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DENSENESS OF THE GENERALIZED EIGENVECTORS OF A DISCRETE OPERATOR IN A BANACH SPACE

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ABSTRACT. Let T be a closed, densely defined, linear operator in a separable, reflexive Banach space X, and assume that there exists $\xi \in \rho(T)$ such that $R_\xi(T)$ is a compact operator whose approximation numbers are p-summable, $0 < p < \infty$. The operator T is a special type of discrete operator, a so-called $C_p^{(a)}$-discrete operator. Let $\overline{\mathrm{sp}}\,(T)$ be the smallest closed subspace of X containing the subspace spanned by the generalized eigenvectors of T. Sufficient conditions are introduced which guarantee $\overline{\mathrm{sp}}\,(T) = X$. These conditions require that $\|R_\lambda(T)\|$ exhibit the decay rate $\mathrm{O}(|\lambda|^N)$ on certain rays in the complex plane. This work generalizes past Hilbert space theory developed by Dunford and Schwartz.

KEYWORDS: *Generalized eigenvector, discrete operator, compact operator, Carleman's inequality, s-number, approximation number, $C_p^{(a)}$ operator, $C_p^{(c)}$ operator.*

AMS SUBJECT CLASSIFICATION: Primary 47B99; Secondary 47B05, 46B10.

1. INTRODUCTION

Let T be a closed, densely defined, linear operator on a Banach space X and let $\overline{\mathrm{sp}}\,(T)$ be the smallest closed subspace of X containing the subspace spanned by the generalized eigenvectors of T. In their celebrated treatise ([1], vol. II, p. 1088–1116) Dunford and Schwartz used a generalized Carleman inequality and the Phragmán-Lindelöff Theorem to establish sufficient conditions for $\overline{\mathrm{sp}}\,(T)$ to coincide with X in the case X is a separable Hilbert space. Their results depend on two assumptions: (1) that there exists a point ξ in the resolvent set $\rho(T)$ of T such that the resolvent $R_\xi(T) = (\xi I - T)^{-1}$ is a compact operator whose singular values are p-summable, $0 < p < \infty$, and (2) that for $|\lambda|$ sufficiently large all points on $n > 2p$ appropriately located rays belong to $\rho(T)$, and that the resolvent satisfies an inequality $||R_\lambda(T)|| = O(|\lambda|^N)$ as $\lambda \to \infty$ along each ray, where N is a positive integer.

These results do not apply on a general Banach space, nor can their proofs be easily modified to do so, because they both rely on the use of singular values, a strictly Hilbert space concept. In this paper, s-numbers, which are numbers that measure the degree of approximability or compactness of a bounded linear operator, are used to obtain these results. By using an interplay of two distinct sets of s-numbers, the approximation numbers and the Gelfand numbers, sufficient conditions are established in Theorem 4.5 for $\overline{\mathrm{sp}}(T) = X$. This result contains the previously mentioned results, is valid on a general, reflexive separable Banach space, and applies to many linear differential and partial differential operators T whose resolvents are compact.

Mathematical notation, terminology and some results are introduced in Section 2. In particular the approximation numbers and Gelfand numbers are defined, and are used to subdivide the compact operators on X into smaller subclasses, the $C_p^{(a)}$ operators and the $C_p^{(c)}$ operators. The properties of the $C_p^{(c)}$ operators are outlined and then used, in Section 3, to obtain a generalized Carleman inequality, which is essential to the final arguments of Section 4, where Theorem 4.5 is proved. This result applies to $C_p^{(a)}$-discrete operators, a class of closed, densely defined, linear operators having a compact resolvent. These $C_p^{(a)}$-discrete operators are defined and briefly studied in the first part of Section 4.

2. MATHEMATICAL PRELIMINARIES

This section sets forth the notation, terminology, and results used in this paper.

Let T denote a linear operator acting on a Banach space X with values in a Banach space Y. Throughout this paper $D(T), N(T)$, and $R(T)$ will denote the domain, nullspace, and range of T, respectively, $\mathcal{B}(X, Y)$ will denote the Banach space of all bounded linear operators T, with norm $|| \cdot ||$, such that $D(T) = X$ and $R(T) \subseteq Y, T'$ will denote the Banach space adjoint of T, and X' will denote the dual space of X. $\mathcal{F}(X, Y)$ will denote the subset of $\mathcal{B}(X, Y)$ consisting of all operators with finite dimensional range. To simplify notation, $\mathcal{B}(X)$ will denote $\mathcal{B}(X, X)$.

Unless stated otherwise, all spaces X, Y, Z, W, etc., will be assumed to be complex Banach spaces.

DEFINITION 2.1. Let $T \in \mathcal{B}(X,Y)$, and $n = 1,2,\ldots$. The *n-th approximation number* $a_n(T)$ of T is defined by

$$a_n(T) := \inf\{\|T - L\| \,|L \in \mathcal{B}(X,Y),\ \operatorname{rank} L\ < n\},$$

where $\operatorname{rank} L\ = \dim R(L)$.

DEFINITION 2.2. Let $T \in \mathcal{B}(X,Y)$, and $n = 1,2,\ldots$. The *n-th Gelfand number* $c_n(T)$ of T is defined by

$$c_n(T) := \inf\{\|TJ_M^X\| \,|M \subseteq X, \operatorname{codim} M < n\},$$

where M is a closed linear subspace of X, and J_M^X is the embedding map from M to X.

DEFINITION 2.3. A map

$$s : T \to (s_n(T)),$$

which assigns a non-negative scalar sequence to each operator, is called an *s-scale* if for all Banach spaces X, Y, Z, W the following conditions are satisfied:

(i) $\|T\| = s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0$ for all $T \in \mathcal{B}(X,Y)$.

(ii) $s_{n+m-1}(S+T) \leqslant s_m(S) + s_n(T)$ for $S, T \in \mathcal{B}(X,Y)$.

(iii) $s_n(RST) \leqslant \|R\| s_n(S) \|T\|$ for all $R \in \mathcal{B}(Z,W)$, $S \in \mathcal{B}(Y,Z)$, $T \in \mathcal{B}(X,Y)$.

(iv) If $\operatorname{rank} T < n$ then $s_n(T) = 0$, for all $T \in \mathcal{B}(X,Y)$.

(v) $s_n(I_n) = 1$, where I_n is the identity map of $l_n^2 := \{x \in l^2 : x_i = 0$ if $i > n\}$ to itself.

