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CANONICAL $*$-ENDOMORPHISMS AND SIMPLE C^*-ALGEBRAS

MARIE CHODA

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ABSTRACT. Canonical property for a unit preserving $*$-endomorphism γ of a unital C^*-algebra A and the extension algebra $\langle A, \gamma\rangle$ of A via γ are introduced through the method of the crossed products by $*$-endomorphisms. The relation between Cuntz's canonical $*$-endomorphisms and Ocneanu's canonical shifts are obtained.

KEYWORDS: *$*$-endomorphisms, simple, crossed product.*

AMS SUBJECT CLASSIFICATION: Primary 46L35; Secondary 46L37.

1. INTRODUCTION

Let A be a unital C^*-algebra and γ a unit preserving $*$-endomorphism of A. When γ satisfies a certain condition which naturally comes from the Jones index theory ([13]), we say that γ has the canonical property. For such a $*$-endomorphism γ of A, we define a $*$-endomorphism ρ of A which is a $*$-isomorphism from A onto eAe for some non zero projection $e \in A$. The crossed product $M = A \triangleleft \langle\rho\rangle$ is defined as the C^*-algebra which is generated by A and an isometry w with the following relations :

$$\text{(I)} \qquad waw^* = \rho(a),\ (a \in A), \quad w^*Aw \subset A$$

and has a faithful conditional expectation E onto A with $E(w^k) = 0$ for all $k \geqslant 1$. If the $*$-endomorphism ρ is induced (in the method described below) by γ with the canonical property, then there exist a representation π of A and an isometry W

with the relation (I), that is, $W\pi(a)W^* = \pi(\rho(a))$, $(a \in A)$ and $W^*\pi(A)W \subset \pi(A)$ so that the C^*-algebra $C^*(A, W)$ generated by $\pi(A)$ and W is isomorphic to $A \lhd \langle\rho\rangle$.

The C^*-algebra $\langle A, \gamma\rangle$ generated by $\pi(A)$ and W is called the extension algebra of A via the $*$-endomorphism γ with the canonical property. A $*$-endomorphism γ of A with the canonical property is always extended to a $*$-endomorphism $\hat{\gamma}$ of $\langle A, \gamma\rangle$ which has also the canonical property.

In the category of $*$-algebras, the terminology "canonical" is used by Ocneanu ([17]) for $*$-endomorphisms on some important $*$-algebras in the classification theory of subfactors of the hyperfinite II_1 factor and his canonical $*$-endomorphisms are generalization of Φ^2 of the $*$-endomorphism Φ in [11], Section 4.4, which has an important role to give the irreducible subfactor of the hyperfinite II_1 factor. Same terminology "canonical" is used by Cuntz ([8]) for a $*$-endomorphism on the Cuntz algebra ([7]) induced naturally from the definition of the Cuntz algebra. These "canonical" $*$-endomorphisms are also canonical in our sense. We show that the extension algebra $\langle A, \gamma\rangle$ is simple for Ocneanu's canonical shift γ on an approximately finite dimensional C^*-algebra A and that Cuntz's "canonical" inner $*$-endomorphism Φ is the extension $\hat{\gamma}$ of a special Ocneanu's canonical shift γ.

Our extension algebra $\langle A, \gamma\rangle$ is generated (under some identification) by A and an isometry W with the relation (I). Paschke ([19]) proved that if A is strongly amenable and simple, then the C^*-algebra $C^*(A, W)$ generated by A and W with relations (I) is always simple. The extension algebra $\langle O_n, \Phi\rangle$ is generated by O_n and an isometry W with the relation (I) but $\langle O_n, \Phi\rangle$ is not simple.

In the index theory of infinite factors, Longo's "canonical" $*$-endomorphism (for example see [16], [12]) plays an important role. We investigate in [6] a similar subject for factors as in this paper and some parts in this paper are essentially the same as the results for factors in [6] but we denote them for the sake of completeness.

2. CROSSED PRODUCTS BY $*$-ENDOMORPHISMS

In this section, we define the crossed product of a C^*-algebra A by a $*$-endomorphism ρ and investigate some properties of the crossed products which we need in the next section. Our method is somewhat similar to the method for von Neumann algebras in [1] and [22]. Related topics on the crossed products of C^*-algebras by $*$-endomorphisms are investigated in [9], [19], [20], [21].

Let A be a unital C^*-algebra acting on a Hilbert space H_0 and ρ a $*$-endomorphism of A (that is, a $*$-homomorphism from A into A) with an isometry w on H_0 such that

$$\text{(I)} \qquad \rho(a) = waw^* \quad (a \in A), \quad w^*Aw \subset A.$$

In the next section, we show that the $*$-endomorphism ρ induced by a $*$-endomorphism γ with the canonical property satisfies automatically these conditions.

