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K-THEORY FOR CERTAIN REDUCED FREE PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. A six-term exact sequence of K-groups for certain reduced free products of C^*-algebras is derived. This sequence is used to show that $\mathrm{K}_*(C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n)$ is isomorphic to $\mathrm{K}_*(C^*_{\mathrm{red}}(G))$ for any countable discrete group satisfying property Λ of Lance. This result is then used to compute the K-groups of the reduced noncommutative unitary C^*-algebra $U^{\mathrm{nc}}_{n,\mathrm{red}}$ and the reduced noncommutative Grassmanian C^*-algebra $G^{\mathrm{nc}}_{n,\mathrm{red}}$. It is shown that if G is a nontrivial countable discrete group with property Λ such that the range of the homomorphism from $\mathrm{K}_0(C^*_{\mathrm{red}}(G))$ into $\mathbf{R}$ induced by the usual trace on $C^*_{\mathrm{red}}(G)$ is contained in $\frac{1}{n}\mathbf{Z}$, then the relative commutant of M_n in $C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n$ is a simple projectionless C^*-algebra.

KEYWORDS: *C^*-algebras, Free products, K-theory.*

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1. INTRODUCTION

The notion of free products of C^*-algebras has been used frequently in recent literature [1], [2], [4], [5], [6], [10], [11], [12], [16]. The free product $A *_{\mathbf{C}} B$ of two unital C^*-algebras A and B (over the complex numbers $\mathbf{C}$) can be thought of as a generalization of the group C^*-algebra of the free product of two discrete groups G_1 and G_2 over the trivial group $\{e\}$ as follows:

$$C^*(G_1 *_{\{e\}} G_2) \cong C^*(G_1) *_{\mathbf{C}} C^*(G_2). \tag{1.1}$$

It was shown by J. Cuntz in [4] that if unital $*$-homomorphisms from A and B into $\mathbf{C}$ exist, then there is a six term exact sequence of K-groups of the following form:

$$
\begin{array}{ccccc}
\mathrm{K}_0(\mathbf{C}) & \xrightarrow{(\chi^1_*,\,-\chi^2_*)} & \mathrm{K}_0(A)\oplus \mathrm{K}_0(B) & \xrightarrow{\varepsilon^1_*+\varepsilon^2_*} & \mathrm{K}_0(A *_{\mathbf{C}} B) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(A *_{\mathbf{C}} B) & \xleftarrow{\varepsilon^1_*+\varepsilon^2_*} & \mathrm{K}_1(A)\oplus \mathrm{K}_1(B) & \xleftarrow{(\chi^1_*,\,-\chi^2_*)} & \mathrm{K}_1(\mathbf{C}).
\end{array} \tag{1.2}
$$

The maps $\chi^1\colon \mathbf{C} \to A$, $\chi^2\colon \mathbf{C} \to B$, $\varepsilon^1\colon A \to A *_{\mathbf{C}} B$, and $\varepsilon^2\colon B \to A *_{\mathbf{C}} B$ are the inclusion maps. In fact, it was shown that the free product can be replaced by an amalgamated product $A *_C B$ over a common subalgebra C of A and B provided retractions from A and B onto C exist. In particular, there is an exact sequence as in (1.2) if $A = C^*(G_1)$ and $B = C^*(G_2)$ for discrete groups G_1 and G_2.

The isomorphism in (1.1) led to Avitzour's notion of reduced free products of unital C^*-algebras. If A and B are unital C^*-algebras with states φ and ψ respectively, a cyclic representation $(\pi_{\varphi*\psi}, H_{\varphi*\psi}, \xi_{\varphi*\psi})$ can be defined so that the state $\varphi * \psi(x) = \langle \pi_{\varphi*\psi}(x)\xi_{\varphi*\psi}, \xi_{\varphi*\psi}\rangle$ extends both φ and ψ. The reduced free product of A and B relative to φ and ψ is defined to be $\pi_{\varphi*\psi}(A *_{\mathbf{C}} B)$. We will denote this by $A *^{\mathrm{red}}_{\mathbf{C}} B$ and keep in mind that the definition of the reduced free product depends on the states φ and ψ. The reduced free product generalizes the notion of the reduced group C^*-algebra of the free product of two discrete groups as follows:

$$
C^*_{\mathrm{red}}(G_1 *_{\{e\}} G_2) \cong C^*(G_1) *^{\mathrm{red}}_{\mathbf{C}} C^*(G_2) \cong C^*_{\mathrm{red}}(G_1) *^{\mathrm{red}}_{\mathbf{C}} C^*_{\mathrm{red}}(G_2). \tag{1.3}
$$

The reduced free products on the right are relative to the faithful traces φ on $C^*(G)$ and φ_r on $C^*_{\mathrm{red}}(G)$ defined by

$$
\varphi(g) = \varphi_r(\lambda(g)) = \langle \lambda(g)\delta(e), \delta(e)\rangle = \delta_{ge}, \quad g \in G \tag{1.4}
$$

where λ is the left regular representation on G, e is the identity of G, and $\delta(e)$ is the characteristic function of $\{e\}$.

A natural question to is whether or not the exact sequence in (1.2) holds in the case of reduced free products. In the case where $A = C^*_{\mathrm{red}}(G_1)$ and $B = C^*_{\mathrm{red}}(G_2)$ for discrete groups G_1 and G_2, the answer is yes. This was shown to be the case by E. Lance in [10] under the assumption that one of the two groups satisfies property Λ. A discrete group G is said to have property Λ if the left regular representation of G on $l^2(G)$ is K-homotopic to a representation of $C^*(G)$ having a fixed point. It follows from the more recent work of Pimsner that the property Λ assumption can be dropped ([14]).

We show that the reduced version of the exact sequence in (1.2) holds in a more general context than in the case considered by Lance. Let B be a separable, unital C^*-algebra with trace ψ. Suppose B has a unitary element of trace zero. Let $A = C^*(G)$ for a countable discrete group G with property Λ and let φ be the trace on $C^*(G)$ as defined in (1.4). Under these assumptions we show that the reduced version of (1.2) holds.

As corollaries to the above results we obtain the K-groups of the reduced noncommutative unitary group $U^{\rm nc}_{n,\rm red}$ and the reduced noncommutative Grassmanian $G^{\rm nc}_{n,\rm red}$ discussed in [11]. $U^{\rm nc}_{n,\rm red}$ is defined to be the relative commutant M_n^c of the n by n matrices M_n in the reduced free product $C^*(\mathbb{Z}) *^{\rm red}_{\mathbf{C}} M_n$ using the usual traces on M_n and $C^*(\mathbb{Z})$. $G^{\rm nc}_{n,\rm red}$ is defined similiarly except with $C^*(\mathbb{Z}_2)$ in place of $C^*(\mathbb{Z})$. The computation of these K-groups gives another proof of the projectionlessness of $U^{\rm nc}_{n,\rm red}$ than the one given in [11] as well as the projectionlessness of $G^{\rm nc}_{n,\rm red}$ in the case where n is even. More generally, we show the following: if G is a nontrivial discrete group, φ is the usual trace on $C^*_{\rm red}(G)$, and φ_* is the induced map from $\mathrm{K}_0(C^*_{\rm red}(G))$ into $\mathbb{R}$, then $\varphi_*(\mathrm{K}_0(C^*_{\rm red}(G))) \subset \frac{1}{n}\mathbb{Z}$ implies that $M_n^c \subset C^*(G) *^{\rm red}_{\mathbf{C}} M_n$ is a simple, projectionless C^*-algebra for $n > 1$.

2. FREE PRODUCTS AND REDUCED FREE PRODUCTS

Let A and B be unital C^*-algebras. Let $A *^{\rm alg}_{\mathbf{C}} B$ denote the algebraic free product of A and B, over the complex numbers. That is, $A *^{\rm alg}_{\mathbf{C}} B$ is the $*$-algebra of formal finite sums of monomials $a_1b_1a_2b_2\cdots a_nb_n$ with $\mathbf{1}_A$ and $\mathbf{1}_B$ identified. Define a seminorm on $A *^{\rm alg}_{\mathbf{C}} B$ as follows:

$$\|w\| = \sup\{\|\pi(w)\| \colon \pi \text{ is a unital } *\text{-representation of } A *^{\rm alg}_{\mathbf{C}} B\}.$$

The fact that $\|\cdot\|$ is actually a norm will be demonstrated shortly. $A *_{\mathbf{C}} B$ is defined to be the completion of $A *^{\rm alg}_{\mathbf{C}} B$. $A *_{\mathbf{C}} B$ is called the C^*-algebraic free product of A and B and can be described by the following universal property. If π_A and π_B are unital $*$-homomorphisms from A and B respectively into a unital C^*-algebra E, then there is a unique unital $*$-homomorphism

$$\pi_A * \pi_B \colon A *_{\mathbf{C}} B \to E$$

extending both π_A and π_B.

