J. OPERATOR THEORY
33(1995), 353–361

AN APPLICATION OF TAUBERIAN THEOREMS TO TOEPLITZ OPERATORS

BORIS KORENBLUM and KEHE ZHU

Communicated by Norberto Salinas

ABSTRACT. We prove two Tauberian theorems and then use them to obtain characterizations of compact Toeplitz operators with radial symbols on the Bergman space of the unit disk.

KEYWORDS: *Tauberian theorems, Toeplitz operators, Bergman spaces, Berezin trasform.*

AMS SUBJECT CLASSIFICATION: Primary 47B35, 40E05; Secondary 44A15, 44A60.

1. INTRODUCTION

Let $L^2_a(\mathbf{D})$ be the Bergman space on the open unit disk $\mathbf{D}$. Thus $L^2_a(\mathbf{D})$ consists of analytic functions f in $\mathbf{D}$ with

$$\|f\|^2 = \int_{\mathbf{D}} |f(z)|^2 \, \mathrm{d}A(z) < +\infty,$$

where $\mathrm{d}A$ is the normalized area measure on $\mathbf{D}$. The Bergman projection, denoted P, is then the orthogonal projection from $L^2(\mathbf{D}, \mathrm{d}A)$ onto $L^2_a(\mathbf{D})$.

For f in $L^\infty(\mathbf{D})$ we consider the Toeplitz operator $T_f : L^2_a(\mathbf{D}) \to L^2_a(\mathbf{D})$ defined by $T_f g = P(fg)$, $g \in L^2_a(\mathbf{D})$. It is clear that T_f is bounded with $\|T_f\| \leqslant \|f\|_\infty$. A natural and fundamental question arises: When is the Toeplitz operator T_f compact on $L^2_a(\mathbf{D})$?

Although a complete answer to the question above is still lacking, several special cases have been well understood; these earlier results (outlined below) also serve as the motivation for the present note.

First, if $f : \mathbf{D} \to \mathbf{C}$ extends continuously to the maximal ideal space of $H^\infty(\mathbf{D})$, then T_f is compact on $L^2_a(\mathbf{D})$ if and only if $f \in C_0(\mathbf{D})$; see [1]. Here the space $C_0(\mathbf{D})$ consists of continuous functions g in $\mathbf{D}$ with $g(z) \to 0$ as $|z| \to 1^-$. In particular, if f extends continuously to the closed disk $\overline{\mathbf{D}}$, or if f is harmonic in $\mathbf{D}$, then T_f is compact on $L^2_a(\mathbf{D})$ if and only if $f \in C_0(\mathbf{D})$.

Second, if $f \geqslant 0$, then the following conditions are equivalent (see [3], [7] or [8]):

(i) T_f is compact on $L^2_a(\mathbf{D})$.

(ii) $\widehat{f}$ is in $C_0(\mathbf{D})$, where

$$\widehat{f}(z) = \frac{1}{|S_z|} \int_{S_z} f(w)\, \mathrm{d}A(w), \qquad z \in \mathbf{D}.$$

Here

$$S_z = \{w \in \mathbf{D} : |z| < |w| < 1, |\arg z - \arg w| < 2\pi(1 - |z|)\}$$

is the Carleson square at z and $|S_z|$ is the $\mathrm{d}A$-measure of S_z.

(iii) $\widetilde{f}$ is in $C_0(\mathbf{D})$, where

$$\widetilde{f}(z) = \int_{\mathbf{D}} f(w) \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4}\, \mathrm{d}A(w), \qquad z \in \mathbf{D}.$$

The function $\widetilde{f}$ is called the Berezin transform of f.

(iv) For every (or some) $0 < r < 1$ the function $\widehat{f}_r$ is in $C_0(\mathbf{D})$, where

$$\widehat{f}_r(z) = \frac{1}{|D(z,r)|} \int_{D(z,r)} f(w)\, \mathrm{d}A(w), \qquad z \in \mathbf{D}.$$

Here

$$D(z,r) = \left\{ w \in \mathbf{D} : \left| \frac{z - w}{1 - z\overline{w}} \right| < r \right\}$$

is the pseudo-hyperbolic disk "centered at" z with "radius" r.

