J. OPERATOR THEORY
33(1995), 299–315

ON THE SPECTRA OF THE POSITIVE COMPLETIONS FOR OPERATOR MATRICES

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Communicated by Florian-Horia Vasilescu

ABSTRACT. Let A and B be positive (or selfadjoint) operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. In this paper, the union and the intersection of spectra of all positive (or Hermitian) completions of the partially specified operator matrix $\begin{pmatrix} A & ? \\ ? & B \end{pmatrix}$ are completely characterized.

KEYWORDS: *Operator matrix, positive operator, selfadjoint operator, spectrum.*
AMS SUBJECT CLASSIFICATION: Primary 47A10; Secondary 47B15, 15A48, 15A57.

1. INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex separable Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H},\mathcal{K})$ the Banach spaces of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ and from $\mathcal{H}$ into $\mathcal{K}$, respectively. Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be positive, denoted by $T \geqslant 0$, if $\langle Tx, x\rangle \geqslant 0$ for all $x \in \mathcal{H}$, and to be selfadjoint if $T^* = T$, where $\langle \cdot,\cdot\rangle$ is the inner product. It is known that for positive operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, and for an operator $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, the operator matrix $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ is positive on $\mathcal{H} \oplus \mathcal{K}$ if and only if there exists a contraction $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, i.e., $\|X\| \leqslant 1$, so that $C = A^{1/2}XB^{1/2}$. Given a pair (A, B) of positive operators in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{K})$, let

$$M_X(A,B) = \begin{pmatrix} A & A^{\frac12}XB^{\frac12} \\ B^{\frac12}X^*A^{\frac12} & B \end{pmatrix}$$

(for short, M_X) be a positive completion of the partially specified operator matrix $\begin{pmatrix} A & ? \\ ? & B \end{pmatrix}$, where $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an arbitrary contraction. It is clear that

$$\{M_X(A,B);\ \|X\| \leqslant 1\}$$

is the set of all positive completions of $\begin{pmatrix} A & ? \\ ? & B \end{pmatrix}$.

The purpose of this paper is to characterize the sets $\bigvee(A,B)$ and $\bigwedge(A,B)$, the union and the intersection of the spectra of all $M_X(A,B)$, that is,

$$\bigvee(A,B) = \bigcup\{\sigma(M_X(A,B));\ \|X\| \leqslant 1\}$$

and

$$\bigwedge(A,B) = \bigcap\{\sigma(M_X(A,B));\ \|X\| \leqslant 1\},$$

where $\sigma(T)$ stands for the spectrum of the operator T.

This paper is a continuation of [3] in which we showed that

$$\max\{\|A\|, \|B\|\} \leqslant \|M_X(A,B)\| \leqslant \|A\| + \|B\| \tag{1}$$

and described the sets

$$\mathcal{L}(A,B) = \{X;\ \|M_X(A,B)\| = \max\{\|A\|, \|B\|\}\},$$

and

$$\mathcal{S}(A,B) = \{X;\ \|M_X(A,B)\| = \|A\| + \|B\|\}.$$

Similar problems were studied by H. Du and C. Gu in [2] for the completions

$$M_X(A,B,C) = \begin{pmatrix} A & C \\ X & B \end{pmatrix}$$

with A, B, C fixed and X arbitrary. They obtained the complete characterizations for $\bigcup_X \sigma(M_X(A,B,C))$ and $\bigcap_X \sigma(M_X(A,B,C))$ in finite dimensional case and partial descriptions in infinite dimensional case.

The present paper is organized as follows. In Section 2–4 we discuss the spectra of positive completions: Section 2 deals with the case when both $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional; Section 3 considers the case when one of the spaces $\mathcal{H}$ and $\mathcal{K}$ is finite dimensional and the other is infinite dimensional; while in Section 4 the case $\dim \mathcal{H} = \dim \mathcal{K} = \infty$ is considered. Finally, in Section 5, we consider the similar questions for Hermitian completions of $\begin{pmatrix} A & ? \\ ? & B \end{pmatrix}$ for a given pair (A, B)

of selfadjoint operators and also get the complete descriptions of the union and the intersection of the spectra of all Hermitian completions.

Now we introduce some notation which are used in this paper. For an operator T, we shall denote by $\ker T$, $\mathcal{R}(T)$, $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{e}}(T)$, $\rho(T)$ and $\operatorname{Lat} T$ the null space, the range, the spectrum, the point spectrum, the essential spectrum, the resolvent set and the invariant subspaces lattice of T, respectively. For $\mathcal{S}$ a subset in $\mathcal{H}$, $\bigvee\{x;\ x \in \mathcal{S}\}$ will denote the minimal linear subspace containing $\mathcal{S}$. For a positive operator A, $A = \int\limits_0^{\|A\|} t\,\mathrm{d}\mathbf{E}_t$ will denote the spectral decomposition of A, respectively. The terminology used in this paper agrees with [5] and [8].

2. FINITE DIMENSIONAL CASE

In this section we assume that $\dim\mathcal{H} = n < \infty$ and $\dim\mathcal{K} = m < \infty$. Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be positive (i.e., positive semidefinite).

THEOREM 2.1. *If* $\dim\mathcal{H} = n$ *and* $\dim\mathcal{K} = m$, *then*

$$\bigvee(A, B) = [0, \|A\| + \|B\|] \setminus (\max\sigma(T), \min\sigma(S)),$$

where $T = A$ *or* B *and* $S = B$ *or* A.

Proof. Notice first that if $\dim\mathcal{H} = \dim\mathcal{K} = 1$ and $A = a$, $B = b$, then

$$\bigvee(A, B) = \bigvee(a, b) = [0, \min\{a, b\}] \cup [\max\{a, b\}, a + b]. \tag{2}$$

In fact, for any $c \in \mathbb{C}$ with $|c| \leqslant 1$, we have

$$\sigma(M_c(a, b)) = \{\varphi_+(t), \varphi_-(t)\}$$

with $t = |c| \in [0, 1]$ and

$$\varphi_\pm(t) = \frac{1}{2}\left(a + b \pm \sqrt{(a-b)^2 + 4abt^2}\right).$$

The formula (2) is implied by the continuity of the functions $\varphi_\pm(t)$.