In order to define the reduced free product of A and B, we must fix states φ on A and ψ on B. Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ and $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS representations associated with (A,φ) and (B,ψ). Let $H_\varphi^0=\xi_\varphi^\perp$ and $H_\psi^0=\xi_\psi^\perp$. Define the Hilbert space $H_{\varphi*\psi}$ as follows:

$$H_{\varphi*\psi}=\mathbf{C}\xi_{\varphi*\psi}\oplus H_\varphi^0\oplus H_\psi^0\oplus(H_\varphi^0\otimes H_\psi^0)\oplus(H_\psi^0\otimes H_\varphi^0)\oplus(H_\varphi^0\otimes H_\psi^0\otimes H_\varphi^0)\oplus\cdots$$

where all possible finite alternating tensor products are considered and $\xi_{\varphi*\psi}$ is a unit vector. We will frequently identify H_φ and $\mathbf{C}\xi_{\varphi*\psi}\oplus H_\varphi^0$ as well as H_ψ and $\mathbf{C}\xi\oplus H_\psi^0$ by identifying $\xi_{\varphi*\psi}$ with ξ_φ and ξ_ψ. Define a representation $\pi_{\varphi*\psi}$ of $A*_{\mathbf{C}}B$ on $H_{\varphi*\psi}$ as follows. Let $a\in A$, $h_\varphi^0\in H_\varphi^0$, $h_\psi^0\in H_\psi^0$. Now since $H_\varphi=H_\varphi^0\oplus\mathbf{C}\xi_\varphi$, we have

$$\begin{aligned}\pi_\varphi(a)h_\varphi^1&=(\pi_\varphi(a)h_\varphi^1)_0+\lambda\xi_\varphi\\ \pi_\varphi(a)\xi_\varphi&=(\pi_\varphi(a)\xi_\varphi)_0+\mu\xi_\varphi,\end{aligned}$$

where $(\pi_\varphi(a)h_\varphi^1)_0$, $(\pi_\varphi(a)\xi_\varphi)_0\in H_\varphi^0$ and $\lambda,\mu\in\mathbf{C}$. Then define $\pi_{\varphi*\psi}(a)$ as follows:

$$\begin{aligned}\pi_{\varphi*\psi}(a)\xi_{\varphi*\psi}&=(\pi_\varphi(a)\xi_\varphi)_0+\mu\xi_{\varphi*\psi}\\ \pi_{\varphi*\psi}(a)(h_\varphi^1\otimes h_\psi^1\otimes\cdots)&=(\pi_\varphi(a)h_\varphi^1)_0\otimes h_\psi^1\otimes\cdots+\lambda h_\psi^1\otimes\cdots\\ \pi_{\varphi*\psi}(a)(h_\psi^1\otimes h_\varphi^2\otimes\cdots)&=(\pi_\varphi(a)\xi_\varphi)_0\otimes h_\psi^1\otimes h_\varphi^2\otimes\cdots+\mu h_\psi^1\otimes h_\varphi^2\otimes\cdots.\end{aligned}$$

Similiarly define $\pi_{\varphi*\psi}(b)$ for $b\in B$. Then let $\pi_{\varphi*\psi}$ be the representation on $A*_{\mathbf{C}}B$ determined by $(\pi_{\varphi*\psi}|A,\pi_{\varphi*\psi}|B)$. Notice that $(\pi_{\varphi*\psi}|B,\mathbf{C}\xi_{\varphi*\psi}\oplus H_\varphi^0,\xi_{\varphi*\psi})$ is unitarily equivalent to $(\pi_\varphi,H_\varphi,\xi_\varphi)$ and similiarly for $(\pi_\psi,H_\psi,\xi_\psi)$. The vector $\xi_{\varphi*\psi}$ was shown to be cyclic in [1]. It was also shown that if φ and ψ are faithful, then $\xi_{\varphi*\psi}$ is separating for the algebraic free product $A*_{\mathbf{C}}^{\mathrm{alg}}B$. Hence $\pi_{\varphi*\psi}$ is faithful on $A*_{\mathbf{C}}^{\mathrm{alg}}B$ when φ and ψ are faithful. It follows from this that if A and B have faithful states (in particular, if A and B are separable) then $\|\cdot\|$ is a norm on the algebraic free product $A*_{\mathbf{C}}^{\mathrm{alg}}B$. By restricting one's attention to an algebraic free product of separable subalgebras of A and B which contains $w\in A*_{\mathbf{C}}^{\mathrm{alg}}B$ it is easy to see that $\|\cdot\|$ is a norm on $A*_{\mathbf{C}}^{\mathrm{alg}}B$ whether or not A and B are separable.

$A*_{\mathbf{C}}^{\mathrm{red}}B$ is defined to be $\pi_{\varphi*\psi}(A*_{\mathbf{C}}B)$. Define a state $\varphi*\psi$ on $B(H_{\varphi*\psi})$ by $(\varphi*\psi)(T)=\langle T\xi_{\varphi*\psi},\xi_{\varphi*\psi}\rangle$. Then $\varphi*\psi$ is faithful on $A*_{\mathbf{C}}^{\mathrm{alg}}B$ since $\xi_{\varphi*\psi}$ is separating for $A*_{\mathbf{C}}^{\mathrm{alg}}B$. If $a\in A$, then a routine computation shows that $(\varphi*\psi)(\pi_{\varphi*\psi}(a))=\varphi(a)$. So by identifying a and $\pi_{\varphi*\psi}(a)$ it follows that $\varphi*\psi$ extends φ and similiarly ψ. The cyclic representation $(\pi_{\varphi*\psi},H_{\varphi*\psi},\xi_{\varphi*\psi})$ is the GNS representation of $(A*_{\mathbf{C}}B,(\varphi*\psi)\circ\pi_{\varphi*\psi})$. In the case that φ and ψ are traces, Avitzour showed that $\varphi*\psi$ is a trace ([1], Proposition 1.4). We say that a vector ξ is a trace vector for a $*$-algebra $E\subset B(H)$ if the state $\omega_\xi(x)=\langle x\xi,\xi\rangle$ is a trace on E. A trace vector ξ which is cyclic for a $*$-subalgebra $E\subset B(H)$ is separating for E'' (see [11], Section 3 for a proof). Hence $\xi_{\varphi*\psi}$ is a separating vector for $\pi_{\varphi*\psi}(A *_{\mathbf{C}} B)''$ and consequently for $A *_{\mathbf{C}}^{\mathrm{red}} B$. So $\varphi * \psi$ is a faithful trace on $A *_{\mathbf{C}}^{\mathrm{red}} B$ when φ and ψ are faithful traces. We will also need the following result of Avitzour ([1], Proposition 3.1. and Corollary).

PROPOSITION 2.1. *Let A and B be C^*-algebras with faithful traces φ and ψ respectively. Suppose there are unitaries $a \in A$, $b, c \in B$ such that $\varphi(a) = 0$ and $\psi(b) = \psi(c) = \psi(b^*c) = 0$. Then the reduced free product $A *_{\mathbf{C}}^{\mathrm{red}} B$ relative to (φ, ψ) is simple with $\varphi * \psi$ as its unique trace.*

3. STATEMENT OF THE MAIN RESULT

Let G be a discrete group and let λ denote its left regular representation. Let e be the identity of G and let $\delta(e)$ denote the characteristic function of $\{e\}$. If H is a Hilbert space we let $\mathcal{K}(H)$ denote the ideal of compact operators on H. The following definition was introduced by E. Lance in [10].

DEFINITION 3.1. G has property Λ if there is a one-parameter family (λ_t) of unital $*$-representations of $C^*(G)$ on $l^2(G)$ such that:

(i) $\lambda_0 = \lambda$;

(ii) $\lambda_1(g)\delta(e) = \delta(e)$ for all $g \in G$;

(iii) (λ_t) is a K-homotopy, that is, for each $g \in G$, the map $t \mapsto \|\lambda_t(g)\|$ is continuous and $\lambda_t(g) - \lambda(g) \in \mathcal{K}(l^2(G))$ for all $g \in G$, $t \in [0,1]$.

It was shown by Lance that G has property Λ if G is a discrete countable amenable group. Lance also showed that the class of groups with property Λ is closed under free products and that noncompact groups with property T of Kazhdan ([9]) do not have property Λ.

Our main result is the following.

THEOREM 3.2. *Let (B, ψ) a unital C^*-algebra and state and suppose there is a unitary $b \in B$ such that $\psi(b) = 0$ and $\psi(b^*xb) = \psi(x)$ for all $x \in B$. Let $A = C^*(G)$ for a discrete group G with property Λ and let φ be the canonical trace on A. Let $A *_{\mathbf{C}}^{\mathrm{red}} B = \pi_{\varphi*\psi}(A *_{\mathbf{C}} B)$ (the reduced free product of A and B relative to (φ, ψ)) and $B_{\mathrm{red}} = \pi_{\varphi*\psi}(B)$. Then there is a cyclic six-term exact sequence of K-groups*

$$
(3.1)\qquad
\begin{array}{ccccc}
K_0(\mathbf{C}) & \xrightarrow{(\chi_*^1,\,-\chi_*^2)} & K_0(C^*_{\mathrm{red}}(G)) \oplus K_0(B_{\mathrm{red}}) & \xrightarrow{\epsilon_*^1+\epsilon_*^2} & K_0(C^*_{\mathrm{red}}(G) *_{\mathbf{C}}^{\mathrm{red}} B) \\
\uparrow & & & & \downarrow \\
K_1(C^*_{\mathrm{red}}(G) *_{\mathbf{C}}^{\mathrm{red}} B) & \xleftarrow{\epsilon_*^1+\epsilon_*^2} & K_1(C^*_{\mathrm{red}}(G)) \oplus K_1(B_{\mathrm{red}}) & \xleftarrow{(\chi_*^1,\,-\chi_*^2)} & K_1(\mathbf{C})
\end{array}
$$

where $\chi^1: \mathbf{C} \to C^*_{\mathrm{red}}(G)$, $\chi^2: \mathbf{C} \to B_{\mathrm{red}}$, $\varepsilon^1: C^*_{\mathrm{red}}(G) \to C^*_{\mathrm{red}}(G) *^{\mathrm{red}}_{\mathbf{C}} B$, *and* $\varepsilon^2: B_{\mathrm{red}} \to C^*_{\mathrm{red}}(G) *^{\mathrm{red}}_{\mathbf{C}} B$ *are the natural inclusions. Moreover, the above exact sequence splits into the following two exact sequences:*

$$0 \longrightarrow \mathrm{K}_0(\mathbf{C}) \xrightarrow{(\chi^1_*, -\chi^2_*)} \mathrm{K}_0(C^*_{\mathrm{red}}(G)) \oplus \mathrm{K}_0(B_{\mathrm{red}}) \xrightarrow{\varepsilon^1_* + \varepsilon^2_*} \mathrm{K}_0(C^*_{\mathrm{red}}(G) *^{\mathrm{red}}_{\mathbf{C}} B) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{K}_1(C^*_{\mathrm{red}}(G) *^{\mathrm{red}}_{\mathbf{C}} B) \xrightarrow{\varepsilon^1_* + \varepsilon^2_*} \mathrm{K}_1(C^*_{\mathrm{red}}(G)) \oplus \mathrm{K}_1(B_{\mathrm{red}}) \longrightarrow 0$$

We remark that the last statement of the theorem follows immediately from the fact that $\mathrm{K}_1(\mathbf{C}) = 0$ and the fact that the existence of the homomorphism from $\mathrm{K}_0(C^*_{\mathrm{red}}(G))$ into $\mathbf{R}$ induced by the trace on $C^*_{\mathrm{red}}(G)$ implies that χ^1_* is injective and hence the leftmost vertical map is zero.