Put

$$p_n = \rho^n(1) \quad \text{and} \quad p = p_1.$$

Then $\rho(A) = pAp$ because $\rho(A) = wAw^* = pwAw^*p \subset pAp \subset ww^*Aww^* \subset wAw^* = \rho(A)$.

Put

$$H_k = \begin{cases} H_0, & (k \leqslant 0) \\ p_k H_0, & (k \geqslant 1) \end{cases} \quad \text{and} \quad H = \sum_{k \in \mathbf{Z}} \oplus H_k.$$

Let π be the representation of A on H defined by

$$(\pi(x)\eta)_k = \begin{cases} x\eta_k & (k \leqslant 0) \\ \rho^k(x)\eta_k & (k \geqslant 1), \end{cases}$$

and W be the isometry defined by

$$(W(\eta))_k = \begin{cases} w\eta_{k-1} & (k \leqslant 0) \\ w^2\eta_{k-1} & (k \geqslant 1), \end{cases}$$

where $\eta = (\eta_k)_{k\in\mathbf{Z}}$ and $\eta_k \in H_k$. Then W and π preserve the relations:

$$W\pi(a)W^* = \pi(\rho(a)) \quad \text{and} \quad W^*\pi(a)W = \pi(w^*aw) \quad (a \in A).$$

Under the assumption (I), let $A \lhd \langle\rho\rangle$ be the C^*-algebra generated by $\pi(A)$ and the isometry W. We call $A \lhd \langle\rho\rangle$ the crossed product of A by ρ. As we show later, $A \lhd \langle\rho\rangle$ does not depend essentially on w nor $\{\pi, W\}$.

LEMMA 2.1. (i) *There exists a faithful conditional expectation E of $A \lhd \langle\rho\rangle$ onto $\pi(A)$ such that*

$$E(W^k) = 0 \quad \textit{for all} \quad k \geqslant 1.$$

(ii) *Let ψ be a faithful state on A and B the $*$-algebra generated by $\pi(A)$ and W. Put $\varphi = \psi \cdot E$. Each x in the $*$-algebra B is written for some integer k in the form:*

$$x = \sum_{i=1}^{k} W^{*i}x_{-i} + x_0 + \sum_{i=1}^{k} x_i W^i \quad (x_i \in \pi(A))$$

and

$$\|x\|_\varphi^2 = \sum_{k\geqslant 1} \|p_k x_k\|_\psi^2 + \sum_{k\geqslant 0} \|w^{*k} x_k^* x_k w^k\|_\psi^2.$$

(iii) *For the form of $x, y \in B$ in* (ii), *we have $x = y$ if and only if $x_0 = y_0, x_i\pi(p_i) = y_i\pi(p_i)$ and $\pi(p_i)x_{-i} = \pi(p_i)y_{-i}$ for all $i \geqslant 1$.*

Proof. (i) For each integer k, let q_k be the projection of H onto H_k (identified as a subspace of H). For each $y \in B(H)$, we put $E(y) = \sum_{k\in\mathbf{Z}} q_k y q_k$ (the sum taken in the strong operator topology). Then E is a faithful normal conditional expectation of $B(H)$ onto $\{q_k; k \in \mathbf{Z}\}'$. Since $\pi(A)$ is contained in $\{q_k; k \in \mathbf{Z}\}'$, $q_k W^n q_k = 0$ for all $k \in \mathbf{Z}$, $n \neq 0$ and E has norm one, we have that the restriction of E to $A \triangleleft \langle\rho\rangle$ (which we denote by the same notation E) has the desired property (cf. [2]).

(ii) For an $x \in B$, put $x_{-i} = E(W^i x)$, $x_0 = E(x)$ and $x_i = E(xW^{*i})$. Since $W^i\pi(a)W^{*j} = \pi(\rho^i(a))W^{i-j}$ and $W^{*i}\pi(a)W^j = W^{*(i-j)}(W^{*j}\pi(a)W^j)$ for all i, j and $a \in A$, it implies that $x \in B$ can be written in the form. The relation about $\|\cdot\|_\varphi$ comes from the property of the conditional expectation E.

(iii) Since $x_{-i} = E(W^i x) = E(\pi(p_i)W^i x) = \pi(p_i)x_{-i}$ and $x_i = x_i\pi(p_i)$, we have $x = y$ if and only if the coefficients for x and y satisfy the properties. ∎

The crossed product $A \triangleleft \langle\rho\rangle$ is defined by the above representations π of A and W of w. However, the algebra does not depend on the representations as follows.