$s_n(T)$ is called the n-th *s-number of* T. In addition to the above conditions, if the following condition is satisfied, then the map is called a *multiplicative s-scale*:

(vi) $s_{m+n-1}(ST) \leqslant s_m(S) s_n(T)$ for all $T \in \mathcal{B}(X,Y)$, $S \in \mathcal{B}(Y,Z)$.

If the following condition is satisfied, then the map is called an *injective s-scale*:

(vii) $s_n(T) = s_n(JT)$ for all $T \in \mathcal{B}(X,Y)$, and all $J \in \mathcal{B}(Y,Z)$, where J is an isometry.

THEOREM 2.4. *The map $a : T \to \{a_n(T)\}_{n=1}^\infty$ is a multiplicative s-scale. The map $c : T \to \{c_n(T)\}_{n=1}^\infty$ is an injective, multiplicative s-scale.*

Proof. [2], p. 79–80, 83, 90–91. See [3], p. 53–73, [4], p. 68–70 and [5], p. 26–37 for related results. ∎

In a general Banach space setting there exists a number of distinct s-scales, e.g., the approximation, Gelfand, Kolmogorov, Weyl, Chang and Hilbert numbers all generate distinct s-scales, of which the approximation numbers are the largest. If $T \in \mathcal{B}(X, Y)$, where X and Y are Hilbert spaces, then the singular values of T form a multiplicative s-scale. In this case the s-scale is unique: it coincides with all other s-scales. For more on these numbers see [2], [3], [4].

DEFINITION 2.5. Let $T \in \mathcal{B}(X)$ have approximation number sequence $\{a_n(T)\}_{n=1}^{\infty}$. If $\sum_{n=1}^{\infty} a_n(T)^p < \infty$ for some $0 < p < \infty$, then T is a $C_p^{(a)}$ operator.

DEFINITION 2.6. Let $T \in \mathcal{B}(X)$ have Gelfand number sequence $\{c_n(T)\}_{n=1}^{\infty}$. If $\sum_{n=1}^{\infty} c_n(T)^p < \infty$ for some $0 < p < \infty$, then T is a $C_p^{(c)}$ operator.

In all that follows, the symbol $C_p^{(a)}(X)$ will denote the set of all $C_p^{(a)}$ operators acting on X, $0 < p < \infty$. The symbol $C_p^{(c)}(X)$ will denote the set of all $C_p^{(c)}$ operators acting on X, $0 < p < \infty$.

Let $K(X)$ denote the set of compact operators acting from X into X. The Gelfand operators, approximation operators, and compact operators are related in the following way.

THEOREM 2.7. *Let* $0 < p < \infty$. *Then* $C_p^{(a)}(X) \subseteq C_p^{(c)}(X) \subseteq K(X)$.

Proof. [2], p. 83, 92. ∎

The property of $C_p^{(c)}(X)$ operators found in Lemma 2.8 is crucial to the results of Section 4.

LEMMA 2.8. *Let* $T \in \mathcal{B}(X)$. *Let* Z *be a closed invariant subspace of* X. *Let* $T \mid Z$ *denote the operator such that* $T \mid Z(x) = T(x)$ *for all* $x \in Z$ *and* $T \mid Z \in \mathcal{B}(Z)$. *Then* $c_n(T \mid Z) \leqslant c_n(T)$.

Proof. Let J_Z^X be the embedding map from Z to X. Then $c_n(T \mid Z) = c_n(TJ_Z^X)$ since the Gelfand number sequence is an injective map. Combine this with the observation that $c_n(TJ_Z^X) \leqslant c_n(T)$ to complete the proof. ∎

The property of $C_p^{(a)}(X)$ operators found in Theorem 2.9 is crucial to the results of Section 4.

THEOREM 2.9. *Let* $0 < p < \infty$ *and* $T \in C_p^{(a)}(X)$. *Then* $a_n(T) = a_n(T')$.

Proof. [3], p. 55. ∎

In the remainder of this section, properties of $C_p^{(c)}(X)$ operators are introduced. These properties are required for what takes place in Section 3: the development of a generalized Carleman inequality defined on the class of $C_p^{(c)}(X)$ operators.

By a *quasinorm* $|||\cdot|||$ defined on a complex vector space V, we mean a real-valued function defined on V having the following properties: (1) $|||x||| = 0$ iff $x = 0$, (2) $|||x+y||| \leqslant \eta(|||x||| + |||y|||)$ for all $x, y \in V$, where $\eta \geqslant 1$, and (3) $|||\lambda x||| = |\lambda| \cdot |||x|||$ for all $x \in V, \lambda \in \mathbb{C}$. If $\eta = 1$, then $|||\cdot|||$ is a norm. Every quasinorm $|||\cdot|||$ defined on a complex vector space V induces a metrizable topology such that the algebraic operations are continuous. A fundamental system of neighborhoods of the zero element is formed by the subsets εU, $\varepsilon > 0$, where $U = \{x \in V \mid |||x||| \leqslant 1\}$ [7]. A *quasi-Banach space* is a vector space V equipped with a quasinorm $|||\cdot|||$, which becomes complete with respect to the associated metrizable topology.

Let $T \in C_p^{(c)}(X)$, $0 < p \leqslant \infty$, with Gelfand number sequence $\{c_n(T)\}_{n=1}^{\infty}$. Define the nonnegative function $|\cdot|_p$: $C_p^{(c)}(X) \to [0, \infty)$ by

$$|T|_p = \begin{cases} \left(\sum\limits_{n=1}^{\infty} c_n(T)^p\right)^{\frac{1}{p}}, & 0 < p < \infty \\ c_1(T), & p = \infty. \end{cases} \tag{2.1}$$

THEOREM 2.10. *$C_p^{(c)}(X)$ with the function $|\cdot|_p : C_p(X)^{(c)} \to [0,\infty)$ is a quasi-Banach space which is also a two-sided operator ideal.*

Proof. [2], p. 90. ∎

LEMMA 2.11. *If $0 < p < q < \infty$, then $C_p^{(c)}(X) \subseteq C_q^{(c)}(X)$, with $|T|_q \leqslant |T|_p$, for all $T \in C_p^{(c)}(X)$. If $p, q, r > 0$ be such that $1/p + 1/q = 1/r$, then $C_p^{(c)}(X) \circ C_q^{(c)}(X) \subseteq C_r^{(c)}(X)$, with $|TS|_r \leqslant 2^{1/r}|T|_p|S|_q$ for all $T \in C_p^{(c)}(X)$, $S \in C_q^{(c)}(X)$.*

Proof. [2], p. 81. ∎

To every $T \in C_p^{(c)}(X)$, $0 < p < \infty$, we associate the *eigenvalue sequence* $\{\lambda_n(T)\}_{n=1}^{\infty}$ defined in the following way: (1) if $\lambda \neq 0$ is an eigenvalue of multiplicity k, then it occurs in the sequence k times, one after the other. (2) The eigenvalues are arranged in order of non-increasing magnitude: $|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \cdots \geqslant 0$. In case there are distinct eigenvalues having the same modulus these can be written in any order. (3) If T possesses less than n eigenvalues $\lambda \neq 0$, then we set $\lambda_n(T) = \lambda_{n+1}(T) = \cdots = 0$.

