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# SPECTRAL PROPERTIES OF A CLASS OF RATIONAL OPERATOR VALUED FUNCTIONS

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ABSTRACT. We consider a selfadjoint operator function $L$ of the form $L(\lambda) := \lambda - A \pm B^*(C-\lambda)^{-1}B$ under the assumption that the spectrum of $L$ splits into two parts. In case of the sign $+$ with the pencil $L$ there is associated a selfadjoint operator $\widetilde{A}$ in some Hilbert space $\widetilde{\mathcal{H}} \supset \mathcal{H}$, in case of the sign $-$ with $L$ there is associated a selfadjoint $\widetilde{B}$ in a Kreĭn space $\widetilde{\mathcal{K}} \supset \mathcal{H}$. Spectral properties of these associated operators are crucial for the study of the spectral properties of $L$. Sufficient conditions for the fact that the eigenvectors corresponding to certain parts of the spectrum of $L$ form a Riesz basis in $\mathcal{H}$ are given.

KEYWORDS: *Operator pencil, spectrum, eigenvector, Riesz basis.*

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Let $A$ and $C$ be (possibly unbounded) selfadjoint operators in some Hilbert spaces $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, and let $B \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{H}})$, that is, $B$ is a bounded linear operator from $\mathcal{H}$ into $\widehat{\mathcal{H}}$. In this note we consider operator valued functions

$$(1) \qquad L(\lambda) := \lambda - A \pm B^*(C-\lambda)^{-1}B,$$

defined for all $\lambda \in \rho(C)$ (hence at least for $\lambda \in \mathbb{C}^+ \cup \mathbb{C}^-$) and with the $\lambda$-independent domain $\mathcal{D}(A)$.

If in (1) the sign $+$ holds on the right hand side, then $L$ is an operator valued Nevanlinna function, that is

$$\frac{\Im(L(\lambda)x, x)}{\Im\lambda} \geqslant 0 \quad \text{if } x \in \mathcal{D}(A),\ \lambda \in \mathbb{C}^+ \cup \mathbb{C}^-.$$

Starting from this simple observation, we establish some spectral properties of $L$, considering $L(\lambda)$ as an operator pencil. So, e.g., an eigenvalue $\lambda_0$ of $L$, $\lambda_0 \in \rho(C)$, is a complex number for which there exists a nonzero vector $x_0 \in \mathcal{H}$ such that $L(\lambda_0)x_0 = 0$.

The main result of this note is Theorem 3.5, where it is shown that under some assumptions the eigenvectors of $L$, corresponding to eigenvalues in a certain subset of $\mathbf{R}$, can be chosen to form a Riesz basis of $\mathcal{H}$. As a main tool, we represent the operator function $-L(\lambda)^{-1}$, which is again a Nevanlinna function, in the form

$$-L(\lambda)^{-1} = \int_{-\infty}^{\infty} \frac{\mathrm{d}F(t)}{t-\lambda} \tag{2}$$

with a (unique) selfadjoint nondecreasing operator function $F$ on $\mathbf{R}$ such that $F(-\infty) = 0$, $F(+\infty) = I$ and $F(t+) = F(t)$ $(t \in \mathbf{R})$; here the limits are to be understood in the strong operator topology. If there is a point $\alpha \in \mathbf{R}$ such that $\sigma(A) > \alpha$ and $\sigma(C) < \alpha$, it is shown that

$$F((\alpha,\infty)) := F(\infty) - F(\alpha) \geqslant \frac{1}{2} + \delta \tag{3}$$

for some $\delta > 0$.

The operator function $-L(\lambda)^{-1}$ can be considered as the compressed resolvent of some selfadjoint operator $\widetilde{A}$ in some Hilbert space $\widetilde{\mathcal{H}} \supset \mathcal{H}$, that is

$$-L(\lambda)^{-1} = P_0(\widetilde{A} - \lambda)^{-1}|\mathcal{H}, \quad \lambda \in \mathbf{C}^+ \cup \mathbf{C}^-,$$

where $P_0$ is the orthogonal projection from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$. In fact, $\widetilde{A}$ can be chosen to be

$$\widetilde{A} := \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \quad \text{in } \widetilde{\mathcal{H}} := \mathcal{H} \oplus \widehat{\mathcal{H}}.$$

The relation (2) implies that the spectral subspaces $\widetilde{\mathcal{L}}_+$, $\widetilde{\mathcal{L}}_-$ of $\widetilde{A}$, corresponding to the spectrum $\sigma(\widetilde{A}) \cap (\alpha,\infty)$ and $\sigma(\widetilde{A}) \cap (-\infty,\alpha)$, respectively, can be represented by means of an "angular operator". E. g., $\widetilde{\mathcal{L}}_-$ admits the representation

$$\widetilde{\mathcal{L}}_- = \left\{ \begin{pmatrix} K\widehat{x} \\ \widehat{x} \end{pmatrix} : \widehat{x} \in \widehat{\mathcal{H}} \right\}$$

with a contraction $K \in \mathcal{L}(\widehat{\mathcal{H}}, \mathcal{H})$. Using this contraction, the restriction $\widetilde{A}|\widetilde{\mathcal{L}}_+$ turns out to be unitarily equivalent to the operator $A - BK^*$, which is selfadjoint with respect to the inner product in $\mathcal{H}$ defined by

$$[x,y] := ((I + KK^*)x, y) \quad (x, y \in \mathcal{H}).$$

In the last section we show that under certain assumptions similar results can be proved for operator functions (1) with the sign $-$ on the right hand side. Then $L(\lambda)$ is not a Nevanlinna function. In this case, however, a natural "linearization" is given by the operator

$$\widetilde{B} = \begin{pmatrix} A & B^* \\ -B & C \end{pmatrix}$$

which is selfadjoint in $\widetilde{\mathcal{H}}$ with respect to a suitably chosen Krein space inner product.

The results of this note can immediately be applied to eigenvalue problems of the form

$$y'' + \lambda y + \frac{qy}{u - \lambda} = 0 \quad \text{on } [0,1],\ y(0) = y(1) = 0, \tag{4}$$

where $q$, $u$ are real summable functions on $[0,1]$, $u < 0$, $q > 0$ (see [2], [6]) or to the corresponding problem for an elliptic partial differential operator (see [3]). It follows that under these conditions on $u$, $q$ the eigenfunctions of the problem (4), corresponding to the eigenvalues in $(0,\infty)$, can be chosen to form a Riesz basis in $L^2(0,1)$. If $u$ is a step function, also the eigenfunctions corresponding to the negative eigenvalues have this property.

After this manuscript was completed we came to know about the paper [8] (see also [9]) by A. K. Motovilov. He proves expansion theorems and develops a scattering theory for an eigenvalue problem of the form (1). Although there is a nonvoid intersection of his results and ours, the methods are different.

## 1. THE OPERATOR NEVANLINNA FUNCTION $L$

In this section we consider the operator function

$$L(\lambda) := \lambda - A + B^*(C - \lambda)^{-1}B, \tag{1.1}$$

where $A, B, C$ satisfy the assumptions mentioned at the beginning of the introduction.