In what follows we prove Theorem 3.4 and deduce several corollaries concerning C^*-algebras of the form $C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n$. Much of what follows in Sections 4 through 8 is a straightforward generalization of the work of [10] and of Natsume [12] and consequently the presentation will be sketchy at times. The key idea is that the group structure of the group which is not assumed to have property Λ is not really used in Lance's proof in the case where A and B are both group C^*-algebras.

4. TOEPLITZ EXTENSION

Let (B, ψ) be a separable unital C^*-algebra and state. Let $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS representation associated with (B, ψ). Let S be a countable subset of B with dense linear span chosen so that the corresponding elements of B/N_ψ are linearly independent, where $N_\psi = \{x \in B: \psi(x^*x) = 0\}$. By using the Gram-Schmidt process relative to the inner product $\langle x, y\rangle_\psi = \psi(y^*x)$ on B/N_ψ, we can assume that $\{\pi_\psi(s)\xi_\psi: s \in S\}$ is an orthonormal basis for H_ψ. Assume $\mathbf{1}_B \in S$. Suppose also that B has a unitary element t with $\psi(t) = 0$ for which $\psi(t^*xt) = \psi(x)$ for all $x \in B$. Let $A = C^*(G)$ for a nontrivial countable discrete group G and let φ be the trace on $C^*(G)$ as defined in (1.4). Let A_{red} and B_{red} denote $\pi_{\varphi*\psi}(A) \cong C^*_{\mathrm{red}}(G)$ and $\pi_{\varphi*\psi}(B)$ respectively. Let $\Gamma \subset A *_{\mathbf{C}} B$ denote the following set:

$$\Gamma = \{g_1 s_1 \dots g_n s_n: g_i \in G, s_j \in S, g_i \neq \mathbf{1}_B \text{ for } i \neq 1,\ s_j \neq \mathbf{1}_A \text{ for } j \neq n,\ n \geqslant 1\}.$$

By identifying $\xi_{\varphi*\psi}$ with ξ_φ and ξ_ψ we will identify H_φ with $\mathbf{C}\xi_{\varphi*\psi} \oplus H^0_\varphi$ and H_ψ with $\mathbf{C}\xi_{\varphi*\psi} \oplus H^0_\psi$. By identifying $\mathbf{C}\xi_{\varphi*\psi} \otimes H^0_\psi \otimes \cdots$ with $H^0_\psi \otimes \cdots$ as well as

$\cdots \otimes H_\varphi^0 \otimes \mathbb{C}\xi_{\varphi*\psi}$ with $\cdots \otimes H_\varphi^0$ we obtain the isomorphism below:

$$\begin{aligned} H_\varphi \otimes H_\psi^0 \otimes \cdots \otimes H_\varphi^0 \otimes H_\psi \cong H_\varphi^0 &\otimes H_\psi^0 \otimes \cdots \otimes H_\varphi^0 \otimes H_\psi^0 \\ &\oplus H_\psi^0 \otimes \cdots \otimes H_\varphi^0 \otimes H_\psi^0 \\ &\oplus H_\varphi^0 \otimes H_\psi^0 \otimes \cdots \otimes H_\varphi^0 \\ &\oplus H_\psi^0 \otimes \cdots \otimes H_\varphi^0. \end{aligned}$$

The map which sends $\pi_{\varphi*\psi}(w)\xi_{\varphi*\psi} \in H_\varphi \otimes H_\psi^0 \otimes \cdots \otimes H_\varphi^0 \otimes H_\psi$ to $\delta(w) \in l^2(\Gamma)$ for $w \in \Gamma$ induces an isomorphism $H_{\varphi*\psi} \cong l^2(\Gamma)$. Let $\mathbf{1}$ denote the identity element of $A *_{\mathbb{C}} B$. Let $G^* = G\backslash\{\mathbf{1}\}$ and $S^* = S\backslash\{\mathbf{1}\}$. Let Γ_1^* denote the set of all words in Γ ending in G^* and Γ_2^* denote the set of all words in Γ ending in S^*. Let $\Gamma_1 = \Gamma_1^* \cup \{\mathbf{1}\}$ and $\Gamma_2 = \Gamma_2^* \cup \{\mathbf{1}\}$. Under the isomorphism $H_{\varphi*\psi} \cong l^2(\Gamma)$ we have the corresponding isomorphisms $H_i^* \cong l^2(\Gamma_i^*)$ where

$$\begin{aligned} H_1^* &\cong H_\varphi^0 \oplus (H_\psi^0 \otimes H_\varphi^0) \oplus (H_\varphi^0 \otimes H_\psi^0 \otimes H_\varphi^0) \oplus \cdots \\ H_2^* &\cong H_\psi^0 \oplus (H_\varphi^0 \otimes H_\psi^0) \oplus (H_\psi^0 \otimes H_\varphi^0 \otimes H_\psi^0) \oplus \cdots. \end{aligned}$$

Let $H_i = \mathbb{C}\xi_{\varphi*\psi} \oplus H_i^*$ so that $H_i \cong l^2(\Gamma_i)$. Define Γ^1 as the set of words in Γ beginning in G^* and Γ^2 as the set of words beginning in S^*. Let $\Gamma_i^j = \Gamma_i \cap \Gamma^j$. Define H^j and H_i^j in the obvious manner. For $a \in A$ and $b \in B$ we have $\pi_{\varphi*\psi}(a)H_1 \subset H_1$ and $\pi_{\varphi*\psi}(b)H_1^* \subset H_1^*$. For $a \in A$ let $\mu(a)$ denote the restriction of $\pi_{\varphi*\psi}(a)$ on H_1. For $b \in B$, let $\nu(b)$ denote the operator $\pi_{\varphi*\psi}(b)P(H_1^*)$ where $P(H_0)$ denotes the orthogonal projection onto H_0 for any closed subspace H_0 of H_1. The representations μ and ν can be factored through $\pi_{\varphi*\psi}$ and can be considered as representations of A_{red} and B_{red} respectively. For $w = (a_1)b_1 \cdots a_n(b_n) \in A *_{\mathbb{C}} B$ let $\sigma(w) = (\mu(a_1))\nu(b_1)\cdots\mu(a_n)(\nu(b_n))$. The parentheses around a term indicate that the term may or may not be present. Let $\mathcal{T}$ be the C^*-algebra generated by $\mu(A)$ and $\nu(B)$. Notice that $q \stackrel{\text{def}}{=} \mu(\mathbf{1}) - \nu(\mathbf{1}) \in \mathcal{T}$ and $q = P(\mathbb{C}\xi_{\varphi*\psi})$. Let J denote the closed ideal in $\mathcal{T}$ generated by q. Since q has rank one it follows that $J \subset \mathcal{K}(H_1)$. We assert that $J = \mathcal{K}(H_1)$. This will be shown by producing a system of matrix units for $\mathcal{K}(H_1)$ in J. For $w_1, w_2 \in \Gamma_1$ define $e(w_1, w_2)$ as follows:

$$e(w_1, w_2) = \sigma(w_1)q\sigma(w_2)^*, \quad w_1, w_2 \in \Gamma.$$

The relation

$$e(w_1, w_2)e(w_3, w_4) = \delta_{w_2,w_3}e(w_1, w_4)$$

will follow if we verify the relation

$$q\sigma(w_2)^*\sigma(w_3)q = \delta_{w_2,w_3}q.$$

It is enough to show that these two operators agree on $\xi_{\varphi*\psi}$. For any $\eta \in H_{\varphi*\psi}$ we have $q\eta = \langle \eta, \xi_{\varphi*\psi}\rangle \xi_{\varphi*\psi}$. Hence

$$\begin{aligned} q\sigma(w_2)^*\sigma(w_3)q\xi_{\varphi*\psi} &= \langle \sigma(w_2)^*\sigma(w_3)\xi_{\varphi*\psi}, \xi_{\varphi*\psi}\rangle \xi_{\varphi*\psi} \\ &= \langle \sigma(w_3)\xi_{\varphi*\psi}, \sigma(w_2)\xi_{\varphi*\psi}\rangle \xi_{\varphi*\psi} \\ &= \langle \pi_{\varphi*\psi}(w_3)\xi_{\varphi*\psi}, \pi_{\varphi*\psi}(w_2)\xi_{\varphi*\psi}\rangle \xi_{\varphi*\psi} \\ &= \delta_{w_2,w_3} q\xi_{\varphi*\psi}. \end{aligned}$$

It is clear that $e(w_i, w_j)^* = e(w_j, w_i)$ and that the net of finite sums $\sum e(w,w)$ converges strongly to the identity on $H_1 = l^2(\Gamma_1)$. Thus $\{e(w_i, w_j)\}$ is a system of matrix units for $\mathcal{K}(H_1)$ in J and hence $\mathcal{K}(H_1) \subset J$. We have the following analog of [10], Lemma 3.1.

LEMMA 4.1. *There exists a surjective homomorphism $\pi: T \to A *^{\mathrm{red}}_{\mathbf{C}} B$ such that*

$$\begin{aligned} \pi(\mu(a)) &= \pi_{\varphi*\psi}(a), \quad a \in A \\ \pi(\nu(b)) &= \pi_{\varphi*\psi}(b), \quad b \in B \end{aligned}$$

and $\ker \pi = J$.

