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BIG HANKEL OPERATOR AND $\overline{\partial}_b$-EQUATION

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ABSTRACT. We establish links between Hankel operators theory and $\overline{\partial}_b$ equation in one and several variables. This leads to proofs of the classical Nehari's theorem in the unit disc $\mathbf{D}$ and Corona's theorem in sBMO($\mathbf{T}^n$) togather with the failure of Nehari's theorem for the Bergman class on $\mathbf{D}$ and for the Hardy class on the unit polydisc or the unit ball in $\mathbf{C}^n$, $n \geqslant 2$.

KEYWORDS: *Nehari's Theorem, Hankel Operators, Corona Problem,* BMO *Spaces.*

AMS SUBJECT CLASSIFICATION: 47B35, 32A35.

1. INTRODUCTION

Let $\mathbf{B}_n$ be the unit ball of $\mathbf{C}^n$ and $\mathbf{D}^n$ be the unit polydisc of $\mathbf{C}^n$ and let Ω be either $\mathbf{B}_n$ or $\mathbf{D}^n$ and the Hardy spaces of Ω will be those of $\mathbf{B}_n$ or $\mathbf{D}^n$, the same for the space BMOA(Ω) of holomorphic functions in Ω whose boundary values are in BMO($\partial\Omega$), where again $\partial\Omega$ is $\mathbb{S} = \partial\mathbf{B}_n$ or $\mathbf{T}^n$.

If $\varphi \in L^\infty(\partial\Omega)$ we define the Hankel operator of symbol φ as the operator:

$$\forall\, h \in H^2(\Omega), \gamma_\varphi h := P_{\overline{H}_0^2(\Omega)}(\varphi \cdot h) \in \overline{H}_0^2(\Omega)$$

where $\overline{H}_0^2(\Omega)$ is the space of complex conjugate of functions in $H^2(\Omega)$ which are zero at the origin of $\mathbf{C}^n$.

Clearly, γ_φ depends only on the class of φ modulo the functions orthogonal to the anti-holomorphic functions, hence one can choose as a representative the orthogonal projection of φ on $\overline{H}^2(\Omega)$.

We say that $H^1(\Omega)$ has the factorization property if: $\forall\, h \in H^1(\Omega)$, $\exists\, v_i, w_i \in H^2(\Omega)$ s.t. $h = \sum_i v_i \cdot w_i$ and $\sum_i \|v_i\|_2 \cdot \|w_i\|_2 \preceq \|h\|_1$. Then we have the following:

PROPOSITION 1.1. *If $\overline{\varphi} \in$ BMOA then γ_φ is bounded from $H^2(\Omega)$ to $\overline{H}^2(\Omega)$ and $\|\gamma_\varphi\| \preceq \|\overline{\varphi}\|_{\mathrm{BMOA}}$. $\|\gamma_\varphi\| \simeq \|\overline{\varphi}\|_{\mathrm{BMOA}}$ iff $H^1(\Omega)$ has the factorization property.*

This was already noticed in [6].

Proof. The first assumption is just the fact that $v, w \in H^2(\Omega) \Rightarrow v \cdot w \in H^1(\Omega)$ and BMOA $= (H^1)^*$. For the second one, let

$$\mathcal{E} := \{h \in H^1 \text{ s.t. } \exists\, v_i, w_i \in H^2(\Omega) \text{ and } h = \sum_i v_i \cdot w_i\}$$

with the norm:

$$\|h\|_{\mathcal{E}} := \inf\{\sum_i \|v_i\|_2 \cdot \|w_i\|_2, \text{ over all decompositions of } h\}.$$

Then we have:

$$\|\varphi\|_{\mathcal{E}^*} = \sup_{\substack{h\in\mathcal{E}\\ \|h\|_{\mathcal{E}}=1}} \left|\int \varphi \cdot h\right| \geqslant \sup_{\substack{v,w\in H^2\\ \|v\|^2=\|w\|_2=1}} \left|\int \varphi \cdot vw\right| \geqslant \|\gamma_\varphi\|$$

and

$$\|\varphi\|_{\mathcal{E}^*} = \sup_{\substack{h\in\mathcal{E}\\ \|h\|_{\mathcal{E}}=1}} \left|\int \varphi \cdot h\right| \leqslant \sup_{\substack{h\in\mathcal{E}\\ \|h\|_{\mathcal{E}}=1}} \left|\int \varphi \cdot \sum_i v_i w_i\right| < \|\gamma_\varphi\|$$