LEMMA 1.1. *Suppose $\lambda \in \mathbb{C}^+ \cup \mathbb{C}^-$. Then the following relations hold:*

(i) $$\frac{\Im(L(\lambda)x, x)}{\Im\lambda} > 0 \quad \textit{if } x \in \mathcal{D}(A), \quad x \neq 0;$$

(ii) $$\mathcal{R}(L(\lambda)) = \mathcal{H};$$

(iii) $$\|L(\lambda)^{-1}\| \leqslant |\Im\lambda|^{-1};$$

(iv) $$\lim_{\eta\uparrow\infty} \|\mathrm{i}\eta L(\mathrm{i}\eta)^{-1}x - x\| = 0 \quad \text{if } x \in \mathcal{H}.$$

*Proof.* For $x \in \mathcal{D}(A)$ and $\lambda \neq \bar\lambda$ we have

$$\Im(L(\lambda)x, x) = (\Im\lambda)(\|x\|^2 + \|(C-\lambda)^{-1}Bx\|^2), \tag{1.2}$$

hence (i) follows. Further, the relation (1.2) implies

$$\|L(\lambda)x\|\,\|x\| \geqslant |(L(\lambda)x, x)| \geqslant |\Im(L(\lambda)x, x)| \geqslant |\Im\lambda|\,\|x\|^2,$$

therefore $L(\lambda)$ is injective and the range $\mathcal{R}(L(\lambda))$ is closed. As

$$\mathcal{R}(L(\lambda)) = (\ker L(\lambda)^*)^\perp = \left(\ker L(\bar\lambda)\right)^\perp = \mathcal{H},$$

also (ii) and (iii) follow. Finally, if $x \in \mathcal{D}(A)$ we have

$$\|\mathrm{i}\eta L(\mathrm{i}\eta)^{-1}x - x\| = \|L(\mathrm{i}\eta)^{-1}(Ax - B^*(C - \mathrm{i}\eta)^{-1}Bx)\|,$$

and the expression on the right hand side tends to zero if $\eta \uparrow \infty$. As $\mathcal{D}(A)$ is dense in $\mathcal{H}$ and $\|\mathrm{i}\eta L(\mathrm{i}\eta)^{-1}\| \leqslant 1$ also (iv) is proved. ∎

On the set $\mathbf{C}^+ \cup \mathbf{C}^-$ we consider the operator function

$$N(\lambda) := -L(\lambda)^{-1}$$

with values in $\mathcal{L}(\mathcal{H})$. Evidently, if $x \neq 0$,

$$\frac{\Im(N(\lambda)x, x)}{\Im\lambda} = \frac{\Im(L(\lambda)y_\lambda, y_\lambda)}{\Im\lambda} > 0$$

where $y_\lambda := L(\lambda)^{-1}x$, hence $N$ is an operator valued Nevanlinna function. Using the relation (iv) of Lemma 1 it follows that $N$ admits the representation

$$N(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}F(t)}{t-\lambda} \quad (\lambda \neq \bar\lambda) \tag{1.3}$$

with a (unique) selfadjoint operator function $F$ on $\mathbf{R}$ with the properties mentioned after the relation (2), see, e.g., [10], [1]. If we suppose for simplicity that $F$ is continuous at $t = \alpha$, the Stieltjes inversion formula yields the relation

$$-\frac{1}{2\pi\mathrm{i}}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} N(\lambda)\,\mathrm{d}\lambda = -\int_{t=-\infty}^{\infty} \frac{1}{2\pi\mathrm{i}}\int_{\lambda=\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} (t-\lambda)^{-1}\,\mathrm{d}\lambda\,\mathrm{d}F(t)$$
$$= F((-\infty,\alpha)) - F((\alpha,+\infty))$$

where the singular integrals along the imaginary axis are to be understood in the sense of Cauchy principal values at infinity and at $\alpha$ in the strong operator topology.

The main result of this section is the following

THEOREM 1.2. *Suppose that, additionally to the assumptions about $A$, $B$, $C$ at the beginning of the introduction, there exists an $\alpha \in \mathbb{R}$ such that $\sigma(C) < \alpha$ and $\alpha < \sigma(A)$. Then the operator function $F$ in the integral representation*

$$-L(\lambda)^{-1} = \int_{-\infty}^{\infty} \frac{\mathrm{d}F(t)}{t-\lambda}$$

*has the property*

$$F((\alpha, +\infty)) - F((-\infty, \alpha)) \geqslant \delta_1 \tag{1.4}$$

*for some $\delta_1 > 0$.*

In the proof of Theorem 1.2 we shall use the following

LEMMA 1.3. *Let $A$ be a selfadjoint operator in $\mathcal{H}$, $\sigma(A) > \gamma$ with some $\gamma > 0$ and let $D(\eta)$, $\eta > 0$, be an $\mathcal{L}(\mathcal{H})$-valued continuous function of $\eta$ such that $\|D(\eta)\| \to 0$ if $\eta \uparrow \infty$. Then there exists an $\eta_0 > 0$ such that*

$$\begin{aligned}(\mathrm{i}\eta - A + D(\eta))^{-1}A(-\mathrm{i}\eta - A + D(\eta)^*)^{-1} \geqslant \\ \geqslant \frac{1}{2}(\mathrm{i}\eta - A)^{-1}A(-\mathrm{i}\eta - A)^{-1} \quad \text{if} \quad \eta \geqslant \eta_0.\end{aligned} \tag{1.5}$$

*Proof.* Choose $d > 0$ such that $2d + d^2 \leqslant \gamma$ and $\eta_1 \geqslant 1$ such that $\|D(\eta)\| \leqslant d$ if $\eta > \eta_1$. Observing that $\|A(\mathrm{i}\eta - A)^{-1}\| \leqslant 1$ it follows that for $\eta > \eta_1$ we have

$$\begin{aligned}\|D(\eta)(\mathrm{i}\eta - A)^{-1}A(-\mathrm{i}\eta - A)^{-1}D(\eta)^* + D(\eta)(\mathrm{i}\eta - A)^{-1}A + \\ + A(-\mathrm{i}\eta - A)^{-1}D(\eta)^*\| \leqslant d^2 + 2d \leqslant \gamma,\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{2}(D(\eta)(\mathrm{i}\eta - A)^{-1}A(-\mathrm{i}\eta - A)^{-1}D(\eta)^* + D(\eta)A(\mathrm{i}\eta - A)^{-1} + \\ + A(-\mathrm{i}\eta - A)^{-1}D(\eta)^*) \leqslant \frac{\gamma}{2} \leqslant \frac{1}{2}A\end{aligned}$$

and therefore

$$\frac{1}{2}(D(\eta) + \mathrm{i}\eta - A)(\mathrm{i}\eta - A)^{-1}A(-\mathrm{i}\eta - A)^{-1}(D(\eta)^* - \mathrm{i}\eta - A) \leqslant A.$$

This inequality is equivalent to (1.5) if the inverse $(D(\eta) + \mathrm{i}\eta - A)^{-1}$ exists, which is true for sufficiently large $\eta$. The lemma is proved. ∎

*Proof of Theorem* 1.2. Without loss of generality we suppose that $\alpha = 0$. Then