Proof. Fix $h \in G^*$ and a unitary element t of B with $\psi(t) = 0$. For $n \geqslant 1$ let $E_n = \{w(ht)^n : w \in \Gamma_1\} \subset A *_{\mathbf{C}} B$. Let H_n denote the closed subspace $[\pi_{\varphi*\psi}(E_n)\xi_{\varphi*\psi}]$ of $H_{\varphi*\psi}$. It is easily verified that $H_n \subset H_{n+1}$ and $H_0 \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} H_n$ is a dense subspace of $H_{\varphi*\psi}$.

Let $v_n : H_1 \cong l^2(\Gamma_1) \to H_{\varphi*\psi}$ denote the linear map

$$v_n \pi_{\varphi*\psi}(w)\xi_{\varphi*\psi} = \pi_{\varphi*\psi}(w(ht)^n)\xi_{\varphi*\psi}, \; w \in \Gamma_1.$$

It follows from the fact that h, t are unitaries for which the state determined by $\xi_{\varphi*\psi}$ is conjugation invariant that v_n is an isometry. It is easily verified for $\eta \in H_n$, $a \in A$, $b \in B$, that

$$\begin{aligned} v_m \mu(a) v_m^* \eta &= \pi_{\varphi*\psi}(a)\eta \\ v_m \nu(b) v_m^* \eta &= \pi_{\varphi*\psi}(b)\eta \end{aligned}$$

holds for all $m \geqslant n$. Hence $v_n \mu(a) v_n^* \eta$ converges to $\pi_{\varphi*\psi}(a)\eta$ and $v_n \nu(b) v_n^*$ converges to $\pi_{\varphi*\psi}(b)\eta$ for all $\eta \in H_0$. Since $\|v_n \mu(a) v_n^*\| \leq \|a\|$ and $\|v_n \nu(b) v_n^*\| \leq \|b\|$

holds for all n and H_0 is dense in $H_{\varphi*\psi}$, it follows that

$$v_n\mu(a)v_n^* \longrightarrow \pi_{\varphi*\psi}(a) \text{ strongly } (a \in A)$$
$$v_n\nu(b)v_n^* \longrightarrow \pi_{\varphi*\psi}(b) \text{ strongly } (b \in B).$$

Hence the strong-limit $\pi(x) = \lim_{n\to\infty} v_n x v_n^*$ exists for all $x \in \mathcal{T}$ and $\pi: \mathcal{T} \to A *_{\mathbf{C}}^{\text{red}} B$ is a surjective homomorphism with the desired values on $\mu(A)$ and $\nu(B)$. Since $q_1 = \mu(\mathbf{1}) - \nu(\mathbf{1})$ and $\pi(\mu(\mathbf{1})) = \pi_{\varphi*\psi}(\mathbf{1}) = \pi(\nu(\mathbf{1}))$ it follows that $\pi(q_1) = 0$ and so $J \subset \ker\pi$.

It remains to be shown that $\ker\pi \subset J$. Let $p(x) = P(H_1)xP(H_1)$ for $x \in A *_{\mathbf{C}}^{\text{red}} B$. Then the linear map p satisfies $p(\pi_{\varphi*\psi}(g)) = \mu(g)$ for $g \in G$ and $p(\pi_{\varphi*\psi}(s)) = \nu(s)$ for $s \in S^*$. Then by induction on the length of $w \in \Gamma$ one can show $p\pi_{\varphi*\psi}(w) - \sigma(w) \in J$. Since $\pi\sigma = \pi_{\varphi*\psi}$ we have $p\pi\sigma(w) - \sigma(w) \in J$. Thus $p\pi(y) - y \in J$ for $y \in \mathcal{T}$. Hence if $y \in \ker\pi$ then $y \in J$. ∎

We now have a short exact sequence

$$0 \longrightarrow J \xrightarrow{i} \mathcal{T} \xrightarrow{\pi} A *_{\mathbf{C}}^{\text{red}} B \longrightarrow 0. \tag{4.1}$$

Let $\kappa: J = \mathcal{K}(H_1) \to \mathcal{K}(l^2(\Gamma_1))$ be the isomorphism induced by the isomorphism of H_1 and $l^2(\Gamma_1)$ described previously. We will show that $\mathcal{T}$ is KK-equivalent to $A_{\text{red}} \oplus B_{\text{red}}$. That is, we will show that there are elements $\xi \in \mathrm{KK}(A_{\text{red}} \oplus B_{\text{red}}, \mathcal{T})$ and $\eta \in \mathrm{KK}(\mathcal{T}, A_{\text{red}} \oplus B_{\text{red}})$ so that $\xi\eta = \mathbf{1}_{A_{\text{red}}\oplus B_{\text{red}}}$ and $\eta\xi = \mathbf{1}_{\mathcal{T}}$. Let $\eta_*: \mathrm{K}_*(\mathcal{T}) \to \mathrm{K}_*(A_{\text{red}} \oplus B_{\text{red}})$ be the isomorphism induced by η. We will also show that the diagram below commutes:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{K}_*(J) & \xrightarrow{i_*} & \mathrm{K}_*(\mathcal{T}) & \xrightarrow{\pi_*} & \mathrm{K}_*(A *_{\mathbf{C}}^{\text{red}} B) & \longrightarrow \\
 & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \mathrm{Id}_*} & \\
\longrightarrow & \mathrm{K}_*(\mathbf{C}) & \xrightarrow{(\chi_*^1, -\chi_*^2)} & \mathrm{K}_*(A_{\text{red}}) \oplus \mathrm{K}_*(B_{\text{red}}) & \xrightarrow{\epsilon_*^1 + \epsilon_*^2} & \mathrm{K}_*(A *_{\mathbf{C}}^{\text{red}} B) & \longrightarrow .
\end{array} \tag{4.2}$$

The top row is the periodic six term exact sequence of K-groups induced by (4.1). The bottom row is then exact since all of the vertical maps are isomorphisms and Theorem 3.4 follows.

5. CONSTRUCTION OF VARIOUS HOMOMORPHISMS

We now construct some homomorphisms which will be used in defining the elements ξ and η described in the previous section. Much of what is done here consists of translating Lance's terminology in [10], Section 4, into the setting of reduced free products. We assume from now on that G has property Λ with (λ_t) as the homotopy of representations of G on $l^2(G)$. The map from $H_{\varphi*\psi}$ into $H_2 \otimes H_\varphi$ induced by

$$\begin{aligned} \xi_{\varphi*\psi} &\mapsto \xi_{\varphi*\psi} \otimes \xi_{\varphi*\psi} \\ h_\varphi^0 &\mapsto \xi_{\varphi*\psi} \otimes h_\varphi^0 \\ \cdots \otimes h_\psi^0 \otimes h_\varphi^0 &\mapsto (\cdots \otimes h_\psi^0) \otimes h_\varphi^0 \\ \cdots \otimes h_\psi^0 &\mapsto (\cdots \otimes h_\psi^0) \otimes \xi_{\varphi*\psi} \qquad (h_\varphi^0 \in H_\varphi^0, h_\psi^0 \in H_\psi^0) \end{aligned}$$

is an isomorphism. The restriction of the above map to H_1^* is an isomorphism $H_1^* \cong H_2 \otimes H_\varphi^0$. Define a map

$$u\colon H_{\varphi*\psi} \otimes H_\varphi^0 \cong H_2 \otimes H_\varphi \otimes H_\varphi^0 \to H_1^* \otimes H_\varphi \cong H_2 \otimes H_\varphi^0 \otimes H_\varphi$$

as follows:

$$u(h_2 \otimes \delta(g) \otimes \delta(h)) = \sum_{k \in G^*} \langle \lambda_1(gh)\delta(h^{-1}), \delta(k) \rangle (h_2 \otimes \delta(k) \otimes \delta(k^{-1}gh)), \tag{5.1}$$

(where $g \in G$, $h \in G^*$, $h_2 \in H_2$). We have used the identifications $H_\varphi \cong l^2(G)$ and $H_\varphi^0 \cong l^2(G^*)$ given by $\pi_{\varphi*\psi}(g)\xi_{\varphi*\psi}$
$\leftrightarrow \delta(g)$. The following lemma is identical to a lemma in [10] after a minor change in notation.

LEMMA 5.1. ([10], Lemma 4.1) *The mapping u is an isometry from $H_{\varphi*\psi} \otimes H_\varphi^0$ onto $H_1^* \otimes H_\varphi$ with inverse given by the following formula for $h_2 \in H_2$, $h \in G^*$, $g \in G$:*

$$u^*(h_2 \otimes \delta(h) \otimes \delta(g)) = \sum_{k \in G^*} \langle \delta(h), \lambda_1(hg)\delta(k) \rangle (h_2 \otimes \delta(hgk) \otimes \delta(k^{-1})). \tag{5.2}$$

For $x \in \mathcal{T}$, $\pi(x) \otimes \mathbf{1} \in B(H_{\varphi*\psi}) \otimes B(H_\varphi)$. Regard u as a map into $H_1 \otimes H_\varphi$. Then $u^*\colon H_1 \otimes H_\varphi \to H_{\varphi*\psi} \otimes H_\varphi^0$ is given by (5.2) on $H_1^* \otimes H_\varphi$ and is zero on $\mathbf{C}\xi_{\varphi*\psi} \otimes H_\varphi$. It follows that $uu^* = P(H_1^*) \otimes \mathbf{1}$. Let

$$\omega,\ \bar{\omega}\colon \mathcal{T} \to B(H_1 \otimes H_\varphi)$$

be defined by

$$\begin{aligned} \omega(x) &= u(\pi(x) \otimes \mathbf{1})u^* \\ \bar{\omega}(x) &= x \otimes \mathbf{1} \qquad (x \in \mathcal{T}). \end{aligned} \tag{5.3}$$

For a C^*-algebra A, we let $\mathcal{M}(A)$ denote its multiplier algebra. The following lemma can be proved exactly as in [10] with a few minor changes in notation.

