J. OPERATOR THEORY
33(1995), 327–351

REPRESENTATIONS OF OPERATOR SPACES

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Communicated by Norberto Salinas

ABSTRACT. Let V be any abstract operator space. We represent it completely isometrically into some $\mathcal{B}(H)$ in various ways, then examine the different C^*-algebras and different operator systems it generates. In particular, we construct two C^*-envelopes of an operator space. Using the off-diagonal representation $v \mapsto \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$, from any operator space we are able to build two C^*-algebras which are Morita equivalent C^*-algebras. As an application, we compute the C^*-envelope of $\mathrm{MIN}(X)$, which turns out to be a function algebra over the set of extreme points of $\mathrm{Ball}(X')$ modulo the action of the unit circle. Finally, we introduce a partial ordering on the operator systems spanned by an operator space. We show that there is a maximal element with respect to this ordering.

KEYWORDS: *Operator space, operator system, C^*-envelope, C^*-algebra.*

AMS SUBJECT CLASSIFICATION: 47D25.

1. INTRODUCTION

Throughout this paper, V denotes an abstract operator space and $\mathcal{B}(H)$ denotes the von Neumann algebra of all bounded operators on the Hilbert space H. The letter $\mathbb{C}$ denotes our scalar field — all complex numbers. M_n is the set of all $n \times n$ scalar matrices. For any set X, we use $M_n(X)$ to denote the set of all $n \times n$ matrices with entries from X. $C^*(S)$ stands for the unital C^*-algebra generated by the set S of operators. We will let $C(Y)$ denote the space of continuous functions on a compact Hausdorff space Y.

An *operator space* is a subspace of some C^*-algebra together with the inherited matrix norms. An abstract characterization of operator spaces is given by Ruan in [16]. Since then some effort has been made to study the structure of operator spaces. In this paper, we discuss the behavior of abstract operator spaces by embedding a fixed operator space in different ways concretely into $\mathcal{B}(H)$ and comparing the C^*-algebras and operator systems that it will generate.

A *representation* of V is a complete isometry $\kappa : V \mapsto \mathcal{B}(H)$ for some Hilbert space. For different representation κ, $\kappa(V)$ will generate different C^*-algebras and different operator systems. What we are looking for are the canonical ones. It is fairly easy to construct the greatest C^*-algebra generated by an operator space, which will have $C^*(\kappa(V))$ as a quotient for all representations κ. That is so called the free C^*-algebra generated by V. To obtain its construction, let $\mathbf{F}$ be the free $*$-algebra (no topology) generated by the set V, and define a norm on $\mathbf{F}$ via

$$\|a\|_* = \sup\{\|\rho(a)\|\},$$

where the supremum is over all possible $*$-homomorphisms $\rho : \mathbf{F} \mapsto \mathbf{A}$, induced by representations $\kappa : V \mapsto \mathbf{A}$, and $\mathbf{A}$ is any C^*-algebra. It is easy to check that $\|\cdot\|_*$ is a C^*-algebra norm on $\mathbf{F}$.

However, for many questions, a *smallest* C^*-algebra is important. For the case of abstract operator systems, Hamana ([8]) proved the existence and uniqueness of the C^*-envelope, which in a certain precise sense is the smallest C^*-algebra generated by an operator system. Motivated by his work, in Section 2 we study the off diagonal representation

$$\begin{bmatrix} 0 & \kappa \\ 0 & 0 \end{bmatrix} : V \longmapsto \mathcal{B}(H^2)$$

where κ is any representation. We will see that the operator system $\mathcal{S}$ corresponding to this kind of representation is independent of the choice of κ. Using Hamana's idea of C^*-envelope, we study the C^*-envelope of an operator space by making use of the off-diagonal representation. Also, by examining the four entries of $C^*(\mathcal{S}) \subseteq M_2(\mathcal{B}(H))$, we obtain a very broad class of examples of Morita equivalence, and we are able to construct a C^*-algebra generated by an operator space which is in some sense minimal.

The technique of the representations of Section 2 is used in Section 4 to consider the representations of $\mathrm{MIN}(X)$ for any finite dimensional normed space X. We will exactly compute the C^*-envelope of $\mathrm{MIN}(X)$ and as a consequence study when $\mathrm{MIN}(X)$ can be imbedded into a finite dimensional C^*-algebra. The answer to this question is related to the number of extreme points of X'. (To avoid confusion with operator adjoints, we will use X', instead of X^*, for the dual space of X.)

In Section 5, we discuss the ways in which an operator space can span an operator system. We introduce a partial ordering between these operator system spans. We prove that there is a maximal operator system spanned by an operator space, but in general no minimal one exists.

This paper is part of the author's Ph.D. thesis directed by professor Vern Paulsen. The author wishes to express his profound gratitude.

We thank D. Blecher and J. Froelich who suggested numerous refinements.

2. C^*-ENVELOPES OF OPERATOR SPACES

For a fixed operator space V, our tool of constructing C^*-envelopes of V is the following operator system,

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda & v \\ w^* & \mu \end{pmatrix} : \lambda,\ \mu \in \mathbb{C},\ \ v,\ w \in V \right\}.$$

Accordingly, for a representation κ of V, we let

$$\mathcal{S}_\kappa = \left\{ \begin{pmatrix} \lambda & \kappa(v) \\ \kappa(w)^* & \mu \end{pmatrix} : \lambda,\ \mu \in \mathbb{C},\ \ v,\ w \in V \right\}.$$

The following proposition is the first step in the course of obtaining C^*-envelopes.

PROPOSITION 1. *For any representations κ_1, κ_2 of V, the map*

$$\varphi : \begin{pmatrix} \lambda & \kappa_2(v) \\ \kappa_2(w)^* & \mu \end{pmatrix} \longmapsto \begin{pmatrix} \lambda & \kappa_1(v) \\ \kappa_1(w)^* & \mu \end{pmatrix}$$

defines a unital complete order isomorphism from $\mathcal{S}_{\kappa_2}$ onto $\mathcal{S}_{\kappa_1}$.

Proof. φ is clearly well-defined, unital and invertible. The only non-trivial thing to show is that both φ and φ^{-1} are order preserving.

For any $\varepsilon > 0$, and any operator A, we set $A_\varepsilon = A + \varepsilon I$. Let

$$\left(\begin{bmatrix} \lambda_{ij} & \kappa_2(v_{ij}) \\ \kappa_2(w_{ij})^* & \mu_{ij} \end{bmatrix} \right)_{n \times n}$$

be any positive element in $M_n(\mathcal{S}_{\kappa_2})$. By canonical shuffle this is equivalent to $(\lambda_{ij}) \geqslant 0$, $(\mu_{ij}) \geqslant 0$ and

$$\begin{bmatrix} (\lambda_{ij}) & (\kappa_2(v_{ij})) \\ (\kappa_2(w_{ij}))^* & (\mu_{ij}) \end{bmatrix} \geqslant 0.$$

The latter inequality is equivalent to, for any $\varepsilon > 0$,

$$\begin{bmatrix} I_n & (\lambda_{ij})_\varepsilon^{-\frac{1}{2}}(\kappa_2(v_{ij}))(\mu_{ij})_\varepsilon^{-\frac{1}{2}} \\ (\mu_{ij})_\varepsilon^{-\frac{1}{2}}(\kappa_2(w_{ij}))^*(\lambda_{ij})_\varepsilon^{-\frac{1}{2}} & I_n \end{bmatrix} \geqslant 0,$$

which is true if and only if $(v_{ij}) = (w_{ji})$ and

$$\|(\lambda_{ij})_\varepsilon^{-\frac{1}{2}}(\kappa_2(v_{ij}))(\mu_{ij})_\varepsilon^{-\frac{1}{2}}\| \leqslant 1 \qquad \text{for all } \varepsilon > 0.$$

But κ_1, κ_2 are both representations of the same operator space V, so it is easy to verify that the above inequality is equivalent to

$$\|(\lambda_{ij})_\varepsilon^{-\frac{1}{2}}(\kappa_1(v_{ij}))(\mu_{ij})_\varepsilon^{-\frac{1}{2}}\| \leqslant 1 \qquad \text{for all } \varepsilon > 0.$$

By a similar argument as above, this is equivalent to

$$\left(\begin{bmatrix} \lambda_{ij} & \kappa_1(v_{ij}) \\ \kappa_1(w_{ij})^* & \mu_{ij} \end{bmatrix}\right)_{n\times n} \geqslant 0$$

in $M_n(\mathcal{S}_{\kappa_1})$. ∎

Proposition 1 says that $\mathcal{S}$ is independent of the choice of representation of V. Thus we now unambiguously have a C^*-envelope of $\mathcal{S}$.

But at this point we have to notice that $C^*(\mathcal{S}_\kappa)$ does depend on the representation κ.

If $\kappa : V \mapsto \mathcal{B}(H)$, then $C^*(\mathcal{S}_\kappa)$ is a C^*-subalgebra of $\mathcal{B}(H^2) = M_2(\mathcal{B}(H))$. A moment's thought confirms that all the elements appearing as (1,1)-entries of $C^*(\mathcal{S}_\kappa)$ comprise a C^*-algebra contained in $\mathcal{B}(H)$, and so do the elements of the (2,2)-entries. Thus, $C^*(\mathcal{S}_\kappa)$ has the form

$$C^*(\mathcal{S}_\kappa) = \left\{ \begin{pmatrix} a & x \\ y^* & b \end{pmatrix} : a \in A,\ b \in B,\ x, y \in X \right\},$$

where A, B are C^*-subalgebras of $\mathcal{B}(H)$ and $X \subseteq \mathcal{B}(H)$ is a subspace.