hence we always have:

$$\|\varphi\|_{\mathcal{E}^*} \simeq \|\gamma_\varphi\|.$$

If $\|\gamma_\varphi\| \simeq \|\overline{\varphi}\|_{\mathrm{BMOA}}$ then $H^1(\Omega) = \mathcal{E}$ because $\mathcal{E}$ is dense in $H^1(\Omega)$ and the two norms are equivalent. If $H^1(\Omega)$ has the factorization property, $H^1(\Omega) = \mathcal{E}$ then $\mathcal{E}^* =$ BMOA and $\|\gamma_\varphi\| \simeq \|\overline{\varphi}\|_{\mathrm{BMOA}}$. ∎

In the case of the ball we have [6]: $\overline{\varphi} \in \mathrm{BMOA}(\mathbf{B}_n) \Rightarrow \varphi = \alpha + P_{H^2}\beta$ with $\alpha, \beta \in L^\infty(\partial\mathbf{B}_n)$, $\|\alpha\|_\infty \lesssim \|\varphi\|_{\mathrm{BMOA}}$, $\|\beta\|_\infty \lesssim \|\varphi\|_{\mathrm{BMOA}}$, hence $P_{\overline{H}^2}\varphi = P_{\overline{H}^2}\alpha$. hence $\gamma_\varphi = \gamma_\alpha$. In the case of the polydisc we still have [4]: $\varphi \in \mathrm{BMOA}(\mathbf{D}^n) \Rightarrow \varphi = \alpha + \sum_{i=1}^n \beta_i$ where α is bounded on $\mathbf{T}^n$ and β_i is a BMO function which is holomorphic in z_i; hence again we get:

$$P_{\overline{H}^2}\varphi = P_{\overline{H}^2}\alpha.$$

Now for $n = 1$, the factorization property for $H^1(\mathbf{D})$ is well known hence we made a proof of the famous theorem of Nehari:

THEOREM 1.2. ([9]) *The Hankel operator γ_φ is bounded from $H^2(\mathbb{D})$ to $\overline{H}_0^2(\mathbb{D})$ iff there is a bounded function α on the circle such that $\gamma_\alpha = \gamma_\varphi$. Moreover, if γ_φ is bounded, we have $\|\alpha\|_\infty \simeq \|\gamma_\varphi\|$.*

Hence, we have the characterization of the bounded Hankel operators in the case of the disc:

THEOREM 1.3. *Let $\varphi \in \overline{H}^2(\mathbb{D})$, the following are equivalent:*
(i) *γ_φ is a bounded map of $H^2(\mathbb{D})$ into $\overline{H}_0^2(\mathbb{D})$;*
(ii) *$\varphi = P_{\overline{H}_0^2(\mathbb{D})}\alpha$ for some $\alpha \in L^\infty(\mathbb{T})$;*
(iii) *φ is in* BMO($\mathbb{T}$).
If any of these conditions hold the α can be chosen so that $\|\alpha\|_\infty \simeq \|\varphi\|_{\mathrm{BMO}} \simeq \|\gamma_\varphi\|$.

2. HANKEL OPERATORS IN SEVERAL VARIABLES

For this operator in the ball $\mathbb{B}_n$ we have the theorem of Coifman, Rochberg and Weiss ([6]), which provides a complete analogue of the case $n = 1$:

THEOREM 2.1. *For $\varphi \in H^2(\mathbb{B})$ the following are equivalent:*
(i) *γ_φ is a bounded map from $H^2(\mathbb{B})$ into $\overline{H}^2(\mathbb{B})$;*
(ii) *there is an $F \in L^\infty(\partial\mathbb{B})$ such that $\gamma_F = \gamma_\varphi$;*
(iii) *$\overline{\varphi}$ is in* BMOA.
If any of these conditions hold, F can be chosen so that $\|F\|_\infty \simeq \|\varphi\|_{\mathrm{BMO}} \simeq \|\Gamma_\varphi\|$.

Proof. In fact they proved the factorization property for $H^1(\mathbb{B}_n)$ which, together with the duality between $H^1(\mathbb{B}_n)$ and BMOA($\mathbb{B}_n$) gives the theorem. ∎

For the polydisc, the factorization property for $H^1(\mathbb{D}^n)$ is still an open question and we only have:

PROPOSITION 2.2. *The following are equivalent for $\overline{\varphi} \in H^2(\mathbb{D}^n)$:*
(i) *γ_φ is bounded;*
(ii) $\|\varphi\|_{\mathcal{E}^*} < +\infty$.