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} N(\lambda)\,d\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} L(i\eta)^{-1} i\,d\eta$$

$$= \frac{1}{2\pi}\left[\int_0^{\infty}(i\eta - A + B^*(C-i\eta)^{-1}B)^{-1}\,d\eta + \int_{-\infty}^{0}(i\eta - A + B^*(C-i\eta)^{-1}B)^{-1}\,d\eta\right]$$

$$= \frac{1}{\pi}\int_0^{\infty} L(i\eta)^{-1}(-A + B^*(C+i\eta)^{-1}C(C-i\eta)^{-1}B)L(-i\eta)^{-1}\,d\eta$$

$$< -\frac{1}{\pi}\int_0^{\infty} L(i\eta)^{-1}AL(-i\eta)^{-1}\,d\eta < 0.$$

Putting $D(\eta) := B^*(C+i\eta)^{-1}C(C-i\eta)^{-1}B$, it holds

$$\|D(\eta)\| \leqslant \frac{\|B\|^2}{\eta},$$

and we can choose an $\eta_0$ according to Lemma 1.3. It follows from (1.5) that

$$\frac{1}{\pi}\int_{\eta_0}^{\infty} L(i\eta)^{-1}AL(-i\eta)^{-1}\,d\eta \geqslant \frac{1}{2\pi}\int_{\eta_0}^{\infty}(i\eta - A)^{-1}A(-i\eta - A)^{-1}\,d\eta. \tag{1.6}$$

With the spectral function $E$ of $A$ and a positive lower bound $\varepsilon$ of $A$ we can write

$$(i\eta - A)^{-1}A(-i\eta - A)^{-1} = \int_{\varepsilon}^{\infty}\frac{t}{t^2+\eta^2}\,dE(t),$$

and the expression on the right hand side of (1.6) becomes

$$\frac{1}{2\pi}\int_{\varepsilon}^{\infty}\int_{\eta_0}^{\infty}\frac{t}{t^2+\eta^2}\,d\eta\,dE(t) = \frac{1}{2\pi}\int_{\varepsilon}^{\infty}\left(\frac{\pi}{2} - \arctan\frac{\eta_0}{t}\right)dE(t)$$

$$\geqslant \frac{1}{2\pi}\left(\frac{\pi}{2} - \arctan\frac{\eta_0}{\varepsilon}\right) > 0.$$

Theorem 1.2 is proved. ∎

COROLLARY 1.4. *Under the assumptions of Theorem 1.2 it holds*

$$F((\alpha,\infty)) \geqslant \frac{1}{2} + \delta \tag{1.7}$$

*for some $\delta > 0$.*

Indeed, $F((-\infty,\alpha)) + F((\alpha,\infty)) = I$, hence (1.4) implies

$$2F((\alpha,\infty)) = F((\alpha,\infty)) + (I - F((-\infty,\alpha))) \geqslant 1 + \delta_1.$$

## 2. THE LINEARIZATION OF $L$

We consider again the operator function $L(\lambda)$ in (1.1), where $A$, $B$, $C$ satisfy the assumptions formulated at the beginning of the introduction. In the Hilbert space $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \widehat{\mathcal{H}}$ we define the selfadjoint operator

$$\widetilde{A} := \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \quad \text{on} \quad \mathcal{D}(A) \oplus \mathcal{D}(C). \tag{2.1}$$

At least for nonreal $\lambda$ the resolvent of $\widetilde{A}$ can be written as

$$(\widetilde{A} - \lambda)^{-1} = \begin{pmatrix} -L(\lambda)^{-1} & L(\lambda)^{-1}B^*(C-\lambda)^{-1} \\ (C-\lambda)^{-1}BL(\lambda)^{-1} & -(\lambda - C + B(A-\lambda)^{-1}B^*)^{-1} \end{pmatrix}. \tag{2.2}$$

THEOREM 2.1. *If there exist $\beta, \gamma \in \mathbf{R}$ such that $\sigma(C) \leqslant \gamma < \beta \leqslant \sigma(A)$, then the interval $(\gamma, \beta)$ belongs to $\rho(\widetilde{A})$.*

*Proof.* If $\lambda \in (\gamma, \beta)$ then $\lambda \in \rho(C)$ and we have

$$L(\lambda) = \lambda - A + B^*(C-\lambda)^{-1}B \leqslant \lambda - \beta < 0,$$

hence the selfadjoint operator $L(\lambda)$ has a bounded everywhere defined inverse. By the same reasoning, for these $\lambda$ also $(\lambda - C + B(A-\lambda)^{-1}B^*)^{-1}$ exists as a bounded everywhere defined operator and the claim of Theorem 2.1 follows from (2.2). ∎

If $P_0$ denotes the orthogonal projection in $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$, we have

$$-L(\lambda)^{-1} = P_0(\widetilde{A} - \lambda)^{-1}|\mathcal{H}, \tag{2.3}$$

that is, $-L(\lambda)^{-1}$ is a compressed resolvent of the operator $\widetilde{A}$. The operator function $F$ in the representation (1.3) can therefore be expressed as

$$F(t) = P_0\widetilde{E}(t)|\mathcal{H}, \tag{2.4}$$

where $\widetilde{E}$ denotes the spectral function of $\widetilde{A}$.

In the following we represent the elements of $\widetilde{\mathcal{H}}$ as column vectors : $\widetilde{x} = \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$ with $x \in \mathcal{H}$, $\widehat{x} \in \widehat{\mathcal{H}}$. We shall show that in this representation, as a consequence of Theorem 1.2, the spectral subspaces of $\widetilde{A}$:

$$\widetilde{\mathcal{L}}_- := \widetilde{E}((-\infty, \alpha))\widetilde{\mathcal{H}}, \quad \widetilde{\mathcal{L}}_+ := \widetilde{E}((\alpha, +\infty))\widetilde{\mathcal{H}}, \tag{2.5}$$

have contractive angular operators.

LEMMA 2.2. *Let $A, B, C$ be as in Theorem 1.2. Then there exists a contraction $K \in \mathcal{L}(\widehat{\mathcal{H}}, \mathcal{H})$, $\|K\| < 1$, such that*

$$\widetilde{\mathcal{L}}_- = \left\{ \begin{pmatrix} K\widehat{y} \\ \widehat{y} \end{pmatrix} : \widehat{y} \in \widehat{\mathcal{H}} \right\}, \tag{2.6}$$

$$\widetilde{\mathcal{L}}_+ = \left\{ \begin{pmatrix} x \\ -K^* x \end{pmatrix} : x \in \mathcal{H} \right\}. \tag{2.7}$$

*Proof.* We suppose again without loss of generality $\alpha = 0$. The Corollary 1.4 implies that for $\widetilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \widetilde{\mathcal{H}}$ we have

$$\|x\|^2 \geqslant \|P_0 \widetilde{E}((0,\infty))\widetilde{x}\|^2 \geqslant \left(\frac{1}{2} + \delta\right)^2 \|x\|^2. \tag{2.8}$$