LEMMA 5.2. ([10], Lemma 4.2, [12], Lemma A5) *For* $x \in \mathcal{T}$,
(i) $\omega(x), \bar{\omega}(x) \in \mathcal{M}(\mathcal{K}(H_1) \otimes A_{\text{red}})$;
(ii) $\omega(x) - \bar{\omega}(x) \in \mathcal{K}(H_1) \otimes A_{\text{red}}$.

There is a unitary isomorphism $v: H_{\varphi * \psi} \to H_1 \otimes H_\psi$ defined in the same manner as the isomorphism $H_{\varphi * \psi} \cong H_2 \otimes H_\varphi$ in the beginning of this section. Define

$$\theta,\ \bar{\theta}: \mathcal{T} \to B(H_1 \otimes H_\psi)$$

as follows:

$$\theta(x) = v\pi(x)v^*$$
$$\bar{\theta}(x) = x \otimes \mathbf{1} \qquad (x \in \mathcal{T}).$$

One can show by direct computation that

$$\theta(\mu(a)) = \bar{\theta}(\mu(a)) \quad (a \in A)$$
$$\bar{\theta}(\nu(b)) - \bar{\theta}(\nu(b)) = q \otimes \pi_{\varphi * \psi}(b) \quad (b \in B).$$

This gives the following lemma.

LEMMA 5.3. *For* $x \in \mathcal{T}$
(i) $\theta(x), \bar{\theta}(x) \in \mathcal{M}(\mathcal{K}(H_1) \otimes B_{\text{red}})$;
(ii) $\theta(x) - \bar{\theta}(x) \in \mathcal{K}(H_1) \otimes B_{\text{red}}$.

6. KK-EQUIVALENCE OF $A_{\text{red}} \oplus C^*_{\text{red}}(G)$ AND $\mathcal{T}$.

We will use the quasihomomorphism picture of $\text{KK}(A, B)$ introduced by J. Cuntz ([3]). A quasihomomorphism from A to B is a pair of homomorphisms (φ_1, φ_2) satisfying

$$\varphi_j: A \to \mathcal{M}(\mathcal{K} \otimes B)$$
$$\varphi_1(a) - \varphi_2(a) \in \mathcal{K} \otimes B$$

where $\mathcal{K}$ denotes the ideal of compact operators on a separable infinite dimensional Hilbert space. Notice that $(\varphi, 0)$ is a quasihomomorphism from A to B for any homomorphism from A into $\mathcal{K} \otimes B$. The elements of the abelian group $\text{KK}(A, B)$ are the homotopy classes of quasihomomorphisms from A to B. The homotopy class of (α, β) will be denoted $[\alpha, \beta]$. We will use the following lemma about addition in $\text{KK}(A, B)$.

LEMMA 6.1. ([12], Lemma A4) *Let* $(\alpha, \bar{\alpha})$, $(\beta, \bar{\beta})$ *be quasihomomorphisms from* A *to* B. *Assume* $\alpha(x)\beta(y) = \bar{\alpha}(x)\bar{\beta}(y) = 0$ *for all* $x, y \in A$. *Then*

(i) $(\alpha + \beta, \bar{\alpha} + \bar{\beta})$ *is a quasihomomorphism from* A *to* B;

(ii) $[\alpha + \beta, \bar{\alpha} + \bar{\beta}] = [\alpha, \bar{\alpha}] + [\beta, \bar{\beta}]$.

By identifying $A_{\text{red}} \oplus 0$ with A_{red} and $0 \oplus B_{\text{red}}$ with B_{red} we can regard μ and ν as homomorphisms from $A_{\text{red}} \oplus B_{\text{red}}$ into $\mathcal{T}$ and $\theta, \bar{\theta}, \omega, \bar{\omega}$ as homomorphisms into $A_{\text{red}} \oplus B_{\text{red}}$. Let $j: \mathcal{T} \to \mathcal{K}(H_1) \otimes \mathcal{T}$ be defined by $j(x) = q \otimes x$. Then $(j\mu, 0)$ and $(j\nu, 0)$ are quasihomomorphisms from $A_{\text{red}} \oplus B_{\text{red}}$ into $\mathcal{T}$. By Lemmas 5.2 and 5.3, $(\theta, \bar{\theta})$ and $(\omega, \bar{\omega})$ are quasihomomorphisms from $\mathcal{T}$ into $A_{\text{red}} \oplus B_{\text{red}}$. Let

$$\xi = [j\mu, 0] + [j\nu, 0] \in \mathrm{KK}(A_{\text{red}} \oplus B_{\text{red}}, \mathcal{T})$$
$$\eta = [\theta, \bar{\theta}] - [\omega, \bar{\omega}] \in \mathrm{KK}(\mathcal{T}, A_{\text{red}} \oplus B_{\text{red}}).$$

We will show that $\eta\xi = \mathbf{1}_{A_{\text{red}} \oplus B_{\text{red}}}$ and $\xi\eta = \mathbf{1}_{\mathcal{T}}$. The proofs follow closely those in [12] so we will only give a brief outline of each proof.

PROPOSITION 6.2. ([12], Proposition A6) $\eta\xi = \mathbf{1}_{A_{\text{red}} \oplus B_{\text{red}}}$.

Proof. $\mathbf{1}_{A_{\text{red}} \oplus B_{\text{red}}}$ is represented by $[i_1, 0] + [i_2, 0]$ where $i_1: A_{\text{red}} \to \mathcal{K} \otimes A_{\text{red}}$ is defined by $i_1(x) = q \otimes x$ for $x \in A_{\text{red}}$ and $i_2: B_{\text{red}} \to \mathcal{K} \otimes B_{\text{red}}$ is defined similiarly. Hence

$$\begin{aligned}
\eta\xi - \mathbf{1}_{A_{\text{red}} \oplus B_{\text{red}}} &= ([\theta, \bar{\theta}] - [\omega, \bar{\omega}])([j\mu, 0] + [j\nu, 0]) - \mathbf{1}_{A_{\text{red}} \oplus B_{\text{red}}} \\
&= [\theta, \bar{\theta}][j\nu, 0] - [\omega, \bar{\omega}][j\mu, 0] - \mathbf{1}_{A_{\text{red}} \oplus B_{\text{red}}} \\
&= [\theta\nu, \bar{\theta}\nu] - [\omega\mu, \bar{\omega}\mu] - [i_1, 0] - [i_2, 0] \\
&= -([\bar{\theta}\nu, \theta\nu] + [i_2, 0]) - ([\omega\mu, \bar{\omega}\mu] + [i_1, 0]) \\
&= -[\bar{\theta}\nu + i_2, \theta\nu] - [\omega\mu + i_1, \bar{\omega}\mu].
\end{aligned}$$

The second equality is a consequence of $\theta\mu = \bar{\theta}\mu$ and $\omega\nu = \bar{\omega}\nu$. The last equality follows from Lemma 6.1 and the fact that $(\omega\mu)(x)i_1(y) = 0$ for $x, y \in A_{\text{red}}$ and $(\bar{\theta}\nu)(x)i_2(y) = 0$ for $x, y \in B_{\text{red}}$. It then suffices to show

$$[\bar{\theta}\nu + i_2, \theta\nu] = 0 \quad \text{in } \mathrm{KK}(B_{\text{red}}, B_{\text{red}})$$
$$[\omega\mu + i_1, \bar{\omega}\mu] = 0 \quad \text{in } \mathrm{KK}(A_{\text{red}}, A_{\text{red}}).$$

Direct computation shows that $\bar{\theta}\nu + i_2 = \theta\nu$ and so $[\bar{\theta}\nu + i_2, \theta\nu] = 0$. If we show that $\omega\mu + i_1$ is $\mathcal{K} \otimes A_{\text{red}}$-homotopic to $\bar{\omega}\mu$ then $[\omega\mu + i_1, \bar{\omega}\mu] = 0$ will follow. The proof of this is the same as the proof of [12], Lemma A7, with the appropriate change of notation to the setting of reduced free products of C^*-algebras. ∎

PROPOSITION 6.3. ([12], Proposition A8) $\xi\eta = \mathbf{1}_{\mathcal{T}}$.

Proof. $\mathbf{1}_{\mathcal{T}}$ is represented by $[j, 0]$. Let $\bar{\mu}$ (resp. $\bar{\nu}$) denote the homomorphism of $\mathcal{M}(\mathcal{K}\otimes A_{\text{red}})$ (resp. $\mathcal{M}(\mathcal{K}\otimes B_{\text{red}})$) into $\mathcal{M}(\mathcal{K}\otimes\mathcal{T})$ which extends $\mathbf{1}\otimes\mu: \mathcal{K}\otimes A_{\text{red}} \to \mathcal{K}\otimes\mathcal{T}$ (resp. $\mathbf{1}\otimes\nu: \mathcal{K}\otimes B_{\text{red}} \to \mathcal{K}\otimes\mathcal{T}$). Let $k: \mathcal{T} \to B(H_1)\otimes B(H_1)$ be defined by $k(x) = x \otimes q$. Then

$$\begin{aligned}
\xi\eta - \mathbf{1}_{\mathcal{T}} &= ([j\mu, 0] + [j\nu, 0])([\theta, \bar{\theta}] - [\omega, \bar{\omega}]) - \mathbf{1}_{\mathcal{T}} \\
&= -[\bar{\mu}\omega, \bar{\mu}\bar{\omega}] - [\bar{\nu}\omega, \bar{\nu}\bar{\omega}] + [\bar{\mu}\theta, \bar{\mu}\bar{\theta}] + [\bar{\nu}\theta, \bar{\nu}\bar{\theta}] - \mathbf{1}_{\mathcal{T}} \\
&= -[\bar{\mu}\omega, \bar{\mu}\bar{\omega}] + [\bar{\nu}\theta, \bar{\nu}\bar{\theta}] + [k, k] - [j, 0] \\
&= -[\bar{\mu}\omega + j, \bar{\mu}\bar{\omega}] + [\bar{\nu}\theta + k, \bar{\nu}\bar{\theta} + k] \\
&= -[\bar{\mu}\omega + j, \bar{\mu}\bar{\omega}] + [\bar{\nu}\theta + k, \bar{\mu}\bar{\omega}].
\end{aligned}$$