We first prove a lemma which will be used in Section 4 too. For an abstract operator system S, we use $C_e^*(S)$ to denote the C^*-envelope (in Hamana's sense) of S.

LEMMA 1. *Let $\pi : C^*(\mathcal{S}_K) \mapsto C^*_e(\mathcal{S})$ be an onto $*$-homomorphism, which is an extension of the canonical map $\mathcal{S}_K \mapsto \mathcal{S}$. Then π must be of the form*

$$\pi\left(\begin{pmatrix} a & x \\ y^* & b \end{pmatrix}\right) = \begin{pmatrix} \pi_1(a) & \pi_2(x) \\ \pi_3(y)^* & \pi_4(b) \end{pmatrix}, \quad \begin{pmatrix} a & x \\ y^* & b \end{pmatrix} \in C^*(\mathcal{S}_K),$$

where π_1, π_4 are onto $*$-homomorphisms.

Proof. We suppose $C^*_e(\mathcal{S}) \subseteq \mathcal{B}(K)$. Let E_{ij}'s be 2×2 matrix units. It is easily verified that $\pi(E_{11})$ and $\pi(E_{22})$ are orthogonal projections in K such that $\pi(E_{11})\pi(E_{22}) = 0$ and $\pi(E_{11}) + \pi(E_{22}) = I_K$.

Let $\pi(E_{11}) = P_{K_1}$, $\pi(E_{22}) = P_{K_2}$, where P_{K_1}, P_{K_2} are the projections onto the subspaces K_1, K_2 respectively. Then $K_1 \perp K_2$ and $K_1 + K_2 = K_1 \oplus K_2 = K$. With this decomposition we can write $\pi(E_{11}) = E_{11}$, $\pi(E_{22}) = E_{22}$. Now,

$$\begin{aligned} \pi(E_{11} \otimes a) &= \pi(E_{11}(E_{11} \otimes a)E_{11}) \\ &= E_{11}\pi(E_{11} \otimes a)E_{11} \\ &\stackrel{\text{def}}{=} E_{11} \otimes \pi_1(a). \end{aligned}$$

Similarly, $\pi(E_{12} \otimes x) = E_{12} \otimes \pi_2(x), \pi(E_{21} \otimes y^*) = E_{21} \otimes \pi_3(y)^*, \pi(E_{22} \otimes b) = E_{22} \otimes \pi_4(b)$ for some π_2, π_3, π_4. All of $\pi_1, \pi_2, \pi_3, \pi_4$ are linear onto since π is. Furthermore, it is easy to verify that they are $*$-multiplicative in the following sense:

(i) $\pi_1(aa') = \pi_1(a)\pi_1(a')$;
(ii) $\pi_4(bb') = \pi_4(b)\pi_4(b')$;
(iii) $\pi_1(a)\pi_2(x) = \pi_2(ax)$;
(iv) $\pi_2(x)\pi_3(y)^* = \pi_2(xy^*)$;
(v) $\pi_1(a^*) = \pi_1(a)^*$;
(vi) $\pi_4(b^*) = \pi_4(b)^*$;
(vii) $\pi_2(x^*) = \pi_3(x)^*$;
(viii) $\pi_3(y^*) = \pi_2(y)^*$. ∎

The above ideas necessitate the following notation.

DEFINITION 1. Let

$$C^*_{(ij)}(\mathcal{S}) = E_{ii}C^*(\mathcal{S})E_{jj}, \; C^*_{e(ij)}(\mathcal{S}) = E_{ii}C^*_e(\mathcal{S})E_{jj}, \qquad i,j = 1,2.$$

We call $C^*_{e(1,1)}(\mathcal{S})$, $C^*_{e(2,2)}(\mathcal{S})$ the row, respectively column C^*-envelopes of V respectively.

LEMMA 2. *Let $V \subseteq B(H)$ be an operator space, then*

$$C^*_{(1,1)}(\mathcal{S}) = C^*(VV^*), \quad C^*_{(2,2)}(\mathcal{S}) = C^*(V^*V),$$

where $VV^* = \{vw^*,\ v, w \in V\}$, $V^*V = \{w^*v,\ v, w \in V\}$.

Proof. Let's first note some trivial facts. If A is a set of operators, then

$$C^*(A) = \mathrm{cl}\left(\left\{\lambda + \sum a_1 a_2 \cdots a_n : \lambda \in \mathbf{C},\ \ a_1, a_2, \ldots, a_n \in A \cup A^*\right\}\right).$$

If B is another set of operators, $A \subseteq C^*(B)$, $B \subseteq C^*(A)$, then $C^*(A) = C^*(B)$.

We now prove the lemma by analyzing the entries of any finite product of elements of $\mathcal{S}$. Claims :

(i) The elements in (1,1) entry are in $C^*(VV^*)$;

(ii) The elements in (2,2) entry are in $C^*(V^*V)$;

(iii) The elements in (1,2) entry are of the form

$$\sum_{\text{finite sum}} a_i v_i, \quad a_i \in C^*(VV^*), \quad v_i \in V;$$

(iv) The elements in (2,1) entry are of the form

$$\sum_{\text{finite sum}} w_i^* b_i, \quad b_i \in C^*(V^*V), \quad w_i \in V.$$

The verification is an induction argument over the number of factors of the product. If $n = 1$, then $\begin{pmatrix} \lambda & v \\ w^* & \mu \end{pmatrix}$ clearly satisfies the claims. If the claims are true for $n = k$, let

$$\begin{pmatrix} \lambda_1 & v_1 \\ w_1^* & \mu_1 \end{pmatrix} \cdots \begin{pmatrix} \lambda_k & v_k \\ w_k^* & \mu_k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then for $n = k + 1$

$$\begin{aligned} &\begin{pmatrix} \lambda_1 & v_1 \\ w_1^* & \mu_1 \end{pmatrix} \cdots \begin{pmatrix} \lambda_k & v_k \\ w_k^* & \mu_k \end{pmatrix} \begin{pmatrix} \lambda_{k+1} & v_{k+1} \\ w_{k+1}^* & \mu_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_{k+1} & v_{k+1} \\ w_{k+1}^* & \mu_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{k+1} a + b w_{k+1}^* & a v_{k+1} + \mu_{k+1} b \\ \lambda_{k+1} c + d w_{k+1}^* & c v_{k+1} + \mu_{k+1} d \end{pmatrix}. \end{aligned}$$

Taking the induction hypothesis into account, we easily see that all the four entries satisfy the claims. Consequently, the (1,1), (2,2) entries of $C^*(\mathcal{S})$ are contained in $C^*(VV^*)$, $C^*(V^*V)$, respectively.

Conversely, by the facts :

$$\begin{pmatrix} vw^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ w^* & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & w^*v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ w^* & 0 \end{pmatrix} \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$$

we easily get that $C^*(VV^*)$, $C^*(V^*V)$ are contained in the (1,1), (2,2) entries respectively. ∎

DEFINITION 2. We let

$$C_e^*(VV^*) = C_{e(1,1)}^*(\mathcal{S}), \quad C_e^*(V^*V) = C_{e(2,2)}^*(\mathcal{S})$$

denote the C^*-algebras generated by VV^* and V^*V, respectively.

THEOREM 1. (Universal property) *If $\kappa : V \mapsto \mathcal{B}(H)$ then there is an onto $*$-homomorphism $\pi : C^*(\kappa(V)\kappa(V)^*) \mapsto C_e^*(VV^*)$ such that $\pi(\kappa(v)\kappa(w)^*) = vw^*$. Similarly, there is a $*$-homomorphism from $C^*(\kappa(V)^*\kappa(V))$ onto $C_e^*(V^*V)$.*

Proof. Let $\Pi : C^*(\mathcal{S}_\kappa) \mapsto C_e^*(\mathcal{S})$ be the onto $*$-homomorphism in Hamana's sense. Then by Lemma 1, when Π is restricted to the (1,1) corner of $C^*(\mathcal{S}_\kappa)$, it is also $*$-homomorphic and onto the (1,1) corner of $C_e^*(\mathcal{S})$, and we denote this restriction by π. Now

$$\begin{aligned}\pi(\kappa(v)\kappa(w)^*) &= \Pi\left(\begin{pmatrix}\kappa(v)\kappa(w)^* & 0\\ 0 & 0\end{pmatrix}\right)\\ &= \Pi\left(\begin{pmatrix}0 & \kappa(v)\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & \kappa(w)\\ 0 & 0\end{pmatrix}^*\right)\\ &= \Pi\left(\begin{pmatrix}0 & \kappa(v)\\ 0 & 0\end{pmatrix}\right)\Pi\left(\begin{pmatrix}0 & \kappa(w)\\ 0 & 0\end{pmatrix}\right)^* = vw^*.\end{aligned}$$

The proof of the other half is similar. ∎

Arveson ([1], [2]) defined the C^*-envelope of a unital operator algebra. Let $\mathbf{A}$ be a unital operator algebra, then $\mathbf{A} + \mathbf{A}^*$ is an operator system. It can be shown that $\mathbf{A} + \mathbf{A}^*$ is independent of the unital completely isometric representation of $\mathbf{A}$. Indeed, it is known (see [11], 2.12) that if $\varphi : B \mapsto C$ is a unital contraction from a unital operator space B into a C^*-algebra C, then the natural extension $\tilde{\varphi} : B + B^* \mapsto C$ is well-defined and positive. Hence a unital completely contractive map has a completely positive natural extension. Considering the inverse map, we see that a unital complete isometry has a completely order isomorphic natural extension. Thus $C_e^*(\mathbf{A}+\mathbf{A}^*)$ is well-defined and it is defined to be the C^*-envelope of $\mathbf{A}$.