Third, if φ_a is the Möbius map of the disk that interchanges the origin and the point a, then T_f is compact if and only if $\|P(f \circ \varphi_a)\| \to 0$ as $|a| \to 1^-$; see [5]. Although this characterizes the compactness of T_f for arbitrary $f \in L^\infty(\mathbf{D})$, it is of limited use because the condition involved is hardly checkable except in very special cases.

In this note we look at another special class of bounded functions in $\mathbf{D}$, namely, the class of bounded radial functions, for which we shall be able to characterize the compactness of Toeplitz operators in terms of a geometric condition. The following is our main result.

THEOREM. *Let f be a bounded radial function in $\mathbb{D}$. Then the following conditions are equivalent:*

(i) $T_f : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ *is compact.*

(ii) $\widetilde{f}(z) \to 0$ *as* $|z| \to 1^-$.

(iii) $\frac{1}{1-x}\int_x^1 f(t)\,dt \to 0$ *as* $x \to 1^-$.

2. TWO TAUBERIAN THEOREMS

The proof of our main theorem will depend on two Tauberian type theorems which we prove in this section. The quantities $\widetilde{f}(z)$ and $\frac{1}{1-x}\int_x^1 f(t)\,dt$ are certain averages of f, so Tauberian theorems naturally play a role in the study of them. After all, Tauberian theorems are most effective in dealing with averages.

LEMMA 1. *Suppose $\lambda \geqslant 0$ and*

$$\lim_{t\to 1^-}(1-t)^\lambda \sum_{n=0}^{\infty} a_n t^n = 0.$$

If $a_n \geqslant -C(n+1)^{\lambda-1}$ for some constant $C > 0$ and all $n \geqslant 0$, then

$$\lim_{n\to+\infty} \frac{\sum\limits_{k=0}^{n} a_k}{(n+1)^\lambda} = 0.$$

Proof. . This is well-known; see page 32 of [4]. ∎

THEOREM 2. *Suppose $f \in L^\infty[0,1)$ and*

$$a_n(f) = (n+1)\int_0^1 f(r)\,r^n\,dr, \qquad n \geqslant 0.$$

Then $a_n(f) \to 0$ as $n \to +\infty$ if and only if

$$\lim_{t\to 1^-}(1-t)^2 \sum_{n=0}^{\infty}(n+1)\,a_n(f)\,t^n = 0.$$

Proof. Since

$$(1-t)^2\sum_{n=0}^{\infty}(n+1)\,t^n = 1$$

for every $t \in [0,1)$, an easy ε-δ-N argument shows that $a_n(f) \to 0$ $(n \to +\infty)$ implies

$$\lim_{t\to 1^-}(1-t)^2\sum_{n=0}^{\infty}(n+1)\,a_n(f)\,t^n = 0.$$

To prove the other implication, we need to use Lemma 1. So assume

$$\lim_{t\to 1^-}(1-t)^2\sum_{n=0}^{\infty}(n+1)\,a_n(f)\,t^n = 0.$$

Rearranging terms, we obtain

$$\lim_{t\to 1^-}(1-t)\left[a_0(f)+\sum_{n=1}^{\infty}\left[(n+1)a_n(f)-na_{n-1}(f)\right]t^n\right] = 0.$$

Recall from the definition of $a_n(f)$ that

$$\begin{aligned}(n+1)a_n(f)-na_{n-1}(f) &= (n+1)^2\int_0^1 f(r)\,r^n\,dr - n^2\int_0^1 f(r)\,r^{n-1}\,dr\\ &= (2n+1)\int_0^1 f(r)\,r^n\,dr + n^2\int_0^1 f(r)(r-1)\,r^{n-1}\,dr.\end{aligned}$$