The next result is a generalized version of the Weyl inequality ([1], p. 154).

THEOREM 2.12. *Let $T \in C_p^{(c)}(X)$, $0 < p < \infty$. Then*

$$\sum_{n=1}^{\infty} |\lambda_n(T)|^p \leqslant \beta\, |T|_p^p$$

where β is a constant independent of T.

Proof. [2], p. 157. ∎

3. A GENERALIZED CARLEMAN INEQUALITY

In this section a generalization of Carleman's inequality ([1], vol. II, p. 1088; [2], p. 220), is obtained for all $T \in C_p^{(c)}(X)$, $0 < p < \infty$. This result, given as Theorem 3.16, is obtained through the use of generalized trace and generalized determinant functions introduced below. The arguments presented rely on material found in Section 2 and in [2], Chapter 4.

The following definition, due to Pietsch ([2], p. 170, 171, 185), is based on the properties of the familiar trace and determinant functions used in linear algebra.

DEFINITION 3.1. Let $\mathcal{O}(X)$ be an operator ideal such that $\mathcal{O}(X) \subseteq \mathcal{B}(X)$. A *trace* is a complex valued function tr defined on $\mathcal{O}(X)$, which assigns to $T \in \mathcal{O}(X)$ a complex number $\operatorname{tr}(T)$, that satisfies for all $T, S \in \mathcal{O}(X)$ and all $R \in \mathcal{B}(X)$

(t_1) $\operatorname{tr}(T + S) = \operatorname{tr}(T) + \operatorname{tr}(S)$,

(t_2) $\operatorname{tr}(\alpha T) = \alpha \operatorname{tr}(T)$ for all $\alpha \in \mathbf{C}$,

(t_3) $\operatorname{tr}(TR) = \operatorname{tr}(RT)$,

(t_4) if $T \in \mathcal{O}(X)$ is a rank 1 operator defined by $Tx = \xi(x)y$, where $x \in X$, and ξ, y are fixed elements of X', X, respectively, then $\operatorname{tr}(T) = \xi(y)$.

A *determinant* is a complex valued function det defined on $\mathcal{O}(X)$, which assigns to operators of the form $I + T$, $T \varepsilon \mathcal{O}(X)$, a complex number $\det(I + T)$, that satisfies for all $T, S \in \mathcal{O}(X)$

(d_1) $\det[(I + T)(I + S)] = \det(I + T)\det(I + S)$,

(d_2) $\det(I + \alpha T)$ is entire in α for fixed T,

(d_3) $\det(I + ST) = \det(I + TS)$,

(d_4) if T is as in t_4, then $\det(I + T) = 1 + \xi(y)$.

Determinant property (d_2) implies that the *Gâteaux derivative*

$$\det{}'(T) := \lim_{z \to 0} \frac{\det(I + zT) - 1}{z}$$

exists for all $T \in C_1^{(c)}(X)$.

THEOREM 3.2. *Let $T \in C_1^{(c)}(X)$. Then*

$$\lambda(T) := \sum_{n=1}^{\infty} \lambda_n(T)$$

is a continuous trace. It is called the spectral trace.

Proof. [2], p. 176, 180. ∎

NOTE: it is unknown whether $\lambda(T)$ is the only continuous trace on $C_p^{(c)}(X)$.

THEOREM 3.3. *Let $T \in C_1^{(c)}(X)$. Then*

$$\pi(I+T) := \prod_{n=1}^{\infty} (1 + \lambda_n(T))$$

is a continuous determinant on $C_1^{(c)}(X)$. It is called the spectral determinant.

Proof. [2], p. 194, 210. ∎

The spectral trace and the spectral determinant have the following relationship:

THEOREM 3.4. *Let $T \in C_1^{(c)}(X)$. Then*

$$\lambda(T) = \pi'(T).$$

where $\pi'(T)$ is the Gâteaux derivative.

Proof. [2], p. 194, 206, 210. ∎

LEMMA 3.5. *Let z be a complex number. Let k be a positive integer. Then the entire function $\varepsilon_k(z)$ defined by*

$$1 + \varepsilon_k(z) := \begin{cases} (1+z)\exp\left(\sum\limits_{n=1}^{k-1} \frac{(-1)^n}{n} z^n\right), & \text{for } k \geqslant 2 \\ 1+z, & \text{for } k = 1. \end{cases}$$

has representation $\varepsilon_k(z) = z^k \varphi(k)$ where $\varphi(k)$ is entire.

Proof. $\log(1+z) = -\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n} z^n$ if $|z| < 1$. This implies that

$$1 + \varepsilon_k(z) = \exp\left(-\sum_{n=k}^{\infty} \frac{(-1)^n}{n} z^n\right) = 1 - \frac{(-1)^k}{k} z^k + \dots .$$

This expansion shows that $\varphi(z) := \frac{\varepsilon_k(z)}{z^k}$ has a removable singularity at 0. ∎

DEFINITION 3.6. Let k be a positive integer. Let $T \in C_k^{(c)}(X)$. Let $\varepsilon_k(T)$ be defined by

$$\varepsilon_k(T) := T^k \varphi(T)$$

where φ is defined as in Lemma 3.5.

Note that $\varepsilon_k(T)$ is entire and has alternative representation given by

$$I + \varepsilon_k(T) := \begin{cases} (I+T)\exp\left(\sum_{n=1}^{k-1} \frac{(-1)^n}{n} T^n\right), & \text{for } k \geqslant 2 \\ I+T, & \text{for } k = 1. \end{cases}$$

LEMMA 3.7. *Let k be a positive integer. Let $T \in C_k^{(c)}(X)$. Then $\varepsilon_k(T) \in C_1^{(c)}(X)$.*

Proof. Lemma 2.11 implies that if $T \in C_k^{(c)}(X)$, then $T^k \in C_1^{(c)}(X)$. Since $C_1^{(c)}(X)$ is an ideal, then $T^k\varphi(T) \in C_1^{(c)}(X)$. ∎

In Theorem 3.9 a useful expression for $\pi(I+\varepsilon_k(T))$ is obtained. The corollary to the following theorem is essential to Theorem 3.9.