PROPOSITION 2.2. *Let M_1 be the C^*-algebra generated by a C^*-subalgebra A and an isometry w such that $wAw^* \subset A$ and $w^*Aw \subset A$. Assume that there exists a conditional expectation E_1 of M_1 onto A with $E_1(w^k) = 0$ for all $k \geqslant 1$. Put $\rho(a) = waw^*$, $(a \in A)$. Let $W \in M_2 = A \triangleleft \langle\rho\rangle$ be the isometry corresponding to w. Then there exists a $*$-isomorphism Ψ from M_1 onto M_2 with*

$$\Psi(a) = \pi(a),\ (a \in A) \quad \text{and} \quad \Psi(w^k) = W^k,\ (k \geqslant 1).$$

Proof. Let ψ be a fixed faithful state on A and E_2 the conditional expectation of M_2 onto $\pi(A)$ with $E_2(W^k) = 0$ for all $k \geqslant 1$. Let φ_j be the faithful state on M_j defined by $\varphi_j = \psi \cdot E_j$. For each x in the $*$-algebra generated by A and w, the relations (ii) and (iii) in Lemma 2.1 hold by the same proof as Lemma 2.1. Hence we have an isometry T from $L^2(M_1, \varphi_1)$ onto $L^2(M_2, \varphi_2)$ defined by $T(w^{*k}x\xi_1) = W^{*k}\pi(x)\xi_2$ and $T(xw^k\xi_1) = xW^k\xi_2$ for all $k \geqslant 0$ and $x \in A$, where ξ_j is a cyclic and separating vector for $M_j, (j = 1, 2)$. Put $\Psi(x) = TxT^*$ $(x \in M_1)$. Then Ψ is the desired isomorphism. ∎

Let us consider the following condition $(*)$ for a $*$-endomorphism ρ of A, which is introduced by Kishimoto ([15]) to investigate the simplicity for the crossed product of a C^*-algebra by an outer automorphism group.

CONDITION $(*)$. For an $a \in A$, $\{a_i; i = 1, 2, \ldots, n\} \subset A$, $\{k_1, k_2, \ldots, k_n\}$ and $\varepsilon > 0$, there exists a positive $x \in A$ with $\|x\| = 1$ such that

$$\|xax\| \geqslant \|a\| - \varepsilon, \quad \|xa_i\rho^{k_i}(x)\| \leqslant \varepsilon \quad (i = 1, 2, \ldots, n).$$

THEOREM 2.3. *If ρ satisfies the condition $(*)$, then $\mathcal{A} = A \lhd \langle\rho\rangle$ is simple when the only proper ideal J of A for which $\rho(J) \subset J$ is the zero ideal.*

Proof. By a similar method as [10], [15], we prove the statement. Let $W \in \mathcal{A}$ be the isometry which corresponds w and E the conditional expectation of $\mathcal{A}$ onto $\pi(A)$ obtained in Lemma 2.1. Let J be a proper closed two sided ideal of $\mathcal{A}$. Since E is faithful, it is enough to show that $E(J) = \{0\}$. Since $J \cap \pi(A)$ is a closed ideal in $\pi(A)$ such that $W(J \cap \pi(A))W^* \subset J \cap \pi(A)$, $J \cap \pi(A) = 0$. Hence to prove that $E(J) = 0$ it is sufficient to prove that $E(J) \subset J$. For an $x \in \mathcal{A}$, put $|||x||| = \inf\{\|x + j\|; j \in J\}$. Since $J \cap \pi(A) = 0$, we see that the restriction to $\pi(A)$ of the canonical homomorphism from $\mathcal{A}$ onto $\mathcal{A}/J$ is injective, so that $|||x||| = \|x\|$ for all $x \in \pi(A)$. We show that $\|E(x)\| \leqslant |||x|||$ for all $x \in \mathcal{A}$. Since $|||y||| = 0$, for all $y \in J$, this implies that $E(J) = 0$.

By a density argument, we may assume that $x \in \mathcal{A}$ is of form $x = a + \sum_{i=1}^{n} W^{*i}x_{-i} + \sum_{i=1}^{n} x_i W^i$, for some integer n, where $a, x_i \in \pi(A)$ for all i $(1 \leqslant |i| \leqslant k)$. Let $\varepsilon > 0$. Then there exists a positive $y \in \pi(A)$ which satisfies the condition $(*)$ for a, $\{x_i; 1 \leqslant i \leqslant k\}$, $\{k_i = |i|; 1 \leqslant i \leqslant n\}$ and ε. Hence

$$\begin{aligned}\left|\left|\left| y\Big(\sum_{i=1}^{n} W^{*i}x_{-i} + \sum_{i=1}^{n} x_i W^i\Big)y \right|\right|\right| &\leqslant \sum_{i=1}^{n} \|W^{*i}\rho^i(y)x_{-i}y\| + \sum_{i=1}^{n} \|yx_i\rho^i(y)W^i\| \\ &\leqslant \sum_{i=1}^{n} \|\rho^i(y)x_{-i}y\| + \sum_{i=1}^{n} \|yx_i\rho^i(y)\| \leqslant 2n\varepsilon.\end{aligned}$$