On the other hand, since $\mathbf{A}$ is unital, $\mathbf{A} \subseteq \mathbf{A}\mathbf{A}^*$, $\mathbf{A} \subseteq \mathbf{A}^*\mathbf{A}$. $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{A}$ are both operator systems. Thus, there is a C^*-envelope $C_e^*(\mathbf{A}\mathbf{A}^*)$ (and a $C_e^*(\mathbf{A}^*\mathbf{A})$).

PROPOSITION 2. *$C_e^*(\mathbf{A} + \mathbf{A}^*)$, $C_e^*(\mathbf{A}^*\mathbf{A})$ and $C_e^*(\mathbf{A}\mathbf{A}^*)$ are $*$-isomorphic.*

Proof. For any unital representation $\rho : \mathbf{A} \mapsto \mathcal{B}(H)$, it can be easily seen that $C^*(\rho(\mathbf{A}) + \rho(\mathbf{A})^*) = C^*(\rho(\mathbf{A})\rho(\mathbf{A})^*)$, therefore the conclusion follows. The proof of the other isomorphism is similar. ∎

This proposition says that $C_e^*(\mathbf{AA}^*)$ ($= C_e^*(\mathbf{A}^*\mathbf{A})$) is also the C^*-envelope of $\mathbf{A}$. Thus our definition of C^*-envelopes for an operator space is consistent with that of an operator algebra.

An operator system has one C^*-envelope which contains the operator system completely order isomorphically. In contrast, an operator space V has two C^*-envelopes which may not contain V completely isometrically. However, in the next theorem we will see how to recover for operator spaces the pleasant property of operator systems.

THEOREM 2. *Suppose that an operator space V has a representation κ such that $1 \in \kappa(V)$, then $C_e^*(V^*V) \cong C_e^*(VV^*)$, and they contain V completely isometrically.*

Proof. It is easy to see by observing the proof of Lemma 2 that $C^*(\mathcal{S}_\kappa) = M_2(C^*(\kappa(V)))$ since $\kappa(V)$ contains the identity. By Hamana's theorem there exists an onto $*$-homomorphism

$$\tilde{\pi} : M_2(C^*(\kappa(V))) \mapsto C_e^*(\mathcal{S}),$$

where $C_e^*(\mathcal{S}) \subseteq \mathcal{B}(K)$. As mentioned before, $\tilde{\pi}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$ and $\tilde{\pi}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$ are projections. Let them be P_{K_1}, P_{K_2} respectively, then $\mathcal{B}(K) = \mathcal{B}(K_1 \oplus K_2)$. Suppose

$$\tilde{\pi}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}, \qquad Q : K_1 \mapsto K_2,$$

then

$$\tilde{\pi}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ Q^* & 0 \end{pmatrix}, \qquad Q^* : K_2 \mapsto K_1.$$

It is easy to see that $QQ^* = I_{K_1}$, $Q^*Q = I_{K_2}$. This means that Q and Q^* are inverse of each other, which implies that K_1, K_2 are of the same dimension, i.e. $\mathcal{B}(K) = \mathcal{B}(K_1^2)$ and Q is a unitary. We now build a new $*$-homomorphism

$$\pi = \begin{pmatrix} Q^* & 0 \\ 0 & 1 \end{pmatrix} \tilde{\pi} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easily checked that

$$\pi(E_{ij}) = E_{ij}, \quad i,\ j = 1,\ 2,$$

and that if

$$\pi\begin{pmatrix} 0 & V_i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & W_i \\ 0 & 0 \end{pmatrix}, \quad \text{then} \quad \pi\begin{pmatrix} 0 & 0 \\ V_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ W_i & 0 \end{pmatrix}.$$

The above arguments have insured that there is a unital C^*-algebra homomorphism π_1 such that

$$\pi = \begin{pmatrix} \pi_1 & \pi_1 \\ \pi_1 & \pi_1 \end{pmatrix},$$

and thus $C^*_e(S) = M_2(C^*(\pi_1(\kappa(V))))$. Hence, $C^*_e(VV^*) = C^*_e(V^*V) = C^*(\pi_1(\kappa(V)))$ which contains a copy $\kappa(V)$ of V. ∎

PROPOSITION 3. *If* $\mathbf{A}$ *is a unital* C^**-algebra, then*

$$C^*_e(\mathbf{A}\mathbf{A}^*) \cong C^*_e(\mathbf{A}^*\mathbf{A}) \cong \mathbf{A}.$$

Proof. Let $\kappa_e : \mathbf{A} \mapsto \mathcal{B}(H)$ be a unital representation of $\mathbf{A}$ (as an operator space) such that

$$C^*(\kappa_e(\mathbf{A})\kappa_e(\mathbf{A})^*) = C^*_e(\mathbf{A}\mathbf{A}^*) = C^*(\kappa_e(\mathbf{A})^*\kappa_e(\mathbf{A})) = C^*_e(\mathbf{A}^*\mathbf{A}).$$

Let $\pi : C^*(\mathbf{A}\mathbf{A}^*)(= \mathbf{A}) \mapsto C^*(\kappa_e(\mathbf{A})\kappa_e(\mathbf{A})^*)$ be the $*$-homomorphism, then π is one-one since $\pi|\mathbf{A}$ is a complete isometry. ∎

The author thanks Roger Smith for asking the question which leads to this proposition.

PROPOSITION 4. *If* S *is an operator system, then* $C^*_e(SS^*)$ *and* $C^*_e(S^*S)$ *are both equal to Hamana's envelope* $C^*_e(S)$.

Proof. Let φ be a unital complete order isomorphism of S such that

$$C^*(\varphi(S)) = C^*_e(S).$$

Remark that φ is a complete isometry and $C^*(\varphi(S)\varphi(S)^*) = C^*(\varphi(S)) = C^*_e(S)$, thus by Theorem 1, there is a

$$\pi : C^*(\varphi(S)\varphi(S)^*) \mapsto C^*_e(SS^*).$$

Suppose κ is a completely isometric representation of S such that $C^*_e(SS^*) = C^*(\kappa(S)\kappa(S)^*)$. κ is in fact unital by the arguments in Theorem 2. Consequently, κ is a unital complete order isomorphism. Noticing that $C^*(\kappa(S)\kappa(S)^*) = C^*(\kappa(S))$, we see that the map π is an onto $*$-homomorphism

$$\pi : C^*_e(S) \mapsto C^*(\kappa(S))$$

which has to be a $*$-isomorphism.

By a similar argument one proves the other isomorphism. ∎

3. OPERATOR SPACE AND MORITA EQUIVALENCE

There are more facts connecting the four entries of $C^*(\mathcal{S})$. For convenience, in this section we let $A = C^*_{(1,1)}(\mathcal{S})$, $B = C^*_{(2,2)}(\mathcal{S})$, $X = C^*_{(1,2)}(\mathcal{S})$, then $X^* = C^*_{(1,2)}(\mathcal{S})^* = C^*_{(2,1)}(\mathcal{S})$. By the observation in the proof of the previous lemma, we see that X is a left A right B bimodule. And hence X^* is a left B right A bimodule. This leads to the question of whether or not A and B are Morita equivalent. The answer turns out to be "yes", with a minor amendment of A and B.

EXAMPLE. Let $V = R_n$ be the n dimensional row operator space. If we assume the representation

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda & x_1 & \cdots & x_n \\ y_1 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \cdots & \mu \end{pmatrix}, \quad \lambda, \mu \in \mathbf{C}, \quad x, y \in l^2_n \right\},$$

then $C^*(\mathcal{S}) = C^*_e(\mathcal{S})$, $C^*_{(1,1)}(\mathcal{S}) = \mathbf{C}$, $C^*_{(2,2)}(\mathcal{S}) = M_n$. Therefore $C^*_{(1,1)}(\mathcal{S})$ and $C^*_{(2,2)}(\mathcal{S})$ are Morita equivalent.

Indeed, since $C^*_{(2,2)}(\mathcal{S})$ contains all rank one elements of M_n, $C^*_{(2,2)}(\mathcal{S}) = M_n$. If $C^*_e(\mathcal{S}) = C^*(\mathcal{S}_\kappa)$, let $\pi : C^*(\mathcal{S}) \mapsto C^*(\mathcal{S}_\kappa)$ be the onto $*$-homomorphism. π is obviously one-to-one onto the (1,2) and (2,1) entries because $C^*_{(1,2)}(\mathcal{S})$ has the same elements as the (1,2) entry of $\mathcal{S}$, and a similar thing for the (2,1) entry. π is one-to-one on the (1,1), (2,2) entries since $\mathbf{C}$, M_n have no non-trivial ideal.

The following is an example where $C^*_{(1,1)}$ and $C^*_{(2,2)}$ are not Morita equivalent.