Now consider a sequence of elements $\widetilde{y}_n \in \widetilde{\mathcal{L}}_-$, $n = 1, 2, \ldots$, such that with $\widetilde{y}_n = \begin{pmatrix} y_n \\ \widehat{y}_n \end{pmatrix}$ we have $\|y_n\| = 1$, $\|\widehat{y}_n\| \to 0$ $(n \to \infty)$. For arbitrary elements $\widetilde{z}_n := \begin{pmatrix} z_n \\ 0 \end{pmatrix} \in \widetilde{\mathcal{H}}$, $n = 1, 2, \ldots$, $\|z_n\| \to 1$ $(n \to \infty)$ it follows that

$$0 = \left( \begin{pmatrix} y_n \\ \widehat{y}_n \end{pmatrix}, \widetilde{E}((0,+\infty))\widetilde{z}_n \right) = \left( y_n, P_0 \widetilde{E}((0,+\infty))\widetilde{z}_n \right) + \\ + \left( \widehat{y}_n, (I - P_0)\widetilde{E}((0,+\infty))\widetilde{z}_n \right).$$

The second term on the right hand side tends to zero if $n \to \infty$ as $\|\widehat{y}_n\| \to 0$ and $\|\widetilde{z}_n\| \to 1$. Therefore also the first term tends to zero, or

$$(y_n, F((0,+\infty))z_n) \to 0 \quad (n \to \infty)$$

for each sequence $(z_n) \subset \mathcal{H}$, $\|z_n\| \to 1$. Choosing $z_n = y_n$, the relation (1.7) with $\alpha = 0$ implies $\|\widetilde{y}_n\| \to 0$ $(n \to \infty)$, a contradiction to the assumption $\|\widetilde{y}_n\| = 1$. Therefore, if $\widetilde{y} \in \widetilde{\mathcal{L}}_-$, $\widetilde{y} = \begin{pmatrix} y \\ \widehat{y} \end{pmatrix}$, the first component $y$ can be written as $K\widehat{y}$ with some bounded linear operator $K$ from $\widehat{\mathcal{H}}_0$ into $\mathcal{H}$, where $\widehat{\mathcal{H}}_0$ is a closed subspace of $\widehat{\mathcal{H}}$ : $\widehat{\mathcal{H}}_0 = (I - P_0)\widetilde{\mathcal{L}}_-$.

Next we show that $\widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}$. Consider $\widehat{y} \in \widehat{\mathcal{H}}$, $\widehat{y} \perp \widehat{\mathcal{H}}_0$ such that $\widehat{y} \neq 0$. Then

$$\left( \begin{pmatrix} 0 \\ \widehat{y} \end{pmatrix}, \widetilde{\mathcal{L}}_- \right) = \{0\},$$

hence $\begin{pmatrix} 0 \\ \widehat{y} \end{pmatrix} \in \widetilde{\mathcal{L}}_+$. It follows that

$$\left(\widetilde{A}^{-1}\begin{pmatrix} 0 \\ \widehat{y} \end{pmatrix}, \begin{pmatrix} 0 \\ \widehat{y} \end{pmatrix}\right) > 0,$$

as $\widetilde{\mathcal{L}}_+$ is the spectral subspace of $\widetilde{A}$ corresponding to $(0, \infty)$. On the other hand we have from (2.2) and the assumptions $\sigma(C) < 0$, $\sigma(A) > 0$ :

$$\left(\widetilde{A}^{-1}\begin{pmatrix} 0 \\ \widehat{y} \end{pmatrix}, \begin{pmatrix} 0 \\ \widehat{y} \end{pmatrix}\right) = ((C - BA^{-1}B^*)^{-1}\widehat{y}, \widehat{y}) < 0,$$

a contradiction. Thus the representation (2.6) of $\widetilde{\mathcal{L}}_-$ with some bounded linear operator $K$ is shown. The representation (2.7) of $\widetilde{\mathcal{L}}_+$ follows immediately from the fact that $\widetilde{\mathcal{L}}_+ = \widetilde{\mathcal{L}}_-^{\perp}$.

It remains to show that $K$ is a contraction. To this end we introduce the operator $Q := \widetilde{E}((0, \infty))P_0$, mapping $\mathcal{H}$ onto $\widetilde{\mathcal{L}}_+$. The representation (2.7) implies $\mathcal{H} = P_0\widetilde{\mathcal{L}}_+$, therefore for arbitrary $\widetilde{x} \in \widetilde{\mathcal{L}}_+$ we have

$$\|P_0\widetilde{x}\|^2 = \|P_0\widetilde{E}((0, \infty))\widetilde{x}\|^2 > 0.$$

Since the relation (2.8) implies

$$Q^*Q \geqslant \frac{1}{2} + \delta \quad \text{on } \mathcal{H} \tag{2.9}$$

it follows that

$$QQ^* = \widetilde{E}((0, \infty))P_0\widetilde{E}((0, \infty)) \geqslant \frac{1}{2} + \delta \quad \text{on } \widetilde{\mathcal{L}}_+.$$

If $\widetilde{x} = \begin{pmatrix} x \\ -K^*x \end{pmatrix} \in \widetilde{\mathcal{L}}_+$ is arbitrary, we get

$$\|x\|^2 \geqslant (\|x\|^2 + \|K^*x\|^2)\left(\frac{1}{2} + \delta\right) \quad (x \in \mathcal{H}),$$

which implies $\|K^*\| < 1$. The Lemma 2.2 is proved. ∎

It is easy to check that the orthogonal projections $\widetilde{E}((\alpha, \infty))$ and $\widetilde{E}((-\infty, \alpha))$ admit the following matrix representations by means of the angular operator $K$ from Lemma 2.2:

$$\widetilde{E}((\alpha, \infty)) = \begin{pmatrix} (I + KK^*)^{-1} & -(I + KK^*)^{-1}K \\ -K^*(I + KK^*)^{-1} & K^*(I + KK^*)^{-1}K \end{pmatrix},$$

$$\widetilde{E}((-\infty,\alpha)) = \begin{pmatrix} K(K^*K+I)^{-1}K^* & K(K^*K+I)^{-1} \\ (K^*K+I)^{-1}K^* & (K^*K+I)^{-1} \end{pmatrix}.$$

The main result of this note is the following theorem. For simplicity we assume that the operator $C$ is bounded.