Let $\sigma(A) = \{a_i\}_{i=1}^n$ and $\sigma(B) = \{b_j\}_{j=1}^m$. It is easy to see from (2) that

$$\bigcup_{i,j}[0, \min\{a_i, b_j\}] \subset \bigvee(A, B),$$

$$\bigcup_{i,j}[\max\{a_i, b_j\}, a_i + b_j] \subset \bigvee(A, B)$$

and hence we must have

$$[0, \min\{\|A\|, \|B\|\}] \cup [\max\{\min\{a_i\}, \min\{b_j\}\}, \|A\| + \|B\|] \subseteq \bigvee(A, B),$$

that is,

$$[0, \|A\| + \|B\|] \setminus (\max \sigma(T), \min \sigma(S)) \subseteq \bigvee(A, B), \tag{3}$$

where $T = A$ or B and $S = B$ or A.

If $\max \sigma(T) \geqslant \min \sigma(S)$, (3) will imply that

$$[0, \|A\| + \|B\|] \subseteq \bigvee(A, B)$$

and by virtue of (1), we get

$$\bigvee(A, B) = [0, \|A\| + \|B\|]. \tag{4}$$

If $\max \sigma(T) < \min \sigma(S)$, for the sake of preciseness, say $T = A$ and $S = B$, i.e., if $\|A\| = \max \sigma(A) < \min \sigma(B)$, applying Theorem 4.3.15 in [7], it is not difficult to check that for any contraction X,

$$\sigma(M_X) \subset [0, \|A\|] \cup [\min \sigma(B), \|A\| + \|B\|] \tag{5}$$

and $\sigma(M_X) \cap [0, \|A\|] \neq \emptyset$, $\sigma(M_X) \cap [\min \sigma(B), \|A\| + \|B\|] \neq \emptyset$. Hence we have

$$\bigvee(A, B) \subseteq [0, \|A\|] \cup [\min \sigma(B), \|A\| + \|B\|],$$

completing the proof. ∎

THEOREM 2.2. *Let* $\dim \mathcal{H} = n$ *and* $\dim \mathcal{K} = m$. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})$ *be positive. Then*

$$\begin{aligned}\bigwedge(A, B) = &\Big(\{0\} \cap (\sigma(A) \cup \sigma(B))\Big) \\ &\cup \{\lambda; \dim\ker(A - \lambda) > \dim(\ker B)^\perp \text{ or } \dim\ker(B - \lambda) > \dim(\ker A)^\perp\}.\end{aligned}$$

Proof. Since $M_X(A, B)$ is invertible (i.e., nonsingular) if and only if both A and B are invertible and $\|X\| < 1$, it is obvious that $0 \in \bigwedge(A, B)$ if and only if $0 \in \sigma(A) \cup \sigma(B)$. Denote $A|(\ker A)^\perp$ and $B|(\ker B)^\perp$ by A_1 and B_1, respectively. A_1 and B_1 are invertible. According to the space decompositions $\mathcal{H} = (\ker A) \oplus (\ker A)^\perp$ and $\mathcal{K} = (\ker B)^\perp \oplus (\ker B)$, $A = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$. If $X = (X_{ij})_{2\times 2} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is a contraction, and if $\lambda \neq 0$, then

$\lambda \in \rho(M_X(A,B))$ if and only if $\lambda \in \rho(M_{X_{21}}(A_1,B_1))$. So, $\lambda \in \bigwedge(A,B)$ if and only if $\lambda \in \bigwedge(A_1,B_1)$ whenever $\lambda \neq 0$. By this observation, we can, without loss of generality, assume that A and B are invertible. Notice also that we always have

$$\bigwedge(A,B) \subset \sigma(A) \cup \sigma(B).$$

Let $\lambda \neq 0$ and $\lambda \in \sigma(A) \cup \sigma(B)$.

If $\dim\ker(A-\lambda) > \dim(\ker B)^\perp = m$ or $\dim\ker(B-\lambda) > \dim(\ker A)^\perp = n$, then $M_X - \lambda$ is not surjective for any X. This implies that $\lambda \in \bigwedge(A,B)$.

Now assume that $\dim\ker(A-\lambda) \leqslant m$ and $\dim\ker(B-\lambda) \leqslant n$. We have to prove that $\lambda \notin \bigwedge(A,B)$, or equivalently, there exists a contraction X for which $\lambda \in \rho(M_X)$. To do this, diagonalize A and B, i.e. write $A = UD_AU^*$, $B = VD_BV^*$. Here U and V are unitary matrices, D_A and D_B diagonal matrices. Instead of considering M_X one may consider the analogous matrix given by

$$\begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} A & A^{\frac{1}{2}}XB^{\frac{1}{2}} \\ B^{\frac{1}{2}}X^*A^{\frac{1}{2}} & B \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} D_A & D_A^{\frac{1}{2}}X'D_B^{\frac{1}{2}} \\ D_B^{\frac{1}{2}}X'^*D_A^{\frac{1}{2}} & D_B \end{pmatrix}$$

where $X' = UXV^*$. Since D_A and D_B are diagonal, some simple choices of X' are sufficient to construct the necessary examples. For instance, if $\lambda \in \sigma(A) \cap \sigma(B)$ and has multiplicity one in both of these spectra, an operator unitarily equivalent to M_X is in the form

$$\left(\begin{array}{ccc|ccc} \ddots & & 0 & & \vdots & \\ & \lambda & & \cdots & \sqrt{\lambda}x_{ij}\sqrt{\lambda} & \cdots \\ 0 & & \ddots & & \vdots & \\ \hline & \vdots & & \ddots & & 0 \\ \cdots & \sqrt{\lambda}\overline{x}_{ij}\sqrt{\lambda} & \cdots & & \lambda & \\ & \vdots & & 0 & & \ddots \end{array}\right).$$

Choosing all entries, except (i,j), of X' to be zero, one obtain $\lambda \notin \sigma(M_X)$. This construction can be modified to cover all the other cases one needs, and we will leave it to the reader.

3. THE CASE ONE SPACE IS INFINITE DIMENSIONAL AND THE OTHER IS FINITE DIMENSIONAL

We begin with two lemmas which are useful in the sequel.

LEMMA 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces, let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be two positive operators. Then*

$$[\max\{\|A\|, \|B\|\}, \|A\| + \|B\|) \subset \bigvee(A, B) \subseteq [0, \|A\| + \|B\|].$$

Furthermore,

$$\|A\| + \|B\| \in \bigvee(A, B)$$

if and only if

(i) $0 \in \sigma_p(\|A\| - A) \cap \sigma_p(\|B\| - B)$, *or*
(ii) $0 \in \sigma_e(\|A\| - A) \cap \sigma_e(\|B\| - B)$.