LEMMA 3.8. *Let k be a positive integer. Then the eigenvalue sequence of T and $\varepsilon_k(T)$ can be arranged in such a way that $\lambda_n(\varepsilon_k(T)) = \varepsilon_k(\lambda_n(T))$, for $n = 1, 2, \ldots$.*

Proof. An extension of the version of the Spectral Mapping Theorem given by Pietsch ([2], p. 147–148) yields the results. ∎

THEOREM 3.9. *Let k be a positive integer. Let $T \in C_p^{(c)}(X)$ where $0 < p \leqslant k$. Let $\{\lambda_i(T)\}_{i=1}^{\infty}$ denote the eigenvalues of T arranged in the usual manner. Then*

$$\pi(I+\varepsilon_k(T)) = \prod_{i=1}^{\infty}(1+\lambda_i(T))\exp\left(\sum_{n=1}^{k-1}\frac{(-1)^n}{n}(\lambda_i(T))^n\right).$$

Proof. Since $T \in C_p^{(c)}(X) \subseteq C_k^{(c)}(X)$, then $\varepsilon_k(T) \in C_1^{(c)}(X)$, by Lemma 3.7. Hence $\pi(I+\varepsilon_k(T))$ makes sense. Lemma 3.8 and the definition of π on $C_1^{(c)}(X)$ combine to show that

$$\pi(I+\varepsilon_k(T)) = \prod_{i=1}^{\infty}(1+\varepsilon_k(\lambda_i(T))). \quad ∎$$

Lemmas 3.14 and 3.15 establish bounds on the above expression, which may be thought of as a generalized determinant. These bounds are then used to establish the generalized Carleman inequality given in Theorem 3.16. The next four results are necessary to establish these bounds. In all that follows, if $n < m$, then the sum $\sum_{j=m}^{n}$ equals zero.

LEMMA 3.10. *Let k be a positive integer. Let $T \in C_k^{(c)}(X)$ be such that $-1 \notin \sigma(T)$. Let $S \in C_1^{(c)}(X)$. Then*

$$\lambda\left[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_k(T+zS)\Big|_{z=0}\left(I+\varepsilon_k(T)\right)^{-1}\right]=\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\right)S\right].$$

Proof. (*Proof by Induction*). Before beginning the proof, note that the existence of $\left(I+\varepsilon_k(T)\right)^{-1}$ is assured by the existence of $(I+T)^{-1}$. For $k=1$, and $T \in C_1^{(c)}(X)$, the equation is obviously true. Assume the results for the integer k. Let $T \in C_{k+1}^{(c)}(X)$. Use the fact that $I+\varepsilon_{k+1}(T)=\left(I+\varepsilon_k(T)\right)\mathrm{e}^{\frac{(-1)^k}{k}T^k}$ and the product rule for differentiation to yield

$$\begin{aligned}
&\lambda\left[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_{k+1}(T+zS)\Big|_{z=0}\left(I+\varepsilon_{k+1}(T)\right)^{-1}\right]\\
&=\lambda\left[\frac{\mathrm{d}}{\mathrm{d}z}\left\{\left((I+\varepsilon_k(T+zS))\mathrm{e}^{\frac{(-1)^k}{k}(T+zS)^k}\right)-I\right\}\Big|_{z=0}\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\left(I+\varepsilon_k(T)\right)^{-1}\right]\\
&=\lambda\Bigg[\Bigg\{\frac{\mathrm{d}}{\mathrm{d}z}(I+\varepsilon_k(T+zS))\Big|_{z=0}\mathrm{e}^{\frac{(-1)^k}{k}T^k}\\
&\qquad+\left(I+\varepsilon_k(T)\right)\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\frac{(-1)^k}{k}(T+zS)^k}\Big|_{z=0}\Bigg\}\left(\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\left(I+\varepsilon_k(T)\right)^{-1}\right)\Bigg]\\
&=\lambda\Bigg[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_k(T+zS)\Big|_{z=0}\left(I+\varepsilon_k(T)\right)^{-1}\\
&\qquad+\left(\left(I+\varepsilon_k(T)\right)\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\frac{(-1)^k}{k}(T+zS)^k}\Big|_{z=0}\right)\left(\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\left(I+\varepsilon_k(T)\right)^{-1}\right)\Bigg].
\end{aligned}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}z}\exp\left(\frac{(-1)^k}{k}(T+zS)^k\right)\big|_{z=0}\in C_1^{(c)}(X)$. To see this, expand the exponential as a power series, take the derivative term by term, evaluate at $z=0$, and note that S is a factor of every term of the power series. Use the ideal property of $C_1^{(c)}(X)$ to conclude that both terms in the last expression are members of $C_1^{(c)}(X)$ and so λ is linear in both terms due to trace property (t_1). Use this fact, the induction hypothesis and the commutativity of the operators allowed by trace property (t_3) in the next step to yield

$$\lambda\left[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_k(T+zS)\Big|_{z=0}\left(I+\varepsilon_k(T)\right)^{-1} + \left(\left(I+\varepsilon_k(T)\right)\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\frac{(-1)^k}{k}(T+zS)^k}\Big|_{z=0}\right)\left(\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\left(I+\varepsilon_k(T)\right)^{-1}\right)\right]$$
$$= \lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\right)S\right] + \lambda\left[\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\frac{(-1)^k}{k}(T+zS)^k}\Big|_{z=0}\right].$$