THEOREM 2.3. *Let $A$ be a selfadjoint operator in $\mathcal{H}$, $C$ a bounded selfadjoint operator in $\widehat{\mathcal{H}}$, $B \in \mathcal{L}(\mathcal{H},\widehat{\mathcal{H}})$ and suppose that there exists an $\alpha \in \mathbf{R}$ such that $\sigma(A) > \alpha$ and $\sigma(C) < \alpha$. Then there exists a contraction $K \in \mathcal{L}(\widehat{\mathcal{H}},\mathcal{H})$, $\|K\| < 1$, such that:*

(i) *The subspaces $\widetilde{\mathcal{L}}_\pm$, defined in (2.5), have the representations*

$$\widetilde{\mathcal{L}}_- = \left\{\begin{pmatrix} K\widehat{y} \\ \widehat{y}\end{pmatrix} : \widehat{y} \in \widehat{\mathcal{H}}\right\}, \quad \widetilde{\mathcal{L}}_+ = \left\{\begin{pmatrix} x \\ -K^*x\end{pmatrix} : x \in \mathcal{H}\right\}.$$

(ii) *The operator $K$ has the property $\mathcal{R}(K) \subset \mathcal{D}(A)$ and it satisfies the Riccati equation*

$$KBK - B^* - AK + KC = 0. \tag{2.10}$$

(iii) *The restriction $\widetilde{A}|\widetilde{\mathcal{L}}_+$ is unitarily equivalent to the operator $A - B^*K^*$, which is selfadjoint in the Hilbert space $(\mathcal{H},[\cdot,\cdot])$, where $[\cdot,\cdot]$ denotes the inner product*

$$[x,y] := ((I+KK^*)x,y) \quad (x,y \in \mathcal{H}).$$

(iv) *The restriction $\widetilde{A}|\widetilde{\mathcal{L}}_-$ is unitarily equivalent to the operator $C + BK$, which is selfadjoint in the Hilbert space $(\mathcal{H},[\cdot,\cdot]_\wedge)$, where $[\cdot,\cdot]_\wedge$ denotes the inner product*

$$[\widehat{x},\widehat{y}]_\wedge := ((I+K^*K)\widehat{x},\widehat{y}) \quad (\widehat{x},\widehat{y} \in \widehat{\mathcal{H}}).$$

*If the resolvent of the operator $A$ is compact then also $K$ is a compact operator.*

*Proof.* The existence of $K$ and the statement (i) follow from Lemma 2.2. The operator $\widetilde{A}$ is bounded from below and hence $\widetilde{\mathcal{L}}_- \subset \mathcal{D}(\widetilde{A})$. On the other hand we have

$$\mathcal{D}(\widetilde{A}) = \left\{\begin{pmatrix} x \\ \widehat{x}\end{pmatrix} : x \in \mathcal{D}(A),\ \widehat{x} \in \widehat{\mathcal{H}}\right\},$$

and the representation of $\widetilde{\mathcal{L}}_-$ by means of $K$ yields $\mathcal{R}(K) \subset \mathcal{D}(A)$. The Riccati equation is equivalent to the fact that $\widetilde{A}$ maps $\widetilde{\mathcal{L}}_-$ into itself. Indeed, if $\widetilde{x} = \begin{pmatrix} K\widehat{x} \\ \widehat{x}\end{pmatrix} \in \widetilde{\mathcal{L}}_-$, $\widetilde{A}\widetilde{x} = \widetilde{y} = \begin{pmatrix} K\widehat{y} \\ \widehat{y}\end{pmatrix} \in \widetilde{\mathcal{L}}_-$, the relation

$$\begin{pmatrix} A & B^* \\ B & C\end{pmatrix}\begin{pmatrix} K\widehat{x} \\ \widehat{x}\end{pmatrix} = \begin{pmatrix} K\widehat{y} \\ \widehat{y}\end{pmatrix}$$

is equivalent to $AK + B^* = K(BK + C)$. Thus (ii) is proved. If $\widetilde{x}, \widetilde{y} \in \widetilde{\mathcal{L}}_-$ the equation $\widetilde{A}\widetilde{x} = \widetilde{y}$ is equivalent to $(C + BK)\widehat{x} = \widehat{y}$ and the norm $\|\cdot\|_\Lambda$ in $\widehat{\mathcal{H}}$ generated by the inner product $[\cdot,\cdot]_\Lambda$ is just the norm of the corresponding element $\widetilde{x} = \begin{pmatrix} K\widehat{x} \\ \widehat{x} \end{pmatrix} \in \widetilde{\mathcal{L}}_-$ :

$$[\widehat{x}, \widehat{x}]_\Lambda = ((I + K^*K)\widehat{x}, \widehat{x}) = \left\| \begin{pmatrix} K\widehat{x} \\ \widehat{x} \end{pmatrix} \right\|^2 .$$

The selfadjointness of the operator $C + BK$ in this inner product, that is the relation

$$(I + K^*K)(C + BK) = (C + K^*B^*)(I + K^*K),$$

follows either from this isomorphism or it can be checked directly using (2.10). Thus (iv) is proved; the proof of (iii) is analogous.

Finally, the Riccati equation (2.10) can be written in the form

$$KBK - B^* - (A - \mu)K + K(C - \mu) = 0$$

with an arbitrary complex number $\mu$. Choosing $\mu \in \rho(A)$ and multiplying this relation from the left by $(A - \mu)^{-1}$, it follows that $K$ is compact if $(A - \mu)^{-1}$ is compact. Theorem 2.3 is proved. ∎

REMARK 2.4. The above construction implies that under the assumptions of Theorem 2.3 the contractive solution $K$ of the Riccati equation (2.10) can be represented as

$$K = Q_2 Q_1^{-1} \tag{2.11}$$

with

$$Q_1 := -\frac{1}{2\pi i} \oint_\Gamma (\lambda - C + B(A - \lambda)^{-1}B^*)^{-1}\, d\lambda,$$

$$Q_2 := -\frac{1}{2\pi i} \oint_\Gamma (A - \lambda)^{-1}B^*(\lambda - C + B(A - \lambda)^{-1}B^*)^{-1}\, d\lambda,$$

where $\Gamma$ is a closed contour in $\rho(\widetilde{A})$ which surrounds $\sigma(\widetilde{A}) \cap (-\infty, \alpha)$ and does not surround any point of $\sigma(\widetilde{A}) \cap (\alpha, \infty)$. Here also the invertibility of the operator $Q_1$ follows from the above consideration.

This formula resembles the well-known form of the solution $K_0$ of the equation

$$K_0 C - AK_0 = B^*;$$

namely, if $\sigma(A) \cap \sigma(C) = \emptyset$ then

$$K_0 = -\frac{1}{2\pi i} \oint_{\Gamma_C} (A-\lambda)^{-1} B^* (C-\lambda)^{-1} \, d\lambda,$$

where $\Gamma_C$ is a Cauchy contour which surrounds $\sigma(C)$ and does not surround any point of $\sigma(A)$.

## 3. THE SPECTRUM OF $L$

In this section we consider the spectrum of the operator function $L$ in (1.1):

$$L(\lambda) := \lambda - A + B^*(C-\lambda)^{-1} B,$$

where $A, B$ and $C$ are again supposed to satisfy the assumptions formulated at the beginning of the introduction.

While in the common definition of the spectrum etc. of the operator pencil $L$ there would be considered only those points $\lambda$ which belong to $\rho(C)$, that is where $L(\lambda)$ is defined and holomorphic, it seems to be more natural here to define the *resolvent set* $\rho(L)$ as the set of those $\lambda \in \mathbb{C}$ into which $L(\lambda)^{-1}$ can be continued analytically, and to put $\sigma(L) := \mathbb{C} \setminus \rho(L)$. Evidently, $\mathbb{C}^+ \cup \mathbb{C}^- \subset \rho(L)$, and it is easy to see that also points of $\sigma(C)$ may belong to $\rho(L)$, e.g. if they are isolated eigenvalues of $C$ or if $B$ has a nontrivial kernel.

