{d}z}{|\mathrm{d}z|}\Big[\int_{z}^{z_0} f(\zeta)\,\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)\,\mathrm{d}\nu(\zeta)\Big]|\mathrm{d}z| + g(z)\,\mathrm{d}\nu(z)\Big\}.
\end{aligned} \tag{4.6.6}$$

Taking $p \equiv 1$ in (4.6.6) gives (4.6.2). Considering all polynomials p' with $p(z_0) = 0$, we get by the F. and M. Riesz Theorem ([33], p. 47) that the measure

$$\frac{\mathrm{d}z}{|\mathrm{d}z|}\Big[\int_{z}^{z_0} f(\zeta)\,\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)\mathrm{d}\nu(\zeta)\Big]|\mathrm{d}z| + g(z)\,\mathrm{d}\nu(z)$$

is absolutely continuous with respect to $|\mathrm{d}z|$. Since the first summand is absolutely continuous with respect to $|\mathrm{d}z|$, it follows that $g(z)\mathrm{d}\nu(z) \ll |\mathrm{d}z|$. This gives (4.6.1).

Finally (4.6.3) follows from the F. and M. Riesz Theorem by again considering all polynomials p' with $p(z_0) = 0$ in (4.6.6).

The converse is obtained by simply reversing the argument above. We should note that since we could have chosen z_0 as any point on $\mathbf{T}$ in the beginning of the proof, necessarily condition (4.6.3) is independent of $z_0 \in \mathbf{T}$. To see this directly, note that a change of z_0 to z_0' amounts to perturbing the expression in (4.6.3) by a term of the form $c\frac{\mathrm{d}z}{|\mathrm{d}z|}$ (where c is some constant) which is an element of H_0^1.

COROLLARY 4.7. *$P_2^2(\mu,\nu,\theta) = P_2^2(\mu,\nu_a,\theta) \oplus [0 \oplus L^2(\nu_s)]$ where $\mathrm{d}\nu = \mathrm{d}\nu_a \oplus \mathrm{d}\nu_s$ is the Lebesgue decomposition of $\mathrm{d}\nu$ with respect to $|\mathrm{d}z|$ with $\mathrm{d}\nu_a \ll |\mathrm{d}z|$ and $\mathrm{d}\nu_s \perp |\mathrm{d}z|$.*

Proof. By Proposition 4.6, if $k \in L^2(\nu)$ and $k = k_a \oplus k_s$ is the decomposition of k with respect to $\mathrm{d}\nu_a \oplus \mathrm{d}\nu_s$, then $0 \oplus k_s \in \{P_2^2(\mu,\nu,\theta)^\perp\}^\perp = P_2^2(\mu,\nu,\theta)$. ∎

COROLLARY 4.8. *If $T(\mu,\nu,\theta)$ is pure, then $\mathrm{d}\nu$ is absolutely continuous with respect to $|\mathrm{d}z|$.*

Proof. This follows immediately from Theorem 3.4 and Corollary 4.7. ∎

COROLLARY 4.9. *If $\mathbf{T} \setminus \sigma(J) \neq \emptyset$, then $f \oplus g \in P_2^2(\mu,\nu,\theta)^\perp$ if and only if (4.6.1) and (4.6.2) hold in addition to*

$$\text{(4.9.1)} \qquad \frac{\mathrm{d}z}{|\mathrm{d}z|}\int_z^{z_0}\Big[f(\zeta)\,\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)\,\mathrm{d}\zeta\Big] = -g(z)w(z)$$

for all $z_0 \in \mathbf{T}$.

Proof. If $\mathbf{T} \setminus \sigma(J) \neq \emptyset$, then this set contains an arc I disjoint from $\operatorname{supp}\mu$ and hence from $\operatorname{supp}(w|\mathrm{d}z|)$. Thus by taking $z_0 \in I$, we get from (4.6.3) that

$$\frac{\mathrm{d}z}{|\mathrm{d}z|}\int_z^{z_0}\Big[f(\zeta)\,\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}z|\Big] + g(z)w(z)$$

is an H^1 function which vanishes on a set of positive Lebesgue measure. Thus by a corollary on page 52 of [33], this function must vanish identically. ∎

If $T(\mu,\nu,\theta)$ is pure, we may write $\mathrm{d}\nu = w|\mathrm{d}z|$ by Corollary 4.8. Let

$$M = \bigcup\Big\{ I : I \text{ is a maximal subarc of } \mathbf{T} \text{ with the property that for every compact subarc } J \subseteq I, \int_J w^{-1}|\mathrm{d}z| < \infty \Big\}.$$

Then $K = \mathbf{T} \setminus M$ consists precisely of those points z on $\mathbf{T}$ for which

$$\int_z^{z+\delta} w^{-1}|\mathrm{d}z| = \int_{z-\delta}^{z} w^{-1}|\mathrm{d}z| = \infty$$

for all $\delta > 0$. This set K corresponds to the set K defined by (2.12.1), p. 105 of [6]. We cite Lemmas 2.11 and 2.13 of [6] whose statements and proofs hold with the following modifications:

1. Consider all intervals as subarcs of $\mathbf{T}$ and integrals as line integrals along $\mathbf{T}$ in a positive direction.

2. When the proofs refer to Lemma 2.7, use Corollary 4.9 instead.

3. Use $\lambda_1(A) = \int_A f(\zeta)\,\mathrm{d}\mu(\zeta)$, $\lambda_2(A) = \int_A \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|$ and $\lambda = \lambda_1 + \lambda_2$.