In order to simplify the second term in the above expression, first expand $\exp\left(\frac{(-1)^k}{k}(T+zS)^k\right)$ as a power series, take the derivative term by term and evaluate at 0. The problem of simplifying this result is of course caused by the fact that, in general, $ST \neq TS$. This problem is overcome in the following manner. Use the continuity of function composition on $\mathcal{B}(X)\times C_1^{(c)}(X)$ into $C_1^{(c)}(X)$ guaranteed by part (iii) of Definition 2.3 to multiply $\exp\left(-\frac{(-1)^k}{k}T^k\right)$ by each term of the power series expansion described above, then use the continuity of λ to observe that the trace of the sum is the sum of the traces of individual terms. On each individual term use the commutativity of operators allowed by trace property (t_3) together with the fact that T commutes with $\exp\left(-\frac{(-1)^k}{k}T^k\right)$ to achieve the desired simplification. This procedure yields

$$\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\right)S\right] + \lambda\left[\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\frac{(-1)^k}{k}(T+zS)^k}\Big|_{z=0}\right]$$
$$= \lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\right)S\right] + \lambda\left[\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{m=0}^{\infty}\frac{(-1)^{km}}{m!k^m}(T+zS)^{km}\right)\Big|_{z=0}\right]$$
$$= \lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\right)S\right] + \sum_{m=1}^{\infty}\lambda\left[\mathrm{e}^{-\frac{(-1)^k}{k}T^k}\frac{(-1)^{km}}{m!k^m}km(T)^{km-1}S\right].$$

Use the properties listed in the paragraph above to reverse the process: the sum of the traces of individual terms becomes the trace of the sum, and then simplify to yield

$$\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^j T^j\right)S\right]+\sum_{m=1}^{\infty}\lambda\left[e^{-\frac{(-1)^k}{k}T^k}\frac{(-1)^{km}}{m!k^m}km(T)^{km-1}S\right]$$

$$=\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^j T^j\right)S\right]$$

$$+\lambda\left[e^{-\frac{(-1)^k}{k}T^k}\left(\sum_{m=1}^{\infty}\frac{\left[\frac{(-T)^k}{k}\right]^{m-1}}{(m-1)!}\right)\left(-(-T)^{k-1}S\right)\right]$$

$$=\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^j T^j\right)S\right]+\lambda\left[-(-T)^{k-1}S\right]$$

$$=\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-1}(-1)^j T^j\right)S\right]. \quad \blacksquare$$

THEOREM 3.11. [Pietsch]. *Let* det *be a continuous determinant on a quasi-Banach operator ideal $\mathfrak{U}$. Let $T(z)$ be a function from a domain of the complex plane with values in $\mathfrak{U}$. If $T(z)$ is differentiable at z_0, then so is* $\det\left(I+T(z)\right)$. *If $I+T(z_0)$ is invertible then*

$$\frac{d}{dz}\det\left(I+T(z)\right)\Big|_{z=z_0}=\det{}'\left[\frac{d}{dz}T(z)\Big|_{z=z_0}\left(I+T(z_0)\right)^{-1}\right]\det\left(I+T(z_0)\right).$$

Proof. [2], p. 193. ∎

LEMMA 3.12. *Let $T\in C_k^{(c)}(X)$ where k is a positive integer. Let $S\in C_1^{(c)}(X)$. Then $\pi\left(I+\varepsilon_k(T+zS)\right)$ is an entire function of the complex variable z. Furthermore, if $-1\notin\sigma(T)$ then*

$$\frac{d}{dz}\pi\left(I+\varepsilon_k(T+zS)\right)\Big|_{z=0}=\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^j T^j\right)S\right]\pi\left(I+\varepsilon_k(T)\right).$$

Proof. Let $T\in C_k^{(c)}(X)$ be such that $-1\notin\sigma(T)$ and let $S\in C_1^{(c)}(X)$. Since $(I+T)$ is invertible, then $\left(I+\varepsilon_k(T)\right)$ is invertible. Apply Theorem 3.11 with $T(z)=\varepsilon_k(T+zS), z_0=0$, and det $=\pi$ to yield

$$\frac{d}{dz}\pi\left(I+\varepsilon_k(T+zS)\right)\Big|_{z=0}=\pi'\left[\frac{d}{dz}\varepsilon_k(T+zS)\Big|_{z=0}\left(I+\varepsilon_k(T)\right)^{-1}\right]\pi\left(I+\varepsilon_k(T)\right).$$

Use Theorem 3.4 to yield

$$\pi'\left[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_k(T+zS)\Big|_{z=0}(I+\varepsilon_k(T))^{-1}\right]\pi(I+\varepsilon_k(T))$$
$$=\lambda\left[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_k(T+zS)\Big|_{z=0}(I+\varepsilon_k(T))^{-1}\right]\pi(I+\varepsilon_k(T))$$

and then use Lemma 3.10 to yield

$$\lambda\left[\frac{\mathrm{d}}{\mathrm{d}z}\varepsilon_k(T+zS)\Big|_{z=0}(I+\varepsilon_k(T))^{-1}\right]\pi(I+\varepsilon_k(T))$$
$$=\lambda\left[\left((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\right)S\right]\pi(I+\varepsilon_k(T)).\quad ∎$$

LEMMA 3.13. *Let* $T\in\mathcal{B}(X)$ *and* $0<p<\infty$. *Then* $\|T\|\leqslant|T|_I$, *where*

$$|T|_I=\sup_{|B|_p=1}\{|\lambda(TB)|\,|\,B\in\mathcal{F}(X)\}.$$

Proof. Assume $T\neq 0$; the case when $T=0$ is trivial. Let $\varepsilon>0$ be given and choose $x_0\in X$ such that $\|x_0\|=1$, $Tx_0\neq 0$, and $\|T\|<\|Tx_0\|+\varepsilon$. The Hahn-Banach theorem implies that there exists $f\in X'$ such that $f(Tx_0)=\|Tx_0\|$ and $\|f\|=1$. Let $L\in\mathcal{B}(X)$ be defined by $Lx=f(x)x_0$. Note that $|L|_p=\|L\|$ since L is a rank one operator. Also,

$$\|L\|=\sup_{\|x\|=1}\|Lx\|=\sup_{\|x\|=1}\|f(x)x_0\|=\sup_{\|x\|=1}|f(x)|=\|f\|=1,$$

showing that $|L|_p=1$. Let S be the rank one operator defined by $Sx=f(x)Tx_0$. Then (t_4) of Definition 3.1 implies that $\|Tx_0\|=f(Tx_0)=\lambda(S)=\lambda(TL)=|\lambda(TL)|$. Consequently, $\|T\|<\|Tx_0\|+\varepsilon=|\lambda(TL)|+\varepsilon\leqslant|T|_I+\varepsilon$. Since ε was arbitrary, the lemma follows. ∎