By the fact $|||a||| = \|a\|$ for all $a \in \pi(A)$,

$$\|E(x)\| = \|a\| \leqslant \|yay\| + \varepsilon \leqslant |||yxy||| + (2n+1)\varepsilon \leqslant |||x||| + (2n+1)\varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have the desired inequality. ∎

REMARK 2.4. If the C^*-algebra A is strongly amenable, then we do not need the condition $(*)$ for simplicity of $\mathcal{A} = A \lhd \langle\rho\rangle$ by [19]. However, if A is not strongly amenable, $\mathcal{A}$ is not always simple. We show in Section 3 an example of a pair of an amenable C^*-algebra A and a canonical $*$-endomorphism γ which implies non simple $\mathcal{A}$.

3. EXTENSION VIA CANONICAL $*$-ENDOMORPHISMS

In this section, we consider a canonical property for a unit preserving $*$-endomorphism γ of a unital C^*-algebra A which induces a $*$-endomorphism ρ of A to give the crosed product $A \triangleleft \langle\rho\rangle$. Applying the method of crossed products in Section 2 to the $*$-endomorphism ρ induced from such a $*$-endomorphism γ with the canonical property, we investigate relations between the crossed products by such $*$-endomorphisms ρ and $*$-endomorphisms γ with the canonical property.

DEFINITION 3.1. Let A be a unital C^*-algebra. A unit preserving $*$-endomorphism γ of A is said to have the *canonical property*, if there exists a γ invariant faithful state φ of A, an isometry w on the Hilbert space H_φ (the GNS representation space by φ) and a projection $f \in \gamma^2(A)' \cap A$ with the following relations (i) and (ii) :

$$\text{(i)} \qquad \gamma(a)w = wa \quad \text{for all } a \in A \quad \text{and} \quad ww^* = e \in A,$$

$$\text{(ii)} \qquad eAe = \gamma(A)e, \quad \begin{Bmatrix} efe = \lambda e \\ fef = \lambda f \end{Bmatrix} \quad \begin{Bmatrix} f\gamma(e)f & = \lambda f \\ \gamma(e)f\gamma(e) & = \lambda\gamma(e) \end{Bmatrix},$$

for some λ $(0 < \lambda < 1)$.

The projection $e = ww^*$ is called a *basic* projection for γ. The $*$-endomorphism ρ_γ on A is defined by

$$(\dagger) \qquad \rho_\gamma(a) = e\gamma(a), \quad \text{for all } a \in A.$$

We remark that the relation (i) implies $e \in \gamma(A)' \cap A$.

PROPOSITION 3.2. *Let γ be a unit preserving $*$-endomorphism of A. If there exist projections $e \in \gamma(A)' \cap A$, $f \in \gamma^2(A)' \cap A$ with the relation* (ii) *and a faithful state φ of A with*

$$\text{(iii)} \qquad \varphi \cdot \gamma = \varphi \quad \textit{and} \quad \varphi(\gamma(a)e) = \lambda\varphi(a), \quad \textit{for all } a \in A,$$

then γ has the canonical property.

Furthermore, if A has a unique faithful tracial state φ and projections $e \in \gamma(A)' \cap A$, $f \in \gamma^2(A)' \cap A$ with the relation (ii), *then φ satisfies the condition* (iii), *so that γ has the canonical property.*

Proof. Let us consider A as the C^*-algebra of left multiplication operators on $L^2(A, \varphi)$ for the state φ satisfying (iii). Let w be the isometry on $L^2(A, \varphi)$

defined by

$$w(a\xi_0) = \frac{1}{\sqrt{\lambda}} e\gamma(a)\xi_0 \quad \text{for all } a \in A,$$

where ξ_0 is the cyclic separating vector $1 \in A \subset L^2(A, \varphi)$. Then $\gamma(a)w = wa$ for all $a \in A$ and $ww^* = e \in \gamma(A)' \cap A$. Hence all conditions are satisfied.