EXAMPLE. Let $V = R_\infty$ be the infinite dimensional row operator space. We can prove that

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda & x_1 & x_2 & x_3 & \cdots \\ y_1 & \mu & 0 & 0 & \cdots \\ y_2 & 0 & \mu & 0 & \cdots \\ y_3 & 0 & 0 & \mu & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \lambda,\ \mu \in \mathbf{C},\ x,\ y \in l^2 \right\}$$

satisfies $C^*_{(2,2)}(\mathcal{S}) = \mathbf{C}I + \mathcal{K}$, where $\mathcal{K}$ is the set of all compact operators. As in the above example, we can show that $C^*(\mathcal{S}) = C^*_e(\mathcal{S})$. It is well known that $\mathbf{C}$ is Morita equivalent to $\mathcal{K}$, but not to $\mathbf{C}I + \mathcal{K}$.

In fact $C^*_{(1,1)}$, $C^*_{(2,2)}$ are "almost" Morita equivalent. As a matter of fact we do not have to require that they contain units.

LEMMA 3. *Let $V \subseteq \mathcal{B}(H)$ be a concrete space of operators. Let*

$$\overline{V} = \left\{ \begin{pmatrix} 0 & v \\ w^* & 0 \end{pmatrix}, \quad v, w^* \in V \right\}$$

then

$$C^*_{(1,1)}(\overline{V}) \cong C^*_0(VV^*); \; C^*_{(2,2)}(V^*) \cong C^*_0(V^*V)$$

and

$$C^*_{(1,2)}(\overline{V}) = C^*_{(1,2)}(\mathcal{S}) = X; \quad C^*_{(2,1)}(\overline{V}) = C^*_{(2,1)}(\mathcal{S}).$$

Here $C^*_0(S)$ denotes the possibly non-unital C^*-algebra generated by S. The proof is similar to that of Lemma 2, we omit it.

For convenience, we let $A_0 = C^*_0(VV^*)$, $B_0 = C^*_0(V^*V)$. Then $X = C^*_{(1,2)}(\overline{V})$ ($X^* = C^*_{(2,1)}(\overline{V})$) is an A_0-B_0 (B_0-A_0) bimodule. We now define an A_0-valued inner product $\langle v, w\rangle_{A_0} = vw^*$, and a B_0-valued inner product $\langle v, w\rangle = v^*w$, where $v, w \in X$. Then it is easy to verify that

(i) $\langle x, x\rangle_{A_0} \geqslant 0$, $\langle x, x\rangle_{B_0} \geqslant 0$;

(ii) $\langle x, y\rangle^*_{A_0} = \langle y, x\rangle_{A_0}$, $\langle x, y\rangle^*_{B_0} = \langle y, x\rangle_{B_0}$;

(iii) for any $a \in A_0$, $b \in B_0$, $\langle ax, y\rangle_{A_0} = a\langle x, y\rangle_{A_0}$, $\langle x, yb\rangle_{B_0} = \langle x, y\rangle_{B_0} b$;

(iv) $\langle x, y\rangle_{A_0} z = x\langle y, z\rangle_{B_0}$;

(v) $\langle xb, xb\rangle_{A_0} \leqslant \|b\|^2 < x, x >_{A_0}$, $\langle ax, ax\rangle_{B_0} \leqslant \|a\|^2\langle x, x\rangle_{B_0}$.

THEOREM 3. *Let $V \subseteq \mathcal{B}(H)$ be a concrete operator space. Then $C^*_0(VV^*)$ and $C^*_0(V^*V)$ are Morita equivalent.*

Proof. By the above remarks about the inner products, the only thing left to verify is that

(1) $\left\{\sum\limits_{i=1}^{n} x_i y_i^*, \; x_i, \; y_i \in X, \; n = 1, 2, \ldots\right\}$ is dense in $C^*_0(VV^*)$;

(2) $\left\{\sum\limits_{i=1}^{n} x_i^* y_i, \; x_i, \; y_i \in X, \; n = 1, 2, \ldots\right\}$ is dense in $C^*_0(V^*V)$.

But (1) contains VV^*, and (2) contains V^*V. So, the closure of (1) is a C^*-algebra containing $C^*_0(VV^*)$, and the closure of (2) is a C^*-algebra containing $C^*_0(V^*V)$. Consequently, $C^*_0(VV^*)$ and $C^*_0(V^*V)$ are Morita equivalent. ∎

The significance of this theorem is that from any operator space V, we can always construct a pair of C^*-algebras which are Morita equivalent to each other.

4. REPRESENTATIONS OF $MIN(X)$

For any Banach space X, there are two special operator space structures $\mathrm{MIN}(X)$ and $\mathrm{MAX}(X)$ that can be assigned on X. See [3], [12] for definitions and properties. Looking at the definition of its matrix norm, it is not hard to see that $\mathrm{MIN}(X)$ can in fact be regarded as a space of functions defined on the unit ball $B_1(X')$ of X', where X' is endowed with the weak*-topology.

In this section we suppose that $\dim(X)$ is finite. We are going to compute

$$C_e^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}) \quad (= C_e^*(\overline{\mathrm{MIN}(X)}\mathrm{MIN}(X)))$$

and then we consider whether or not $\mathrm{MIN}(X)$ can be imbedded into a finite dimensional C^*-algebra. We prove that the "finiteness" is somehow related to the number of extreme points of $B_1(X')$.

Following the notation before, we still set

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda & f \\ \overline{g} & \mu \end{pmatrix}, \quad \lambda, \mu \in \mathbf{C}, \quad f, g \in \mathrm{MIN}(X) \right\} \subseteq M_2(C(B_1(X'))). \tag{1}$$

NOTATION. Let X be a Banach space, then we will use the following notation in the sequel: $B_1(X)$ = unit ball of X; X_1 = unit sphere of X; $E(X)$ = the set of all extreme points of $B_1(X)$; $\mathbf{T}$ = unit circle in $\mathbf{C}$.

It is easy to see that

$$\begin{aligned} \|(x_{ij})\|_{\min} &= \sup\{\|(f(x_{ij}))\|_n : f \in X_1'\} \\ &= \sup\{\|(f(x_{ij}))\|_n : f \in \mathrm{cl}(E(X'))\}. \end{aligned}$$

Thus we now have obtained another two natural representations of $\mathrm{MIN}(X)$:

$$\begin{aligned} \kappa_1 &: \mathrm{MIN}(X) \mapsto C(X_1'), \\ \kappa_2 &: \mathrm{MIN}(X) \mapsto C(\mathrm{cl}(E(X'))). \end{aligned}$$

In this section we use κ_1, κ_2 to denote these two representations only.

DEFINITION 3. For any Banach space X, we define a relation on X_1 as follows. For any $x, y \in X_1$, $x \sim y$ if there is an $e^{i\theta} \in \mathbf{T}$ such that $x = e^{i\theta} y$.

Since $\mathbf{T}$ is a group under multiplication, we easily see that $\sim$ is an equivalence relation. We will use $S/\sim$ to denote the quotient space of S induced from the equivalence $\sim$.

PROPOSITION 5. *If* $E(X')/\sim$ *has* n *elements, then* $\mathrm{MIN}(X)$ *can be imbedded into* M_n.

Proof. Let $x'_1, \dots, x'_n$ be all the inequivalent extreme points. Define $\kappa : \mathrm{MIN}(X) \mapsto M_n$ by

$$\kappa(f) = \begin{pmatrix} f(x'_1) & & 0 \\ \cdot & \ddots & \\ 0 & & f(x'_n) \end{pmatrix}.$$

Then

$$\begin{aligned} \|(\kappa(f_{ij}))\|_n &= \|(\mathrm{diag}\{f_{ij}(x'_1), \dots, f_{ij}(x'_n)\})\| \\ &= \left\| \begin{pmatrix} (f_{ij}(x'_1)) & & 0 \\ & \ddots & \\ 0 & & (f_{ij}(x'_n)) \end{pmatrix} \right\| \\ &= \max\{\|(f_{ij}(x'_k))\|,\ 1 \leqslant k \leqslant n\} \\ &= \sup\{\|(f_{ij}(x'))\|,\ \|x'\| \leqslant 1\} \\ &= \|(f_{ij})\|. \quad \blacksquare \end{aligned}$$

PROPOSITION 6. $C^*(\kappa_1(\mathrm{MIN}(X))\overline{\kappa_1(\mathrm{MIN}(X))})$ *is* $*$*-isomorphic to* $C(X'_1/\sim)$.

Proof. By the maximal modulus theorem we see that $\kappa_1 : \mathrm{MIN}(X)\overline{\mathrm{MIN}(X)} \mapsto C(X'_1)$ such that $\kappa_1(f\,\overline{g}) = \kappa_1(f)\overline{\kappa_1(g)}$ is a complete isometry.

For any generator $f\,\overline{g} \in \mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}$,

$$(f\,\overline{g})(\mathrm{e}^{\mathrm{i}\theta}x') = f(\mathrm{e}^{\mathrm{i}\theta}x')\overline{g}(\mathrm{e}^{\mathrm{i}\theta}x') = \mathrm{e}^{\mathrm{i}\theta}f(x')\mathrm{e}^{-\mathrm{i}\theta}g(x') = (fg)(x').$$

So $h(\mathrm{e}^{\mathrm{i}\theta}x') = h(x')$ for all θ and for any fixed $h \in C^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$. Consequently, we may regard $C^*(\kappa_1(\mathrm{MIN}(X))\overline{\kappa_1(\mathrm{MIN}(X))}) \subseteq C(X'_1/\sim)$.