LEMMA 3.14. *Let k be a positive integer and let p be such that $k-1<p\leqslant k$. Then there exists a constant Γ independent of T such that for all* $T\in C_p^{(c)}(X)$

$$|\pi(I+\varepsilon_k(T))|\leqslant\exp(\Gamma|T|_p^p).$$

Proof. For any $c\in\mathbf{C}$, and $k-1<p\leqslant k$,

$$\left|(1+c)\exp\Big(\sum_{n=1}^{k-1}\frac{(-c)^n}{n}\Big)\right|\leqslant\exp(l|c|^p),$$

where l is constant independent of c ([1], vol. II, p. 1107). Thus by Theorem 2.12 and Theorem 3.9,

$$\begin{aligned}|\pi(I+\varepsilon_k(T))| &\leqslant \prod_{n=1}^{\infty}\exp(l|\lambda_n(T)|^p) = \exp\Big(l\sum_{n=1}^{\infty}|\lambda_n(T)|^p\Big)\\ &\leqslant \exp\Big(l\beta\sum_{n=1}^{\infty}c_n(T)^p\Big) = \exp\big(l\beta|T|_p^p\big). \quad \blacksquare\end{aligned}$$

LEMMA 3.15. *Let k be a positive integer with $k-1<p\leqslant k$. If $T\in C_p^{(c)}(X)$ is such that $-1\notin\sigma(T)$, then there exists a constant Γ depending only on p such that*

$$\Big\|\pi(I+\varepsilon_k(T))\Big[(I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\Big]\Big\| \leqslant \exp(\Gamma(|T|_p^p+1)).$$

Proof. Cauchy's theorem, together with Lemma 3.12 imply that

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\pi(I+\varepsilon_k(T+zS))}{z^2}\,\mathrm{d}z = \pi(I+\varepsilon_k(T))\cdot\lambda\Big[\Big((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\Big)S\Big],$$

where $T\in C_p^{(c)}(X)$ with $-1\notin\sigma(T)$, $S\in C_1^{(c)}(X)$, and γ is the circle given by $\exp(\mathrm{i}t), 0\leqslant t\leqslant 2\pi$. Lemma 3.14, the quasinorm inequality, and the inequality $(|a|+|b|)^r\leqslant 2^r(|a|^r+|b|^r), r>0, a,b\in\mathbb{C}$ imply that

$$\begin{aligned}\Big|\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\pi(I+\varepsilon_k(T+zS)}{z^2}\,\mathrm{d}z\Big| &\leqslant \max_{|z|=1}|\pi(I+\varepsilon_k(T+zS))|\\ &\leqslant \exp(\Gamma\eta^p2^p(|T|_p^p+|S|_p^p)),\end{aligned}$$

where Γ,η are independent of T. These two observations imply that

$$\sup_{\substack{S\in\mathcal{F}(X)\\ |S|_p=1}}\Big|\pi(I+\varepsilon_k(T))\cdot\lambda\Big[\Big((I+T)^{-1}-\sum_{j=0}^{k-2}(-1)^jT^j\Big)S\Big]\Big| \leqslant \exp(\Gamma\eta^p2^p(|T|_p^p+1)).$$

This fact combined with Lemma 3.13 proves the lemma. $\blacksquare$

We can now state and prove a generalized Carleman's inequality.

THEOREM 3.16. *Let k be a positive integer with $k-1 < p \leqslant k$ and $T \in C_p^{(c)}(X)$ with $-1 \notin \sigma(T)$. Then there exists constants c, Γ depending on p such that*

$$\|\pi(I+\varepsilon_k(T))(I+T)^{-1}\| \leqslant c\exp(\Gamma|T|_p^p).$$

Proof. Lemma 3.15 implies this result when $k=1$. Assume $k \geqslant 2$. When $|T|_p \geqslant 1$ it follows that $\|T^j\| \leqslant \|T\|^j \leqslant |T|_p^j \leqslant |T|_p^p \leqslant \exp(|T|_p^p)$ for all $j = 0,1,\ldots,k-1$, and when $|T|_p < 1$, observe that $\|T^j\| \leqslant \|T\|^j \leqslant |T|_p^j < 1 \leqslant \exp(|T|_p^p)$, for $j=0,1,\ldots,k-1$, thereby showing $\|T^j\| \leqslant \exp(|T|_p^p)$ for $j = 0,1,\ldots,k-1$. Now Lemmas 3.14 and 3.15 imply that

$$\begin{aligned}
&\left\|\pi(I+\varepsilon_k(T))(I+T)^{-1}\right\| \\
&\quad = |\pi(I+\varepsilon_k(T))| \cdot \left\|(I+T)^{-1} - \sum_{j=0}^{k-2}(-1)^j T^j + \sum_{j=0}^{k-2}(-1)^j T^j\right\| \\
&\quad \leqslant |\pi(I+\varepsilon_k(T))| \left(\left\|(I+T)^{-1} - \sum_{j=0}^{k-2}(-1)^j T^j\right\| + \left\|\sum_{j=0}^{k-2}(-1)^j T^j\right\|\right) \\
&\quad \leqslant \exp(\Gamma_1(|T|_p^p + 1)) + \exp(\Gamma_2(|T|_p^p)p\exp(|T|_p^p),
\end{aligned}$$

where Γ_1 and Γ_2 are constants independent of T. The above implies that $\|\pi(I+\varepsilon_k(T))(I+T)^{-1}\| \leqslant c\exp(\Gamma|T|_p^p)$, where c, Γ are constants independent of T and dependent upon p. ∎

The next result, which is the last result of this section, plays a key role in the next section of this paper. It follows easily from the work preceding it.

COROLLARY 3.17. *Let $0 < p < \infty$ and $N \in C_p^{(c)}(X)$ be quasinilpotent. Then*

$$\|(\lambda I - N)^{-1}\| \leqslant \frac{c}{|\lambda|}\exp(\Gamma|\lambda|^{-p}),$$

for all nonzero $\lambda \in \mathbf{C}$, where c, Γ are constants depending on N and p only.