4. Define a maximal interval of local integrability (MILI) for $\frac{1}{w}$ to be any of the components of M.

With these changes we then have the following analogous result which has as an immediate consequence the piece necessary for the proof of Theorem 4.1.

PROPOSITION 4.10. *If $T(\mu,\nu,\theta)$ is pure, if $\sigma(J) \neq \mathbf{T}$ and if $f \oplus g \in P_2^2(\mu,\nu,\theta)^\perp$, then $f = 0$ μ-a.e. and $g = 0$ ν-a.e. on $\mathbf{T} \setminus M$.*

PROPOSITION 4.11. *If $T(\mu,\nu,\theta)$ is pure and $\sigma(J) \neq \mathbf{T}$ then* $\operatorname{supp}\nu$ *is regularly closed in* $\mathbf{T}$.

Proof. Since T is pure, $d\nu = w|\mathrm{d}z|$. Furthermore, by Proposition 4.10, if $K = \mathbf{T} \setminus M$ then $0 \oplus \chi_K$ is orthogonal to $P_2^2(\mu,\nu,\theta)^\perp$ and hence belongs to $P_2^2(\mu,\nu,\theta)$. Therefore, $0 \oplus L^2(\mu|K) \subset P_2^2(\mu,\nu,\theta)$. Since this contradicts Theorem 3.4, we conclude $\nu(K) = 0$. Hence ν is carried by M, a union of subarcs. It follows that $\operatorname{supp}\nu = M^-$ is a regularly closed subset of $\mathbf{T}$. ∎

Proof of Theorem 4.1. By Proposition 4.2 it suffices to consider the case when $\sigma(J) \neq \mathbf{T}$. If T is cyclic we may assume that $T = T(\mu, \nu, \theta)$. As noted earlier, $\sigma(J) = \operatorname{supp} \mu$. By Lemma 4.4, we have that $\sigma(J) = \operatorname{supp} \nu$ is a regularly closed subset (relative to $\mathbf{T}$) via Proposition 4.11. Finally, by Theorem 3.8 of [3], $\sigma(J) = \sigma(T)$ since the only component of $\mathbb{C} \setminus \sigma(J)$ is $\mathbb{C} \setminus \sigma(J)$ itself. The general case can be reduced to the cyclic case by the same procedure as in [6] for the real case; here one must use that a normal operator with spectrum equal to a proper subset of $\mathbf{T}$ is *reductive*, i.e. any invariant subspace is reducing. This also can be seen as a consequence of Szegö's theorem, for example. ∎

5. THE SPECTRA OF PURE COMPLEX SUB-JORDAN OPERATORS

The result of Section 4 can be viewed as saying that the spectrum of a pure complex sub-Jordan operator must be locally thick in a certain precise sense, at least for the case where the minimal Jordan extension has spectrum inside the unit circle. In this section we present a plausible way of making this statement precise for general pure sub-Jordan operators, and check its validity for some computable special cases. The conjectured characterization of spectra of pure sub-Jordan operators is function algebraic in nature, and is modeled on the characterization of spectra of pure subnormal operators by Clancey and Putnam [17].

We begin by recalling the result of Clancey and Putnam. For K a compact subset of the plane, $C(K)$ denotes the algebra of continuous complex valued functions on K and $R(K)$ is the uniform closure over K of rational functions with poles off K.

THEOREM 5.1. (see [17]). *A compact set E of the complex plane is the spectrum of some pure subnormal operator if and only if for every open disk Δ with $\Delta \cap E \neq \emptyset$.*

$$R(E \cap \bar{\Delta}) \neq C(E \cap \bar{\Delta}).$$

To present a possible analogue for complex sub-Jordan operators, we introduce the following.

DEFINITION 5.2. Let K be a compact subset of the complex plane. We define $C_2(K)$ as $C(K) \oplus C(K)$ and $R_2(K)$ to be the subspace of $C_2(K)$ equal to the uniform closure in $C_2(K)$ of the manifold

$$\{r \oplus r' : r \text{ a rational function with no poles in } K\}.$$

CONJECTURE 5.3. *A compact subset E of the complex plane is the spectrum of some pure complex sub-Jordan operator T if and only if*

$$R_2(E \cap \bar{\Delta}) \neq C_2(E \cap \bar{\Delta})$$

for all open disks Δ such that $\Delta \cap E \neq \emptyset$.

First we notice that in at least one direction this conjecture is consistent with Theorem 5.1. Indeed a compact set which is the spectrum of a pure subnormal operator in particular is the spectrum of a pure complex sub-Jordan operator. By the result of Clancey-Putnam,

$$R(E \cap \bar{\Delta}) \neq C(E \cap \bar{\Delta})$$

for each open disk Δ for which $\Delta \cap E \neq \emptyset$. But it is easily seen directly that

$$\begin{aligned} R_2(E \cap \bar{\Delta}) &= C_2(E \cap \bar{\Delta}) \\ &\Rightarrow R(E \cap \bar{\Delta}) = C(E \cap \bar{\Delta}); \end{aligned}$$

for if $r_n \oplus r_n' \to h \oplus k$, then in particular $r_n \to h$. Then by the contrapositive, $R(E \cap \bar{\Delta}) \neq C(E \cap \bar{\Delta})$ implies that $R_2(E \cap \bar{\Delta}) \neq C_2(E \cap \bar{\Delta})$. Thus the condition of Conjecture 5.3 holds for a subset E which is the spectrum of a pure subnormal operator.

The proof of Theorem 5.1 in [17] relies on the Cauchy transform; this technique does not extend in any obvious way to give insight on Conjecture 5.3. Rather than tackling the conjecture in full generality, we will verify its validity in a number of computable special cases.