If $x',\ y' \in X'_1$ are linearly independent, then there exists $f \in X$ such that $f(x') = 0,\quad f(y') \neq 0$. Therefore $(f\overline{f})(x') = 0,\quad (f\overline{f})(y') > 0$. Thus $\kappa_1(\mathrm{MIN}(X))\overline{\kappa_1(\mathrm{MIN}(X))}$ separates points of $X'_1/\sim$, and therefore

$$C^*(\kappa_1(\mathrm{MIN}(X))\overline{\kappa_1(\mathrm{MIN}(X))}) = C(X'_1/\sim). \quad \blacksquare$$

Let $\mathcal{S}$ be as in (1), $\pi : C^*(\mathcal{S}) \mapsto C^*_{\mathrm{e}}(\mathcal{S})$ be the onto $*$-homomorphism. By Lemma 1, $C^*_{\mathrm{e}(1,1)}(\mathcal{S}) = \pi(C^*_{(1,1)}(\mathcal{S}))$ is the same as $C^*_{\mathrm{e}}(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$ which is called the C^*-envelope of $\mathrm{MIN}(X)$. (By the commutativity, the row and column C^*-envelopes are the same in this case.)

LEMMA 4. *$\kappa_2 : \mathrm{MIN}(X)\overline{\mathrm{MIN}(X)} \mapsto C(\mathrm{cl}(E(X'))/\sim)$ defined by*

$$\kappa_2(f\,\overline{g})(\tilde{x}') = f(x')\overline{g}(x'), \qquad f\,\overline{g} \in X\overline{X}$$

is a well-defined complete isometry.

Proof. The well-definedness is already seen in the proof of Proposition 6.

$$\begin{aligned}
\|\kappa_2(f\,\overline{g})\| &= \sup\{|\kappa_2(f\,\overline{g})(\tilde{x}')|,\ x' \in \mathrm{cl}(E(X'))/\sim\} \\
&= \sup\{|f(x')\overline{g}(x')|,\ x' \in \mathrm{cl}(E(X'))\} \\
&= \sup\{|(f\,\overline{g})(x')|,\ \|x'\| \leqslant 1\} = \|f\,\overline{g}\|, \\
\|(\kappa_2(f_{ij}\overline{g_{ij}}))\|_n &= \sup\{\|(\kappa_2(f_{ij}\overline{g_{ij}})(\tilde{x}'))\|,\ \tilde{x}' \in \mathrm{cl}(E(X'))/\sim\} \\
&= \sup\{\|(f_{ij}(x')\overline{g_{ij}(x')})\|,\ x' \in \mathrm{cl}(E(X'))\} \\
&= \sup\{\|(f_{ij}(x')\overline{g_{ij}(x')})\|,\ \|x'\| \leqslant 1\} \\
&= \|(f_{ij}\overline{g_{ij}})\|_{\min}. \quad \blacksquare
\end{aligned}$$

THEOREM 4. *$C^*_e(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$ is $*$-isomorphic to $C(\mathrm{cl}(E(X'))/\sim)$.*

Proof. By Lemma 4 we have a representation κ_2 of $\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}$ such that $C^*(\kappa_2(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}))$ is commutative. If

$$\pi : C^*(\kappa_2(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})) \mapsto C^*_e(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$$

is the $*$-homomorphism onto map, then we easily see that $C^*_e(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$ is commutative too. Therefore, there is a compact Hausdorff space Y such that

$$C^*_e(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}) = C(Y)$$

$*$-isomorphically. Hence there is by Hamana ([8]) an onto $*$-homomorphism

$$\pi : C(\mathrm{cl}(E(X'))/\sim) \mapsto C(Y).$$

It is well known that for such a π, there is a continuous map $\eta : Y \mapsto X'_1/\sim$ such that $\eta^* = \pi$ which means that $\pi(f)(y) = f(\eta(y))$ for all $y \in Y$ and $f \in C^*(\kappa_2(\mathrm{MIN}(X))\overline{\kappa_2(\mathrm{MIN}(X))}$. η is one-to-one since π is onto. If $\eta(Y)$ is not onto $\mathrm{cl}(E(X'))/\sim$, let $\tilde{x}'$ be an element in $E(X')/\sim$ (notice $\eta(Y)$ is compact) but not in $\eta(Y)$, and let $q : X'_1 \mapsto X'_1/\sim$ be the quotient map. Since $\eta(Y)$ is compact, $q^{-1}(\eta(Y)) \subseteq X'_1$ is a compact set and x' is not in $q^{-1}(\eta(Y))$. This implies that x' is not in the convex hull $\mathrm{conv}(q^{-1}(\eta(Y)))$ because x' is an extreme point. And in fact $\mathrm{conv}(q^{-1}(\eta(Y)))$ is also a compact set. Therefore, there is an

$f \in X$ such that $\mathrm{Re}(f(y')) < \mathrm{Re}(f(x'))$ for all $y' \in \mathrm{conv}(q^{-1}(\eta(Y)))$. This implies that $|f(y')| < |f(x')|$ for all y' because $f(\mathrm{conv}(q^{-1}(\eta(Y))))$ is a disk centered at the origin. Now

$$\begin{aligned} \|\pi(f\overline{f})\| &= \sup\{|\pi(f\overline{f})(y)| : y \in Y\} \\ &= \sup\{|(f\overline{f})(\eta(y))| : y \in Y\} \\ &= \sup\{|(f\overline{f})(y')| : y' \in q^{-1}(\eta(Y))\} \\ &= \sup\{|f(y')|^2 : y' \in \mathrm{conv}(q^{-1}(\eta(Y)))\} \\ &< |f(x')|^2 \leqslant \|f\|^2 = \|f\overline{f}\|. \end{aligned}$$

This contradicts the fact that $\pi|\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}$ is a complete isometry. Thus $E(X')/\sim$ (thus $\mathrm{cl}(E(X'))/\sim$) is contained in $\eta(Y)$. On the other hand $\eta(Y)$ is a quotient of $\mathrm{cl}(E(X'))/\sim$, thus by Proposition 4 we have $\mathrm{cl}(E(X'))/\sim = \eta(Y)$. This completes the proof. ∎

THEOREM 5. $\mathrm{MIN}(X)$ *can be imbedded completely isometrically into* M_n *for some* n *if and only if* $E(X')/\sim$ *has finitely many elements.*

Proof. The "if" part is proved by Proposition 5. Now the "only if" part. Let $\kappa : \mathrm{MIN}(X) \mapsto M_n$ be a complete isometry. Then there exists an onto $*$-homomorphism

$$\pi : C^*(\kappa(\mathrm{MIN}(X))\kappa(\mathrm{MIN}(X))^*) \mapsto C_e^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}).$$

Since $C^*(\kappa(\mathrm{MIN}(X))\kappa(\mathrm{MIN}(X))^*) \subseteq M_n$, we have that $C_e^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$ is finite dimensional. By the $*$-isomorphism of the above result, $C(\mathrm{cl}(E(X'))/\sim)$ is finite dimensional and hence $E(X')/\sim$ must be a finite set. ∎

Timur Oikberg recently has proved in his thesis that if $\mathrm{MIN}(X)$ is contained in $\mathcal{K}(l^2)$ (compact operators on l^2) completely isometrically, then X can be imbedded into some finite dimensional l_n^∞ isometrically. The basic tool of his proof is some Banach space techniques. Here we indicate how to prove this result using our previous two theorems. Since $\mathrm{MIN}(X) \subseteq \mathcal{K}$,

$$C^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}) \subseteq \mathcal{K} + \mathbb{C}I.$$

There is an onto $*$-homomorphism

$$\pi : C^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)}) \mapsto C(\mathrm{cl}(E(X'))/\sim) \quad (\cong C_e^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})\ ).$$

If $E(X')/\sim$ is an infinite set, choose an $f \in C(\mathrm{cl}(E(X'))/\sim)$ such that the spectrum $\sigma(f)$ is an infinite set bounded away from 0. Since π is onto, $f = \pi(T)$ for some $T \in C^*(\mathrm{MIN}(X)\overline{\mathrm{MIN}(X)})$. Then $\sigma(T) \supseteq \sigma(f)$ which is impossible because T is in $\mathcal{K} + \mathbf{C}I$. This means that $E(X')/\sim$ is a finite set. Thus X is isometrically contained in some l_n^∞.

We now study $C^*(\mathcal{S})$ as a C^*-subalgebra of $M_2(C(X_1'))$. We already have seen what the elements in (1,1) entry and (2,2) entry look like. They are in fact all the elements f of $C^*(X_1')$ satisfying $f(e^{i\theta}x') = f(x')$. To characterize all elements of $C^*(\mathcal{S})$, we want to know the relations that characterize $C^*_{(1,2)}(\mathcal{S})$ and $C^*_{(2,1)}(\mathcal{S})$. By observing the proof of Lemma 2, it is easy to see that any element $f \in C^*_{(1,2)}(\mathcal{S})$ satisfies

$$f(e^{i\theta}x') = e^{i\theta}f(x') \qquad e^{i\theta} \in \mathbf{T}, \quad x' \in X_1',$$

and any element $g \in C^*_{(2,1)}(\mathcal{S})$ satisfies

$$g(e^{i\theta}x') = e^{-i\theta}g(x') \qquad e^{i\theta} \in \mathbf{T}, \quad x' \in X_1'.$$

The following proposition proves that $C^*_{(1,2)}(\mathcal{S})$ and $C^*_{(2,1)}(\mathcal{S})$ contain exactly all those kinds of elements respectively.

