J. OPERATOR THEORY
33(1995), 317–325

COMPACTNESS OF DOMINATED OPERATORS AND THE (CRP) IN SPACES OF COMPACT OPERATORS

GIOVANNI EMMANUELE

Communicated by Şerban Strătilă

ABSTRACT. We present results showing that sometimes the (CRP) lifts from two Banach spaces E^*, F to the Banach space $K(E, F)$. They essentially depend on the compactness of certain dominated operators.

KEYWORDS: *Spaces of compact operators, compact range property.*

AMS SUBJECT CLASSIFICATION: Primary 46A32; Secondary 46B20.

Let E, F be two Banach spaces. Several papers have been devoted to the question of when an isomorphic property lifts from the spaces E^*, F to the space of compact operators $K(E, F)$ (see [1], [5], [8], [12], [20] and References); following the same line of research we devote this paper to the question of when the so called *Compact Range Property*, in symbols (CRP), is enjoyed by $K(E, F)$. From our results it follows that this question is heavily depending on the question of when each *dominated operator* from the space $C(S, E)$, S a Hausdorff compact space, to F is compact. In the case $E = \mathbb{R}$ the results in [11] show that these two questions actually are equivalent, but in the infinite dimensional case we do not know if the same happens or not, except in the case of F a dual Banach space in which we still have equivalence.

In order to start we need three definitions

DEFINITION 1. ([17]) A Banach space E is said *to possess the (CRP)* if any E-valued countably additive measure μ, defined on a σ-field of subsets of an arbitrary set S, with finite variation $|\mu|(S)$, has relatively compact range.

DEFINITION 2. ([7]) An operator $T : C(S,E) \to F$, S a Hausdorff compact space, is called a *dominated operator* if there exists a countably additive regular positive Borel measure μ such that

$$\|T(f)\|_F \leq \int_S \|f(t)\|_E d\mu \qquad f \in C(S,E).$$

DEFINITION 3. Let E be a Banach space. A bounded subset X of E is a *Dunford-Pettis set* if $\lim_n \sup_X |x_n^*(x)| = 0$ for every w-null sequence (x_n^*) in E^*.

The first result, which is also the main result of the paper, clarifies the role of dominated operators in the investigation of the (CRP) in $K(E,F)$. In it (and in the sequel too) we shall make use of the following well known equivalence: *E does not contain copies of l_1 if and only if E^* has the* (CRP) (see, for instance, [11]).

THEOREM 1. *Let E^* have the* (CRP). *If, for any Hausdorff compact space S, each dominated operator $T : C(S,E) \to F$ is compact, we get*

(i) $L(E,F) = K(E,F)$

(ii) *$K(E,F)$ has the (CRP).*

Proof. (i) If H is not a compact operator from E into F, then for an arbitrary S and an arbitrary μ (as in the definition above) we can define a dominated operator $T : C(S,E) \to F$ by putting

$$T(f) = \int_S Hf(t) d\mu \qquad f \in C(S,E).$$

Using constant functions we see very easily that T is not compact.

(ii) We still argue by contradiction. Let S be a set, Σ a σ-algebra of subsets of S and μ a countably additive measure with bounded variation from Σ into the space $K(E,F)$ without relatively compact range. Using the Stone representation Theorem ([7]) we can assume that S is a Hausdorff compact space and μ is a regular Borel measure on Σ. Since μ does not have relatively compact range there is a sequence (A_n) in $\mathrm{Bo}(S)$ such that $(\mu(A_n))$ converges weakly to some $H \in K(E,F)$ (because the range of any vector measure is relatively weakly compact, [6]), but not strongly. So there is a sequence $(x_n) \subset B_E$ such that $(\mu(A_n)(x_n) - H(x_n))$ does not go to zero strongly in F. Since E does not contain l_1 we can assume that (x_n) is a weak Cauchy sequence in B_E, otherwise we pass to a subsequence (use the famous Rosenthal Theorem, [19]). Let $x^{**} \in E^{**}$ be the w*-limit of (x_n) in E^{**}. We want to show that $(\mu(A_n)(x_n) - H(x_n))$ goes to θ weakly in F. Take $y^* \in F^*$. We consider the following equality

$$(1) \quad \begin{aligned} \langle \mu(A_n)(x_n) - H(x_n), y^* \rangle &= \langle [\mu(A_n)]^*(y^*), x_n - x^{**} \rangle - \langle H^*(y^*), x_n - x^{**} \rangle \\ &+ \langle \mu(A_n) - H, x^{**} \otimes y^* \rangle \qquad n \in \mathbb{N}. \end{aligned}$$