In order to do this we need to be able to compute $R_2(K)$ for as many compact sets K as possible. In general this is difficult.

For the case where one omits consideration of derivatives, there is a characterization of when $R(K) = C(K)$ in terms of analytic capacity due to Vitushkin (see [26], Chapter 10) and more recent tests involving generalized Cauchy-Green formulas and distributional $\bar{\partial}$-derivatives due to Khavinson ([37]) and extended by Ferry ([22]). For the space $R_2(K)$ considered here, we mention [9], [10], [25] and [7]. The following theorem summarizes some special cases where the question of whether $R_2(K) = C_2(K)$ can be settled definitively.

THEOREM 5.4. *Let K be a compact subset of the plane.*

(i) *If $K \subset \mathbf{R}$, then $R_2(K) = C_2(K)$ if and only if K has empty one dimensional interior.*

(ii) *If $K \subset \mathbf{T}$, then $R_2(K) = C_2(K)$ if and only if K has empty interior relative to $\mathbf{T}$.*

(iii) *If K is totally disconnected, then $R_2(K) = C_2(K)$.*

(iv) *If K is a smooth simple curve without critical points, then $R_2(K) \neq C_2(K)$.*

Proof. (i) By Theorem 3.2 in [7], we know that $P_2(K) = C_2(K)$ (where $P_2(K)$ is the uniform closure of $\{p \oplus p' : p \text{ a polynomial}\}$ in $C_2(K)$) if and only if K has empty one dimensional interior. To verify (i) it suffices to check that $P_2(K) = R_2(K)$ if $K \subset \mathbf{R}$. To do this we need only check that any $r \oplus r'$ (where r is rational with poles off K) can be approximated uniformly on K by $p \oplus p'$ where p is a polynomial.

So let r be rational with poles in $\mathbf{C} \setminus E$. Since r, and hence r', can only have a finite number of discontinuities, the poles of r form a compact set in $\mathbf{C}$ disjoint from E. Thus, since by assumption E is compact, we can cover E by a finite set of closed finite intervals disjoint from the poles of r and r'. Let $\widehat{E}$ denote the union of these closed intervals. Then $\widehat{E}$ is compact and if $a = \min\{x : x \in \widehat{E}\}$ and $b = \max\{x : x \in \widehat{E}\}$, then $(b-a) < \infty$. Furthermore both r and r' are continuous on $\widehat{E}$.

Now let f be a smooth continuous extension of $r|\widehat{E}$ to $[a,b]$. Then f' is defined on $[a,b]$ and $r = f$ and $r' = f'$ on $\widehat{E}$. Let $\varepsilon > 0$ be given and choose a polynomial p' so that

$$\sup_{x\in[a,b]} |p'(x) - f'(x)| < \frac{\varepsilon}{2}(\max\{1, b-a\})^{-1}.$$

Define the polynomial p via

$$p(x) = \int_a^x p'(t)\,dt + f(a).$$

Noticing that

$$f(x) = \int_a^x f'(t)\,dt + f(a)$$

we have

$$\begin{aligned}\|p - r\|_{L^\infty(\widehat{E})} + \|p' - r'\|_{L^\infty(\widehat{E})} &= \|p - f\|_{L^\infty(\widehat{E})} + \|p' - f'\|_{L^\infty(\widehat{E})} \\ &\leqslant \sup_{x\in[a,b]} \int_a^x |p'(t) - f'(t)|\,dt + \frac{\varepsilon}{2}\Big(\max\{1, b-a\}\Big)^{-1} \\ &< (b-a)\frac{\varepsilon}{2}\Big(\max\{1, b-a\}\Big)^{-1} + \frac{\varepsilon}{2}\Big(\max\{1, b-a\}\Big)^{-1} \\ &< \varepsilon.\end{aligned}$$

(ii) The result can be obtained by using the same duality technique as used in [7] for the real line case; we omit the details.

(iii) This is essentially the result from [10].

(iv) This is essentially the result from [25]. ∎

We now list special cases where one can verify Conjecture 5.3 by direct checking.

THEOREM 5.5. *Conjecture* 5.3 *is valid if E satisfies any of the following additional hypotheses:*

(i) $E \subset \mathbf{R}$.

(ii) $E \subset \mathbf{T}$ *or* $E = \mathbf{D}$.

(iii) E *is totally disconnected.*

(iv) E *is a smooth simple curve without critical points.*

Proof of Theorem 5.5. (i) By Corollary 3.16 in [6], a compact subset E of $\mathbf{R}$ is the spectrum of a pure (real) sub-Jordan operator if and only if E is the closure of its one dimensional interior (i.e. E is regularly closed relative to $\mathbf{R}$). By Theorem 5.4 (i), Conjecture 5.3 for this case follows if we show that $(E \cap \Delta)^-$ has nonempty one dimensional interior for all open disks Δ with $E \cap \Delta \neq \emptyset$ if and only if E is regularly closed with respect to $\mathbf{R}$. But this is straightforward.

(ii) This follows in the same way as (i), based on Theorem 4.1 and part (ii) of Theorem 5.4.

(iii) If E is totally disconnected and Δ is any open disk such that $E \cap \Delta \neq \emptyset$, then $K = E \cap \bar{\Delta}$ is also totally disconnected. Then by part (iii) of Theorem 5.4, $R_2(E \cap \bar{\Delta}) = C_2(E \cap \bar{\Delta})$. On the other hand, J. Agler [3] has shown that any complex sub-Jordan operator with totally disconnected spectrum is actually Jordan, so in particular cannot be pure. Thus this case is consistent with Conjecture 5.3.