PROPOSITION 7.

$$C^*(\mathcal{S}_{\kappa_1}) = \left\{ \begin{pmatrix} f & h \\ l & g \end{pmatrix} \in M_2(C(X_1')) : \begin{array}{ll} f(e^{i\theta}x') = f(x'), & g(e^{i\theta}x') = g(x'), \\ h(e^{i\theta}x') = e^{i\theta}h(x'), & l(e^{i\theta}) = e^{-i\theta}l(x'), \end{array} \quad e^{i\theta} \in \mathbf{T}, \quad x' \in X_1' \right\}.$$

Proof. Suppose $h \in C(X_1')$ such that $h(e^{i\theta}x') = e^{i\theta}h(x')$. Let $f_1, f_2, \ldots, f_m$ be a linear basis of X, and $U_i = \{x' \in X_1',\ f_i(x') \neq 0\}$, $i = 1, 2, \ldots, m$. Then $\{U_i\}_{i=1}^m$ is an open covering of the compact Hausdorff space X_1'. If q is the quotient map with respect to Definition 3, then $\{q(U_i)\}_{i=1}^m$ is an open covering of $X_1'/\sim$. Let $\{\alpha_i\}_{i=1}^m$ be a partition of unity subordinated to $\{q(U_i)\}_{i=1}^m$, then $\{\alpha_i\}_{i=1}^m$ can be considered as a partition of unity subordinated to $\{U_i\}_{i=1}^m$ with $\{\alpha_i\}_{i=1}^m \subset C(X_1')$. Define

$$h_i(x') = \begin{cases} \frac{\alpha_i(x')}{f_i(x')}h(x') & x' \in U_i \\ 0 & \text{otherwise}. \end{cases}$$

Then $\{h_i\}_{i=1}^m$ are continuous functions in $C^*(\kappa_1(\mathrm{MIN}(X))\overline{\kappa_1(\mathrm{MIN}(X))})$. Notice that

$$h(x') = \sum_{i=1}^m f_i(x')h_i(x')$$

$$\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sum_{i=1}^{m} f_i h_i \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{m} \begin{pmatrix} h_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & f_i \\ 0 & 0 \end{pmatrix}.$$

So h is in the (1,2) entry of $C^*(\mathcal{S}_{K_1})$.

In a similay fashion we can show that any function l satisfying $l(e^{i\theta}x') = e^{-i\theta}l(x')$ is in $C^*_{(2,1)}(\mathcal{S}_{K_1})$. ∎

PROPOSITION 8. *Let*

$$\mathbf{C} = \left\{ \begin{pmatrix} f & h \\ l & g \end{pmatrix} \in M_2(C(E(X'))) : \begin{array}{ll} f(e^{i\theta}x') = f(x'), & g(e^{i\theta}x') = g(x'), \\ h(e^{i\theta}x') = e^{i\theta}h(x'), & l(e^{i\theta}x') = e^{-i\theta}l(x'), \end{array} \quad e^{i\theta} \in \mathbf{T}, \quad x' \in \mathrm{cl}(E(X')) \right\}.$$

Then $C^*_e(\mathcal{S})$ and $\mathbf{C}$ are $*$-isomorphic.

Proof. Let $\pi : \mathbf{C} \mapsto C^*_e(\mathcal{S})$ be the onto $*$-homomorphism in the sense of Hamana. By Theorem 4, π is one-to-one when restricted to (1,1) or (2,2) entries. Now if $\pi\left(\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}\right) = 0$, then

$$\pi\left(\begin{pmatrix} f\bar{f} & 0 \\ 0 & 0 \end{pmatrix}\right) = \pi\left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\right)\pi\left(\begin{pmatrix} 0 & 0 \\ \bar{f} & 0 \end{pmatrix}\right) = 0.$$

Hence $f\bar{f} = 0$, i.e. $f = 0$. So π is one-to-one when restricted to (1,2) entry. Similarly π is one-to-one when restricted to (2,1) entry. We conclude that π is a $*$-isomorphism. ∎

The (1,1) and (2,2) corners of $C^*_e(\mathcal{S})$ are actually both $*$-isomorphic to $C(\mathrm{cl}(E(X'))/\sim)$. This makes it natural to try and represent $C^*_e(\mathcal{S})$ as a subalgebra of $M_2(C(\mathrm{cl}(E(X'))/\sim))$. In the following results we discuss when this is possible.

THEOREM 6. *Suppose that there is a continuous function $s : \mathrm{cl}(E(X'))/\sim \mapsto \mathrm{cl}(E(X'))$ such that $q \circ s = i_{\mathrm{cl}(E(X'))/\sim}$, the identity map on $\mathrm{cl}(E(X'))/\sim$. Then $C^*_e(\mathcal{S}) \cong M_2(C(\mathrm{cl}(E(X'))/\sim))$.*

Proof. Let $\pi : C^*_e(\mathcal{S}) \mapsto M_2(C(\mathrm{cl}(E(X'))/\sim))$ be defined by

$$\pi\left(\begin{pmatrix} f & h \\ l & g \end{pmatrix}\right) = \begin{pmatrix} f\circ s & h\circ s \\ l\circ s & g\circ s \end{pmatrix}, \quad \begin{pmatrix} f & h \\ l & g \end{pmatrix} \in C^*_e(\mathcal{S}).$$

To verify that π is $*$-homomorphic is trivial. One-to-one : If

$$\begin{pmatrix} f\circ s & h\circ s \\ l\circ s & g\circ s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then $f(s(\tilde{x}')) = 0$ for all $\tilde{x}' \in \mathrm{cl}(E(X'))/\sim$. By the property $f(e^{i\theta}x') = f(x')$ we get $f(x') = 0$ for all $x \in \mathrm{cl}(E(X'))$, i.e. $f = 0$. Similar arguments prove that $g = 0$, $h = 0$, $l = 0$. Onto: Let $\begin{pmatrix} \tilde{f} & \tilde{h} \\ \tilde{l} & \tilde{g} \end{pmatrix}$ be any element in $M_2(C(\mathrm{cl}(E(X'))))$. Let $f(x') = \tilde{f}(\tilde{x}')$, $g(x') = \tilde{g}(\tilde{x}')$, $h(e^{i\theta}s(\tilde{x}')) = e^{i\theta}\tilde{h}(\tilde{x}')$, $l(e^{i\theta}s(\tilde{x}')) = e^{-i\theta}\tilde{l}(\tilde{x}')$ for all $e^{i\theta} \in \mathbf{T}$ and $x' \in \mathrm{cl}(E(X'))$. Then

$$\pi\left(\begin{pmatrix} f & h \\ l & g \end{pmatrix}\right)(\tilde{x}') = \begin{pmatrix} f\circ s & h\circ s \\ l\circ s & g\circ s \end{pmatrix}(\tilde{x}') = \begin{pmatrix} f(s(\tilde{x}')) & h(s(\tilde{x}')) \\ l(s(\tilde{x}')) & g(s(\tilde{x}')) \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{f}(\tilde{x}') & \tilde{h}(\tilde{x}') \\ \tilde{l}(\tilde{x}') & \tilde{g}(\tilde{x}') \end{pmatrix} = \begin{pmatrix} \tilde{f} & \tilde{h} \\ \tilde{l} & \tilde{g} \end{pmatrix}(\tilde{x}'). \quad \blacksquare$$

COROLLARY 1. *If $E(X')/\sim$ is finite, then $C_e^*(\mathcal{S}) \cong M_2(C(\mathrm{cl}(E(X'))/\sim))$.*

PROPOSITION 9. *If $C_e^*(\mathcal{S})$ is $*$-isomorphic to $M_2(C(Y))$, then Y is homeomorphic to $\mathrm{cl}(E(X'))/\sim$.*

Proof. The center of $C_e^*(\mathcal{S})$ is

$$\left\{\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : f \in C(E(X')/\sim)\right\} \cong C(\mathrm{cl}(E(X'))/\sim),$$

while the center of $M_2(C(Y))$ is clearly isomorphic to $C(Y)$. Therefore $\mathrm{cl}(E(X'))/\sim$ is homeomorphic to Y. $\blacksquare$

5. PARTIAL ORDERING OF REPRESENTATIONS

In this section we study the problem of finding "canonical" operator systems associated with a fixed operator space. Given a representation $\kappa : V \mapsto \mathcal{B}(H)$, we use SP_κ to denote the operator system spanned by $\kappa(V)$, i.e.

$$SP_\kappa = \{\lambda I + \kappa(v) + \kappa(w)^*,\ \lambda \in \mathbf{C},\ v, w \in V\}.$$

We start our discussion by observing that for a fixed V, SP_{κ_1} and SP_{κ_2} may be different operator systems (i.e. they may not be completely order isomorphic) for different κ_1 and κ_2. For example:

Let $V = \mathbb{C}$ (the usual complex plane), $\kappa_1(\lambda) = \lambda$, $\kappa_2(\lambda) = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}$, then $SP_{\kappa_1} = \mathbb{C}$, and $SP_{\kappa_2} = \left\{ \begin{bmatrix} \mu & \lambda \\ \overline{\gamma} & \mu \end{bmatrix}, \lambda, \mu, \gamma \in \mathbb{C} \right\}$. SP_{κ_1}, SP_{κ_2} are clearly not completely order isomorphic.