Proof. If the factorization property for $H^1(\mathbb{D}^n)$ is true, then it will exist a function α in $L^\infty(\mathbb{T}^n)$ such that $\gamma_\alpha = \gamma_\varphi$ and $\|\gamma_\varphi\| \simeq \|\overline{\varphi}\|_{\mathrm{BMOA}} \simeq \|\alpha\|_\infty$. ∎

2.1. BIG HANKEL OPERATOR IN $\mathbf{B}$. The big Hankel operator of symbol φ is $\Gamma_\varphi : H^2(\mathbf{B}) \longrightarrow H^2(\mathbf{B})^\perp$ defined by:

$$\forall h \in H^2(\mathbf{B}) \quad \Gamma_\varphi h = P_{H^{2\perp}}\varphi h.$$

This operator is the other possible generalization of the Hankel operator of the disc in $\mathbf{C}$.

In fact, still in [6], the authors prove that the commutator of φ and the orthogonal projection on $H^2(\mathbf{B})$ is bounded on $L^2(\partial\mathbf{B})$ if $\varphi \in \mathrm{BMO}(\partial\mathbf{B})$; but here we have:

$$\Gamma_\varphi h = P_{H^2(\mathbf{B})^\perp}(\varphi h) = \varphi h - P_{H^2(\mathbf{B})} = \left[\varphi, P_{H^2(\mathbf{B})}\right] \cdot h$$

because h is already in $H^2(\mathbf{B})$; hence they proved:

THEOREM 2.3. *The big Hankel operator Γ_φ is bounded if*

$$P_{H^2(\mathbf{B})^\perp}\varphi \in \mathrm{BMO}(\partial\mathbf{B}).$$

2.2. LINK WITH THE $\overline{\partial}_b$-EQUATION IN THE BALL. We now establish links between the norm of Γ_φ and a norm of φ in term of the $\overline{\partial}$ of a Stokes extension of φ in $\mathbf{B}$.

PROPOSITION 2.4. *Let $\tilde{\varphi}$ be any Stokes extension of φ in $\mathbf{B}$, then:*

$$\forall h \in H^2(\mathbf{B}), \quad \overline{\partial}_b(\Gamma_\varphi h) = h \cdot \overline{\partial}\tilde{\varphi}.$$

The proof of this proposition will be an easy consequence of the following lemma:

LEMMA 2.5. *The space $H^2(\mathbf{B})^\perp$ can be identified with the space of $(n, n-1)$ forms, $\overline{\partial}_b$-closed and in $L^2(\partial\mathbf{B})$ in such a way that:*

$$\forall f \in H^2(\mathbf{B})^\perp \longrightarrow \Omega(\overline{f}) \in L^2_{(n,n-1)}(\mathbf{B}), \quad \overline{\partial}_b\Omega(\overline{f}) = 0$$

and

$$\forall h \in L^2(\partial\mathbf{B}), \quad \int_{\partial\mathbf{B}} h \cdot \overline{f}\, d\sigma = \int_{\partial\mathbf{B}} h \cdot \Omega(\overline{f}).$$

Proof. Let us prove it in $\mathbf{C}^2$ for simplicity; the space $H^2(\mathbf{B})^\perp$ can be decomposed in the direct sum of 3 terms:

$H_0 := \{f \mid \overline{f} \in H^2(\mathbf{B}),\ f(0) = 0 \Leftrightarrow \overline{f} = z_1 g_1 + z_2 g_2,\ g_i \in H^2(\mathbf{B}),\ i = 1, 2\}$

$H_1 := \{f \mid \exists g \in H^2(\mathbf{B}) \text{ s.t. } \overline{f}(z_1, z_2) = z_1 g(z_1, \overline{z}_2)\}$

$H_2 := \{f \mid \exists g \in H^2(\mathbf{B}) \text{ s.t. } \overline{f}(z_1, z_2) = z_2 g(\overline{z}_1, z_2)\}$

because Leibenson's decomposition is true for $H^2(\mathbb{B})$.