Assume that φ is the unique faithful tracial state of A, then the restriction of φ to $\gamma(A)$ is the unique tracial state because $\gamma(A)$ is $*$-isomorphic to A. Hence $\varphi \cdot \gamma = \varphi$. Put $\psi(\gamma(a)e) = \varphi(a)$ for all $a \in A$. Then ψ is a unique faithful tracial state of $\gamma(A)e$. Since $\gamma(A)e$ is isomorphic to A, $\gamma(A)e$ has a unique tracial state φ' defined by $\varphi'(\gamma(a)e) = \lambda^{-1}\varphi(\gamma(a)e)$. Hence $\varphi(\gamma(a)e) = \lambda\varphi'(\gamma(a)e) = \lambda\varphi(\gamma(a)e) = \lambda\varphi(a)$. ∎

LEMMA 3.3. *If γ has the canonical property, then the isometry w in Definition* 3.1 *satisfies* (I) *:*

$$w^*Aw = A \quad \textit{and} \quad \rho_\gamma(a) = waw^* \quad (a \in A).$$

Proof. Since $ww^*Aww^* = \gamma(A)ww^*$, we have $w^*Aw = w^*\gamma(A)w = w^*wA = A$ and $\rho_\gamma(a) = ww^*\gamma(a) = waw^*$ for all $a \in A$ by (i). ∎

DEFINITION 3.4. Let γ be a unit preserving $*$-endomorphism of a unital C^*-algebra A. Assume γ has the canonical property. The *extension algebra* $\langle A, \gamma\rangle$ of A via γ with the canonical property is defined as the crossed product $A \lhd \langle\rho_\gamma\rangle$ of A by ρ_γ.

THEOREM 3.5. *If a unit preserving $*$-endomorphism γ of a unital C^*-algebra A has the canonical property, then γ is extended to a $*$-endomorphism $\hat{\gamma}$ of $\langle A, \gamma\rangle$ which has the canonical property and*

$$\hat{\gamma}(\pi(x)) = \pi(\gamma(x)) \quad \textit{for all } x \in A, \quad \hat{\gamma}(W) = \pi(v)W,$$

where $v = (\lambda)^{-1}\gamma(e)fe$ for projections e and f in Definition 3.1.

Proof. Let φ and w be as in Definition 3.1. Put $\rho = \rho_\gamma$ in (†). Let ψ be the $*$-endomorphism of $\gamma(A)$ defined by $\psi(x) = v\rho(x)v^*$ for all $x \in \gamma(A)$. Then,

$$\psi(\gamma(a)) = ve\gamma^2(a)v^* = \gamma(\rho(a)), \ (a \in A)$$

and $\psi(x) = vwx(vw)^*$, $(x \in \gamma(A))$ for the isometry vw with $(vw)^*\gamma(A)vw = \gamma(A)$. Let u be the isometry on $L^2(A, \varphi)$ defined by $u(a\xi_0) = \gamma(A)\xi_0$ for all $a \in A$ where $\xi_0 = 1 \in A \subset H_0$. Then $vw\gamma(a)(vw)^* = \psi(\gamma(a)) = \gamma(\rho(a)) = uwu^*\gamma(a)(uwu)^*$ for all $a \in A$. Let U be the isometry defined by $(U\eta)_k = u\eta_k$, where $\eta = (\eta_k)_{k\in\mathbf{Z}} \in H$ for $\eta_k \in H_k$ in Section 2 for A and ρ. Let s be the isometry defined by $s\gamma(a)\xi_0 = (\lambda)^{-1}\psi(\gamma(a))\xi_0$. Then $\psi(x) = sxs^*$ for all $x \in \gamma(A)$ and $s^*\gamma(A)s = \gamma(A)$. Put $r_0 = uu^*$. Then

$$(U\pi(a)U^*(\eta))_k = \begin{cases} \pi(\gamma(a))r_0\eta_k & (k \leqslant 0) \\ \pi(\psi^n \cdot \gamma(a))r_0\eta_k & (k \geqslant 1), \end{cases}$$

and

$$(UWU^*(\eta))_k = \begin{cases} sr_0\eta_k & (k \leqslant 0) \\ s^2r_0\eta_k & (k \geqslant 1) \end{cases}.$$

Put $v_0 = 1, v_n = v\rho(v)\cdots\rho^{n-1}(v)$ and $r_n = v_n^*uu^*v_n$. The projection r_n commutes with $\rho^n(\gamma(A))$. Let V be the isometry defined by

$$(V(\eta))_k = \begin{cases} \eta_k & (k \leqslant 0) \\ v\rho(v)\cdots\rho^{k-1}(v)\eta_k & (k \geqslant 1), \end{cases}$$

respectively, where $\eta = (\eta_k)_k \in H, (\eta_k \in H_k)$. Then V^*U is an isometry on H. Put $\Psi(x) = V^*UxU^*V$ for all $x \in \langle A, \gamma\rangle$. Then Ψ is a $*$-isomorphism from $\langle A, \gamma\rangle$ onto $\gamma(A) \lhd \langle\psi\rangle$ such that $\Psi(\pi(a)) = \pi(\gamma(a))$ and $\Psi(W)$ is the isometry in $\gamma(A) \lhd \langle\psi\rangle$ corresponding to ψ. By Proposition 2.2, we have the $*$-endomorphism $\hat{\gamma}$ of $\langle A, \gamma\rangle$ with $\hat{\gamma}(\pi(a)) = \pi(\gamma(a))$ and $\hat{\gamma}(W) = \pi(v)W$.