For any vector space V, we abstractly define another vector space V^* such that the map $* : v \mapsto v^*$ is a conjugate linear vector space isomorphism. Let $\tilde{V} = \mathbb{C} \oplus V \oplus V^*$. We define a natural extension $\tilde{\kappa}$ of κ, $\tilde{\kappa} : \tilde{V} \mapsto \mathcal{B}(H)$ via

$$\tilde{\kappa}(\lambda \oplus v \oplus w^*) = \lambda I + \kappa(v) + \kappa(w)^*.$$

DEFINITION 4. Let $\kappa_1 : V \mapsto \mathcal{B}(H_1)$, $\kappa_2 : V \mapsto \mathcal{B}(H_2)$ be two representations of V. We say that $\kappa_1 \preceq \kappa_2$ if there is a unital completely positive map $\varphi : SP_{\kappa_2} \mapsto SP_{\kappa_1}$ such that $\tilde{\kappa_1} = \varphi \circ \tilde{\kappa_2}$.

NOTE. The existence of φ means that the map $\varphi : \lambda + \kappa_2(v) + \kappa_2(w)^* \mapsto \lambda + \kappa_1(v) + \kappa_1(w)^*$ is a well-defined completely positive map. But the abstract definition is easier to use.

PROPOSITION 10. *The ordering "$\preceq$" defined above is a partial ordering on the set of equivalence classes.*

NOTE. Two representations κ_1, κ_2 are said to be equivalent if and only if the above φ is a complete order isomorphism.

Proof of Proposition 10. Reflexivity is true because the identity map $i : SP_\kappa \mapsto SP_\kappa$ is unital completely positive. Transitivity follows from the fact that the composition of two unital completely positive maps is again a unital completely positive map. Now for the anti-symmetry. If $\kappa_1 \preceq \kappa_2$, $\kappa_2 \preceq \kappa_1$, let $\tilde{\kappa_1} = \varphi \circ \tilde{\kappa_2}$, $\tilde{\kappa_2} = \psi \circ \tilde{\kappa_1}$, then, $\tilde{\kappa_1} = (\varphi \circ \psi) \circ \tilde{\kappa_1}$, $\tilde{\kappa_2} = (\psi \circ \varphi) \circ \tilde{\kappa_2}$. Since the image of $\tilde{\kappa_1}$, $\tilde{\kappa_2}$ are dense in SP_{κ_1}, SP_{κ_2} respectively, we conclude that $\varphi \circ \psi$ is the identity map on SP_{κ_1} and $\psi \circ \varphi$ is the identity map on SP_{κ_2}. Hence $\psi = \varphi^{-1}$ and φ is a complete order isomorphism. ∎

EXAMPLE. If T is an isometry on H and κ_1, κ_2 are related by

$$\kappa_1(v) = T^*\kappa_2(v)T \quad v \in V$$

then $\kappa_1 \preceq \kappa_2$.

EXAMPLE. It is proved in [16] that the map $v \mapsto \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$ is a complete isometry from V into $M_2(V)$. So, for any representation $\kappa : V \mapsto \mathcal{B}(H)$

$$\overline{\kappa}(v) = \begin{bmatrix} 0 & \kappa(v) \\ 0 & 0 \end{bmatrix} \quad v \in V$$

defines a new representation.

In the sense of our partial ordering κ, $\overline{\kappa}$ are not comparable in general. For example, $V = \mathbf{C}$, the map

$$\begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \mapsto 1 + \alpha + \beta$$

is not even positive. This means that the off diagonal representation of V is not bigger than the identity representation. But we have the following:

PROPOSITION 11. *Suppose κ_1, κ_2 are any representations of V, then the new representations*

$$\overline{\kappa}_1 = \begin{bmatrix} 0 & \kappa_1 \\ 0 & 0 \end{bmatrix}, \qquad \overline{\kappa}_2 = \begin{bmatrix} 0 & \kappa_2 \\ 0 & 0 \end{bmatrix}$$

are equivalent.

The proof of this proposition is similar to that of Proposition 1 which looks (but is not) more general.

Choi and Effros ([4]) abstractly characterized operator systems as vector spaces W endowed with an order on $M_n(W)$ for each n, satisfying certain axioms. We will use this characterization to prove the main result of this section.

To define an ordering on W as in Choi-Effros' theorem is the same as characterizing all positive elements of $M_n(W)$ for all n. If P_n denotes the cone of all positive elements of $M_n(W)$, then we let the pair $(W, \{P_n\}_{n=1}^{\infty})$ stand for an operator system W endowed with the order $\{P_n\}_{n=1}^{\infty}$.

Now, for any fixed representation $\kappa : V \mapsto \mathcal{B}(H)$, we define an operator system structure on $\tilde{V} = \mathbf{C} \oplus V \oplus V^*$ by the following cones:

$$P_{\kappa}^{(n)} = \left\{((\lambda_{ij} \oplus v_{ij} \oplus w_{ij})) \in M_n(\tilde{V}) \ : \ [\lambda_{ij} + \kappa(v_{ij}) + \kappa(w_{ij})^*] \geqslant 0\right\},$$
$$n = 1, 2, 3, \ldots .$$

It is trivial to verify that $\{P_{\kappa}^{(n)}\}$ does satisfy the axioms needed for an operator system.

NOTE. Let $W_\kappa = \left\{(\lambda \oplus v \oplus w^*) \in \tilde{V} : \lambda + \kappa(v) + \kappa(w)^* = 0\right\}$. One thing to notice is that the operator system $\left(\tilde{V}, \{P_\kappa^{(n)}\}_{n=1}^\infty\right)$ in fact means the quotient $\left(\tilde{V}/W_\kappa, \{P_\kappa^{(n)}\}_{n=1}^\infty\right)$.

It is easily seen that $\kappa_1 \preceq \kappa_2$ if and only if $P_{\kappa_2}^{(n)} \subseteq P_{\kappa_1}^{(n)}$ for all n, if and only if the canonical map $\left(\tilde{V}, \{P_{\kappa_2}^{(n)}\}_{n=1}^\infty\right) \longmapsto \left(\tilde{V}, \{P_{\kappa_1}^{(n)}\}_{n=1}^\infty\right)$ is completely positive. Notice that in the implication of $\kappa_1 \prec \kappa_2$ from $P_{\kappa_2}(n) \subseteq P_{\kappa_1}(n)$ for all n, we need a well-defined φ. This is insured because, if $\lambda + \kappa_2(v) + \kappa_2(w)^* = 0$, then it is in $P_{\kappa_2}^{(1)} \cap (-P_{\kappa_2}^{(1)})$, so $\lambda + \kappa_1(v) + \kappa_1(w)^*$ is in $P_{\kappa_1}^{(1)} \cap (-P_{\kappa_1}^{(1)})$. Thus $\lambda + \kappa_1(v) + \kappa_1(w)^* = 0$.

PROPOSITION 12. *SP_κ is completely order isomorphic to* $\left(\tilde{V}, \{P_\kappa^{(n)}\}_{n=1}^\infty\right)$.

Proof. Notice that $\psi(\lambda \oplus v \oplus w^*) = \lambda + \kappa(v) + \kappa(w)^*$ is well-defined and one-to-one when $\tilde{V}$ is considered to be $\tilde{V}/W_\kappa$. ∎

Before proceeding, we first remark that any completely positive map φ is completely bounded and $\|\varphi\|_{\mathrm{cb}} = \|\varphi\| = \|\varphi(1)\|$ (see [11], 3.5). Hence, unital completely positive maps are completely contractive. The following is well known, we include it for completeness.

LEMMA 5. *A unital linear map between two operator systems $\varphi : S_1 \mapsto S_2$ is a complete order isomorphism if and only if it is a complete isometry.*

Proof. Since both φ and φ^{-1} are completely contractive, φ has to be a complete isometry. The converse follows immediately from [11], 2.12. ∎

We are now going to deal with families of representations of V. For any operator space V, the class of all representations may not be a set. But all possible operator system structures on $\tilde{V}$ is a set. So, if we identify the equivalent representations, the "class" becomes a "set".

THEOREM 7. *Any operator space V has a representation κ such that SP_κ is maximal in the partial order. We denote this operator system by $SP^V_{\max}$.*

Proof. Let $\{\kappa_\alpha\}_{\alpha\in\mathcal{A}}$ be the set of all non-equivalent representations of V. For any $\alpha \in \mathcal{A}$, let $\left(\tilde{V}, \{P_{\kappa_\alpha}^{(n)}\}_{n=1}^\infty\right)$ be the corresponding operator system structure on $\tilde{V}$. Define

$$P^{(n)} = \bigcap_{\alpha\in\mathcal{A}} P_{\kappa_\alpha}^{(n)} \qquad n = 1, 2, 3, \ldots .$$

We verify that $\{P^{(n)}\}_{n=1}^\infty$ will give an operator system structure on $\tilde{V}$.

(i) Since each $P_\alpha^{(n)}$ is a connected positive cone, the intersection $P^{(n)}$ is again a connected positive cone.