(iv) If E is a smooth simple curve without critical points, then part (iv) of Theorem 5.4 implies that $R_2(E\cap\bar{\Delta}) \neq C_2(E\cap\bar{\Delta})$ for any open Δ with $E\cap\Delta \neq \emptyset$.

On the other hand we can construct a pure complex sub-Jordan operator having spectrum equal to E as follows.

Let f be a differentiable complex-valued function defined on $[0,1]$ satisfying

(5.5.1) $$f \text{ is one-to-one,}$$

(5.5.2) $$f' \in C[0,1]$$

(5.5.3) $$f'(t) \neq 0 \text{ for all } t \in [0,1],$$

and

(5.5.4) $$f([0,1]) = E.$$

Let J_0 be multiplication by $\begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix}$ on $\mathcal{K} = L^2(m) \oplus L^2(m)$ where m is Lebesgue measure on $[0,1]$. Let $T_0 = J_0|\mathcal{H}$ where $\mathcal{H}$ is the $\mathcal{K}$-closure of the manifold $\{p \oplus p' \quad p$ a polynomial$\}$. Then by Theorem 3.9, T_0 is a pure (real) sub-Jordan operator with minimal Jordan extension J.

We have a C^1-functional calculus available for use. That is, since both f and f' are continuous on $[0,1]$, then $f(T_0)$ is multiplication by $\begin{bmatrix} f(t) & 0 \\ f'(t) & f(t) \end{bmatrix}$ restricted to $\mathcal{H}$. (See the latter part of the proof of Theorem 3.17 of [6]). If we let $J = M_f + N_f$ where M_f is multiplication by $\begin{bmatrix} f(t) & 0 \\ 0 & f(t) \end{bmatrix}$ on $\mathcal{K}$ and N_f is multiplication by $\begin{bmatrix} 0 & 0 \\ f'(t) & 0 \end{bmatrix}$ on $\mathcal{K}$, then clearly J is Jordan and hence T is sub-Jordan. It is left to show T is pure and $\sigma(T) = E$.

By the Stone-Weierstrass Theorem in the complex plane (see [47], Theorem 7.3.8), the manifold

$$\{p(f, f') : \ p \text{ a polynomial in } z \text{ and } \bar{z}\}$$

is dense in $C[0,1]$. (Note: Here we use 5.5.1 to get that the manifold separates points.)

Now suppose $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ which is invariant for M_f and M_f^*. Then $\mathcal{H}_0$ is invariant for $p(M_f)$ and $p(M_f^*)$, for all polynomials p. Thus $\mathcal{H}_0$ is invariant for $p(M_f, M_f^*)$ where p is any polynomial in z and $\bar{z}$. By the result noted above, there is a sequence of polynomials $\{p_n\}$ in z and $\bar{z}$ so that $\{p_n(f,\bar{f})\}$ converges uniformly to $g(t) = t$ on $[0,1]$. Thus

$$p_n(M_f, M_f^*) = \begin{bmatrix} p_n(f,\bar{f}) & 0 \\ 0 & p_n(f,\bar{f}) \end{bmatrix} \to \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = S,$$

implying that $\mathcal{H}_0$ is invariant for S.

Suppose further that $\mathcal{H}_0$ is invariant for N_f. By 5.5.2 and 5.5.3 we have as before a sequence $\{p_n\}$ of polynomials in two variables so that $\{p_n(f,\bar{f})\}$ converges uniformly to $(f')^{-1}$ on $[0,1]$. Since $\mathcal{H}_0$ is invariant for

$$\begin{aligned} N_f \cdot p_n(M_f, M_f^*) &= \begin{bmatrix} 0 & 0 \\ f'(t) & 0 \end{bmatrix} \begin{bmatrix} p_n(f,\bar{f}) & 0 \\ 0 & p_n(f,\bar{f}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ f'(t)p_n(f,\bar{f}) & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = N, \end{aligned}$$

it follows that $\mathcal{H}_0$ is invariant for N. Therefore $\mathcal{H}_0 \subset \mathcal{H}$ being invariant for M_f, M_f^* and N_f implies $\mathcal{H}_0$ is invariant for both S and N. Since T_0 is pure this says $\mathcal{H}_0 = \{0\}$. Hence T is also pure.

To show $\sigma(T) = E$, we first show $J = f(J_0)$ is the minimal Jordan extension of $T = f(T_0)$. Considering $\mathcal{H} = \left\{ \begin{bmatrix} h \\ h' \end{bmatrix} : h \in AC[0,1] \right\}$, we need to show that $\mathcal{K}$ is the smallest space containing $\mathcal{H}$ that is invariant for both M_f and N_f. Notice that for all $h \in AC[0,1]$, for all polynomials p

$$p(N_f) \begin{bmatrix} h \\ h' \end{bmatrix} = \begin{bmatrix} 0 \\ p(f')h \end{bmatrix} \in \mathcal{K}.$$

But since functions of the form $p(f')$ are dense in $L^2(m)$, this says $\begin{bmatrix} 0 \\ L^2(m) \end{bmatrix} \subseteq \mathcal{K}$. In particular, for all polynomials p,

$$\begin{bmatrix} 0 \\ p(f')h + p(f)h' \end{bmatrix} \in \mathcal{K}.$$

Thus

$$p(J) \begin{bmatrix} h \\ h' \end{bmatrix} - \begin{bmatrix} 0 \\ p(f')h + p(f)h' \end{bmatrix} = \begin{bmatrix} p(f)h \\ 0 \end{bmatrix} \in \mathcal{K}$$

for all polynomials p and all $h \in AC[0,1]$. Since the manifold $\{p(f)h : p$ a polynomial, $h \in AC[0,1]\}$ is dense in $L^2(m)$, we get $\begin{bmatrix} L^2(m) \\ 0 \end{bmatrix} \subseteq \mathcal{K}$. Therefore $\mathcal{K} = \begin{bmatrix} L^2(m) \\ L^2(m) \end{bmatrix}$. Again by Agler's result ([3], Theorem 3.8), we conclude $\sigma(J) = \sigma(T)$, provided $\sigma(J) \subset E$.