Proof. The inclusion $\bigvee(A, B) \subset [0, \|A\| + \|B\|]$ is obvious. Given $\varepsilon > 0$, take unit vectors $x \in \mathcal{H}$ and $y \in \mathcal{K}$ so that $\|A^{1/2}x\|^2 \geqslant \|A\| - \delta$ and $\|B^{1/2}y\|^2 \geqslant \|B\| - \delta$, where $\delta = \varepsilon/3$. Let

$$x_o = \sqrt{\frac{\|A\|}{\|A\| + \|B\|}}\, x \quad \text{and} \quad y_o = \sqrt{\frac{\|B\|}{\|A\| + \|B\|}}\, y.$$

Then, $\|x_o \oplus y_o\| = 1$ and

$$\|A^{\frac{1}{2}}x_o\|^2 \leqslant \frac{\|A\|^2}{\|A\| + \|B\|},$$

$$\|B^{\frac{1}{2}}y_o\|^2 \geqslant \frac{\|B\|}{\|A\| + \|B\|}(\|B\| - \delta).$$

Let $X \in B(\mathcal{K}, \mathcal{H})$ defined by $XB^{1/2}y_o = \frac{\|B\|^{1/2}\sqrt{\|B\| - \delta}}{\|A\|}A^{1/2}x_o$ and $Xz = 0$ if $z \perp B^{1/2}y_o$. It is clear that $\|X\| \leqslant 1$ as

$$\frac{\|B\|(\|B\| - \delta)}{\|A\|^2}\|A^{1/2}x_o\|^2 \leqslant \|B^{1/2}y_o\|^2.$$

But

$$\begin{aligned}\langle M_X(x_o \oplus y_o), (x_o \oplus y_o)\rangle &= \|A^{\frac{1}{2}}x_o\|^2 + \|B^{\frac{1}{2}}y_o\|^2 + 2Re\,\langle XB^{\frac{1}{2}}y_o, A^{\frac{1}{2}}x_o\rangle \\ &\geqslant \frac{1}{\|A\| + \|B\|}[\|A\|^2 - \delta\|A\| + \|B\|^2 - \delta\|B\| + 2(\|B\| - \delta)(\|A\| - \delta)] \\ &\geqslant \|A\| + \|B\| - \varepsilon,\end{aligned}$$

so

$$\|M_X\| \geqslant \|A\| + \|B\| - \varepsilon.$$

Note that $\|M_{tX}\|$, as a function of t, is continuous on $[0,1]$ and $\|M_{tX}\| \in \sigma(M_{tX})$. It follows that

$$[\max\{\|A\|, \|B\|\}, \|A\| + \|B\| - \varepsilon] \subset \bigcup_{0 \leqslant t \leqslant 1} \sigma(M_{tX}) \subset \bigvee(A, B).$$

Let $\varepsilon \to 0$, we get

$$[\max\{\|A\|, \|B\|\}, \|A\| + \|B\|) \subset \bigvee(A, B).$$

Moreover, $\|A\| + \|B\| \in \bigvee(A, B)$ if and only if there is a contraction X such that $\|A\| + \|B\| \in \sigma(M_X)$, which happens if and only if $\|M_X\| = \|A\| + \|B\|$. Therefore, by [3], $\|A\| + \|B\| \in \bigvee(A, B)$ if and only if $0 \in \sigma_p(\|A\| - A) \cap \sigma_p(\|B\| - B)$ or $0 \in \sigma_e(\|A\| - A) \cap \sigma_e(\|B\| - B)$. This completes the proof. ∎

Also observe that $\sigma(A) \cup \sigma(B) \subset \bigvee(A, B)$.

The following lemma was obtained by A. Fragela Kol'yar in [4].

LEMMA 3.2. *If operator matrix* $T = \begin{pmatrix} T_1 & T_3 \\ T_3^* & T_2 \end{pmatrix}$ *is selfadjoint and if* $\alpha = \max \sigma(T_1) < \min \sigma(T_2) = \beta$, *then* $(\alpha, \beta) \subset \rho(T)$.

Now we return to the questions of this section. For the sake of convenience, we suppose that $\dim \mathcal{H} = \infty$ and $\dim \mathcal{K} < \infty$. The other situation can be discussed in the same way.

THEOREM 3.3. *Let* $\dim \mathcal{H} = \infty$ *and* $\dim \mathcal{K} = m < \infty$. *Let A and B be positive operators acting on $\mathcal{H}$ and $\mathcal{K}$, respectively. If* $\|A\| \in \sigma_p(A)$, *then*

$$\bigvee(A, B) = [0, \|A\| + \|B\|] \setminus (\max \sigma(T), \min \sigma(S)),$$

and otherwise,

$$\bigvee(A, B) = [0, \|A\| + \|B\|) \setminus (\max \sigma(T), \min \sigma(S)).$$

where $T = A$ or B while $S = B$ or A.

Proof. Since $\dim \mathcal{K} < \infty$, it follows from Lemma 3.1 that $\|A\| + \|B\| \in \bigvee(A, B)$ if and only if $\|A\| \in \sigma_p(A)$. And by Lemma 3.2, we always have

$$\bigvee(A, B) \subseteq [0, \|A\| + \|B\|] \setminus (\max \sigma(T), \min \sigma(S)).$$

So, we need only to prove that

$$[0, \|A\| + \|B\|) \setminus (\max \sigma(T), \min \sigma(S)) \subseteq \bigvee(A, B), \tag{6}$$

where $T = A$ or B while $S = B$ or A.

Let $a \in \sigma(A)$ and $b \in \sigma(B) = \sigma_p(B)$.

If $a \in \sigma_p(A)$, then similarly to the proof of Theorem 2.1 we have

$$[0, \min\{a, b\}] \cup [\max\{a, b\}, a + b] \subset \bigvee(A, B). \tag{7}$$

Suppose $a \in \sigma(A) \setminus \sigma_p(A)$. For sufficiently small $\varepsilon > 0$, let $A_1 = \int_0^{a+\varepsilon} t\, d\mathbf{E}_t$, $A_2 = \int_{a-\varepsilon}^{\|A\|} t\, d\mathbf{E}_t$. Let $B_1 = B|\ker(B - b)$. Then $\bigvee(A_1, B_1) \subset \bigvee(A, B)$ and $\bigvee(A_2, B_1) \subset \bigvee(A, B)$. Notice that $\|A_1\| \leqslant a + \varepsilon$. So, by Lemma 3.1,

$$[\max\{a + \varepsilon, b\}, a + b) \subset \bigvee(A_1, B_1) \subset \bigvee(A, B).$$