(ii) Let $\gamma = (\gamma_{ij})$ be any $m \times n$ scalar matrix, then

$$\gamma^* P^{(m)}\gamma = \gamma^* \left(\bigcap_{\alpha\in\mathcal{A}} P_{\kappa_\alpha}^{(m)}\gamma\right) = \bigcap_{\alpha\in\mathcal{A}} \gamma^* P_{\kappa_\alpha}^{(m)}\gamma \subseteq \bigcap_{\alpha\in\mathcal{A}} P_{\kappa_\alpha}^{(n)} = P^{(n)}.$$

(iii)

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}$$

is an order unit of $M_n((\tilde{V}, \{P_{\kappa_\alpha}^{(m)}\}_{m=1}^{\infty}))$ for each $\alpha \in \mathcal{A}$, and $P^{(m)} \subseteq P_{\kappa_\alpha}^{(m)}$ for all α, hence the above matrix automatically is an order unit of $M_n((\tilde{V}, \{P^{(m)}\}_{m=1}^{\infty}))$.

(iv) That $P^{(1)} \cap (-P^{(1)}) = \{0\}$ is obvious.

(v) To show that each $P^{(n)}$ is Archimedian, notice that

$$P^{(n)} = \left\{((\lambda_{ij} \oplus v_{ij} \oplus w_{ij}^*)) \in M_n(\tilde{V}) : \ (\lambda_{ij} + \kappa_\alpha(v_{ij}) + \kappa_\alpha(w_{ij})^*) \geqslant 0, \ \alpha \in \mathcal{A}\right\}.$$

For any $(x_{ij}) \in M_n((\tilde{V}, \{P^{(n)}\}_{n=1}^{\infty}))_h$, the subscript h means that (x_{ij}) is self-adjoint, and for any $(y_{ij}) \in P^{(n)}$, if

$$(-t)(y_{ij}) \leqslant (x_{ij})$$

for all $t > 0$, i.e.

$$(\tilde{\kappa}_\alpha(x_{ij})) + t(\tilde{\kappa}_\alpha(y_{ij})) \geqslant 0$$

for all $t > 0$ and for all $\alpha \in \mathcal{A}$, then $(\tilde{\kappa}_\alpha(x_{ij})) \geqslant 0$ for any α. So, $(x_{ij}) \in P^{(n)}$ by the definition of $P^{(n)}$.

Consequently, we obtain an operator system $(V, \{P^{(n)}\}_{n=1}^{\infty})$. Now let Φ : $(\tilde{V}, \{P^{(n)}\}) \mapsto \mathcal{B}(K)$ be a unital complete order isomorphism, then by Lemma 5, Φ is a complete isometry. Let $\kappa = \Phi|V$, then κ is a representation of V and $\tilde{\kappa} = \Phi$, and by the construction we see that $\kappa_\alpha \preceq \kappa$ for all $\alpha \in \mathcal{A}$. ∎

The following is an example of the maximal operator system spanned by an operator space.

EXAMPLE. Let $V = \mathrm{MAX}(\ell_n^1)$, where ℓ_n^1 is the n dimensional space with ℓ^1 norm. We claim that the maximal operator system $SP_{\max}^V$ is the operator system spanned by n free unitaries. First observe that for any representation κ of $\mathrm{MAX}(\ell_n^1)$, there are contractions $T_1, T_2, \ldots, T_n$ such that $\kappa(e_i) = T_i$, $i = 1, 2, \ldots, n$, where $\{e_i\}_{i=1}^n$ is the canonical basis. The T_i's can be dilated by

$$S_i = \begin{pmatrix} T_i & (1 - T_iT_i^*)^{\frac{1}{2}} \\ (1 - T_i^*T_i)^{\frac{1}{2}} & -T_i^* \end{pmatrix}, \quad i = 1, 2, \ldots, n,$$

so that the S_i's are unitaries. Observe that

$$\left\| \sum_{i=1}^n A_i \otimes S_i \right\| \geqslant \left\| \sum_{i=1}^n A_i \otimes T_i \right\|$$

for any $A_1, A_2, \ldots, A_n \in M_m$. So we easily see that the map $e_i \mapsto S_i$ $(i = 1, 2, \ldots, n)$ is also a representation of $\mathrm{MAX}(\ell_n^1)$, furthermore, the map $S_i \mapsto T_i$ $(i = 1, 2, \ldots, n)$ induces a unital completely positive map from the operator system spanned by $\{S_i\}_{i=1}^n$ onto the operator system spanned by $\{T_i\}_{i=1}^n$ since it is a compression. Hence the maximal operator system can be attained by unitary representations (i.e. e_i's go to unitaries).

Now let $F_n = C^*(U_1, U_2, \ldots, U_n)$, where $U_1, U_2, \ldots, U_n$ are free unitaries. Define $\rho : \mathrm{MAX}(\ell_n^1) \mapsto F_n$ via

$$\rho(e_i) = U_i, \quad i = 1, 2, \ldots, n.$$

Then ρ is an isometry : for any $(\lambda_1, \ldots, \lambda_n) \in \ell_n^1$

$$\|\rho(\lambda_1, \ldots, \lambda_n)\| = \left\| \sum_{i=1}^n \lambda_i U_i \right\| \leqslant \sum_{i=1}^n |\lambda_i| = \|(\lambda_1, \ldots, \lambda_n)\|.$$

On the other hand, for any isometry $\varphi : \ell_n^1 \mapsto \mathcal{B}(H)$ with $\varphi(e_i)$ unitaries, there is an onto $*$-homomorphism $\psi : F_n \mapsto C^*(\varphi(\ell_n^1))$ such that the diagram

$$\begin{array}{ccc} \ell_n^1 & & \\ \rho \downarrow & \searrow^{\varphi} & \\ F_n & \underset{\psi}{\longrightarrow} & C^*(\varphi(\ell_n^1)) \end{array}$$

commutes, since F_n is free. But a $*$-homomorphism is necessarily a contraction, thus ρ is an isometry. If the above φ is a representation of $\mathrm{MAX}(\ell_n^1)$, and since

$$\|(\rho(x_{ij}))\| \geqslant \|(\varphi(x_{ij}))\|$$

for all $(x_{ij}) \in M_m(\ell_n^1)$, we see that ρ is also a representation of $\mathrm{MAX}(\ell_n^1)$. By the freeness of F_n, the map

$$\Phi\left(\lambda_0 + \sum_{i=1}^{n} \lambda_i \rho(e_i) + \sum_{i=1}^{n} \mu_i \rho(e_i)^*\right) = \lambda_0 + \sum_{i=1}^{n} \lambda_i \varphi(e_i) + \sum_{i=1}^{n} \mu_i \varphi(e_i)^*$$

is well-defined and in fact it is a restriction of the onto $*$-homomorphism induced from the canonical map $\rho(e_i) \mapsto \varphi(e_i)$. Hence $\Phi|SP_\rho$ is unital completely positive. So, $SP_\rho = SP^V_{\max}$. ∎

THEOREM 8. *Let κ' be a representation of V such that $SP_{\kappa'}$ is completely order isomorphic to $SP^V_{\max}$. Then for any representation $\kappa : V \mapsto B(H)$, there is a Hilbert space $K \supseteq H$ and a $*$-homomorphism $\pi : C^*(SP_{\kappa'}) \mapsto B(K)$ such that*

$$\kappa(v) = P_H \pi(v)\big|H \qquad v \in V.$$

Conversely, if for a representation κ', $C^*(SP_{\kappa'})$ satisfies the above universal property, then $SP_{\kappa'}$ is completely order isomorphic to $SP^V_{\max}$.

Proof. Let $SP^V_{\max} = SP_{\kappa'}$, then

$$\Phi(\lambda + \kappa'(v) + \kappa'(w)^*) = \lambda + \kappa(v) + \kappa(w)^*$$

defines a unital completely positive map. By the extension theorem for completely positive maps, there is an extension $\tilde{\Phi} : C^*(SP_{\kappa'}) \mapsto B(H)$ which is completely positive. By Stinespring's theorem, there is a $*$-homomorphism $\pi : C^*_e(SP_{\kappa'}) \mapsto B(K)$ such that $\tilde{\Phi}(x) = P_H \pi(x)|H$ for all $x \in C^*_e(SP^V_{\max})$. Consequently $\kappa(v) = P_H \pi(v)|H$ for all $v \in V$.

Conversely, if $C^*(SP_\kappa)$ satisfies the universal property, suppose $SP^V_{\max} = SP_{\kappa'}$, where $\kappa' : V \mapsto B(H)$. Suppose $K \supseteq H$ and

$$\kappa'(v) = P_H \pi(v)\big|H \qquad v \in V$$

where $\pi : C^*_e(SP_\kappa) \mapsto B(K)$ is $*$-homomorphism. Then obviously $P_H \pi(SP_\kappa)\big|H = SP_{\kappa'}$. Thus SP_κ is completely order isomorphic to $SP^V_{\max}$. ∎

There is also a natural question about the existence of minimal operator system. The following example gives a negative answer to this question.

EXAMPLE. Let $V = \mathbf{C}$. Let κ be a minimal representation of V. $\kappa_1(\lambda) = \lambda$ defines another representation of V. Then map $\lambda \mapsto \kappa(\lambda)$ is a unital completely positive and onto. On the other hand, $\kappa_2(\lambda) = -\lambda$ also defines a representation of V, but $-\lambda \mapsto \kappa(\lambda)$ can not be a positive map. Hence $\mathbf{C}$ can have no minimal representation.

The author thanks the referee for the helpful criticisms.

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Received June 7, 1994; revised October 24, 1994.