To see this is indeed the case, we show $\mathbb{C}\setminus E \subseteq \rho(J)$. So let $\lambda \in \mathbb{C}\setminus E$. Then both functions $(\lambda - f(t))^{-1}$ and $(\lambda - f(t))^{-2}(\lambda - f(t))'$ are defined and continuous on $[0,1]$. Thus multiplication by

$$\begin{bmatrix} (\lambda - f)^{-1} & 0 \\ (\lambda - f)^{-2}(\lambda - f)' & (\lambda - f)^{-1} \end{bmatrix}$$

is bounded. The following calculation shows this operator is in fact $(\lambda I - f(J_0))^{-1}$:

$$\begin{aligned} &\begin{bmatrix} (\lambda - f) & 0 \\ -f' & (\lambda - f) \end{bmatrix} \begin{bmatrix} (\lambda - f)^{-1} & 0 \\ (\lambda - f)^{-2}(\lambda - f)' & (\lambda - f)^{-1} \end{bmatrix} \\ &\quad = \begin{bmatrix} 1 & 0 \\ -(\lambda - f)'(\lambda - f)^{-1} + (\lambda - f)^{-1}(\lambda - f)' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

In conclusion, we cite Proposition 3.7 of [6] to get that both $\sigma(J_0) = \sigma(S)$ and $\sigma(J) = \sigma(M_f)$. Obviously, $\sigma(S) = [0,1]$, so by the Spectral Mapping Theorem, $E = f([0,1]) = f(\sigma(S)) = \sigma(f(S)) = \sigma(M_f) = \sigma(J) = \sigma(T)$. ∎

6. THE SPACE $P_2^2(\mu, \nu, \theta)$

The model for a complex sub-Jordan operator with cyclic vector presented in Section 3 suggests the general problem of characterizing precisely what are the elements $f \oplus g$ of $P_2^2(\mu,\nu,\theta)$. Here μ and ν are assumed to be compactly supported Borel measures on the plane, θ is assumed to be an element of $L^2(\nu)$ and $P_2^2(\mu,\nu,\theta)$ is defined to be the $L^2(\mu)\oplus L^2(\nu)$ closure of the set $\{p \oplus (\theta p + p') : p \text{ a polynomial}\}$. For the case where μ and ν are supported on the real line, a nearly definitive intrinsic characterization of $P_2^2(\mu,\nu,\theta)$ was given in [6]. For the case where $\nu = 0$ (and hence also $\theta = 0$), J. E. Thomson [48] has recently obtained a complete structure theory for $P^2(\mu) = P^2(\mu, 0, 0)$ in terms of bounded analytic point evaluations. For the case where ν (as well as θ) is zero and μ is supported on the unit circle, $P^2(\mu)$ can be described explicitly as a consequence of the Szegö-Kolmogoroff-Krein theorem (see [33], p. 49). Here we give some results on $P_2^2(\mu,\nu,\theta)$ for the case where both μ and ν are supported on $\mathbf{T}$; these results might be considered as a beginning toward a Szegö-Kolmogoroff-Krein theorem with derivatives. The main tool is the characterization of $P_2^2(\mu,\nu,\theta)^\perp$ given by Proposition 4.6.

We first consider the case where μ and ν have support equal to a proper subset of $\mathbf{T}$. Then the annihilator $P_2^2(\mu,\nu,\theta)^\perp$ is characterized by conditions (4.6.1), (4.6.2) and (4.9.1), just as in the real case analyzed in detail in [6]. As a consequence the space $P_2^2(\mu,\nu,\theta)$ in this case has the same structure as in the real case. To summarize the result we need some notation and terminology. Let $\mu = w|\mathrm{d}z| + w_{\text{sing}}$ be the Lebesgue decomposition with respect to arc length measure on the unit circle. If $w(z) = 0$ set $w(z)^{-1} = \infty$. An arc I on $\mathbf{T}$ is said to be an MALI (maximal arc of local integrability) for w^{-1} if I is a maximal subarc of $\mathbf{T}$ with respect to the property that $\int_J w^{-1}|\mathrm{d}z| < \infty$ for every compact subarc $J \subset I$. The following sums up the results from [6] adapted to the situation where μ and ν are measures with supports properly contained in $\mathbf{T}$.