Since $\mu(A_n) \xrightarrow{w} H$ and $x^{**} \otimes y^* \in (K(E,F))^*$ the last summand of (1) goes to zero as well as the second one does because $x_n \xrightarrow{w^*} x^{**}$ in E^{**} and $H^*(y^*) \in E^*$. Now, we recall that the range of a vector measure μ of bounded variation is a Dunford-Pettis set ([12]). Using the operator $P : K(E,F) \to E^*$ defined by $P(H) = H^*(y^*)$ we see very easily that $([\mu(A_n)]^*(y^*))$ is a Dunford-Pettis set in E^*. From [11] and [12] it follows that E^* has the (CRP) if and only if any Dunford-Pettis set in E^* is relatively compact; since $x_n \xrightarrow{w^*} x^{**}$ we see very easily that also the first summand of (1) goes to zero. This shows that $(\mu(A_n)(x_n) - H(x_n))$ goes to θ weakly in F. Since H is compact, we can assume that there is $y \in F$ such that $\|H(x_n) - y\| \to 0$. Hence $\mu(A_n)(x_n) \xrightarrow{w} y$, but not strongly. Now, define a dominated operator $T : C(S,E) \to F$ by

$$T(f) = \int_S f(t)\, \mathrm{d}\mu \qquad f \in C(S,E).$$

T can be extended to all of $L^1(|\mu|, E)$ because it is dominated and μ is regular. If we consider the functions $g_n = \chi_{A_n} x_n$ in $L^1(|\mu|, E)$, using Lusin Theorem and Borsuk–Dugundji Extension Theorem we can approximate each of them by elements in the unit ball of $C(S,E)$ in the L^1-norm. But T must be compact and so the above remarks imply that $(T(g_n)) = (\mu(A_n)(x_n))$ must be relatively compact in F. This contradicts what we proved. We are done. ∎

QUESTION 1. Is the converse of Theorem 1 true?

Theorem 1 can be used to get the following sufficient conditions for $K(E,F)$ to possess the (CRP). For the first of these results we need one more definition

DEFINITION 4. ([9]) Let E be a Banach space. A bounded subset X of E^* is called a *(L) set* if $\lim_n \sup_X |x_n(x^*)| = 0$ for every w-null sequence (x_n) in E.

THEOREM 2. *Let F be a dual Banach space. If E^*, F have the* (CRP) *and $L(E,F) = K(E,F)$, then $K(E,F)$ has the* (CRP).

Proof. We shall prove that under our assumptions any dominated operator T from $C(S,E)$ into F is compact. We need a representation theorem for dominated operators to be found in [7]. According to it there is $G : S \to L(E,F) = K(E,F)$ such that

(i) $|G(t)| = 1$ almost everywhere on S

(ii) for each $y \in Z$, $Z^* = F$, and $f \in C(S,E)$ the function $\langle G(\cdot)f(\cdot), y\rangle$ is μ-integrable and furthermore

$$\langle T(f), y\rangle = \int_S \langle G(t)f(t), y\rangle \, \mathrm{d}\mu \qquad \forall f \in C(S,E),\ y \in Z \tag{2}$$

where μ is the least positive regular Borel measure dominating T.

We shall prove that any sequence in $T(B_{C(S,E)})$ is an (L)-set. So let us consider $(f_n) \subset B_{C(S,E)}$ and $(y_n) \subset B_Z$ such that $y_n \xrightarrow{\text{w}} \theta$. It is clear that $G^*(t)y_n \xrightarrow{\text{s}} \theta$ on S and so we have $\lim_n \langle G(t)f_n(t), y_n\rangle = 0$. Since $|\langle G(t)f_n(t), y_n\rangle| \leqslant 1$ a. e. on S, for all $n \in \mathbb{N}$, we get

$$\lim_n \int_S \langle G(t)f_n(t), y_n\rangle \, \mathrm{d}\mu = 0. \tag{3}$$

Both (2) and (3) imply that $T(B_{C(S,E)})$ is an (L)-set in F, i. e. a relatively compact subset of F, by virtue of a result in [9]. We are done, thanks to Theorem 1. ∎

REMARK. Theorem 2 was also obtained in [12] with a totally different proof.

In the case considered above of F a dual Banach space we are also able to answer positively Question 1 thanks to the following result in which we use an isomorphic property introduced in our paper [12].