Now if:

$f \in H_0$ we put $\Omega(\overline{f}) := g_2\, d\overline{z}_1 - g_1\, d\overline{z}_2$

$f \in H_1$ we put $\Omega(\overline{f}) := -g(z_1, \overline{z}_2)\, d\overline{z}_2$

$f \in H_2$ we put $\Omega(\overline{f}) := g(\overline{z}_1, z_2)\, d\overline{z}_1$

and we check easily that in any case we have $\overline{\partial}\Omega(f) = 0$.

Conversely if φ is a $\overline{\partial}$-closed (2,1) form in $\mathbb{B}$ then we say that $\varphi \in L^2_{(2,1)}(\mathbb{B})$ if $\forall f \in L^2(\partial\mathbb{B})$

$$\Big|\int_{\partial\mathbb{B}} f\varphi\Big| = \Big|\int_{\partial\mathbb{B}} f\varphi^{\#}\, d\sigma\Big| \leqslant C\|f\|_2;$$

then using Stokes' theorem we get that $\varphi^{\#} \perp H^2(\mathbb{B})$ and the lemma. ∎

Proof of the Proposition 2.4. Let $\varphi \in L^\infty(\partial\mathbb{B})$, $h \in H^2(\mathbb{B})$ and $v := \Gamma_\varphi h$, then:

$$\forall k \in H^2(\mathbb{B})^\perp, \quad \int_{\partial\mathbb{B}} v \cdot \overline{k}\, d\sigma = \int_{\partial\mathbb{B}} v \cdot \Omega(\overline{k}) = \int_{\partial\mathbb{B}} \varphi h \cdot \Omega(\overline{k}) = \int_{\mathbb{B}} h\overline{\partial}\tilde{\varphi} \wedge \Omega(\overline{k})$$

hence $\overline{\partial}_b v = h \cdot \overline{\partial}\tilde{\varphi}$ and the proposition. ∎

I introduced a class of $\overline{\partial}$-closed (0,1) form in $\mathbb{B}$, ([2]), named class A, such that $\omega \in A$ iff $\forall h \in H^2(\mathbb{B})$, $\exists u \in L^2(\partial\mathbb{B})$ $\overline{\partial}_b u = h \cdot \omega$; now the preceding proposition implies that $\|\Gamma_\varphi\| < +\infty \Leftrightarrow \overline{\partial}\tilde{\varphi} \in A$ for any Stokes' extension of φ in $\mathbb{B}$. Before going on we need to recall definitions about Carleson forms in $\mathbb{B}$.

DEFINITION 2.6. Let $z \in \partial\mathbb{B}$, $r > 0$ then *the pseudo-ball* of center z and radius r is:

$$Q(z,r) := \{\zeta \in \mathbb{B} \mid |1 - \overline{\zeta} \cdot z| < r\}.$$

DEFINITION 2.7. Let μ a measure on $\mathbb{B} \subset \mathbb{C}^n$; μ is *a Carleson measure* if:

$$\exists C > 0,\ \forall z \in \partial\mathbb{B},\ \forall r > 0,\ |\mu|(Q(z,r)) \leqslant Cr^n.$$

We note $V^1(\mathbb{B})$ the space of Carleson measures in $\mathbb{B}$.

The Carleson norm of μ is the smallest C in the preceding definition.

DEFINITION 2.8. Let ω a (0,1) form on $\mathbf{B}$, ω is a *Carleson* (0,1) *form* if:
(i) the coefficients of ω are Carleson measures;
(ii) the coefficients of $\frac{\omega\wedge\bar\partial\rho}{\sqrt{-\rho}}$ are Carleson measures.
We note $V^1_{(0,1)}(\mathbf{B})$ the space of *Carleson* (0,1) *forms* in $\mathbf{B}$.

DEFINITION 2.9. Let γ a (1,1) form on $\mathbf{B}$, γ is a Carleson (1,1) form if:
(i) the coefficients of $\gamma\wedge\partial\rho\wedge\bar\partial\rho$ are Carleson measures in $\mathbf{B}$;
(ii) the coefficients of $\sqrt{-\rho}\gamma\wedge\bar\partial\rho$ are Carleson measures in $\mathbf{B}$;
(iii) the coefficients of $\sqrt{-\rho}\ \gamma\wedge\partial\rho$ are Carleson measures in $\mathbf{B}$;
(iv) the coefficients of $-\rho\cdot\gamma$ are Carleson measures in $\mathbf{B}$.
We note $V^1_{(1,1)}(\mathbf{B})$ the space of Carleson (1,1) forms in $\mathbf{B}$.