Let $\varepsilon \to 0$, and note that $a, b \in \bigvee(A, B)$, we get

$$[\max\{a, b\}, a + b) \subset \bigvee(A, B). \tag{8}$$

On the other hand, for any natural N, there exists an $X : \ker(B - b) \to \mathcal{H}_2 = \int_{a-\varepsilon}^{\|A\|} d\mathbf{E}_t \mathcal{H}$ with $\|X\| < 1$ such that

$$\|(1 - X^* X)^{-1}\| \geqslant N.$$

For $t \in [0, 1]$, let

$$M_{tX}(A_2, B_1) = M_{tX}(A_2, b) = \begin{pmatrix} A_2 & t\sqrt{b} A^{\frac{1}{2}} X \\ t\sqrt{b} X^* A^{\frac{1}{2}} & b \end{pmatrix}.$$

$M_{tX}(A_2, B_1)$ is invertible and

$$\begin{aligned} M_{tX}(A_2,B_1)^{-1} &= \begin{pmatrix} A_2^{\frac{1}{2}} & 0 \\ 0 & b^{\frac{1}{2}} \end{pmatrix}^{-1} \begin{pmatrix} 1 & tX \\ tX^* & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_2^{\frac{1}{2}} & 0 \\ 0 & b^{\frac{1}{2}} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} A_2^{-\frac{1}{2}}(1 - t^2 X X^*)^{-1} A_2^{-\frac{1}{2}} & -t A_2^{-\frac{1}{2}} X (1 - t^2 X^* X)^{-1} b^{-\frac{1}{2}} \\ -t b^{-\frac{1}{2}} (1 - t^2 X^* X)^{-1} X^* A_2^{-\frac{1}{2}} & b^{-1}(1 - t^2 X^* X)^{-1} \end{pmatrix}. \end{aligned}$$

So,

$$\begin{aligned} \|M_{tX}(A_2, B_1)^{-1}\| &\geqslant \max\{\|A_2^{-1/2}(1 - t^2 X X^*)^{-1} A_2^{-1/2}\|,\ b^{-1}\|(1 - t^2 X^* X)^{-1}\|\} \\ &= \varphi(t), \qquad t \in [0, 1]. \end{aligned}$$

It is clear that $\|M_{tX}(A_2,B_1)^{-1}\|$ and $\varphi(t)$ are continuous functions of t on $[0,1]$, and

$$\min\sigma(M_0(A_2,B_1)) = \frac{1}{\|M_0(A_2,B_1)^{-1}\|} = \frac{1}{\varphi(0)} \geqslant \min\{a-\varepsilon, b\},$$

$$\min\sigma(M_X(A_2,B_1)) = \frac{1}{\|M_X(A_2,B_1)^{-1}\|} \leqslant \frac{1}{\varphi(1)} \leqslant \frac{1}{b^{-1}\|(1-X^*X)^{-1}\|} \leqslant \frac{b}{N}.$$

Therefore, it follows that

$$\left[\frac{b}{N}, \min\{a-\varepsilon, b\}\right] \subset \bigcup_{t\in[0,1]} \min\sigma(M_{tX}(A_2,B_1)) \subset \bigvee(A_2,B_1).$$

By the arbitrary choice of N and ε, as well as the fact that $0 \in \bigvee(A,B)$ is always the case, we get

$$[0, \min\{a,b\}] \subset \bigvee(A_2,B_1) \subset \bigvee(A,B). \tag{9}$$

Combining (7)–(9), it is easily seen that

$$\begin{aligned}
[0, \min\{\|A\|, \|B\|\}] \cup [\max\{\min\sigma(A), \min\sigma(B)\}, \|A\| + \|B\|) \\
\subseteq \Big(\bigcup_{\substack{a\in\sigma(A)\\ b\in\sigma(B)}} [0, \min\{a,b\}]\Big) \cup \Big(\bigcup_{\substack{a\in\sigma(A)\\ b\in\sigma(B)}} [\max\{a,b\}, a+b)\Big) \\
\subseteq \bigvee(A,B),
\end{aligned}$$

but this is the same as (6). ∎

COROLLARY 3.4. *Let A and B be as in Theorem 3.3. Assume that $\|A\| \geqslant \min\sigma(B)$ and $\|B\| \geqslant \min\sigma(A)$. If $\|A\| \in \sigma_p(A)$, then*

$$\bigvee(A,B) = [0, \|A\| + \|B\|],$$

otherwise,

$$\bigvee(A,B) = [0, \|A\| + \|B\|).$$

Now, we compute the set $\bigwedge(A,B)$. We begin doing this by listing two easy lemmas whose proofs are left to the reader.

LEMMA 3.5. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ a selfadjoint operator which is not algebraic. Then, there is a vector $x \in \mathcal{H}$ such that $\bigvee_{k=0}^{\infty} \{T^k x\}$ is infinite dimensional and $\langle T^n x, x\rangle \neq 0$ for every natural number n.*

LEMMA 3.6. *Let $A = A_1 \oplus A_2 \geqslant 0$ and $B = B_1 \oplus B_2 \geqslant 0$. If $\lambda \notin \bigwedge(A_1,B_1) \cup \bigwedge(A_2,B_2)$, then $\lambda \notin \bigwedge(A,B)$.*

THEOREM 3.7. *Let* $\dim\mathcal{H}=\infty$ *and* $\dim\mathcal{K}=m<\infty$. *Let* $A\in\mathcal{B}(\mathcal{H})$ *and* $B\in\mathcal{B}(\mathcal{K})$ *be positive. Then,*

$$\bigwedge(A,B)=\Big(\{0\}\cap(\sigma(A)\cup\sigma(B))\Big)\\ \cup\{\lambda;\ \lambda\in\sigma_e(A)\text{ or }\dim\ker(A-\lambda)>\dim(\ker B)^{\perp}\}.$$

Proof. We firstly assume that $0\notin\sigma(A)\cup\sigma(B)$, i.e., both A and B are invertible. So, $\dim(\ker B)^{\perp}=m$.

If $\lambda\in\sigma_p(A)$ such that $\dim\ker(A-\lambda)>m$, then $M_X(A,B)-\lambda$ is not surjective for every contraction X. Hence,

$$\lambda\in\bigcap_{\|X\|\leqslant 1}\sigma(M_X(A,B))=\bigwedge(A,B).$$

If $\lambda\in\sigma_e(A)$, then $\lambda\in\sigma_e(A\oplus B)$. For any contraction $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$,

$$M_X=\begin{pmatrix}A&0\\0&B\end{pmatrix}+\begin{pmatrix}0&A^{\frac12}XB^{\frac12}\\B^{\frac12}X^*A^{\frac12}&0\end{pmatrix}$$

is a finite rank perturbation of $A\oplus B$. Therefore $\lambda\in\sigma_e(M_X)\subseteq\sigma(M_X)$ and hence $\lambda\in\bigwedge(A,B)$.