DEFINITION 5. ([12]) A Banach space E is said to *have the (DPrcP)* if any Dunford-Pettis set in E is relatively compact.

THEOREM 3. *Let F be a dual Banach space such that $L(E,F) = K(E,F)$ and $K(E,F)$ has the* (CRP). *Then any dominated operator T from $C(S,E)$ into F is compact.*

Proof. From [12] it follows that the (CRP) and the (DPrcP) are equivalent in dual spaces; so, from our hypotheses, it follows that F has the (DPrcP). To reach our goal it will be enough to prove that a dominated operator $T : C(S,E) \longrightarrow F$ maps $B_{C(S,E)}$ into a Dunford-Pettis subset. This can be done as in Theorem 2. We are done. ∎

Since in the case of F not a dual space the function G takes its values in $L(E,F^{**})$, we have the following result in which we still use Definition 5.

THEOREM 4. *Let E^* have the* (CRP) *and F the* (DPrcP). *If $L(E,F) = K(E,F)$ then $K(E,F)$ has the* (CRP).

Proof. The proof goes as in Theorem 2 with the only change that $T(B_{C(S,E)})$ is now a Dunford-Pettis set. ∎

In the next result we use the definition of Gelfand-Phillips property (see, for instance, [8]).

DEFINITION 6. A Banach space E is said to have the *Gelfand-Phillips property*, in symbols (GPP), if any limited set in E is relatively compact. A bounded subset X of E is a *limited set* if $\lim_n \sup_X |x_n^*(x)| = 0$ for every w*-null sequence (x_n^*) in E^*.

We shall also use the well known definition of Radon-Nikodym property (in symbols (RNP)) for which we refer to [6], where one can also find the well known fact that the (RNP) implies the (CRP).

THEOREM 5. *Let E be a separable complemented Banach space with E^* enjoying the* (CRP) *and F be a Banach space possessing the* (RNP). *If $L(E,F) = K(E,F)$, then $K(E,F)$ has the* (CRP).

Proof. Let T be a dominated operator from $C(S,E)$ into F. If (f_n) is a sequence in $B_{C(S,E)}$ we consider $E_0 = \overline{\text{span}}\{f_n(t) : n \in \mathbb{N},\ t \in S\}$; note that it is a separable Banach space and we can assume that E_0 is (contained in) a complemented and separable subspace of E; hence we have the equality $L(E_0, F) = K(E_0, F)$. Furthermore, following [2], Theorem 8, we can suppose that S is a compact metric space. Hence, $C(S, E_0)$ is separable, which implies that T actually takes its values in a separable subspace of F; our assumptions allow us to suppose that F is separable (and hence that it has the (GPP),[8]) and it enjoys the (RNP). If G is the representing function of T and ρ is a lifting of $L_\infty(\mu)$, μ the least countably additive regular Borel measure dominating T, we can choose G so that

$$\rho(\langle G(\cdot)x, y^*\rangle) = \langle G(\cdot)x, y^*\rangle \qquad \forall x \in E,\ y^* \in F^*.$$

Now, let us consider a countable subset (z_n) dense in E_0; if we denote by $m : \mathrm{Bo}(S) \longrightarrow K(E_0, F)$ the representing measure of T, the measures $m(\cdot)z_n$ take their values in F, that has the (RNP). Theorem III. 2.7 in [6] gives that, for each $\varepsilon > 0$ there exists a set S_ε, with $\mu(S \setminus S_\varepsilon) < \varepsilon$ and such that

$$C_\varepsilon = \overline{\mathrm{co}}\left\{\frac{m(H)z_n}{\mu(H)} : H \in \mathrm{Bo}(S), H \subset S_\varepsilon, \mu(H) > 0\right\}$$

is a compact subset of F, for each $n \in \mathbf{N}$. Repeating the proof of part (4,b) in [7], Chapter II, Section 13.4, Theorem 4, we get the existence of a null subset S_0 of S for which $G(t)z \in F$ for all $t \in S \setminus S_0$ and all $z \in E_0$ (use also the separability of E_0). Now, considering a w*-null sequence (y_n^*) in F^*, as in Theorem 2 we show that $T(B_{C(S,E_0)})$ is a limited subset of F and so relatively compact. We are done. ∎

Theorem 5 generates the following natural double question:

QUESTION 2. Is it possible to eliminate the assumption "E is separably complemented"? Is it possible to improve the assumption on F just assuming that it has the (WRNP) (see [17]) ?

We can partially answer to the first of these questions using different assumptions on E or F.