Next, we show that $\hat{\gamma}$ has the canonical property and $\pi(e)$ is basic for $\hat{\gamma}$. By the relation $\hat{\gamma}(W)\pi(e) = \pi(\gamma(e)e)W = \pi(e)\hat{\gamma}(W)$, we have $\pi(e) \in \hat{\gamma}(\langle A, \gamma\rangle)' \cap \langle A, \gamma\rangle$. The relation $\gamma(e)e = ev$ implies that

$$\pi(e)W^k\pi(e) = \pi(e)\hat{\gamma}(W^k).$$

Hence the basic projection e for γ satisfies $\pi(e)\langle A, \gamma\rangle\pi(e) = \hat{\gamma}(\langle A, \gamma\rangle)\pi(e)$. Let E be the conditional expectation of $\langle A, \gamma\rangle$ onto $\pi(A)$ with $E(W^k) = 0$ for all $k \neq 0$, then $E \cdot \hat{\gamma} = \gamma \cdot E$. Let $\hat{\varphi} = \varphi \cdot E$. Then $\hat{\varphi}$ is a faithful state of $\langle A, \gamma\rangle$ which satisfies $\hat{\varphi} \cdot \hat{\gamma} = \hat{\varphi}$ and $\hat{\varphi}(\hat{\gamma}(x)\pi(e)) = \varphi(\gamma(E(x))e) = \lambda\varphi(E(x)) = \lambda\hat{\varphi}(x)$. It is obvious that the pair $\{\pi(e), \pi(f)\}$ satisfies the relation (ii). Therefore $\hat{\gamma}$ has the canonical property by Proposition 3.2. ∎

4. CUNTZ'S ENDOMORPHISM AND OCNEANU'S ENDOMORPHISM

In this section we shall restrict ourselves to the case of concrete $*$-endomorphisms of C^*-algebras and show relations between $*$-endomorphisms with the canonical property.

PROPOSITION 4.1. *Let A_0 be the set of all $n\times n$ matrices and A be the infinite C^*-tensor product $\otimes_{i=1}^{\infty} A_i$, where $A_i = A_0$ for all i. Then the 1-shift translation γ to the right on A has the canonical property.*

Proof. Let τ be the unique tracial state of A. Then τ is γ invariant. For a matrix unit $e_{i,j}$ of A_0, we identify $e_{i,j}$ and $e_{i,j}\otimes 1\otimes\cdots$. Put $e = e_{1,1}$. Then it is clear that $eAe = \gamma(A)e$ and $\tau(\gamma(a)e) = (1/n)\tau(a)$.

$$u = \sum_{i=1}^{n} e_{i,i-1}, \quad f = \frac{1}{n}\sum_{i,j} u^{j-i}\otimes e_{i,j}.$$

Then γ, τ, e and f satisfies the conditions in Proposition 3.2 for $\lambda = 1/n$. ∎

Cuntz ([7]) defined the simple C^*-algebra O_n generated by isometries $\{S_j; 1\leqslant j\leqslant n\}$ with $S_iS_j = \delta_{i,j}1$ and $\sum_i S_iS_i^* = 1$. He obtained in [8] interesting results based on his "canonical" inner $*$-endomorphism Φ on O_n defined by

$$\Phi(x) = \sum_{j=1}^{n} S_j x S_j^* \quad \text{for all} \quad x\in O_n.$$

PROPOSITION 4.2. *Let A and γ be the same as in Proposition 4.1. Then the extension algebra $\langle A,\gamma\rangle$ is the Cuntz algebra O_n and the extension $\hat{\gamma}$ of γ to $\langle A,\gamma\rangle$ is Cuntz's canonical inner $*$-endomorphism Φ.*

Proof. Put

$$T_j = \pi(e_{j,1})W, \quad \text{for all} \quad 1\leqslant j\leqslant n.$$

Since $e = e_{1,1}$ is basic for γ, T_j is isometry for all j, with $T_j^*T_i = \delta_{1,j}1$ and $\sum_j T_jT_j^* = \sum_j \pi(e_{j,j}) = 1$. The C^*-subalgebra B of $\langle A,\gamma\rangle$ generated by $\{T_j : j = 1,2,\ldots,n\}$ contains $W = T_1$ and $\pi(e_{j,1}) = T_jWW^*$ for all j. Hence B contains $\pi(A)$ and W which generate $\langle A,\gamma\rangle$ so that $\langle A,\gamma\rangle$ is the Cuntz algebra O_n.