To complete the proof of the case $0\notin\sigma(A)\cup\sigma(B)$, we have to show that $\lambda\notin\sigma_e(A)$ and $\dim\ker(A-\lambda)\leqslant m$ will imply $\lambda\notin\bigwedge(A,B)$. As $\bigwedge(A,B)\subset\sigma(A)\cup\sigma(B)$, we need only check this for $\lambda\in\sigma(A)\cup\sigma(B)$.

Case (i). $\lambda\in\sigma(A)\setminus(\sigma_e(A)\cup\sigma(B))$ and $\dim\ker(A-\lambda)\leqslant m$.

Obviously, λ is a isolated point of $\sigma(A)$. Let $A_1=A|\ker(A-\lambda)$ and $A_2=A|\ker(A-\lambda)^{\perp}$. Then, $\lambda\notin\sigma(A_2)$. However, A_1 and B are positive operators acting on finite dimensional spaces. By Theorem 2.2, $\lambda\notin\bigwedge(A_1,B)$ and hence $\lambda\notin\bigwedge(A,B)$ as $A_2-\lambda$ is invertible.

Case (ii). $\lambda\in\sigma(B)\setminus\sigma(A)$.

In this case, $M_X(A,B)-\lambda$ is invertible for some contraction X if and only if

$$\Phi_\lambda(X)=(B-\lambda)-B^{\frac12}X^*A(A-\lambda)^{-1}XB^{\frac12}$$

is invertible.

If there exists a finite dimensional subspace $\mathcal{M}\in\operatorname{Lat}A(A-\lambda)^{-1}$ so that $\dim\mathcal{M}\geqslant\dim\ker(B-\lambda)=n$, then there exists a $\mathcal{N}\in\operatorname{Lat}A(A-\lambda)^{-1}$ with $\dim\mathcal{N}=n$. Let $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$ be a partial isometry with initial space $\ker(B-\lambda)$ and final space $\mathcal{N}$. It is easily seen that $\Phi_\lambda(X)$ is invertible. Hence, $\lambda\notin\bigwedge(A,B)$.

If there is no such finite dimensional invariant subspace of $A(A-\lambda)^{-1}$, then $A(A-\lambda)^{-1}$ is not an algebraic operator. By Lemma 3.5, there is a vector $x_0\in\mathcal{H}$

such that $\dim \bigvee_{k=0}^{\infty} \{A^k(A-\lambda)^{-k}x_\circ\} = \infty$ and $\langle A^n(A-\lambda)^{-n}x_\circ, x_\circ\rangle \neq 0$. Let X be a partial isometry with $\ker(B-\lambda)$ and $\bigvee_{k=0}^{n-1} \{A^k(A-\lambda)^{-k}x_\circ\}$ the initial and final space, respectively. Because that $\mathcal{R}(A(A-\lambda)^{-1}XB^{1/2}) = \bigvee_{k=1}^{n} \{A^k(A-\lambda)^{-k}x_\circ\}$ and $\langle A^n(A-\lambda)^{-n}x_\circ, x_\circ\rangle \neq 0$, we have $\mathcal{R}(X^*A(A-\lambda)^{-1}XB^{1/2}) = \bigvee_{k=0}^{n-1} \{A^k(A-\lambda)^{-k}x_\circ\}$. It follows that $B^{1/2}X^*A(A-\lambda)^{-1}XB^{1/2}|\ker(B-\lambda)$ is invertible. So, writing $B_2 = B|\ker(B-\lambda)^\perp$,

$$\Phi_\lambda(X) = \begin{pmatrix} B^{\frac12}X^*A(A-\lambda)^{-1}XB^{\frac12} & 0 \\ 0 & B_2-\lambda \end{pmatrix}$$

is invertible, and hence, $\lambda \notin \bigwedge(A,B)$.

Case (iii). $\lambda \in (\sigma(A)\setminus\sigma_e(A)) \cap \sigma(B)$ and $\dim\ker(A-\lambda) \leqslant m$.

If $\dim\ker(B-\lambda) \leqslant \dim\ker(A-\lambda)$, let $A_1 = A|\ker(A-\lambda)$, then by Theorem 2.2, $\lambda \notin \bigwedge(A_1, B)$, which implies $\lambda \notin \bigwedge(A, B)$ since obviously the operator $(A-\lambda)|\ker(A-\lambda)^\perp$ is invertible.

If $\dim\ker(B-\lambda) > \dim\ker(A-\lambda)$, write $\ker(B-\lambda) = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\dim\mathcal{M}_1 = \dim\ker(A-\lambda)$. Let $A_1 = A|\ker(A-\lambda)$, $A_2 = A|\ker(A-\lambda)^\perp$, $B_1 = B|\mathcal{M}_1$ and $B_2 = B|\mathcal{M}_1^\perp$. It follows from Theorem 2.2 that $\lambda \notin \bigwedge(A_1, B_1)$ and from what proved in Case (ii) that $\lambda \notin \bigwedge(A_2, B_2)$. Applying Lemma 3.6, we get again that $\lambda \notin \bigwedge(A, B)$.

Combining what proved in Case (i), (ii) and (iii), we can claim that if $0 \notin \sigma(A) \cup \sigma(B)$, then

$$\bigwedge(A, B) = \{\lambda;\ \lambda \in \sigma_e(A) \text{ or } \dim\ker(A-\lambda) > m\}, \tag{10}$$

i.e., the theorem holds in this situation.