The second and third terms of the second line are zero because $\bar{\nu}\omega = \bar{\nu}\bar{\omega}$ and $\bar{\mu}\theta = \bar{\mu}\bar{\theta}$. The fourth equality is a consequence of Lemma 6.1 and the fifth follows from $\bar{\mu}\bar{\omega} = \bar{\nu}\bar{\theta} + k$. The conclusion will follow if we show that $\bar{\mu}\omega + j$ is $\mathcal{K}\otimes\mathcal{T}$-homotopic to $\bar{\nu}\theta + k$. For this we refer the reader to [12], Lemma A9. ∎

7. CONCLUSION OF THE PROOF

All that remains is to prove commutativity of the diagram (4.2). The unlabeled horizontal maps in the bottom row of (4.2) are by definition the maps required in order to make the squares that they are contained in commute. Thus we need only to show that $\pi_* = (\varepsilon_*^1 + \varepsilon_*^2)\eta_*$ and $\eta_* i_* = (\chi_*^1, -\chi_*^2)\kappa_*$. Now since $\eta_* = \xi_*^{-1}$, this is equivalent to showing:

(1) $\pi_*\xi_* = \varepsilon_*^1 + \varepsilon_*^2$;

(2) $\eta_* i_* \kappa_*^{-1} = (\chi_*^1, -\chi_*^2)$.

Since $\xi = [j\mu, 0] + [j\nu, 0]$, (1) follows from $\pi_*\mu_* = \varepsilon_*^1$ and $\pi_*\nu_* = \varepsilon_*^2$. Since $\eta = [\theta, \bar{\theta}] - [\omega, \bar{\omega}]$, (2) will follow if we show

(i) $[\omega, \bar{\omega}][i, 0][\alpha, 0] = [i_1, 0][\chi^1, 0]$ in $\mathrm{KK}(\mathbb{C}, A_{\text{red}})$;

(ii) $[\bar{\theta}, \theta][i, 0][\alpha, 0] = [i_2, 0][\chi^2, 0]$ in $\mathrm{KK}(\mathbb{C}, B_{\text{red}})$;

where $\alpha: \mathbb{C} \to \mathcal{K}(H_1)$ is defined by $\alpha(z) = zq$. This follows from the equalities $\omega i\alpha = i_1\chi^1$, $\bar{\omega} i\alpha = 0$, $\bar{\theta} i\alpha = i_2\chi^2$, and $\theta i\alpha = 0$ which can be verified by direct computation.

8. CONSEQUENCES OF THE MAIN RESULT

In the following section we assume that G is a countable discrete group. We let φ and ψ denote the usual traces on M_n and $C^*(G)$ and $C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n$ denote the reduced free product of $C^*(G)$ and M_n relative to (φ, ψ). χ^1, χ^2, ε^1, and ε^2 will be as in the statement of Theorem 3.2 with $B = B_{\mathrm{red}} = M_n$. We will also let ε^j denote the inclusion maps into the nonreduced product $C^*(G) *_{\mathbf{C}} M_n$ and χ^2 will denote the inclusion map into the full group C^*-algebra $C^*(G)$ as well. Since $(\chi^1_*, -\chi^2_*)$ is injective in this case, the exact sequence in (3.1) splits into two short exact sequences. This is summarized in the following proposition.

PROPOSITION 8.1. *Let G be a countable discrete group with property Λ. Then the following sequence is exact for $j = 0, 1$:*

$$(8.1) \qquad 0 \longrightarrow \mathrm{K}_j(\mathbf{C}) \xrightarrow{(\chi^1_*, -\chi^2_*)} \mathrm{K}_j(C^*_{\mathrm{red}}(G)) \oplus \mathrm{K}_j(M_n) \xrightarrow{\varepsilon^1_* + \varepsilon^2_*} \mathrm{K}_j(C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n) \longrightarrow 0.$$

In particular, for $j = 1$, ε^1_* is an isomorphism of the groups $\mathrm{K}_1(C^*_{\mathrm{red}}(G))$ and $\mathrm{K}_1(C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n)$.

Proposition 8.1 also holds in the setting of full free products. That is, (8.1) holds if $C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n$ is replaced by $C^*(G) *_{\mathbf{C}} M_n$ and $C^*_{\mathrm{red}}(G)$ is replaced by $C^*(G)$. To show this we will need a result of N.C. Phillips. Given a C^*-algebra A and an integer $n \geqslant 1$, let $W_n(A)$ be the universal C^*-algebra generated by the symbols $x_n(a, i, j)$ for $a \in A$, $1 \leqslant i, j \leqslant n$, subject to the relations which we will now describe. Let $x_n(a)$ denote the $n \times n$ matrix $[x_n(a, i, j)]$ and let $x_n^*(a)$ denote the $n \times n$ matrix $[x_n(a, j, i)^*]$. For any polynomial f in $2k$ noncommuting variables with no constant term and every $a_1, \ldots, a_k \in A$ such that

$$f(a_1, a_1^*, \ldots, a_k, a_k^*) = 0$$

we require the n^2 relations on the entries of $x_n(a_i)$ and $x_n^*(a_i)$ which make the following equality hold:

$$f(x_n(a_1), x_n^*(a_1), \ldots, x_n(a_k), x_n^*(a_k)) = 0.$$

Let B be a C^*-algebra and $\psi: A \to M_n(B)$ be a homomorphism. Define a homomorphsim $\bar{\psi}: W_n(A) \to B$ by setting $\bar{\psi}(x_n(a, i, j)) = \psi(a)_{i,j}$. The correspondence $\psi \mapsto \bar{\psi}$ is a bijection of $\mathrm{Hom}(A, M_n(B))$ and $\mathrm{Hom}(W_n(A), B)$ ([13], Proposition 1.5). Let $\varphi: A \to M_n(A)$ be the map defined by $\varphi(a) = a \oplus 0_{n-1}$. Let $m: A \to M_n(W_n(A))$ be the homomorphism defined by $m(a) = x_n(a)$. The following proposition is due to Phillips.

PROPOSITION 8.2. *Let A be a C^*-algebra. The element $\bar{\varphi} \in \mathrm{KK}(W_n(A), A)$ is a* KK*-equivalence with $m \in \mathrm{KK}(A, W_n(A))$ as its inverse.*

Let M_n^c denote the relative commutant of M_n in the free product $A *_{\mathbf{C}} M_n$. Let e_{ij} be a system of matrix units for M_n. The map $x \otimes y \mapsto xy$ is an isomorphism of $M_n \otimes M_n^c \cong M_n(M_n^c)$ and $A *_{\mathbf{C}} M_n$ with the inverse map given by $x \mapsto [x(i,j)]$ where $x(i,j) = \sum_{k=1}^{n} e_{ki} x e_{jk} \in M_n^c$. If $\alpha: M_n^c \to E$ is a unital homomorphism, then $\mathrm{Id} \otimes \alpha: M_n \otimes M_n^c \to M_n \otimes E$ is a homomorphism inducing a unital homomorphism $\alpha_0: A *_{\mathbf{C}} M_n \to M_n(E)$ which restricts to a unital homomorphism $\bar{\alpha}: A \to M_n(E)$. Conversely, any unital homomorphism $\beta: A \to M_n(E)$ determines a unital homomorphism $\mathrm{Id} * \beta: A *_{\mathbf{C}} M_n \to M_n(E)$. Since $(\mathrm{Id} * \beta)(M_n) = M_n \otimes \mathbf{1}_E$, it follows that $(\mathrm{Id} * \beta)(M_n^c) \subset I_n \otimes E \cong E$. Thus the restriction β' of $\mathrm{Id} * \beta$ to M_n^c determines a homomorphism $\beta': M_n^c \to E$. The correspondences $\alpha \mapsto \bar{\alpha}$ and $\beta \mapsto \beta'$ are inverses giving a bijection from $\mathrm{Hom}_1(M_n^c, E)$ to $\mathrm{Hom}_1(A, M_n(E))$ for any unital C^*-algebra E, where $\mathrm{Hom}_1(A, B)$ denotes the set of unital homomorphisms from A to B. Using these observations one sees that M_n^c is the universal C^*-algebra given by generators $a(i,j)$ subject to the condition that

$$f(a_1, a_1^*, \ldots, a_k, a_k^*) = 0, \ a_1, \ldots, a_k \in A$$

implies

$$f([a_1(i,j)], [a_1(j,i)^*], \ldots, [a_k(i,j)], [a_k(j,i)^*]) = 0$$

for any polynomial f in $2k$ noncommuting variables. Suppose A is a unital C^*-algebra and $\omega: A \to \mathbf{C}$ is a unital homomorphism. It follows from the above observations that the maps

$$\varphi: M_n^c \to W_n(\ker \omega)^+$$
$$\varphi(a(i,j)) = x_n(a - \omega(a)\mathbf{1}_A, i, j) + \omega(a)\delta_{ij}\mathbf{1}$$

and

$$\psi: W_n(\ker \omega)^+ \to M_n^c$$
$$\psi(x_n(a,i,j) + \lambda\mathbf{1}) = a(i,j) + \lambda\mathbf{1}$$

are homomorphisms which are mutual inverses. These observations are summarized in the following lemma which is a generalization of [13], Lemma 4.1, where the case $A = C(\mathbf{S}^1)$ is considered.