In the representation (2.3) of $L(\lambda)^{-1}$ the space $\widehat{\mathcal{H}}$ can possibly be reduced without changing the operator function $L$. Indeed, let

$$\widehat{\mathcal{H}}_1 := \text{c.l.s.}\{(C-\lambda)^{-1} B\mathcal{H} \; : \; \lambda \neq \bar{\lambda}\} \; (\subset \widehat{\mathcal{H}}).$$

It is easy to see that $\widehat{\mathcal{H}}_1$ contains $\overline{\mathcal{R}(B)}$ and that $\widehat{\mathcal{H}}_1$ is invariant under $C$. By $C_1$ we denote the restriction of $C$ to $\widehat{\mathcal{H}}_1$, by $B_1$ the operator defined by $B$ as a mapping from $\mathcal{H}$ into $\widehat{\mathcal{H}}_1$. Then, evidently,

$$L(\lambda) := \lambda - A + B^*(C-\lambda)^{-1} B = \lambda - A + B_1^*(C_1-\lambda)^{-1} B_1.$$

Besides the operator $\widetilde{A}$ in $\widetilde{\mathcal{H}}$, defined by (2.1), we consider the operator

$$\widetilde{A}_1 := \begin{pmatrix} A & B_1^* \\ B_1 & C_1 \end{pmatrix} \quad \text{in} \quad \widetilde{\mathcal{H}}_1 := \mathcal{H} \oplus \widehat{\mathcal{H}}_1.$$

Evidently, in (2.3) and (2.4) $\widetilde{A}$ and $\widetilde{E}$ can be replaced by $\widetilde{A}_1$ and its spectral function $\widetilde{E}_1$, respectively.

THEOREM 3.1. *Under the assumptions at the beginning of the introduction it holds*

$$\sigma(L) \subset \sigma(\widetilde{A}) \text{ and } \sigma(L) = \sigma(\widetilde{A}_1).$$

*Proof.* The inclusions $\rho(\widetilde{A}) \subset \rho(L)$ and $\rho(\widetilde{A}_1) \subset \rho(L)$ are clear from (2.2) and a corresponding representation of the resolvent $(\widetilde{A}_1 - \lambda)^{-1}$. In order to prove the converse of the second inclusion it is sufficient to consider a real point $\lambda_0 \in \rho(L)$ and to show that for each connected neighbourhood $\Delta_0$ of $\lambda_0$ with $\overline{\Delta_0} \subset \rho(L)$ we have $\widetilde{E}_1(\Delta_0) = 0$. If $\Delta_0$ has the property that its closure belongs to $\rho(\widetilde{L})$ the Stieltjes inversion formula and (2.2) imply that in the matrix representation of $\widetilde{E}_1(\Delta_0)$ the left upper entry is zero. Hence the matrix representation of the nonnegative operator $\widetilde{E}_1(\Delta_0)$ must be of the form

$$\widetilde{E}_1(\Delta_0) = \begin{pmatrix} 0 & 0 \\ 0 & E_1(\Delta_0) \end{pmatrix}$$

with some selfadjoint projection $E_1(\Delta_0)$ in $\widehat{\mathcal{H}}_1$.

As the range of $\widetilde{E}_1(\Delta_0)$ is invariant under $\widetilde{A}$, the range of $E_1(\Delta_0)$ is invariant under $C_1$ and $B_1^* E_1(\Delta_0) = 0$. Then for nonreal $\lambda$ we find

$$(C_1 - \lambda)^{-1} E_1(\Delta_0)\widehat{\mathcal{H}}_1 \subset E_1(\Delta_0)\widehat{\mathcal{H}}_1,$$

and it follows that

$$\begin{aligned}(E_1(\Delta_0)\widehat{\mathcal{H}}_1, (C_1 - \lambda)^{-1} B_1\mathcal{H}) &\subset (E_1(\Delta_0)\widehat{\mathcal{H}}_1, B_1\mathcal{H}) \\ &= (B_1^* \widehat{E}_1(\Delta_0), \mathcal{H}) = \{0\}.\end{aligned}$$

Therefore $E_1(\Delta_0)\widehat{\mathcal{H}}_1$ is orthogonal on a total subset of $\widehat{\mathcal{H}}_1$, hence $E_1(\Delta_0) = 0$ and $\widetilde{E}_1(\Delta_0) = 0$ follows. Theorem 3.1 is proved. ∎

In the rest of this section we suppose without loss of generality that the space $\widehat{\mathcal{H}}$ is chosen minimal: $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_1$. We define the *point spectrum* or the set of *eigenvalues* $\sigma_p(L)$ of the operator function $L$ as the point spectrum of $\widetilde{A} = \widetilde{A}_1$: $\sigma_p(L) := \sigma_p(\widetilde{A})$.

LEMMA 3.2. *If $\lambda_0 \in \sigma_p(L) \cap \rho(C)$ and $(x_0 \ \widehat{x}_0)^{\mathrm{T}} \in \widetilde{\mathcal{H}}$ is a corresponding eigenvector of $\widetilde{A}$, then $\widehat{x}_0 = -(C - \lambda_0)^{-1} B x_0$ and $x_0 \neq 0$,*

$$L(\lambda_0)x_0 = 0. \tag{3.1}$$

*Conversely, if $\lambda_0 \in \rho(C)$ and there exists an element $x_0 \in \mathcal{H}$ such that $x_0 \neq 0$ and (3.1) holds, then $(x_0 \ \ -(C - \lambda_0)^{-1} B x_0)^{\mathrm{T}}$ is an eigenvector of $\widetilde{A}$ corresponding to the eigenvalue $\lambda_0$ of $\widetilde{A}$.*

The straightforward proof of the lemma is left to the reader. The vector $x_0 \neq 0$, satisfying (3.1) is called an *eigenvector* of the operator function $L$ corresponding to the eigenvalue $\lambda_0 \in \rho(C)$. We mention that also for eigenvalues of $L$ in $\sigma(C)$ the notion of an eigenvector can be introduced as the nontangential boundary value of some vector function which is holomorphic in $\mathbf{C}^+$. This question will be considered elsewhere.

In the sequel we often suppose that for some (and hence for all) $\lambda \in \rho(A)$ the resolvent of $A$ is compact.

LEMMA 3.3. *If, additionally to the assumptions at the beginning of the introduction, the resolvent of $A$ is compact, then $\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(C)$. Moreover, if*

$$c_1 := \sup \sigma(C) \in \sigma_{\mathrm{ess}}(C),$$

*there exists a $\delta > 0$ such that*

$$(c_1, c_1 + \delta) \cap \sigma(\widetilde{A}) = \emptyset.$$

*Proof.* It follows from (2.2) and (2.11), (2.12) that the resolvent of $\widetilde{A}$ is finitely meromorphic outside of $\sigma_{\mathrm{ess}}(C)$. In order to prove the second claim suppose first that $a_0 := \inf \sigma(A) > c_1$. Then the interval $(c_1, a_0)$ belongs to $\rho(L)$ as $L(\lambda) \leqslant -\delta$ for some $\delta > 0$ if $\lambda \in (c_1, a_0)$.