Now assume that $0 \in \sigma(A) \cup \sigma(B)$. It is trivial to see that $0 \in \bigwedge(A, B)$. Suppose $\lambda \neq 0$ and $\lambda \in \sigma(A) \cup \sigma(B)$. Take any real number ε so that $0 < \varepsilon < \lambda$. Let $A_1 = \int_0^\varepsilon t\,d\mathbf{E}_t$, $A_2 = \int_{\varepsilon+0}^{\|A\|} t\,d\mathbf{E}_t$, $B_1 = B|\ker B$ and $B_2 = B|(\ker B)^\perp$. It is clear that $\lambda \notin \bigwedge(A_1, B_1)$ and $\lambda \in \sigma(A_2) \cup \sigma(B_2)$. If $\lambda \notin \sigma_e(A)$ and $\dim\ker(A-\lambda) \leqslant \dim(\ker B)^\perp$, then the same is true for (A_2, B_2). Thus, by (10), $\lambda \notin \bigwedge(A_2, B_2)$, but this implies from Lemma 3.6 that $\lambda \notin \bigwedge(A, B)$. Clearly, $\lambda \in \sigma_e(A)$ will imply $\lambda \in \bigwedge(A, B)$. If $\lambda \in \sigma(A)\setminus\sigma_e(A)$ and if $\dim\ker(A-\lambda) > \dim(\ker B)^\perp$, with respect to the decomposition as above, writing $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$, we have

$$M_X(A,B) - \lambda = \begin{pmatrix} A_1 - \lambda & 0 & 0 & A_1^{\frac{1}{2}} X_{12} B_2^{\frac{1}{2}} \\ 0 & A_2 - \lambda & 0 & A_2^{\frac{1}{2}} X_{22} B_2^{\frac{1}{2}} \\ 0 & 0 & -\lambda & 0 \\ B_2^{\frac{1}{2}} X_{12}^* A_1^{\frac{1}{2}} & B_2^{\frac{1}{2}} X_{22}^* A_2^{\frac{1}{2}} & 0 & B_2 - \lambda \end{pmatrix}.$$

Since $\dim \mathcal{R}(A_2^{1/2} X_{22} B_2^{1/2}) < \dim \ker(A_2 - \lambda)$, it is clear that $M_X(A,B) - \lambda$ can not be surjective and therefore, can not be invertible for any X, but this will mean that $\lambda \in \bigwedge(A,B)$. ∎

4. THE CASE OF INFINITE DIMENSIONS

In this section we assume that both $\mathcal{H}$ and $\mathcal{K}$ are infinite dimensional. Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be positive operators. Similarly to the arguments in Section 2 and Section 3, one can show that

$$[0, \min\{a,b\}] \cup [\max\{a,b\}, a+b) \subset \bigvee(A,B) \tag{11}$$

whenever a and b are satisfying one of the following conditions:

(1) $a \in \sigma_p(A)$ and $b \in \sigma_p(B)$;
(2) $a \in \sigma(A) \setminus \sigma_p(A)$ and $b \in \sigma_p(B)$;
(3) $a \in \sigma_p(A)$ and $b \in \sigma(B) \setminus \sigma_p(B)$.

Now consider the case that $a \in \sigma(A) \setminus \sigma_p(A)$ and $b \in \sigma(B) \setminus \sigma_p(B)$. Replacing $B_1 = B|\ker(B-b)$ by $B_1 = \int_0^{b+\epsilon} s\, d\mathbf{F}_s$ and $B_2 = \int_{b-\epsilon}^{\|B\|} s\, d\mathbf{F}_s$ in the proof of (8) and (9) in Section 3, respectively, a similar demonstration shows that (11) still holds true. Thus, we have proved that

$$[0, \min\{\|A\|, \|B\|\}] \cup [\max\{\min\sigma(A), \min\sigma(B)\}, \|A\| + \|B\|) \subset \bigvee(A,B).$$

Now, the following theorem is an immediate consequence of Lemma 3.1 and Lemma 3.2.

THEOREM 4.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional. Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be positive. If $0 \in \sigma_p(\|A\| - A) \cap \sigma_p(\|B\| - B)$ or $0 \in \sigma_e(\|A\| - A) \cap \sigma_e(\|B\| - B)$, then*

$$\bigvee(A,B) = [0, \|A\| + \|B\|] \setminus (\max\sigma(T), \min\sigma(S));$$

Otherwise,

$$\bigvee(A,B) = [0, \|A\| + \|B\|) \setminus (\max\sigma(T), \min\sigma(S)),$$

where $T = A$ or B, while $S = B$ or A.

As to the intersection of the spectra of all positive completions $M_X(A,B)$, we have

THEOREM 4.2. *Suppose that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are positive operators, and $\dim\mathcal{H} = \dim\mathcal{K} = \infty$. If $0 \notin \sigma(A) \cup \sigma(B)$, then*

$$\bigwedge(A,B) = \emptyset;$$

Otherwise,

$$\bigwedge(A,B) = \{0\} \cup \{\lambda;\ \lambda \in \sigma_e(A) \text{ and } B \text{ is compact}\} \\ \cup \{\lambda;\ \lambda \in \sigma_e(B) \text{ and } A \text{ is compact}\}.$$

Proof. We first assume that $0 \notin \sigma(A) \cup \sigma(B)$.

Claim 1. If $\lambda \in \sigma(B) \setminus \sigma(A)$ or $\lambda \in \sigma(A) \setminus \sigma(B)$, then $\lambda \notin \bigwedge(A,B)$.

Suppose that $\lambda \in \sigma(B) \setminus \sigma(A)$. It is similar to the argument in the proof of Theorem 3.7 that for any contraction $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $\lambda \in \rho(M_X(A,B))$ if and only if the selfadjoint operator

$$\Phi_\lambda(X) = (B-\lambda) - B^{\frac{1}{2}}X^*A(A-\lambda)^{-1}XB^{\frac{1}{2}} \in \mathcal{B}(\mathcal{K})$$

is invertible, and, $\lambda \in \sigma(B) \setminus \sigma_e(B)$ implies $\lambda \notin \bigwedge(A,B)$. Hence, to complete the proof of Claim 1, we may assume that $\lambda \in \sigma_e(B)$. Fix a number $d > 0$ so that $\|A(A-\lambda)^{-1}x\| \geqslant d\|x\|$ for all x in $\mathcal{H}$. Take $\varepsilon > 0$ small enough so that $0 < \min\{\lambda, (\lambda-\varepsilon)^2 d\}$. Let $\mathcal{K}_\varepsilon = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} d\mathbf{F}_s\mathcal{K}$. It is clear that $\dim\mathcal{K}_\varepsilon = \infty$. Let $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be any partial isometry with initial space $\mathcal{K}_\varepsilon$ and final space $\mathcal{H}$. With regard to the decomposition $\mathcal{K} = \mathcal{K}_\varepsilon \oplus \mathcal{K}_\varepsilon^\perp$, and for each $y = y_1 \oplus y_2 \in \mathcal{K}_\varepsilon \oplus \mathcal{K}_\varepsilon^\perp$,

$$\begin{aligned}\|\Phi_\lambda(X)y\|^2 &= \|(B-\lambda)y_1 - B^{\frac{1}{2}}X^*A(A-\lambda)^{-1}XB^{\frac{1}{2}}y_1\|^2 + \|(B-\lambda)y_2\|^2 \\ &\geqslant (\|B^{\frac{1}{2}}X^*A(A-\lambda)^{-1}XB^{\frac{1}{2}}y_1\| - \|(B-\lambda)y_1\|)^2 + \varepsilon^2\|y_2\|^2 \\ &\geqslant ((\lambda-\varepsilon)^2 d - \varepsilon)^2\|y_1\|^2 + \varepsilon^2\|y_2\|^2 \geqslant \delta^2\|y\|^2,\end{aligned}$$

where $\delta = \min\{\varepsilon,\ (\lambda-\varepsilon)^2 d - \varepsilon\} > 0$, which means that $\Phi_\lambda(X)$ is invertible on $\mathcal{K}$. Hence, $\lambda \notin \bigwedge(A,B)$.