THEOREM 6.1. *Let μ and ν be finite measures with supports equal to proper subsets of* $\mathbf{T}$ *and $\theta \in L^2(\nu)$. Then:*

(i) *A given $h \oplus k \in L^2(\mu) \oplus L^2(\nu)$ is in $P_2^2(\mu, \nu, \theta)$ if and only if $(h \oplus k)|I$ is in $P_2^2(\mu|I, \nu|I, \theta|I)$ for each* MALI *for w^{-1}. Moreover $h \oplus k$ is in $P_2^2(\mu, \nu, \theta)$ if and only if $h \oplus k_a$ is in $P_2^2(\mu, w|\mathrm{d}z|, \theta)$, where $k = k_a + k_s$ is the decomposition of $k \in L^2(\nu)$ with $k_a \in L^2(w|\mathrm{d}z|)$ and $k_s \in L^2(\nu_{\text{sing}})$.*

(ii) *If $\nu = w|\mathrm{d}z|$, $\operatorname{supp}\nu$ is the closed proper subarc $I = [z_0, z_1]$ of* $\mathbf{T}$ *and $\int_{z_0}^{z_1} w^{-1}|\mathrm{d}z| < \infty$, then a function pair $h \oplus k \in L^2(\mu) \oplus L^2(\nu)$ is in $P_2^2(\mu, \nu, \theta)$ if and only if there exists a function h_1 such that*

(a) *h_1 is absolutely continuous on I*

(b) *$h = h_1$ μ-a.e.*

(c) *$e^{\Theta(z)}h_1(z) = h_1(z_0) + \int_{z_0}^{z}(e^{\Theta}k)(z)\,\mathrm{d}z$ where $\Theta(z) = \int_{z_0}^{z}\theta(z)\,\mathrm{d}z$ and line integrals are taken along I.*

(iii) *If $\nu = w|\mathrm{d}z|$ and ν is carried by an* MALI *I for w^{-1} whose closure is a proper subarc of* $\mathbf{T}$, *then a given function $h \oplus k \in L^2(\mu) \oplus L^2(w|\mathrm{d}z|)$ which satisfies conditions* (a), (b), (c) *in statement* (ii) *above (where z_0 is some point in I) and which also satisfies*

(d) *$h \oplus k$ has compact support in I*

is in $P_2^2(\mu, \nu, \theta)$.

We next present a special case with $\operatorname{supp}\nu = \mathbf{T}$ where an explicit description of $P_2^2(\mu, \nu, \theta)$ is possible. We first note that if w is a function in $L^1(|\mathrm{d}z|)$ with $w \geqslant 0$ on $\mathbf{T}$ such that $\int_{\mathbf{T}} w^{-1}|\mathrm{d}z| < \infty$, then

$$\int_{\mathbf{T}} \log w|\mathrm{d}z| = -\int_{\mathbf{T}} \log(w^{-1})|\mathrm{d}z| \geqslant -\int_{\mathbf{T}} w^{-1}|\mathrm{d}z| > -\infty.$$

Then, by a consequence of Szegö's theorem (see [33] or [27]) there is an outer function $v \in H^2$ such that $w = |v|^2$ on $\mathbf{T}$.

THEOREM 6.2. *Let μ and ν be finite positive measures on* $\mathbf{T}$ *and* $\theta \in L^2(\nu)$. *Suppose* $\nu = w|\mathrm{d}z|$ *where* $w^{-1} \in L^1(|\mathrm{d}z|)$ *and suppose* θ *is an element of the Hardy space* H^1. *Let* v *be an outer function such that* $|v|^2 = w$. *Then a given function pair* $h \oplus k \in L^2(\mu) \oplus L^2(\nu)$ *is in* $P_2^2(\mu, \nu, \theta)$ *if and only if there exists functions* h_1 *and* k_1 *such that*

(i) h_1 *is absolutely continuous on* $\mathbf{T}$ *and* $h = h_1$ μ*-a.e.*.

(ii) $k_1 \in H^2$ *and* $k = v^{-1}k_1$.

(iii) $k = \theta h_1 + h_1'$ $|\mathrm{d}z|$*-a.e. on* $\mathbf{T}$.

Proof. To verify necessity we first need a couple of observations. First of all, the hypothesis $w^{-1} \in L^1(|\mathrm{d}z|)$ implies that $w^{-1} \in L^2(w|\mathrm{d}z|)$; indeed

$$\int_{\mathbf{T}} |w^{-1}|^2 w|\mathrm{d}z| = \int_{\mathbf{T}} w^{-1}|\mathrm{d}z| < \infty.$$

Hence, for any $f \in L^2(w|\mathrm{d}z|)$, by the Cauchy-Schwarz inequality

$$\int_{\mathbf{T}} |f||\mathrm{d}z| = \int_{\mathbf{T}} |f|w^{-1}(w|\mathrm{d}z|) \leqslant \|f\|_{L^2(w|\mathrm{d}z|)}\|w^{-1}\|_{L^2(w|\mathrm{d}z|)} < \infty,$$

that is, $L^2(w|\mathrm{d}z|) \subset L^1(|\mathrm{d}z|)$. Moreover, if v is an outer function with $|v|^2 = w$, the map $p \to vp$ (where p is an arbitrary analytic polynomial) maps a dense subspace of $P^2(w|\mathrm{d}z|)$ to a dense subspace of $H^2(= P^2(|\mathrm{d}z|))$ isometrically and hence extends to a unitary map of $P^2(w|\mathrm{d}z|)$ onto H^2; as a consequence we see that $P^2(w|\mathrm{d}z|)$ is identical to $v^{-1}H^2$ and that $L^2(w|\mathrm{d}z|) \cap H^1 = v^{-1}H^2$.

Now suppose that $\{p_n\}$ is a sequence of polynomials such that $p_n \oplus (\theta p_n + p_n')$ converges to $h \oplus k$ in $L^2(\mu) \oplus L^2(w|\mathrm{d}z|)$. In particular, since $\theta \in H^1$ by assumption, $\theta p_n + p_n' \in H^1 \cap L^2(w|\mathrm{d}z|)$. This latter space by the discussion above is identical to $P^2(w|\mathrm{d}z|) = v^{-1}H^2$. Hence k, as the limit in $L^2(\nu)$ of $\theta p_n + p_n'$, is in $v^{-1}H^2$; this verifies the necessity of condition (ii).