By the definition of $\hat{\gamma}$,

$$\begin{aligned}\sum_{j=1}^{n} T_j\pi(a)T_j^* &= \sum_j \pi(e_{j,1})W\pi(a)W^*\pi(e_{1,j}) = \pi\Big(\sum_j e_{j,1}\gamma(a)e_{1,j}\Big)\\ &= \pi\Big(\sum_j e_{j,j}\gamma(a)\Big) = \pi(\gamma(a))\\ &= \hat{\gamma}(\pi(a)),\end{aligned}$$

for all $a \in A$. On the other hand,

$$v = n\gamma(e)fe = \sum_{j=1}^{n} e_{j,1} \otimes e_{1,j} = \sum_j e_{j,1}\gamma(e_{1,j}).$$

Hence

$$\sum_{j=1}^{n} T_j W T_j^* = \sum_j \pi(e_{j,1})WWW^*\pi(e_{1,j}) = \sum_j \pi(e_{j,1})\pi(\gamma(e_{1,j}))W = \hat{\gamma}(W).$$

Since these Φ and $\hat{\gamma}$ is norm continuous $*$-homomorphism, we have $\Phi = \hat{\gamma}$. ∎

COROLLARY 4.3. *Let Φ be Cuntz's canonical inner $*$-endomorphism on O_n. Then Φ has the canonical property.*

Proof. By Proposition 4.2, Φ is the extension $\hat{\gamma}$ of γ on A in Proposition 4.1. Since γ has the canonical property, Φ has the canonical property by Theorem 3.5. ∎

PROPOSITION 4.4. *Let Φ be Cuntz's $*$-endomorphism of O_n. Then $\langle O_n, \Phi\rangle$ is isomorphic to the tensor product $O_n \otimes C^*(u)$ of O_n and the C^*-algebra $C^*(u)$ generated by some unitary $u \in \langle O_n, \Phi\rangle$ with $E(u^k) = 0$ for all $k \geqslant 1$, where E is the conditional expectation from $\langle O_n, \Phi\rangle$ onto $\pi(O_n)$.*

Proof. Let ρ be the $*$-endomorphism of O_n defined by (†) for Φ. Let π be the representation of O_n and W be the isometry defined by Φ in Section 2. Put $u = \pi(S_1^*)W$. Then u is a unitary in $\langle O_n, \Phi\rangle$ because e is basic for γ and $\pi(e)$ is basic for γ. The unitary u satisfies $u^k = \pi(S_1)^{*k}W^k$ and $E(u^k) = 0$ for all $k \geqslant 1$. Since

$$\pi(S_1)^* W\pi(x) = \pi(S_1^*\Phi(x))W = \pi(x)\pi(S_1)^*W, \text{ for all } x \in O_n,$$

we have $u \in O_n' \cap \langle O_n, \Phi\rangle$. The algebra $\langle O_n, \Phi\rangle$ is generated by $\pi(O_n), W$ and $W = \pi(S_1)u$. Hence $\langle O_n, \Phi\rangle$ is the C^*-algebra generated by $\pi(O_n)$ and u so that $\langle O_n, \Phi\rangle$ is isomorphic to $O_n \otimes C^*(u)$. ∎

Paschke ([19]) proved that if A is a unital simple strongly amenable C^*-algebra and W is a non unitary isometry with the relation:

$$\text{(II)} \qquad WAW^* \subset A, \quad W^*AW \subset A$$

then the C^*-algebra $C^*(A, W)$ generated by A and W is always simple. The Cuntz algebra O_n is simple by [7] and amenable ([20]) but not strongly amenable. The next corollary shows that his result does not hold in general without strong amenability.

COROLLARY 4.5. *There exist isometries W_1 and W_2 with the relation* (II) *for O_n which give a simple $C^*(O_n, W_1)$ and a non simple $C^*(O_n, W_2)$.*

Proof. Let $\{S_i : 1 \leqslant i \leqslant n\}$ be isometries which generate O_n. Put $W_1 = S_1$. Then $C^*(O_n, W_1) = O_n$, so that $C^*(O_n, W_1)$ is simple. Let W_2 be the isometry in $\langle O_n, \Phi\rangle$ which corresponds the $*$-endomorphism defined by Φ. Then $C^*(O_n, W_2)$ is isomorphic to $O_n \otimes C^*(u)$ by Proposition 4.4. Hence $C^*(O_n, W_2)$ is not simple. ∎

EXAMPLE 4.6. Let $\{e_i; i = 1, 2, 3, \ldots\}$ be a sequence of projections with Jones relations for a λ $(0 < \lambda < 1)$:

$$e_ie_j = e_je_i \ (|i-j| \neq 1), \quad e_ie_je_i = \lambda e_i \ (|i-j| = 1).$$

Let A_k be the algebra generated by $e_1, e_2, \ldots, e_k$ and the unit 1. Then by [12], there exists a faithful state τ with $\tau(xe_{k+1}) = \lambda\tau(x)$ for all $x \in A_k$ and $k \in \mathbf{N}$. Let A be the C^*-algebra obtained from the GNS construction of $\bigcup_k A_k$ by τ. Then the trace τ is extended to a unique trace τ of A. Let γ be the $*$-endomorphism defined by $\gamma(e_i) = e_{i+2}$. Then γ is canonical because τ and e_1, e_2 satisfies the conditions for γ ([3], [4], [5]).