By the symmetry of A and B, we also have that $\lambda \in \sigma(A) \setminus \sigma(B)$ will imply $\lambda \notin \bigwedge(A,B)$. So, Claim 1 is true.

Claim 2. $\lambda \in \sigma(A) \cap \sigma(B)$ implies $\lambda \notin \bigwedge(A, B)$.

We shall prove this claim by considering several cases.

Case (i). $\lambda \in (\sigma(A) \setminus \sigma_e(A)) \cap (\sigma(B) \setminus \sigma_e(B))$.

In this case, λ is an isolated point of $\sigma(A)$ and $\sigma(B)$ and both $\ker(A-\lambda)$ and $\ker(B-\lambda)$ are finite dimensional. Let $A_1 = A|\ker(A-\lambda)$, $A_2 = A|\ker(A-\lambda)^\perp$, $B_1 = B|\ker(B-\lambda)$ and $B_2 = B|\ker(B-\lambda)^\perp$. By virtue of Theorem 3.7, we must have $\lambda \notin \bigwedge(A_1, B_2) \cup \bigwedge(A_2, B_1)$, and hence $\lambda \notin \bigwedge(A, B)$, by Lemma 3.6.

Case (ii). $\lambda \in (\sigma(A) \setminus \sigma_e(A)) \cap \sigma_e(B)$ or $\lambda \in \sigma_e(A) \cap (\sigma(B) \setminus \sigma_e(B))$.

By the symmetry of A and B, it suffices to deal with the case that $\lambda \in (\sigma(A) \setminus \sigma_e(A)) \cap \sigma_e(B)$. Let $\mathcal{K}_\varepsilon = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} d\mathbf{F}_s\mathcal{K}$. Note that $\dim(\ker(A-\lambda)^\perp) = \dim \mathcal{K}_\varepsilon = \infty$. If there is an $\varepsilon > 0$ such that $\dim \mathcal{K}_\varepsilon^\perp \geqslant \dim\ker(A-\lambda)$, then by Theorem 2.2, Theorem 3.7 and Claim 1, we have $\lambda \notin \bigwedge(A|\ker(A-\lambda), B|\mathcal{K}_\varepsilon^\perp) \cup \bigwedge(A|\ker(A-\lambda)^\perp, B|\mathcal{K}_\varepsilon)$. It turns out that $\lambda \notin \bigwedge(A, B)$.

If for every $\varepsilon > 0$, $\dim \mathcal{K}_\varepsilon^\perp < \dim\ker(A-\lambda)$, then we must have $\lambda \in \sigma_p(B)$, and $\dim\ker(B-\lambda)^\perp < \dim\ker(A-\lambda) = m < \infty$. Take a subspace $\mathcal{M} \subset \ker(B-\lambda)$ such that $\dim \mathcal{M} = m - \dim\ker(B-\lambda)^\perp$. Obviously, $\mathcal{M} \oplus \ker(B-\lambda) \in \operatorname{Lat} B$. Let $A_1 = A|\ker(A-\lambda)$, $A_2 = A|\ker(A-\lambda)^\perp$, $B_1 = B|\mathcal{M} \oplus \ker(B-\lambda)^\perp$ and $B_2 = B|\ker(B-\lambda) \ominus \mathcal{M}$. Again, we have $\lambda \notin \bigwedge(A_1, B_1) \cup \bigwedge(A_2, B_2)$ by use of Theorem 2.2 and Claim 1. This leads to the desired assertion that $\lambda \notin \bigwedge(A, B)$.

Case (iii). $\lambda \in \sigma_e(A) \cap \sigma_e(B)$.

For a fixed $\varepsilon > 0$ satisfying $\delta = (\lambda-\varepsilon)^2 - \varepsilon^2 > 0$, let $\mathcal{H}_\varepsilon = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} d\mathbf{E}_t\mathcal{H}$ and $\mathcal{K}_\varepsilon = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} d\mathbf{F}_s\mathcal{K}$. Let $A_1 = A|\mathcal{H}_\varepsilon$, $B_1 = B|\mathcal{K}_\varepsilon$. If $X \in \mathcal{B}(\mathcal{K}_\varepsilon, \mathcal{H}_\varepsilon)$ is a unitary operator, then

$$\begin{aligned} \|B_1X^*A_1x - (B_1-\lambda)X^*(A_1-\lambda)x\| &\geqslant \|B_1X^*A_1x\| - \|(B_1-\lambda)X^*(A_1-\lambda)x\| \\ &\geqslant (\lambda-\varepsilon)^2\|x\| - \varepsilon^2\|x\| = \delta\|x\| \end{aligned}$$

for all x in $\mathcal{H}_\varepsilon$. Similarly,

$$\|A_1XB_1y - (A_1-\lambda)X(B_1-\lambda)y\| \geqslant \delta\|y\|$$

for all y in $\mathcal{K}_\varepsilon$. So, $B_1X^*A_1 - (B_1-\lambda)X^*(A_1-\lambda) \in \mathcal{B}(\mathcal{H}_\varepsilon, \mathcal{K}_\varepsilon)$ is invertible. This implies that ${B_1}^{1/2}X^*{A_1}^{1/2} - (B_1-\lambda){B_1}^{-1/2}X^*{A_1}^{-1/2}(A_1-\lambda)$ is invertible and from it, one can easily check that $M_X(A_1, B_1) - \lambda$ is invertible, i.e., $\lambda \notin \bigwedge(A_1, B_1)$. It is trivial to see that $\lambda \notin \bigwedge(A|\mathcal{H}_\varepsilon^\perp, B|\mathcal{K}_\varepsilon^\perp)$. So we still have that $\lambda \notin \bigwedge(A, B)$, and Claim 2 holds.

Claim 1 and Claim 2, taken together, imply that

$$\bigwedge(A,B)=\emptyset$$

whenever both A and B are invertible.