Now we can state:

THEOREM 2.10. *Let φ be a function on $\partial\mathbf{B}$; if φ admits a Stokes' extension in $\mathbf{B}$, $\tilde\varphi$, such that the form $\omega := \bar\partial\tilde\varphi$ satisfies one of the following conditions, then Γ_φ is a bounded operator:*
(i) $\omega\in V^1_{(0,1)}(\mathbf{B})$;
(ii) $|\omega|^2\in V^1(\mathbf{B})$;
(iii) $(1-|z|^2)\cdot|\omega|^2\in V^1(\mathbf{B})$ *and* $\partial\omega\in V^1_{(1,1)}(\mathbf{B})$.

Now if we compare Theorem 1.3 and Theorem 2.3, we see that the assertion concerning the bounded function is missing and in fact we have:

PROPOSITION 2.11. *There is a function $\varphi\in\mathrm{BMO}(\partial\mathbf{B})$ (hence Γ_φ is bounded) such that there is no function α in $L^\infty(\partial\mathbf{B})$ with $\Gamma_\alpha=\Gamma_\varphi$.*

Proof. To prove this let $\varphi := \log(1-|z_1|^2)$, then it is easy to check that $\varphi\in\mathrm{BMO}(\mathbf{B})$ and if there is an $\alpha\in L^\infty(\partial\mathbf{B})$ such that $\Gamma_\alpha=\Gamma_\varphi$ then there is a holomorphic function h in $\mathbf{B}$ such that $\varphi=\alpha+h$ and this is not possible by the "minimum principle" ([1]).

Of course the same example proves that Nehari's theorem also fails for big Hankel operators in the Bergman space of the unit disc. ∎

3. CASE OF THE POLYDISC

The big Hankel operators were studied by C. Sadosky and M. Cotlar ([7]) and they introduced the

DEFINITION 3.1. The space sBMO($\mathbf{T}^n$) is

$$\text{sBMO}(\mathbf{T}^n) := \{f \in \text{BMO}(\mathbf{T}^n) \mid f = \Phi_1 + H_{z_1}\Psi_1 = \cdots = \Phi_n + H_{z_n}\Psi_n\}$$

with the norm:

$$\|f\|_{\text{sBMO}} := \inf\{\max_i(\|\Phi_i\|_\infty), \text{ on all decompositions of } f\}.$$

This space is substantially smaller than BMO($\mathbf{T}^n$). They proved:

THEOREM 3.2. *Γ_φ is bounded from $H^2(\mathbf{D}^n)$ to $H^2(\mathbf{D}^n)^\perp$ iff $P_{H^{2\perp}}\varphi \in$ sBMO($\mathbf{T}^n$).*

We shall give other conditions linked to the $\overline{\partial}$-equation.

3.1. LINK WITH THE $\overline{\partial}_b$-EQUATION IN THE POLYDISC. We state and prove the theorems in the case of the bidisc only in order to have simpler notations; everythings go the same way in $\mathbf{C}^n$.

The scheme is the same as for the unit ball; first we decompose the orthogonal of $H^2(\mathbf{T}^2)$:

LEMMA 3.3. *Let f in $H^2(\mathbf{T}^2)^\perp$, then we have: $f = \overline{z}_1 f_1 + \overline{z}_2 f_2$ with $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$ and f_i anti-holomorphic in z_i.*

Proof. The proof is just a Fourier series decomposition. ∎

Let $\omega = \omega_1\, d\overline{z}_1 + \omega_2\, d\overline{z}_2$ a (0,1)-form in $\mathbf{D}^2$, we shall say that ω is uniformly Carleson with constant C if:

(i) $\forall\, z_2 \in \mathbf{T}\ \|\omega_1(\cdot, z_2)\|_C \leqslant C$;

(ii) $\forall\, z_1 \in \mathbf{T}\ \|\omega_1(z_1, \cdot)\|_C \leqslant C$;

where $\|\mu\|_C$ is the Carleson norm of the measure μ in the unit disc of $\mathbf{C}$.