Fix a point z_0 on $\mathbf{T}$ and define $\Theta(z) = \int_{z_0}^{z} \theta(\zeta)\,\mathrm{d}\zeta$. Since θ is in H^1, by Cauchy's theorem and the Lebesgue dominated convergence theorem one can see that $\Theta(z)$ is defined and continuous on the whole closed unit disk $\bar{\mathbf{D}}$ and that the integral is independent of path. In fact Θ is in the disk algebra $A(\mathbf{D}) = H^\infty \cap C(\mathbf{T})$ with $\Theta' = \theta$. Moreover, if we set $q_n = \theta p_n + p_n'$ then $(\mathrm{e}^{\Theta} p_n)' = \mathrm{e}^{\Theta} q_n$.

Thus, for z_1 and z_2 points on $\mathbf{T}$,

$$(\mathrm{e}^{\Theta} p_n)(z_2) - (\mathrm{e}^{\Theta} p_n)(z_1) = \int_{z_1}^{z_2} (\mathrm{e}^{\Theta} q_n)(\zeta)\,\mathrm{d}\zeta \tag{6.2.1}$$

where the line integral is taken along an arc of $\mathbf{T}$. Note next that $e^{\Theta}q_n$ converges to $e^{\Theta}k$ in $L^2(w|dz|)$ since e^{Θ} is bounded. Hence

$$\lim_{n\to\infty}\left|\int_{z_1}^{z_2}(e^{\Theta}k - e^{\Theta}q_n)(\zeta)d\zeta\right| \leqslant \lim_{n\to\infty}\|e^{\Theta}k - e^{\Theta}q_n\|_{L^2(w|dz|)}\|w^{-1}\|_{L^2(w|dz|)} = 0.$$

By choosing a subsequence if necessary, we may assume that $\lim_{n\to\infty} p_n(z) = h(z)$ μ-a.e. on $\mathbf{T}$. Hence, we may pass to the limit in (6.2.1) to get $(e^{\Theta}h)(z_2)-(e^{\Theta}h)(z_1) = \int_{z_1}^{z_2}(e^{\Theta}k)(\zeta)d\zeta$ for μ-a.e. z_1 and z_2. Unraveling this condition leads us to the necessity of conditions (i) and (iii).

Now suppose that $h\oplus k \in L^2(\mu)\oplus L^2(w|dz|)$ satisfies conditions (i), (ii), (iii). To show that $h\oplus k \in P_2^2(\mu, w|dz|, \theta)$, by the Hahn-Banach theorem it suffices to show that $h\oplus k$ is annihilated by an arbitrary element $f\oplus g$ of $P_2^2(\mu, w|dz|, \theta)^{\perp}$. In the following calculation, we use the characterization of $P_2^2(\mu, w|dz|, \theta)^{\perp}$ given by Proposition 4.6. Thus assume that $f\oplus g \in L^2(\mu)\oplus L^2(w|dz|)$ satisfies (4.6.1), (4.6.2) and (4.6.3). The problem is to show

$$I := \int_{\mathbf{T}} hf\,d\mu + \int_{\mathbf{T}} kgw|dz| = 0 \tag{6.2.2}$$

for all $h\oplus k$ satisfying (i), (ii), (iii) and $f\oplus g$ satisfying (4.6.1), (4.6.2) and (4.6.3). By (ii) $k \in H^1$ while by (4.6.3)

$$\chi := g(z) + w(z)^{-1}\frac{dz}{|dz|}\int_{z}^{z_0}\Big[f(\zeta)\,d\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|d\zeta|\Big] \tag{6.2.3}$$

is in $w^{-1}H_0^1 \cap L^2(w|dz|)$; hence

$$\int_{\mathbf{T}} k\chi w|d\zeta| = 0$$

and so

$$\int_{\mathbf{T}} kgw|dz| = -\int_{\mathbf{T}} k(z)\int_{z}^{z_0}\Big[f(\zeta)d\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|d\zeta|\Big]dz.$$

Hence we get

$$I = \int_{\mathbf{T}} hf d\mu - \int_{\mathbf{T}} k(z)\int_{z}^{z_0}\Big[f(\zeta)d\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|d\zeta|\Big]dz.$$

Interchange of the order of integration converts this to

$$\int_{\mathbf{T}} hf\,d\mu - \int_{\mathbf{T}} \left[\int_{z_0}^{\zeta} k(z)\mathrm{d}z\right]\left[f(\zeta)\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|\right].$$

But by (i) and (iii)

$$\int_{z_0}^{\zeta} k(z)\,\mathrm{d}z = \int_{z_0}^{\zeta} \theta(z)h_1(z)\,\mathrm{d}z + h_1(\zeta) - h_1(z_0)$$

and hence,

$$\begin{aligned} I = & \int_{\mathbf{T}} h_1(\zeta)f(\zeta)\,\mathrm{d}\mu(\zeta) \\ & - \int_{\mathbf{T}} \left[\int_{z_0}^{\zeta} \theta(z)h_1(z)\,\mathrm{d}z + h_1(\zeta) - h_1(z_0)\right] f(\zeta)\mathrm{d}\mu(\zeta) \\ & - \int_{\mathbf{T}} \left[\int_{z_0}^{\zeta} \theta(z)h_1(z)\,\mathrm{d}z + h_1(\zeta) - h_1(z_0)\right] \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta| \\ = & \int_{\mathbf{T}} h_1(\zeta)f(\zeta)\,\mathrm{d}\mu(\zeta) \\ & - \int_{\mathbf{T}} \left[\int_{z_0}^{\zeta} \theta(z)h_1(z)\,\mathrm{d}z + h_1(\zeta)\right]\left[f(\zeta)\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|\right] \\ & + h_1(z_0)\int_{\mathbf{T}} \left[f(\zeta)\,\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|\right]. \end{aligned}$$