LEMMA 8.3. *Let A be a unital C^*-algebra and $\omega: A \to \mathbf{C}$ a unital homomorphism. Let M_n^c denote the relative commutant of M_n in $A *_{\mathbf{C}} M_n$. Then*

$$M_n^c \cong W_n(\ker\omega)^+.$$

The following theorem is an improvement of [11], Theorem 2.3, where it was shown that the K-groups of A and $A *_{\mathbf{C}} M_n$ are isomorphic assuming the existence of a retraction from A to $\mathbf{C}$.

THEOREM 8.4. *If A is a unital C^*-algebra and a unital homomorphism from A to $\mathbf{C}$ exists, then A and $A *_{\mathbf{C}} M_n$ are KK-equivalent.*

Proof. Let $\omega: A \to \mathbf{C}$ be a unital homomorphism. Let $\approx$ denote KK-equivalence. Then $\ker\omega \approx W_n(\ker\omega)$ by Proposition 8.2. Hence by Lemma 8.3

$$A \cong (\ker\omega)^+ \approx W_n(\ker\omega)^+ \cong M_n^c \approx M_n \otimes M_n^c \cong A *_{\mathbf{C}} M_n. \quad \blacksquare$$

EXAMPLE 8.5. Theorem 8.4 need not hold if no retraction from A to $\mathbf{C}$ exists. To see this take $A = O_2$. O_n denotes the Cuntz algebra generated by isometries $S_1, \ldots, S_n$ satisfying $\sum_{k=1}^{n} S_k S_k^* = 1$ ([8]). Let S_1, S_2 denote the generators of O_2 and T_1, T_2, T_3 denote the generators of O_3. Let

$$\alpha: O_2 \to M_2(O_3)$$

be the homomorphism defined by

$$\alpha(S_1) = \begin{bmatrix} 1 & 0 \\ 0 & T_1 \end{bmatrix}$$
$$\alpha(S_2) = \begin{bmatrix} 0 & 0 \\ T_2 & T_3 \end{bmatrix}.$$

This defines a homomorphism since $\alpha(S_i)$ is an isometry and $\alpha(S_1)\alpha(S_1)^* + \alpha(S_2)\alpha(S_2)^* = I_2$ ([8], 1.12). Let

$$\beta = \mathrm{Id} * \alpha: M_2 *_{\mathbf{C}} O_2 \to M_2(O_3).$$

Since β is unital and $\mathrm{K}_0(O_3) \cong \mathbf{Z}_2$ is generated by $[1]_0$ ([6], Theorem 3.7), it follows that

$$\beta_*: \mathrm{K}_0(M_2 *_{\mathbf{C}} O_2) \to \mathrm{K}_0(O_3) \cong \mathbf{Z}_2$$

is onto. Hence $\mathrm{K}_0(M_2 *_{\mathbf{C}} O_2) \neq 0$. However $\mathrm{K}_0(O_2) = 0$ ([6], Theorem 3.7).

We will now show that Proposition 8.1 holds in the nonreduced setting. That is, (8.1) holds if $C^*(G) *_{\mathbb{C}}^{\mathrm{red}} M_n$ is replaced by $C^*(G) *_{\mathbb{C}} M_n$ and $C^*_{\mathrm{red}}(G)$ is replaced by $C^*(G)$.

PROPOSITION 8.6. *Let B be a unital C^*-algebra which has a unital $*$-homomorphism $\omega: B \to \mathbb{C}$. Then the following sequence is exact for $j = 0, 1$:*

$$(8.2) \qquad 0 \longrightarrow \mathrm{K}_j(\mathbb{C}) \xrightarrow{(\chi^1_*, -\chi^2_*)} \mathrm{K}_j(B) \oplus \mathrm{K}_j(M_n) \xrightarrow{\varepsilon^1_* + \varepsilon^2_*} \mathrm{K}_j(B *_{\mathbb{C}} M_n) \longrightarrow 0.$$

In particular, for $j = 1$, ε^1_* is an isomorphism of $\mathrm{K}_1(B)$ and $\mathrm{K}_1(B *_{\mathbb{C}} M_n)$.

Proof. The isomorphisms

$$B \cong (\ker \omega)^+$$
$$B *_{\mathbb{C}} M_n \cong M_n \otimes W_n(\ker \omega)^+$$

induce isomorphisms

$$\mathrm{K}_j(M_n) \oplus \mathrm{K}_j(B) \cong \mathrm{K}_j(M_n) \oplus \mathrm{K}_j(\ker \omega) \oplus \mathrm{K}_j(\mathbb{C})$$
$$\mathrm{K}_j(B *_{\mathbb{C}} M_n) \cong \mathrm{K}_j(W_n(\ker \omega)) \oplus \mathrm{K}_j(\mathbb{C}).$$

The map

$$(\chi^1_*, -\chi^2_*): \mathrm{K}_j(\mathbb{C}) \to \mathrm{K}_j(M_n) \oplus \mathrm{K}_j(\ker \omega) \oplus \mathrm{K}_j(\mathbb{C})$$

is given by

$$(8.3) \qquad (\chi^1_*, -\chi^2_*)(y) = ny \oplus 0 \oplus -y.$$

The map

$$\varepsilon^1_* + \varepsilon^2_*: \mathrm{K}_j(M_n) \oplus \mathrm{K}_j(\ker \omega) \oplus \mathrm{K}_j(\mathbb{C}) \to \mathrm{K}_j(W_n(\ker \omega)) \oplus \mathrm{K}_j(\mathbb{C})$$

is given by

$$(8.4) \qquad (\varepsilon^1_* + \varepsilon^2_*)(x \oplus y \oplus z) = m_*(y) \oplus (x + nz)$$

where m is the homomorphism $m(b) = x_n(b)$ from $\ker \omega$ to $M_n(W_n(\ker \omega))$. Since m_* is an isomorphism by Proposition 8.2 it follows from (8.4) that $\varepsilon^1_* + \varepsilon^2_*$ is onto. It follows from (8.3) that $(\chi^1_*, -\chi^2_*)$ is one-to-one. This establishes exactness at every term of (8.2) except the middle term. $(\varepsilon^1_* + \varepsilon^2_*)(\chi^1_*, -\chi^2_*) = 0$ is trivial. Suppose $x \oplus y \oplus z$ is in the kernel of $\varepsilon^1_* + \varepsilon^2_*$. Then by (8.4) $m_*(y) = 0 = x + nz$. Since m_* is an isomorphism $y = 0$. So $x \oplus y \oplus z = -nz \oplus 0 \oplus z = (\chi^1_*, -\chi^2_*)(-z)$ by (8.3). ∎

In [4], J. Cuntz defined the notion of a K-amenable discrete group. A discrete group G is K-amenable if the left regular representation λ (considered as a homomorphism from $C^*(G)$ to $C^*_{\mathrm{red}}(G)$) is an invertible element of $\mathrm{KK}(C^*(G), C^*_{\mathrm{red}}(G))$. If G has property Λ and λ_1 is a representation of G on $l^2(G)$ which is K-homotopic to the left regular representation and has δ_e as a fixed point, then the quasihomomorphism $(\lambda_1, \lambda_1|\delta_e^{\perp})$ gives an element $t_{\mathrm{red}} \in \mathrm{KK}(C^*_{\mathrm{red}}(G), \mathbf{C})$ for which $t_{\mathrm{red}}\lambda = t$ in $\mathrm{KK}(C^*(G), \mathbf{C})$, where $t: C^*(G) \to \mathbf{C}$ is induced by trivial representation. According to [4], Theorem 2.1(a), this implies that G is K-amenable. We also remark that if G is K-amenable, then $\lambda_*: \mathrm{K}_j(C^*(G)) \to \mathrm{K}_j(C^*_{\mathrm{red}}(G))$ is an isomorphism.

COROLLARY 8.7. *Suppose G is a countable discrete group with property Λ. Then the representation*

$$\pi_{\varphi*\psi}: C^*(G) *_{\mathbf{C}} M_n \to C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n$$

induces K*-group isomorphisms*

$$(\pi_{\varphi*\psi})_*: \mathrm{K}_j(C^*(G) *_{\mathbf{C}} M_n) \to \mathrm{K}_j(C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n)$$

for $j = 0, 1$.

Proof. The result follows immediately from the fact that the diagram below commutes, has exact rows by Propositions 8.1 and 8.6, and has isomorphisms for the leftmost two vertical maps:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{K}_j(\mathbf{C}) & \longrightarrow & \mathrm{K}_j(C^*(G)) \oplus \mathrm{K}_j(M_n) & \longrightarrow & \mathrm{K}_j(C^*(G) *_{\mathbf{C}} M_n) & \longrightarrow 0 \\
& \Big\downarrow \mathrm{Id}_* & & \Big\downarrow \lambda_* \oplus \mathrm{Id}_* & & \Big\downarrow (\pi_{\varphi*\psi})_* & \\
0 \longrightarrow & \mathrm{K}_j(\mathbf{C}) & \longrightarrow & \mathrm{K}_j(C^*_{\mathrm{red}}(G)) \oplus \mathrm{K}_j(M_n) & \longrightarrow & \mathrm{K}_j(C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n) & \longrightarrow 0.
\end{array}$$ ∎

COROLLARY 8.8. *If G is a countable discrete group with property Λ, then*

$$\mathrm{K}_j(C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n) \cong \mathrm{K}_j(C^*(G) *_{\mathbf{C}} M_n) \cong \mathrm{K}_j(C^*(G)) \cong \mathrm{K}_j(C^*_{\mathrm{red}}(G)).$$

Proof. The first isomorphism follows from Corollary 8.7, the second follows from Theorem 8.4, and the third follows from K-amenability. ∎