THEOREM 3.4. *Let φ a function in $L^\infty(\mathbf{T}^2)$. If $\overline{\partial}_b\varphi$ is uniformly Carleson in $\mathbf{D}^2$ with constant C, then Γ_φ is bounded with a norm controlled by C.*

Proof. Let $h \in H^2(\mathbf{T}^2)$, we have to show that $| < \varphi\cdot h, f > | \leqslant C\|h\|_2\|f\|_2$ for any $f \in H^2(\mathbf{T}^2)^\perp$; because of the lemma it suffices to prove that with $f_i, i = 1, 2$.

Hence let f be anti-holomorphic in z_1, then we have:

$$\mathrm{i}\int_{\mathbf{T}^2} \varphi h z_1 \overline{f}\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 = \int_{\mathbf{T}} \left\{ \int_{\mathbf{T}} \varphi h \overline{f}\, \mathrm{d}z_1 \right\} \mathrm{d}\theta_2 = \int_{\mathbf{T}} \left\{ \int_{\mathbf{D}} \frac{\partial \tilde{\varphi}}{\partial \overline{z}_1} h \overline{f} \mathrm{d}\overline{z} \wedge \mathrm{d}\overline{z}_1 \right\} \mathrm{d}\theta_2$$

where $\tilde{\varphi}$ is any Stokes' extension of φ.

But by hypothesis $\overline{\partial}_b\varphi$ is uniformly Carleson; then the last integral is bounded by:

$$\int_{\mathbf{T}} \left\{ \|\overline{\partial}_1\tilde{\varphi}(\cdot, z_2)\|_C \|h(\cdot, z_2)\|_2 \|\overline{f}(\cdot, z_2)\|_2 \right\} \mathrm{d}|z_2| \leqslant \|\overline{\partial}_b\varphi\|_C \|h\|_2 \|f\|_2.$$

Of course the same is true if we assume that f is anti-holomorphic in z_2, hence the theorem. ∎

In the same vein we shall say that the (0,1) form $\omega = \omega_1\, \mathrm{d}\overline{z}_1 + \omega_2\, \mathrm{d}\overline{z}_2$ verifies uniformly the Wolff's conditions with constant C if:

(i) $\forall\, z_2 \in \mathbf{T},\ \|(1 - |z_1|^2)|\omega_1|^2\|_C \leqslant C$ and $\left\|(1 - |z_1|^2)\left|\frac{\partial \omega_1}{\partial z_1}\right|\right\|_C \leqslant C$;

(ii) $\forall\, z_1 \in \mathbf{T},\ \|(1 - |z_2|^2)|\omega_2|^2\|_C \leqslant C$ and $\left\|(1 - |z_2|^2)\left|\frac{\partial \omega_2}{\partial z_2}\right|\right\|_C \leqslant C$.

Then we have:

THEOREM 3.5. *Let φ a function in $L^\infty(\mathbf{T}^2)$. If $\overline{\partial}_b\varphi$ verifies uniformly the Wolff's conditions with constant C, then Γ_φ is bounded with a norm controlled by C.*

Proof. Let $h \in H^2(\mathbf{D}^2)$ and $\overline{z}_1 f \in H^2(\mathbf{T}^2)^\perp$, f anti-holomorphic in z_1; by the lemma, we again have to show that $| < \varphi \cdot h, \overline{z}_1 f > | \leqslant C\|h\|_2\|f\|_2$. As was done by Wolff, we apply Green's formula in z_1:

$$\int_{\mathbf{T}} \varphi h z_1 \overline{f} \mathrm{d}\theta_1 = -\int_{\mathbf{D}} \log|z_1|^2 \,\Delta\, (z_1 \varphi h \overline{f})\, \mathrm{d}v$$

where we still note φ an extension of φ in $\mathbf{D}^2$ such that $\overline{\partial}\varphi$ verifies uniformly the Wolff's conditions with constant C.

Then $\Delta z_1 \varphi h \overline{f} = 4\partial(z_1 h \overline{f} \overline{\partial}\varphi)$ because $\overline{f}$ is holomorphic in z_1; hence:

$$\frac{1}{4}\cdot \Delta z_1 \varphi h \overline{f} = \overline{f}\partial h' \overline{\partial}\varphi + h' \partial \overline{f} \overline{\partial}\varphi + \overline{f} h' \partial\overline{\partial}\varphi$$

where we put $h' := z_1 h$, and then we have to integrate 3 terms; let see the first:

$$\left| \int_{\mathbf{D}} \overline{f}\partial h' \overline{\partial}\varphi \log|z_1|^2 \, dv \right| \leqslant \left(\int_{\mathbf{D}} \log|z_1|^2 |w_1| \, dv \right)^{\frac{1}{2}} \left(\int_{\mathbf{D}} \log|z_1|^2 |\overline{f}|^2 |w_1|^2 \, dv \right)^{\frac{1}{2}}$$

by Schwarz's lemma, where we note $\overline{\partial}\varphi = \partial\varphi/\partial\overline{z}_1 = \omega_1$.