Now assume that $0\in\sigma(A)\cup\sigma(B)$. It is clear that in this situation we always have $0\in\bigwedge(A,B)$. Let $\lambda\in\sigma(A)\cup\sigma(B)$ and $\lambda\neq 0$. Given ε so that $0<\varepsilon<\lambda$, denote by $\mathcal{H}_1$ and $\mathcal{K}_1$ the subspaces $\int_0^{\varepsilon} \mathrm{d}\mathbf{E}_t\mathcal{H}$ and $\int_0^{\varepsilon} \mathrm{d}\mathbf{F}_s\mathcal{K}$, respectively. Let $A_1=A|\mathcal{H}_1$, $A_2=A|\mathcal{H}_1^{\perp}$, $B_1=B|\mathcal{K}_1$ and $B_2=B|\mathcal{K}_1^{\perp}$.

If neither of A and B is compact, then we may choose ε so that $\dim\mathcal{H}_1^{\perp}=\dim\mathcal{K}_1^{\perp}=\infty$. It is clear that $\lambda\notin\bigwedge(A_1,B_1)$ as $\lambda\notin\sigma(A_1)\cup\sigma(B_1)$, and $\lambda\notin\bigwedge(A_2,B_2)$ by the first half part of the theorem that has just been proved. So, by virtue of Lemma 3.6, we have $\lambda\notin\bigwedge(A,B)$.

If both A and B are compact, then $\dim\mathcal{H}_1=\dim\mathcal{K}_1=\infty$ while $\mathcal{H}_1^{\perp}$ and $\mathcal{K}_1^{\perp}$ are finite dimensional. Since $\lambda\notin\sigma(A_1)$ and $\lambda\notin\sigma(B_1)$, it follows from Theorem 3.7 that $\lambda\notin\bigwedge(A_1,B_2)\cup\bigwedge(A_2,B_1)$, and therefore, $\lambda\notin\bigwedge(A,B)$.

Suppose that A is not compact but B is. If $\lambda\notin\sigma_e(A)$, then $\lambda\notin\sigma(A)$ or λ is a isolated point of $\sigma(A)$ with $\dim\ker(A-\lambda)<\infty$. Again, by Theorem 3.7, $\lambda\notin\bigwedge(A|\ker(A-\lambda),B_1)\cup\bigwedge(A|\ker(A-\lambda)^{\perp},B_2)$. Hence, $\lambda\notin\bigwedge(A,B)$. However, if $\lambda\in\sigma_e(A)$, then $\lambda\in\sigma_e(A\oplus B)$. Since $A^{1/2}XB^{1/2}$ is compact for each X and

$$M_X(A,B)=\begin{pmatrix}A&0\\0&B\end{pmatrix}+\begin{pmatrix}0&A^{1/2}XB^{1/2}\\B^{1/2}X^*A^{1/2}&0\end{pmatrix}$$

is a compact perturbation of $A\oplus B$, so $\lambda\in\sigma_e(M_X)\subseteq\sigma(M_X)$. Hence, we have $\lambda\in\bigwedge(A,B)$.

The case that A is compact while B is not can be treated similarly.

Therefore, we have shown that if $0\in\sigma(A)\cup\sigma(B)$, then $\bigwedge(A,B)$ consists of 0 and those λ that $\lambda\in\sigma_e(A)$ and B is compact or $\lambda\in\sigma_e(B)$ and A is compact. The proof of the theorem is finished. ∎

COROLLARY 4.3. *If neither of the positive operators A and B is compact, then $\bigwedge(A,B)=\{0\}$ if $0\in\sigma(A)\cup\sigma(B)$ and $\bigwedge(A,B)=\emptyset$ if $0\notin\sigma(A)\cup\sigma(B)$.*

COROLLARY 4.4. *If A and B are compact positive operators acting on infinite dimensional Hilbert spaces then $\bigwedge(A,B)=\{0\}$.*

5. THE SPECTRA OF THE HERMITIAN COMPLETIONS

In this final section, we deal with the analogous questions for the Hermitian completions $N_Y(A,B) = \begin{pmatrix} A & Y \\ Y^* & B \end{pmatrix}$ for a given pair (A,B) of selfadjoint operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, where Y runs over all operators in $\mathcal{B}(\mathcal{K},\mathcal{H})$. We denote by $\bigvee_a(A,B)$ and $\bigwedge_a(A,B)$ the union and the intersection of the spectra of all Hermitian completions $N_Y(A,B)$ for (A,B), i.e.,

$$\bigvee_a(A,B) = \cup\{\sigma(N_Y(A,B));\ Y \in \mathcal{B}(\mathcal{K},\mathcal{H})\}$$

and

$$\bigwedge_a(A,B) = \cap\{\sigma(N_Y(A,B));\ Y \in \mathcal{B}(\mathcal{K},\mathcal{H})\}.$$

Since we have more freedom in choosing Y, it is not difficult to find out $\bigvee_a(A,B)$ and $\bigwedge_a(A,B)$. We shall just list our results and leave the proofs to the reader.

THEOREM 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be two selfadjoint operators. Then*

$$\bigvee_a(A,B) = \mathbf{R} \setminus (\max\sigma(T),\ \min\sigma(S)),$$

where $T = A$ or B and $S = B$ or A.

THEOREM 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be selfadjoint operators.*

(1) *If $\dim\mathcal{H} = n < \infty$ and $\dim\mathcal{K} = m < \infty$, then*

$$\bigwedge_a(A,B) = \{\lambda;\ \dim\ker(A-\lambda) > m \text{ or } \dim\ker(B-\lambda) > n\};$$

(2) *If $\dim\mathcal{H} = \infty$ and $\dim\mathcal{K} = m < \infty$, then*

$$\bigwedge_a(A,B) = \{\lambda;\ \dim\ker(A-\lambda) > m \text{ or } \lambda \in \sigma_e(A)\};$$

(3) *If $\dim\mathcal{H} = \dim\mathcal{K} = \infty$, then*

$$\bigwedge_a(A,B) = \emptyset.$$

REMARK. The separability of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is only necessary for Theorem 4.2 and Theorem 5.2 (3).

Acknowledgements. Part of this work was done while the author visited the Institute of Mathematics, Academia Sinica. The author wishes to thank Professor Bingren Li for his hospitality and Professor Hong-Ke Du for his helpful discussions. He also gives thanks to the referee who kindly offered many helpful suggestions to improve the original manuscript, especially in order to simplify the proofs of Theorem 2.1 and 2.2.

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Received February 22, 1994; revised August 8, 1994 and October 24, 1994.