Proof. Let $\lambda \neq 0$. It is clear that $-N/\lambda$ is quasinilpotent and that $(\lambda I - N)^{-1} = \frac{1}{\lambda}(I - N/\lambda)^{-1}$. Since $\sigma(-N/\lambda) = \{0\}$, it follows that $\pi(I + \varepsilon_k(-N/\lambda)) = 1$ for $p \leqslant k$. This fact and Theorem 3.16 imply that $\|(I - (N/\lambda))^{-1}\| \leqslant c\exp(\Gamma|(N/\lambda)|_p^p)$, where c, Γ depend only on p. The result now follows from these observations. ∎

4. $\overline{\text{sp}}\,(T) = X$

A closed densely defined linear operator T in X is *discrete* if there is a number ξ in its resolvent set $\rho(T)$ such that the resolvent operator $R_\xi(T) = (\xi I - T)^{-1}$ is compact. *In all that follows, unless otherwise stated, T will be assumed to be discrete and X will be assumed to be a separable, reflexive Banach space over $\mathbb{C}$.*

The following shows that for all $\lambda \in \mathbb{C}$ the operator $T_\lambda = \lambda I - T$ is *Fredholm*, i.e., that $R(T_\lambda)$ is closed and $N(T_\lambda)$, $N(T'_\lambda)$ are both finite dimensional. Let $\xi \in \rho(T)$ be such that $R_\xi(T)$ is compact. Since $R(T_\xi) = X$, it is closed. Also, since $N(T_\xi) = \{0\}$, it is finite dimensional. Using the equality $\sigma(T) = \sigma(T')$ ([1], vol. III, p. 2354) it follows that $\xi \in \rho(T')$. Consequently, $N(T'_\xi) = \{0\}$, showing T_ξ to be Fredholm. Fix $\lambda \in \mathbb{C}$ with $\lambda \neq \xi$ and note that $T_\lambda = (\xi - \lambda)(T_\xi)(\frac{1}{\xi-\lambda} I - R_\xi(T))$. Since $R_\xi(T)$ is compact, $\frac{1}{\xi-\lambda} I - R_\xi(T)$ is Fredholm ([6], p. 301), which implies that T_λ is the product of Fredholm operators and is therefore Fredholm ([7], p. 103). For $\lambda \in \rho(T)$, $\xi \neq \lambda$, the above resolvent equation together with the fact that the compact operators in $\mathcal{B}(X)$ form an ideal implies that $R_\lambda(T)$ is compact.

The discreteness of T implies that of T'. The converse of this statement also holds ([1], vol. III, p. 2354). In [1], vol. III, p. 2292, it is shown that $\sigma(T)$ is either empty or a denumerable set of points $\{\lambda_i\}_{i=1}^\infty$ (including multiplicities) consisting entirely of eigenvalues and having no finite limit points. Should $\sigma(T) = \emptyset$, it will not have any generalized eigenfunctions and therefore, the question of whether or not $\overline{\text{sp}}\,(T) = X$ is meaningless. For this reason, *it will be assumed in all that follows that $\sigma(T) \neq \emptyset$.*

For $\lambda_i \in \sigma(T)$, T_{λ_i} has finite ascent m_i and finite descent n_i with $m_i = n_i$ ([7], p. 101–103). Also, the subspace $R(T(\lambda_i)) = R(T_{\lambda_i}^{m_i})$ is closed and $X = R(T(\lambda_i)) \oplus N(T(\lambda_i))$ (topological direct sum) for all $\lambda_i \in \sigma(T)$ ([6], p. 290). It should be noted that $N(T(\lambda_i)) = N(T_{\lambda_i}^{m_i})$ is finite dimensional, since $T_{\lambda_i}^{m_i}$ is Fredholm. Let $M_\infty = \{x \in X \mid P_i x = 0, i = 1, 2, \ldots\}$, where P_i is the continuous projection of X onto $N(T(\lambda_i))$ along $R(T(\lambda_i))$. The symbol M'_∞ is defined in an analogous manner. In [1], vol. III, p. 2295, 2355 it is shown that the closed subspaces M_∞ and M'_∞ are either zero or infinite dimensional and that

$$\overline{\text{sp}}\,(T) = M'^{\perp}_\infty \quad \text{and} \quad \overline{\text{sp}}\,(T') = M^{\perp}_\infty. \tag{4.1}$$

In view of this result, $\overline{\text{sp}}\,(T) = X$ whenever $M'_\infty = \{0\}$. Should $M_\infty = \{0\}$, then it does not follow that $\overline{\text{sp}}\,(T) = X$ ([1], vol. III, p. 2555).

The proof of the following theorem is elementary and may be obtained by paralleling the arguments presented in [8], p. 322-323.

THEOREM 4.1. *The operator T_λ maps $D(T) \cap M_\infty$ one-to-one and onto M_∞ for all $\lambda \in \mathbf{C}$.*

The last few results have pertained to a general discrete operator T with nonempty spectrum. The results that follow pertain to special subclasses of discrete T. These subclasses are defined by the following.

DEFINITION 4.2. Let $0 < p < \infty$. A discrete operator T in X is said to be a $C_p^{(a)}$*-discrete operator* if there exists a point $\xi \in \rho(T)$ such that $R_\xi(T) \in C_p^{(a)}(X)$.

THEOREM 4.3. *Let $0 < p < \infty$ and T be a $C_p^{(a)}$-discrete operator. Then T' is a $C_p^{(a)}$-discrete operator.*

Proof. Let $\xi \in \rho(T)$ be such that $R_\xi(T) \in C_p^{(a)}(X)$. Let $\{a_n\}_{n=1}^\infty$ be the approximation number sequence associated with $R_\xi(T)$. In [1], vol. III, p. 2354, it is shown that T' is discrete with $\rho(T') = \rho(T)$. This implies that $\xi \in \rho(T')$. The operator $R_\xi(T')$ is compact and has an approximation number sequence $\{a'_n\}_{n=1}^\infty$. By Theorem 2.9, $a_n = a'_n$ for $n = 1, 2, \ldots$. Thus, $\sum_{n=1}^{\infty} (a'_n)^p < \infty$, implying that T' is a $C_p^{(a)}$-discrete operator. ∎

In all that follows, T will be assumed to be a $C_p^{(a)}$-discrete operator. Let $\lambda \in \mathbf{C}$ and define $S_\lambda = T_\lambda | D(T) \cap M_\infty$. Theorem 4.1 implies that $(S_\lambda)^{-1}$ maps M_∞ one-to-one and onto $D(T) \cap M_\infty$. Let ξ be such that $R_\xi(T) \in C_p^{(a)}(X)$. Since $K \equiv (S_\xi)^{-1} = R_\xi(T)|M_\infty$, it follows from Lemma 2.8 and Theorem 2.7 that $K \in C_p^{(c)}(M_\infty)$. Thus, K is a one-to-one compact operator. To see that K is quasinilpotent, assume that $\lambda \in \sigma(T), \lambda \neq 0$. Since K is compact, λ is an eigenvalue of K and there exists $u \neq 0$ in M_∞ with $Ku = \lambda u$. Thus $u = \lambda T_\xi u$, which makes $\xi - 1/\lambda$ an eigenvalue of T. Denote $\xi - 1/\lambda$ by λ_i. Then $u \in N(T(\lambda_i)) \cap M_\infty \subseteq N(T(\lambda_i)) \cap R(T(\lambda_i)) = \{0\}$, which is a contradiction.