EXAMPLE 4.7. The example in Proposition 4.1 and Example 4.6 are more generalized as Ocneanu's canonical shift Γ on the higher relative commutant algebras for the factor inclusion $N \subset M$ with finite index ([17], [3], [4]) and also Ocneanu's canonical shifts φ^2 ([17]) on the C^*-algebra A obtained from n-string algebras on a finite bipartite graph G with a biunitary connection ([18] or see [14]).

In fact, the first Jones projection e and the second Jones projection f satisfy the relation (ii) in Definition 3.1 and the C^*-algebra A has a faithful tracial state which satisfies (iii) in Proposition 3.2.

In the case that $M = N \rtimes \mathbb{Z}_n$ for a II_1 factor N, Γ is nothing else than γ in Proposition 4.1. In the case that G is the Coxeter graph A_n, φ^2 coincides with γ in Example 4.6.

LEMMA 4.8. *Let A be a unital C^*-algebra generated by an increasing sequence $A_1 \subset A_2 \subset A_3 \subset \cdots$ of C^*-algebras with the same unit. Let γ be a $*$-endomorphism of A with the canonical property. Assume the following conditions:*

(i) *A basic projection e for γ is contained in A_i, for some i.*

(ii) *For each $j, m \in \mathbf{N}$, the $C^*(A_j, \gamma(A_j), \ldots, \gamma^m(A_j))$ is contained in A_{j+2m}.*

(iii) *For each $j \in \mathbf{N}$, there exists an integer $n_j \in \mathbf{N}$ such that A_j commutes with $\gamma^{kn_j}(A_j)$ for all $k \in \mathbf{N}$.*

Then the condition $(*)$ *is satisfied for ρ defined by* (iii).

Proof. Let a, $\{a_i;\ i = 1, 2, \ldots, n\}$, $\{k_1, k_2, \ldots, k_n\}$ and ε be as in the condition $(*)$. Then there exist an integer j and $b \in A_j$, $b_i \in A_j$ with $\|a - b\| < \varepsilon/2$ and $\|a_i - b_i\| < \varepsilon/2$ for all i $(1 \leqslant i \leqslant n)$. We may assume that $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$. Put

$$q = \rho^{n_j}(e)\rho^{k_1 n_j}(1-e)\rho^{k_2 n_j}(1-e)\cdots\rho^{k_n n_j}(1-e).$$

Then $\|qbq\| = \|b\|$ and $\|qb_i\rho^{k_i}(q)\| = \|b_i q\rho^{k_i}(q)\| = 0$. Hence the projection q satisfies the condition $(*)$. ∎

THEOREM 4.9. *Let $\{A, \gamma\}$ be one of the pairs in Examples 4.6 or 4.7 of finite depth $N \subset M$ or finite graph G. Then $\langle A, \gamma\rangle$ is simple, amenable but not strongly amenable.*

Proof. The increasing sequence $\{A_j; j = 1, 2, \ldots\}$ satisfies the condition in Lemma 4.8 for γ. In the case of Example 4.6, if $\lambda > 1/4$ then the algebra A is simple and if $\lambda \leqslant 1/4$ then A has a unique non trivial ideal J (see [13], [24]). But $e\gamma(J)$ is not contained in J. In the case of Example 4.7, the algebra A is simple under the assumption. Hence $\langle A, \gamma\rangle$ is simple by Theorem 2.2 and Lemma 4.8 (or [10]). By the definition, $\langle A, \gamma\rangle$ is generated by $\pi(A)$ and W which satisfy the condition of [20]. Hence $\langle A, \gamma\rangle$ is amenable. Since $\langle A, \gamma\rangle$ does not have any tracial state, $\langle A, \gamma\rangle$ is not strongly amenable. ∎

M. Izumi told me that these C^*-algebras $\langle A, \gamma\rangle$ in Theorem 4.9 are not always Cuntz algebras O_n, because they have different $K_0(\langle A, \gamma\rangle)$ from $K_0(O_n)$. He obtains interesting results on the Cuntz algebras with relations in the index theory of III factors in [12] and in a preparing paper. Also Katayama obtains an genaralization of Cuntz algebras in a different method from in this paper.

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MARIE CHODA
Department of Mathematics
Osaka Kyoiku University
Asahigaoka, Kashiwara 582
JAPAN

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