THEOREM 6. *Let E be a Banach space with E^* enjoying the* (CRP) *and F be a Banach space possessing the* (RNP). *Let us also suppose that at least one of the following conditions is verified:*

(i) *E has property* (u) *due to Pelczynski (see* [18]*),*

(ii) *F is weakly sequentially complete,*

(iii) *E^* has Schur property and in F each Dunford-Pettis set is relatively weakly compact.*

If $L(E,F) = K(E,F)$, *then* $K(E,F)$ *has the* (CRP).

Proof. As in Theorem 5 we may reduce ourselves to the case of a separable subspace E_0 of E, of a compact metric space S and a separable F. Proposition 3.4 in [15] allows us to suppose that there is a isometric embedding J of E_0^* into E^*. Looking at the proof of Theorem 5 we easily realize that it still works once we have shown that $K(E_0, F) = L(E_0, F)$. So let us consider an element $T \in L(E_0, F)$. Under any of the assumptions (i) and (ii) it is easily seen that T is weakly compact. Indeed, if i) is true we get that E_0 has property (V) of Pelczynski ([18]) as any space not containing copies of l_1 with property (u) does; now we are done since F does not contain copies of c_0. Whereas if (ii) is true we are still done since E_0 does not contain copies of l_1 and F is weakly sequentially complete. Hence, the operator T^* is weakly compact and so a weak*-weak continuous operator from F^* into E_0^*, that composed with J still gives a weak*-weak continuous operator from F^* into E^*, i. e. a conjugate operator. But each such an operator must be compact because of our assumption $L(E,F) = K(E,F)$; since J is an isometry we can conclude that T^*, and so T, must be compact. In the case (iii) is true, since J is an isometry from E_0^* into E^* we argue that even E_0^* has Schur property. It is thus very easy to see that $T(B_{E_0})$ is a Dunford-Pettis subset of F and hence a relatively weakly compact set. Hence T is weakly compact and we are done thanks to the previous reasonings. ∎

In all the results above we assumed the most general hypothesis allowed on E, i. e. the (CRP) for E^*, but something less general about F; in the next result we try to reverse this situation, which means that we shall assume that F has the (CRP); but this will force us to choose a very particular E. Nevertheless the next Theorem 7 will have an interesting consequence that will be stated at the end of the paper as Corollary 8.

THEOREM 7. *Let F have the* (CRP). *Then* $K(c_0, F)$ *has the* (CRP).

Proof. From Theorem 1 it is enough to show that any dominated operator T from $C(S, c_0)$ to F is compact. Take $f \in C(S, c_0)$. For each $t \in S$ there is $f_n(t) \in \mathbb{R}$ such that $f(t) = \sum f_n(t) e_n$, (e_n) the unit vector basis of c_0. It is clear that each f_n is continuous on S and that (f_n) is equicontinuous and equibounded in $C(S)$; since (f_n) goes to zero pointwise, it converges in the sup-norm to θ. So any $f \in C(S, c_0)$ can be identified with an element of $c_0(C(S))$ and, actually, this identification is an isometry onto. Now, observe that c_0 is not contained in F and so any operator $T : C(S, c_0) \to F$ is weakly compact since $C(S, c_0)$ has property

(V) of Pelczynski (see [18]). Defining $T_n : C(S) \to F$ by putting $T_n(f) = T((f_h))$, where $f_h = f$ if $h = n$, 0 otherwise, we get that T_n is weakly compact and

$$\lim_m \left\| \sum_{i=1}^{m} T_i \circ P_i - T \right\| = 0 \tag{4}$$

thanks to a result in [3] (in (4) P_i denotes the i-th projection of $C(S, c_0) = c_0(C(S))$ onto its i-th factor). Furthermore, each T_n is clearly dominated and hence compact, since F has the (CRP) ([11]). It follows from (4) above that T is compact. We are done. ∎

It is clear that in Theorem 7 we still have $L(c_0, F) = K(c_0, F)$.