Let $y \in M'_\infty$ and $x \in M_\infty$ be fixed. Let N be a positive integer. Since K is quasinilpotent, the operator $R_{(\xi-\lambda)^{-1}}(K)$ exists for all $\lambda \neq \xi$, which allows $F : \mathbf{C} \to \mathbf{C}$ to be defined as follows: Let $\lambda \in \mathbf{C}, \lambda \neq \xi$

$$F(\lambda) = y\left(\frac{1}{(\xi - \lambda)^{N+2}}\left(R_{(\xi-\lambda)^{-1}}(K)\right)x\right). \tag{4.2}$$

The next lemma uses the important generalized Carleman inequality developed in the last section to show that F has order $\leqslant p$.

LEMMA 4.4. *Let F be defined as in (4.2). Then there exists a real constant β such that*

$$|F(\lambda)| = \mathrm{O}\left(e^{\beta|\lambda|^p}\right) \quad \textit{as} \quad \lambda \to \infty.$$

Proof. Since K is a quasinilpotent, $C_p^{(c)}$ operator, Corollary 3.17 implies that $\|R_{(\xi-\lambda)^{-1}}(K)\| \leqslant c\,|\xi-\lambda|\exp(\Gamma|\xi-\lambda|^p)$, if $\lambda \neq \xi$, where c, Γ are constants depending on K, and p only. Let $|\lambda| > 2|\xi|$. Then $\frac{1}{2}|\lambda| < |\xi-\lambda| < 2|\lambda|$, which implies that $\|R_{(\xi-\lambda)^{-1}}(K)\| \leqslant c\,|\lambda|\exp(2^p\Gamma|\lambda|^p)$. Applying this to the definition of F yields

$$|F(\lambda)| \leqslant 2^{N+3}c\,|\lambda|^{-(N+1)}\|x\|\,\|y\|\exp(2^p\Gamma|\lambda|^p) \quad \text{for all} \quad |\lambda| > 2\,|\xi|$$

and the result follows. ∎

Two alternative representations of F which will be used in the culminating theorem are derived and displayed below.

A consequence of the fact that K is quasinilpotent and compact is that for all $\lambda \neq \xi$ ([6], p. 278)

$$R_{(\xi-\lambda)^{-1}}(K) = \sum_{j=0}^{\infty}(\xi-\lambda)^{j+1}K^j.$$

From this a Laurent series representation for F is derived: Let $\lambda \neq \xi$

$$F(\lambda) = \sum_{j=0}^{\infty}(\xi-\lambda)^{j-N-1}y(K^jx). \tag{4.3}$$

This shows that F is analytic for $\lambda \neq \xi$.

A rudimentary calculation shows that for all $\lambda \neq \xi$

$$R_{(\xi-\lambda)^{-1}}(K) = (\xi-\lambda)I + (\xi-\lambda)^2(S_\lambda)^{-1}.$$

Applying this to the definition of F yields another representation of F:

$$F(\lambda) = \frac{1}{(\xi-\lambda)^{N+1}}\,y(x) + \frac{1}{(\xi-\lambda)^N}\,y(R_\lambda(T)x) \tag{4.4}$$

for $\lambda \neq \xi$.

The next result is the culminating theorem of this section and paper.

THEOREM 4.5. *Let $0 < p < \infty$ be fixed and T be a $C_p^{(a)}$-discrete operator in a separable, reflexive Banach space X with nonempty spectrum.*

(1) *For $p \geqslant \frac{1}{2}$, let n be the smallest integer such that $n > 2p$. Assume that there exists a set of n rays, $\arg \lambda = \theta_j, j = 1, 2, \dots, n$, such that*

(i) *the angles between adjacent rays are less than $\frac{\pi}{p}$,*

(ii) *for $|\lambda|$ sufficiently large, all points on the n rays belong to $\rho(T)$,*

(iii) *there exists a positive integer N such that*

$$\|R_\lambda(T)\| = O(|\lambda|^N)$$

as $\lambda \to \infty$ along each ray.

(2) *For $p < \frac{1}{2}$, assume that there exists a ray such that for all $|\lambda|$ sufficiently large, all points on the ray belong to $\rho(T)$ and that $\|R_\lambda(T)\| = O(|\lambda|^N)$ as $\lambda \to \infty$ along that ray.*

Then

$$\overline{\mathrm{sp}}\,(T) = X \qquad \overline{\mathrm{sp}}\,(T') = X',$$

$$M_\infty = \{0\} \quad \textit{and} \quad M'_\infty = \{0\}.$$

Proof. Theorem 4.3 implies that T' is a $C_p^{(a)}$-discrete operator whenever T is. This observation implies that it is sufficient to prove that $M_\infty = \{0\}$, for equation (4.1) then yields $\overline{\mathrm{sp}}\,(T') = X'$, and applying the same arguments to T' gives $M'_\infty = \{0\}$ and $\overline{\mathrm{sp}}\,(T) = X$.

Let F be the function defined in (4.2). If λ is on one of the rays with $|\lambda| > 2|\xi|$, the representation of F displayed in (4.4) yields

$$|F(\lambda)| \leqslant \frac{2^{N+1}}{|\lambda|^{N+1}} \|x\| \, \|y\| + \frac{2^N}{|\lambda|^N} \|R_\lambda(T)\| \, \|x\| \, \|y\|. \tag{4.5}$$

Condition (iii) of the hypothesis implies that the right hand side of (4.5) remains bounded as $\lambda \to \infty$ on the ray. Therefore F is bounded as $\lambda \to \infty$ on each of the n rays.

This boundedness together with Lemma 4.4 imply, by the Phragmén-Lindelöff Theorem ([9], p. 177-178) that F is bounded on a neighborhood of $\lambda = \infty$. This implies that in the Laurent series expansion of F found in equation (4.3), the coefficients corresponding to the positive powers of $(\xi - \lambda)$ are all zero. Thus, $y(K^{N+2}x) = y(K^{N+3}x) = y(K^{N+4}x) = \cdots = 0$. Since this is true for all $y \in M'_\infty$, then in particular $K^{N+2}x = 0$. But K^{N+2} is one-to-one since K is. This implies $x = 0$. Therefore $M_\infty = \{0\}$. ∎

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