From condition (4.6.2) we see that the last term vanishes. The first term cancels with part of the second term to leave us with

$$\begin{aligned} I = & -\int_{\mathbf{T}} h_1(\zeta)\theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta| \\ & - \int_{\mathbf{T}} \left[\int_{z_0}^{\zeta} \theta(z)h_1(z)\,\mathrm{d}z\right]\left[f(\zeta)\,\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|\right]. \end{aligned}$$

Another interchange in the order of integration brings us to

$$\begin{aligned} I = &- \int_{\mathbf{T}} h_1(\zeta)\theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta| \\ &- \int_{\mathbf{T}} \theta(z)h_1(z)\left[\int_z^{z_0} f(\zeta)\mathrm{d}\mu(\zeta) + \theta(\zeta)g(\zeta)w(\zeta)|\mathrm{d}\zeta|\right]\mathrm{d}z \\ = &- \int_{\mathbf{T}} \theta(z)h_1(z)\chi(z)w(z)|\mathrm{d}z| \end{aligned}$$

where $\chi(z)$ is given by (6.2.3) and is in $w^{-1}H_0^1 \cap L^2(w|\mathrm{d}z|)$. By solving (iii) for h_1 and using that k and θ are in H^1, we see that $h_1 \in A(\mathbf{D})$; hence $\theta h_1 \in H^1 \cap L^2(w|\mathrm{d}z|)$, and therefore $I = -\int_{\mathbf{T}} \theta h_1 \chi w|\mathrm{d}z| = 0$, as required. ∎

For measures μ and ν with supports equal to more general compact subsets of the plane, such an explicit description of $P_2^2(\mu,\nu,\theta)$ would apppear to be very complicated. Rather than seeking such an explicit description, it makes sense to try to understand some qualitative properties. Specifically, Theorems 3.3 and 3.4 suggest the following question: for what measures μ and v compactly supported in the complex plane and functions $\theta \in L^2(v)$, does one have

(1) $P_2^2(\mu, v, \theta) = L^2(\mu) \oplus L^2(v)$

or

(2) $P_2^2(\mu, v, \theta)$ contains a nontrivial subspace of the form $L^2(\alpha)\oplus(0)$ or $(0)\oplus L^2(\alpha)$?

Both of these problems seem difficult in general though complete answers are given in Chapters 2 and 3 of [6] when μ and ν are supported on the real line (see Theorem 2.31, the model construction and Theorem 3.17 of [6]).

When ν is taken to be zero, then the questions above reduce to:

For what measure μ, compactly supported in the plane, does one have

(1_0) $P^2(\mu) = L^2(\mu)$

or

(2_0) $P^2(\mu)$ splits into a direct sum with one summand an L^2-space.

Here $P^2(\mu)$ denotes the closure of the polynomials as a linear manifold in $L^2(\mu)$. These latter questions have been addressed and answered in some cases by various authors. The reader may pursue work done regarding (1_0) in [11], [12], [32], [33], [26], [44], [49], [50], [51], and [52]. As noted previously, a consequence of Szegö's theorem generalized by Kolmogoroff and Krein is that for μ supported on the unit circle, $P^2(\mu) = L^2(\mu)$ provided that

$$\int \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}|z|}\right)\mathrm{d}|z| = -\infty$$

(see [33], [26]). T. T. Trent characterized when $P^2(\mu) \neq L^2(\mu)$ in general with the condition that there exist a finite measure ν singular with respect to μ so that for some positive constant c,

$$\|p\|_{1,\nu} \leqslant c\|p\|_{2,\mu}$$

for all polynomials p. Thomson ([48]) recently characterized the failure of (1_0) in terms of the existence of bounded analytic point evaluations.

Question (2_0) is addressed in [38], [39], and [41] in the case where μ is supported on the closed unit disk $\bar{\mathbb{D}}$ and $\mu = \nu + w\mathrm{d}m$ where ν is carried by the open disc $\mathbb{D}$, m is Lebesgue measure on $\partial\mathbb{D}$, and $w \in L^1(\mathrm{d}m)$. Kriete shows that if $\operatorname{supp}\nu \subseteq \mathbb{D}$ and ν is circular symmetric then

$$P^2(\mu) = P^2(\nu) + L^2(w\mathrm{d}m).$$

Here "circular symmetric" means that $\mathrm{d}\nu = \mathrm{d}m(r)\mathrm{d}\theta$ for some Borel measure m on $[0,1]$. The interplay between m and w is investigated with various results. For example if $\mathrm{d}m(r) = G(r)r\mathrm{d}r$, then the rate of decay of G as $r \nearrow 1$ is a determining factor for splitting, as is the logarithmic integrability of w. In particular, if for small $\delta > 0$

$$\int_{1-\delta}^{1} \log\log\frac{1}{G(r)}\mathrm{d}r = \infty$$

and if $w = 0$ on a set of positive Lebesgue measure in ∂D, then $P^2(\mu)$ splits. Conversely, if the above integral is finite and $\mathbb{T}$ contains a subarc for which $1/w$ is integrable, then $P^2(\mu)$ does not split; this condition is similar to those examined in Section 2 of [6]. Further references concerning when $P^2(\mu)$ splits include [13], [28], [34], [35], [36], [40], [45], [53], and [54].

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