We have $\|h'\|_2 = \|h\|_2$ and

$$-\int_{\mathbf{D}} |\partial_1 h'|^2 \log|z_1|^2 \, dv \leqslant C \int_{\mathbf{T}} |h'|^2 \, d\theta_1$$

and the second factor is bounded by $\int_{\mathbf{D}} |\overline{f}|^2 \, d\theta_1$ because the Carleson condition on $(1-|z_1|^2)|\omega_1|^2$. The same is valid for the second term and for the last one we use the fact that $\overline{f}h' \in H^1$, $\|\overline{f}h'\|_1 \leqslant \|f\|_2 \|h'\|_2$ and the Carleson condition on $(1-|z_1|^2)\partial_1\omega_1$.

Hence we have:

$$\left| \int_{\mathbf{T}} \varphi h z_1 \overline{f} \, d\theta_1 \right| \preceq \left(\int_{\mathbf{T}} |h|^2 \, d\theta_1 \right)^{\frac{1}{2}} \left(\int_{\mathbf{T}} |\overline{f}|^2 \, d\theta_1 \right)^{\frac{1}{2}} .$$

and

$$|\langle \varphi \cdot h, \overline{z}_1 f \rangle| = \left| \int_{\mathbf{T}} \left\{ \int_{\mathbf{T}} \varphi h' \overline{f} \, d\theta_1 \right\} d\theta_2 \right| \preceq \|\overline{f}\|_2 \|h\|_2$$

again by Schwarz's lemma.

We do the same with the part anti-holomorphic in z_2 and then we prove that Γ_φ is bounded. ∎

COROLLARY 3.6. *Let f_1, f_2 in $H^\infty(\mathbf{D}^n)$ be such that $z \in \mathbf{D}^n$, $|f_1(z)|+|f_2(z)| \geqslant \delta > 0$, then there are g_1, g_2 in* sBMOA *verifying $f_1 g_1 + f_2 g_2 = 1$.*

Proof. As done by Wolff, we easily see that the "Corona form" ω verifies the Wolff's conditions uniformly and then we have a solution φ of $\overline{\partial}_b \varphi = \omega$ such that Γ_φ is bounded; hence, by the result of M. Cotlar and C. Sadosky, we get that φ is in sBMO. ∎

This result was proved directly by U. Cegrell for the bidisc ([5]) and improves preceding results of N. Varopoulos ([10]) and S.Y.A. Chang ([3]).

4. THE BERGMAN SPACE

Instead of the Hardy space $H^2(\mathbf{D})$ we may consider the Bergman space $A^2(\mathbf{D})$ and study the boundedness of the Big Hankel operator from $A^2(\mathbf{D})$ into $A^2(\mathbf{D})^\perp$. The space. $A^2(\mathbf{D})$ may be considered as the Hardy space of $\mathbf{B}$ restricted on $z_2 = 0$, or as the Hardy space of $\mathbf{D}^2$ restricted on $z_1 = z_2$.

Now if we compare Theorem 1.3 and Theorem 2.3, we see that the assertion concerning the bounded function is missing and in fact we have:

PROPOSITION 4.1. *There is a function $\varphi \in \mathrm{BMO}(\mathbf{D})$ (hence Γ_φ is bounded) such that there is no function F in $L^\infty(\mathbf{D})$ with $\Gamma_F = \Gamma_\varphi$.*

Proof. To prove this let $\varphi := \log(1 - |z_1|^2)$; then it is easy to check that $\varphi \in \mathrm{BMO}(\mathbf{D})$ and if there is an $F \in L^\infty(\mathbf{D})$ such that $\Gamma_F = \Gamma_\varphi$ then there is a holomorphic function h in $\mathbf{D}$ such that $\varphi = F + h$ and this is not possible by the "minimum principle" ([1]). ∎

Hence Nehari's theorem fails for Big Hankel operators in the Bergman space of the unit disc and also for Big Hankel operators in $\mathbf{B}$ and in $\mathbf{D}^n$.

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