If $a_0 \leqslant c_1$, we denote by $P$ the orthogonal projection onto the linear span of all the eigenspaces of $A$ corresponding to eigenvalues $\leqslant c_1$ and put $Q = I - P$. Then we have $\sigma(AQ + (c_1 + 1)Q) > \sigma(C)$, hence the operator

$$\widetilde{A}_0 = \begin{pmatrix} AQ + (c_1 + 1)Q & B^* \\ B & C \end{pmatrix}$$

has all the properties of the operator $\widetilde{A}$ considered in connection with Theorem 2.3. According to what has been shown already, the spectrum of $\widetilde{A}_0$ has a gap of the form $(c_1, c_1 + \delta)$. As $\widetilde{A}$ is a finite dimensional perturbation of $\widetilde{A}_0$, it can have only a finite number of eigenvalues in this gap. The lemma is proved. ∎

COROLLARY 3.4. *Under the assumptions of Lemma 3.3, the spectrum of $L$ in $\mathbf{C} \setminus \sigma_{\mathrm{ess}}(C)$ consists of isolated eigenvalues and for some $\delta > 0$ we have $(c_1, c_1 + \delta) \subset \rho(L)$.*

THEOREM 3.5. *Let $A$ be a selfadjoint operator in $\mathcal{H}$ with a compact resolvent, $C$ a bounded selfadjoint operator in $\widehat{\mathcal{H}}$, $B \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{H}})$ and suppose that there exists an $\alpha \in \mathbf{R}$ such that $\sigma(A) > \alpha$, $\sigma(C) < \alpha$. Then the spectrum of $L$ in $(\alpha, \infty)$*

*consists only of isolated eigenvalues; the corresponding eigenvectors can be chosen to form a Riesz basis of $\mathcal{H}$.*

*Proof.* It follows from Lemma 3.3 and Corollary 3.4, that $\sigma(L)$ consists outside of the essential spectrum of $C$ only of isolated eigenvalues and that an interval around $\alpha$ belongs to $\rho(L)$.

We shall show that the eigenvectors of $L$ corresponding to an eigenvalue in $(\alpha,\infty)$ coincide with the eigenvectors of the operator $A - B^*K^*$ in Theorem 2.3 to the same eigenvalue. As $A - B^*K^*$ is similar to a selfadjoint operator in $\mathcal{H}$ (see Theorem 2.3 (iii)), these eigenvectors can be chosen to form a Riesz basis.

Let $x_0$ be an eigenvector of $L$ corresponding to $\lambda_0 \in (\alpha,\infty)$. Then, according to Lemma 3.2, $\big(x_0 \;\; -(C-\lambda_0)^{-1}Bx_0\big)^{\mathrm{T}}$ is an eigenvector of $\widetilde{A}$ to $\lambda_0$ and

$$\big(x_0 \;\; -(C-\lambda_0)^{-1}Bx_0\big)^{\mathrm{T}} \in \widetilde{\mathcal{L}}_+.$$

Therefore $(C-\lambda_0)^{-1}Bx_0 = K^*x_0$ and we find

$$(A - B^*K^* - \lambda_0)x_0 = (A - B^*(C-\lambda_0)^{-1}B - \lambda_0)x_0 = 0.$$

Conversely, if $(A-B^*K^*-\lambda_0)x_0 = 0$ then the reasoning in the proof of Theorem 2.3 implies

$$(\widetilde{A}-\lambda_0)\begin{pmatrix} x_0 \\ -K^*x_0 \end{pmatrix} = 0,$$

and Lemma 3.2 yields $L(\lambda_0)x_0 = 0$. Theorem 3.5 is proved. ∎

REMARK 3.6. For the last statement of Theorem 3.5 the assumption that the resolvent of $A$ is compact can be replaced by the assumption that the essential spectrum of $A$ consists only of a finite number of points. If (under the assumption $\sigma(C) < \alpha$ and $\sigma(A) > \alpha$) the essential spectrum of the (bounded selfadjoint) operator $C$ consists only of a finite number of points, also the eigenvectors of $L$ corresponding to eigenvalues in $(-\infty,\alpha)$ can be chosen to form a Riesz basis of $\mathcal{H}$.

REMARK 3.7. Under the assumptions of Theorem 3.5, the eigenvalues of $L$ in $(\alpha,\infty)$ can also be characterized by a minimum–maximum principle. To this end we consider on $[\alpha,\infty)$ the scalar functions

$$\varphi_x(\lambda) := (L(\lambda)x, x) \quad (x \in \mathcal{D}(A),\ \|x\| = 1).$$

Then $\varphi_x(\alpha) < 0$, $\lim\limits_{\lambda\uparrow\infty} \varphi_x(\lambda) = \infty$ and $\varphi_x'(\lambda) > 0$ if $\lambda \in [\alpha,\infty)$. Denote the unique zero of the function $\varphi_x$ in $(\alpha,\infty)$ by $p(x)$. Then, if we denote the nondecreasing sequence of eigenvalues of $L$ in $(\alpha, \infty)$, counted according to their multiplicities, by $(\lambda_j(L))$, we have

$$\lambda_j(L) = \min_{\mathcal{L}:\dim \mathcal{L}=j} \max_{x\in\mathcal{L}\subset\mathcal{D}(A),\|x\|=1} p(x)$$

(see [7]). Moreover, with $p_0(x) := (Ax, x)$ we have

$$\varphi_x(p_0(x)) = \big((C - p_0(x))^{-1}Bx, Bx\big) \leqslant 0,$$

hence $p(x) \geqslant p_0(x)$. It follows that the eigenvalues of $L$ satisfy the inequalities

$$\lambda_j(L) \geqslant \lambda_j(A), \quad j = 1, 2, \ldots,$$

where $(\lambda_j(A))_1^\infty$ denotes the nondecreasing sequence of eigenvalues of $A$, again counted according to their multiplicities.

## 4. A KREIN SPACE SITUATION

In this final section we make some remarks about the operator function

$$M(\lambda) := \lambda - A - B^*(C - \lambda)^{-1}B \quad (\lambda \in \rho(C)),$$

where, again, $A$ and $C$ are selfadjoint operators in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, and $B \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{H}})$. With $M$ we associate the operator

$$\widetilde{B} := \begin{pmatrix} A & B^* \\ -B & C \end{pmatrix},$$

which is selfadjoint in the Krein space $\widetilde{\mathcal{K}} := \mathcal{H} \oplus \widehat{\mathcal{H}}$ with inner product

$$[\widetilde{x}, \widetilde{y}] := (x, y) - (\widehat{x}, \widehat{y}), \text{ where } \widetilde{x} = (x, \widehat{x})^{\mathrm{T}}, \ y = (y, \widehat{y})^{\mathrm{T}} \in \widetilde{\mathcal{K}}.$$

Suppose first that additionally $C$ is bounded, $A$ is semibounded from below and has a compact resolvent. Then it follows as in [6], Section 2.2, that the operator $\widetilde{B}$ is definitizable (for the definition and properties of definitizable operators see [5]). Indeed, let $c_1 = \sup \sigma(C)$ and, if $(\lambda_j)_1^\infty$ is the sequence of the eigenvalues of $A$ arranged in nondecreasing order and according to their multiplicities, let $\lambda_n$ be the first eigenvalue with the property

$$\lambda_n - c_1 > \|B\|.$$

Denote by $P$ the orthogonal projection onto the linear span of the eigenspaces of $A$ corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, $Q := I - P$. We consider the operators

$$\widetilde{B}_0 := \begin{pmatrix} QAQ + \lambda_n P & B^* \\ -B & C \end{pmatrix},$$

$$\widetilde{B}_1 := \begin{pmatrix} QAQ + \lambda_n P & 0 \\ 0 & C \end{pmatrix}.$$