In Theorems 2–7 we used the assumption $L(E, F) = K(E, F)$ in order to guarantee that any dominated operator from $C(S, E)$ into F is compact and, consequently, that $K(E, F)$ has the (CRP), thanks to Theorem 1. This assumption is, in several many cases, necessary for $K(E, F)$ to possess the (CRP) as may be deduced from a number of results from the papers [10], [14], [16], because if it does not hold then c_0 embeds into $K(E, F)$ that it is not allowed to possess the (CRP). However we must underline that, as remarked in the paper [13], at least in one case $K(E, F)$ has the (CRP), actually the (RNP), even if $L(E, F) \neq K(E, F)$; it is enough to take $E = F =$ a $\mathcal{L}_\infty$-space with the (RNP) and with dual isomorphic to l_1 constructed in [4] by Bourgain and Delbaen. We also remark that the same Bourgain-Delbaen space furnishes an example of pair E, F such that $K(E, F)$ has the (CRP), $K(E, F) \neq L(E, F)$ and there exists a not compact dominated operator from $C(S, E)$ to F, so that it cannot be used to answer Question 1. Following the lines for the proof of this fact, contained in [13], we can get the last result of the paper that also uses Theorem 7 as already announced.

COROLLARY 8. *Let E be a Banach space such that E^* is isomorphic to l_1. If F has the* (CRP), *then $K(E, F)$ has the same property.*

Proof. $K(E, F)$ is isomorphic to $E^* \otimes_\varepsilon F$, that in turn is isomorphic to $l_1 \otimes_\varepsilon F$, that in turn is isomorphic to $K(c_0, F)$. It is now enough to apply Theorem 7. ∎

Once more, we remark that the assumptions of Corollary 8 do not necessarily imply that $L(E, F) = K(E, F)$.

REFERENCES

1. K.T. ANDREWS, The (RNP) for spaces of operators, *J. London Math. Soc. (2)* **28**(1983), 113–122.

2. J. BATT, E.J. BERG, Linear bounded transformations on the space of continuous functions, *J. Funct. Anal.* **4**(1969), 215–239.
3. F. BOMBAL, Operators on spaces of vector sequences, in *Geometric Aspects of Banach Spaces*, LN **140**, London Math. Soc., Cambridge University Press 1989.
4. J. BOURGAIN, F. DELBAEN, A class of special $\mathcal{L}_\infty$-spaces, *Acta Math.* **145**(1980), 155–176.
5. J. DIESTEL, T.J. MORRISON, The Radon-Nikodym property for the space of operators, *Math. Nachr.* **92**(1979), 7–12.
6. J. DIESTEL, J.J. UHL JR., *Vector Measures*, Math. Surveys Monographs, vol. **15**, Amer. Math. Soc. 1977.
7. N. DINCULEANU, *Vector Measures*, Pergamon Press 1967.
8. L. DREWNOWSKI, G. EMMANUELE, On Banach spaces with the Gelfand-Phillips property II, *Rend. Circ. Mat. Palermo (2)* **38**(1989), 377–391.
9. G. EMMANUELE, A dual characterization of Banach spaces not containing l_1, *Bull. Acad. Pol. Sci.* **34**(1986), 155–160.
10. G. EMMANUELE, On the containment of c_0 by spaces of compact operators, *Bull. Sci. Math. (2)* **115**(1991), 177–184.
11. G. EMMANUELE, Banach spaces with the (CRP) and dominated operators on $C(K)$, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **16**(1991), 243–248.
12. G. EMMANUELE, Banach spaces in which Dunford-Pettis sets are relatively compact *Arc. Math. (Basel)* **58**(1992), 477–485.
13. G. EMMANUELE, On Banach spaces with the Gelfand-Phillips property. III, *J. Math. Pures Appl. (9)* **72**(1993), 327–333.
14. G. EMMANUELE, K. JOHN, Uncomplementability of spaces of compact operators in larger spaces of operators, *Czechoslovak Math. J.*, to appear.
15. R. HEINRICH, P. MANKIEWICZ, Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces, *Studia Math.* **73**(1982), 225–251.
16. N.J. KALTON, Spaces of compact operators, *Math. Ann.* **208**(1974), 267–278.
17. K. MUSIAL, Martingales of Pettis integrable function in *Measure Theory*, Oberwolfach 1979, LNM 794, Springer Verlag 1980, pp. 324–329.
18. A. PELCZYNSKI, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Pol. Sci.* **10**(1962), 641–648.
19. H.P. ROSENTHAL, A characterization of Banach spaces containing l_1, *Proc. Natl. Acad. Sci. U.S.A.* **71**(1974), 2411–2413.
20. W. RUESS, Duality and geometry of spaces of compact operators, in *Functional Analysis: Surveys and Recent Results. III*, Math. Studies, vol. **90**, North Holland 1984.

G. EMMANUELE
Department of Mathematics
University of Catania
Viale A. Doria 6, 95125 Catania
Italy

Received March 17, 1994; revised May 12, 1994.