As a consequence of the above corollaries one can compute the K-groups of the C^*-algebras $U^{\mathrm{nc}}_{n,\mathrm{red}}$ and $G^{\mathrm{nc}}_{n,\mathrm{red}}$ studied in [11]. $U^{\mathrm{nc}}_{n,\mathrm{red}}$ and $G^{\mathrm{nc}}_{n,\mathrm{red}}$ are defined to be the reduced versions of the noncommutative unitary C^*-algebras U^{nc}_n and the noncommutative Grassmanian C^*-algebra G^{nc}_n introduced by L. Brown ([2]). U^{nc}_n is defined to be the universal C^*-algebra generated by elements u_{ij}, $1 \leqslant i, j \leqslant n$, satisfying the relations which make the matrix $[u_{ij}]$ a unitary matrix. U_n^{nc} is isomorphic to the relative commutant M_n^c of M_n in the free product $M_n *_{\mathbf{C}} C^*(\mathbf{Z})$. $U_{n,\mathrm{red}}^{\mathrm{nc}}$ is defined to be the relative commutant of $\pi_{\varphi*\psi}(M_n) \cong M_n$ in $M_n *_{\mathbf{C}}^{\mathrm{red}} C^*(\mathbf{Z})$. It follows that $M_n \otimes U_{n,\mathrm{red}}^{\mathrm{nc}} \cong M_n *_{\mathbf{C}}^{\mathrm{red}} C^*(\mathbf{Z})$. G_n^{nc} is defined to be the universal C^*-algebra generated by elements p_{ij}, $1 \leqslant i,j \leqslant n$, and a multiplicative identity satisfying the relations which make the matrix $[p_{ij}]$ a projection. G_n^{nc} is isomorphic to the relative commutant M_n^c of M_n in the free product $M_n *_{\mathbf{C}} C^*(\mathbf{Z}_2)$. $G_{n,\mathrm{red}}^{\mathrm{nc}}$ is defined to be the relative commutant M_n^c of M_n in the reduced free product $M_n *_{\mathbf{C}}^{\mathrm{red}} C^*(\mathbf{Z}_2)$. It follows that $M_n \otimes G_{n,\mathrm{red}}^{\mathrm{nc}} \cong M_n *_{\mathbf{C}}^{\mathrm{red}} C^*(\mathbf{Z}_2)$. Since $\mathbf{Z}$ and $\mathbf{Z}_2$ are amenable, they are K-amenable and satisfy property A. Hence by Corollary 8.8 we have the following proposition.

PROPOSITION 8.9.

$$\mathrm{K}_j(U_{n,\mathrm{red}}^{\mathrm{nc}}) \cong \mathrm{K}_j(U_n^{\mathrm{nc}}) \cong \mathrm{K}_j(C^*(\mathbf{Z})) \cong \mathbf{Z},\ j = 0,1$$
$$\mathrm{K}_0(G_{n,\mathrm{red}}^{\mathrm{nc}}) \cong \mathrm{K}_0(G_n^{\mathrm{nc}}) \cong \mathrm{K}_0(C^*(\mathbf{Z}_2)) \cong \mathbf{Z}^2$$
$$\mathrm{K}_1(G_{n,\mathrm{red}}^{\mathrm{nc}}) \cong \mathrm{K}_1(G_n^{\mathrm{nc}}) \cong \mathrm{K}_1(C^*(\mathbf{Z}_2)) = 0.$$

The K-groups of U_n^{nc} were previously computed in [13], Lemmas 4.1 and 4.3, and the K-groups of $G_{n,\mathrm{red}}^{\mathrm{nc}}$ were computed in [11], Corollary 2.5. Let $\tau^{\mathbf{Z}}$ and $\tau^{\mathbf{Z}_2}$ denote $\varphi * \psi$ in the cases $C^*(\mathbf{Z}) *_{\mathbf{C}}^{\mathrm{red}} M_n$ and $C^*(\mathbf{Z}_2) *_{\mathbf{C}}^{\mathrm{red}} M_n$ respectively. Let

$$\tau_*^G : \mathrm{K}_0(C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n) \to \mathbf{R}$$

denote the group homomorphism induced by τ^G. The generator of the group $\mathrm{K}_0(C^*(\mathbf{Z}) *_{\mathbf{C}}^{\mathrm{red}} M_n)$ can be taken to be $[e_{11}]_0$ and the generators of the group $\mathrm{K}_0(C^*(\mathbf{Z}_2) *_{\mathbf{C}}^{\mathrm{red}} M_n)$ can be taken to be $[e_{11}]_0$ and $[\frac{a+b}{2}]_0$ where a and b are the elements of $\mathbf{Z}_2$. Since $\tau^{\mathbf{Z}}(e_{11}) = \tau^{\mathbf{Z}_2}(e_{11}) = \frac{1}{n}$ and $\tau^{\mathbf{Z}_2}(\frac{a+b}{2}) = \frac{1}{2}$ it follows that

$$\tau_*^{\mathbf{Z}}(\mathrm{K}_0(C^*(\mathbf{Z}) *_{\mathbf{C}}^{\mathrm{red}} M_n)) = \frac{1}{n}\mathbf{Z}$$
$$\tau_*^{\mathbf{Z}_2}(\mathrm{K}_0(C^*(\mathbf{Z}_2) *_{\mathbf{C}}^{\mathrm{red}} M_n)) = \frac{1}{[n,2]}\mathbf{Z}$$

where $[x,y]$ denotes the least common multiple of x and y. If P is a projection in $M_n^c \subset C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n$, then $P\pi_{\varphi*\psi}(e_{11})$ is a projection in $C^*(G) *_{\mathbf{C}}^{\mathrm{red}} M_n$ satisfying $\varphi * \psi(P) = n\varphi * \psi(P\pi_{\varphi*\psi}(e_{11}))$. Thus

$$\tau^{\mathbf{Z}}(\mathrm{Proj}(U_{n,\mathrm{red}}^{\mathrm{nc}})) \subset \mathbf{Z}$$
$$\tau^{\mathbf{Z}_2}(\mathrm{Proj}(G_{n,\mathrm{red}}^{\mathrm{nc}})) \subset \frac{n}{[n,2]}\mathbf{Z},$$

where $\mathrm{Proj}(A)$ denotes the set of projections of a C^*-algebra A. These observations lead to the following propostion.

PROPOSITION 8.10. *$U^{\mathrm{nc}}_{n,\mathrm{red}}$ is a simple projectionless C^*-algebra for $n \geqslant 2$. $G^{\mathrm{nc}}_{n,\mathrm{red}}$ is a simple projectionless C^*-algebra if n is even.*

Proof. The simplicity of $C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n$ follows from Proposition 2.1 if $n > 1$ and $|G| > 1$. The simplicity of $U^{\mathrm{nc}}_{n,\mathrm{red}}$ and $G^{\mathrm{nc}}_{n,\mathrm{red}}$ follows from the isomorphisms $M_n \otimes U^{\mathrm{nc}}_{n,\mathrm{red}} \cong C^*(\mathbf{Z}) *^{\mathrm{red}}_{\mathbf{C}} M_n$ and $M_n \otimes G^{\mathrm{nc}}_{n,\mathrm{red}} \cong C^*(\mathbf{Z}_2) *^{\mathrm{red}}_{\mathbf{C}} M_n$. The discussion preceding the statement of the proposition shows that $\tau^{\mathbf{Z}}(\mathrm{Proj}(U^{\mathrm{nc}}_{n,\mathrm{red}}))$ and $\tau^{\mathbf{Z}_2}(\mathrm{Proj}(G^{\mathrm{nc}}_{n,\mathrm{red}}))$ are contained in $\mathbf{Z}$ under the hypotheses on n. The projectionlessness of $U^{\mathrm{nc}}_{n,\mathrm{red}}$ and $G^{\mathrm{nc}}_{n,\mathrm{red}}$ then follows from the faithfulness of $\tau^{\mathbf{Z}}$ and $\tau^{\mathbf{Z}_2}$. ∎

Proposition 8.10 can be generalized as follows.

THEOREM 8.11. *Let G be a countable discrete group with property Λ. Let $\varphi\colon C^*_{\mathrm{red}}(G) \to \mathbf{C}$ denote the usual faithful trace. Suppose $\varphi_*(\mathrm{K}_0(C^*_{\mathrm{red}}(G)) \subset \frac{1}{n}\mathbf{Z}$, where φ_* is the homomorphism from $\mathrm{K}_0(C^*_{\mathrm{red}}(G))$ into $\mathbf{R}$ induced by φ. Then the relative commutant M_n^c of M_n in $C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n$ is a simple projectionless C^*-algebra if $n > 1$ and $|G| > 1$.*

Proof. The simplicity of $C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n$ follows from Proposition 2.1. The isomorphism $M_n \otimes M_n^c \cong C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n$ implies that M_n^c is simple. Let $\tau = \varphi * \psi$ where φ is the trace on M_n. Let

$$\tau_*\colon \mathrm{K}_0(C^*_{\mathrm{red}}(G)) \to \mathbf{R}$$

denote the induced group homomorphism. It follows from the exact sequence (3.1) that

$$\begin{aligned}\tau_*(\mathrm{K}_0(C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n)) &\subset \tau_*(\varepsilon^1_*(\mathrm{K}_0(C^*_{\mathrm{red}}(G)))) + \tau_*(\varepsilon^2_*(\mathrm{K}_0(M_n)))\\ &= \psi_*(\mathrm{K}_0(M_n)) + \varphi_*(\mathrm{K}_0(C^*_{\mathrm{red}}(G)))\\ &\subset \frac{1}{n}\mathbf{Z} + \frac{1}{n}\mathbf{Z} = \frac{1}{n}\mathbf{Z}.\end{aligned}$$

The argument preceding Proposition 8.10 shows that

$$\tau(\mathrm{Proj}(M_n^c)) \subset \tau(\mathrm{Proj}(C^*(G) *^{\mathrm{red}}_{\mathbf{C}} M_n)) \subset \mathbf{Z}.$$

The faithfulness of the traces φ and ψ imply that τ is faithful and hence M_n^c is projectionless. ∎

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