As $\sigma(QAQ + \lambda_n P) \geqslant \lambda_n$ and $\sigma(C) \leqslant c_1$ it follows that the operator $\widetilde{B}_1 - \frac{c_1+\lambda_n}{2}$ is nonnegative in the Krein space $\widetilde{\mathcal{K}}$. Then, as the gap in $\sigma(\widetilde{B}_1)$ is greater than $2\|B\|$, also the operator $\widetilde{B}_0 - \frac{c_1+\lambda_n}{2}$ is nonnegative in $\widetilde{\mathcal{K}}$. Therefore (see [4]) the operator $\widetilde{B}$, which is a finite dimensional perturbation of $\widetilde{B}_0$, is definitizable.

We shall not formulate the consequences of the definitizability of $\widetilde{B}$. Instead, we consider a situation where a complete analogue of Theorem 3.5 and hence also analogues of the results in Section 3 can be formulated.

In Theorem 4.1, a subspace $\widetilde{\mathcal{M}}$ is called invariant under the (unbounded) operator $\widetilde{B}$ if $\mathcal{D}(\widetilde{B}) \cap \widetilde{\mathcal{M}}$ is dense in $\widetilde{\mathcal{M}}$ and $\widetilde{B}(\mathcal{D}(\widetilde{B}) \cap \widetilde{\mathcal{M}}) \subset \widetilde{\mathcal{M}}$; by $\sigma(\widetilde{B}|\widetilde{\mathcal{M}})$ we denote the spectrum of the restriction $\widetilde{B}|\mathcal{D}(\widetilde{B}) \cap \widetilde{\mathcal{M}}$, considered as an operator in $\widetilde{\mathcal{M}}$.

THEOREM 4.1. *Let $A$ and $C$ be selfadjoint operators in $\mathcal{H}$ and $\widehat{\mathcal{H}}$, respectively, $\sigma(A) \geqslant \alpha$, $\sigma(C) \leqslant \alpha$ for some $\alpha \in \mathbf{R}$, $B \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{H}})$, and suppose additionally that*

$$|(Bx, \widehat{x})|^2 \leqslant ((A-\alpha)x, x)((\alpha - C)\widehat{x}, \widehat{x}) \quad (x \in \mathcal{D}(A), \widehat{x} \in \mathcal{D}(C)). \tag{4.1}$$

*Then the operator $\widetilde{B} - \alpha$ is nonnegative in the Krein space $\widetilde{\mathcal{K}}$. If, additionally, $\alpha \in \rho(C)$ and $M(\alpha)$ is boundedly invertible, then there exists a contraction $K \in \mathcal{L}(\widehat{\mathcal{H}}, \mathcal{H})$, $\|K\| < 1$, such that the following statements hold:*

(i) *The (maximal uniformly negative) subspace*

$$\widetilde{\mathcal{M}}_- := \left\{ \begin{pmatrix} K\widehat{x} \\ \widehat{x} \end{pmatrix} : \widehat{x} \in \widehat{\mathcal{H}} \right\}$$

*is invariant under $\widetilde{B}$ and $\sigma(\widetilde{B}|\widetilde{\mathcal{M}}_-) = \sigma(B) \cap (-\infty, \alpha)$, the (maximal uniformly positive) subspace*

$$\widetilde{\mathcal{M}}_+ := \left\{ \begin{pmatrix} x \\ K^* x \end{pmatrix} : x \in \mathcal{H} \right\}$$

*is invariant under $\widetilde{B}$, $\sigma(\widetilde{B}|\widetilde{\mathcal{M}}_+) = \sigma(B) \cap (\alpha, +\infty)$ and $\widetilde{\mathcal{K}} = \widetilde{\mathcal{M}}_+ [+] \widetilde{\mathcal{M}}_-$.*

(ii) *The restriction* $\widetilde{B}|\widetilde{\mathcal{M}}_+$ *is unitarily equivalent to the operator* $A + B^*K^*$, *which is selfadjoint in the Hilbert space* $(\mathcal{H}, [\cdot,\cdot])$ *where* $[\cdot,\cdot]$ *denotes the inner product*

$$[x, y] := ((I - KK^*)x, y) \quad (x, y \in \mathcal{H}).$$

(iii) *The restriction* $\widetilde{B}|\widetilde{\mathcal{M}}_-$ *is unitarily equivalent to the operator* $C + BK$, *which is selfadjoint in the Hilbert space* $(\widehat{\mathcal{H}}, [\cdot,\cdot]_\wedge)$ *where* $[\cdot,\cdot]_\wedge$ *denotes the inner product*

$$[\widehat{x}, \widehat{y}]_\wedge := ((I - K^*K)\widehat{x}, \widehat{y}) \quad (\widehat{x}, \widehat{y} \in \widehat{\mathcal{H}}).$$

*Proof.* If $\widetilde{x} = (x \;\; \widehat{x})^{\mathrm{T}} \in \mathcal{D}(\widetilde{B})$, then $x \in \mathcal{D}(A)$, $\widehat{x} \in \mathcal{D}(C)$ and we get

$$\begin{aligned}[(\widetilde{B} - \alpha)\widetilde{x}, \widetilde{x}] &= ((A - \alpha)x, x) + 2\Re(Bx, \widehat{x}) - ((C - \alpha)\widehat{x}, \widehat{x}) \\ &\geqslant ((A - \alpha)x, x) - 2((A - \alpha)x, x)^{\frac{1}{2}}|((C - \alpha)\widehat{x}, \widehat{x})|^{\frac{1}{2}} + |((C - \alpha)\widehat{x}, \widehat{x})| \geqslant 0,\end{aligned}$$

where the assumption (4.1) has been used. If $M(\alpha)$ is boundedly invertible, then $\alpha \in \rho(\widetilde{B})$ hence $\widetilde{B}$ has the only possible critical point $\infty$. This is a regular critical point of the operator

$$\widetilde{B}_0 := \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}.$$

According to a criterion of K. Veselić, it is also a regular critical point of the operator $\widetilde{B}$ as the perturbation $\begin{pmatrix} 0 & B^* \\ -B & 0 \end{pmatrix}$ is bounded. It follows that the spectral subspaces $\widetilde{\mathcal{M}}_+$ and $\widetilde{\mathcal{M}}_-$ of $\widetilde{B}$, corresponding to $(\alpha, \infty)$ and $(-\infty, \alpha)$, respectively, are uniformly positive and uniformly negative, respectively. Therefore they admit a representation with a strictly contractive angular operator. The other statements follow as the corresponding ones in Theorem 3.5. ∎

The formulation of the analogues of the results in Section 3 for the operator function $M$ in $\mathcal{H}$ is left to the reader.

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