Elementary calculus shows that the above is bounded in n if f is bounded. Using Lemma 1 (with $\lambda = 1$) we see that the condition

$$\lim_{t\to 1^-}(1-t)^2\sum_{n=0}^{\infty}(n+1)\,a_n(f)\,t^n = 0$$

implies

$$\lim_{n\to+\infty}\frac{a_0(f)+\sum_{k=1}^{n}\left[(k+1)a_k(f)-ka_{k-1}(f)\right]}{n+1} = 0,$$

or

$$\lim_{n\to+\infty} a_n(f) = 0.$$

This finishes the proof of the theorem. ∎

LEMMA 3. *Let* $K_1, K_2 \in L^1[0,+\infty)$. *If*

$$\int_0^{+\infty} K_j(t)t^{ix}\,dt \neq 0$$

for $j = 1, 2$ and all real x, then the two conditions

$$\lim_{\varepsilon\to 0+} \int_0^{+\infty} K_1(t)g(\varepsilon t)\,\mathrm{d}t = 0, \quad \lim_{\varepsilon\to 0+} \int_0^{+\infty} K_2(t)g(\varepsilon t)\,\mathrm{d}t = 0$$

are equivalent for $g \in L^\infty[0, +\infty)$.

Proof. This is a version of the Tauberian theorem of Karamata-Wiener. For example, it is a consequence of Theorem VIII in [6] via a logarithmic change of variables. ∎

THEOREM 4. *Suppose f is in $L^\infty[0, 1)$. Then*

$$\lim_{x\to 1-} \frac{1}{1-x} \int_x^1 f(t)\,\mathrm{d}t = 0$$

if and only if

$$\lim_{n\to+\infty} (n+1) \int_0^1 f(t)\,t^n\,\mathrm{d}t = 0.$$

Proof. Let

$$h(x) = \frac{1}{1-x} \int_x^1 f(t)\,\mathrm{d}t, \qquad x \in [0, 1).$$

An application of Fubini's theorem shows that

$$(n+2)(n+1) \int_0^1 h(x)\,x^n(1-x)\,\mathrm{d}x = (n+2) \int_0^1 f(t)\,t^{n+1}\,\mathrm{d}t$$

for all $n \geqslant 0$. This clearly gives the "only if" part of the result.

To prove the "if" part of the desired result, we shall need to use Lemma 3. Since f is bounded and the $L^1[0, 1)$-norm of $t^n - t^{n+1}$ is $\mathrm{O}(\frac{1}{n^2})$, the condition

$$\lim_{n\to+\infty} (n+1) \int_0^1 f(t)\,t^n\,\mathrm{d}t = 0$$

is equivalent to (just compare t^s and t^n for $n < s \leqslant n+1$)

$$\lim_{s\to+\infty} s \int_0^1 f(t)\,t^s\,\mathrm{d}t = 0.$$

Changing variables, we get

$$s\int_0^1 f(t)\,t^s\,\mathrm{d}t = s\int_0^1 f(1-t)(1-t)^s\,\mathrm{d}t$$

$$= \int_0^s f\left(1-\frac{t}{s}\right)\left(1-\frac{t}{s}\right)^s\,\mathrm{d}t = \int_0^{+\infty} K_\varepsilon(t)g(\varepsilon t)\,\mathrm{d}t,$$

where $\varepsilon = 1/s$,

$$g(t) = \begin{cases} f(1-t), & 0 \leqslant t \leqslant 1 \\ 0, & t > 1, \end{cases}$$

and

$$K_\varepsilon(t) = \begin{cases} (1-\varepsilon t)^{\frac{1}{\varepsilon}}, & 0 \leqslant t \leqslant \frac{1}{\varepsilon} \\ 0, & t > \frac{1}{\varepsilon}. \end{cases}$$

Since $0 \leqslant K_\varepsilon(t) \leqslant \mathrm{e}^{-t}$, dominated convergence implies that $K_\varepsilon(t) \to \mathrm{e}^{-t}$ in $L^1[0,+\infty)$ as $\varepsilon \to 0^+$. Therefore, the condition

$$\lim_{s\to+\infty} s\int_0^1 f(t)\,t^s\,\mathrm{d}t = 0$$

is equivalent to

$$\lim_{\varepsilon\to 0+} \int_0^{+\infty} \mathrm{e}^{-t}\,g(\varepsilon t)\,\mathrm{d}t = 0.$$

Similarly, the condition

$$\lim_{x\to 1-} \frac{1}{1-x}\int_x^1 f(t)\,\mathrm{d}t = 0$$

is equivalent to

$$\lim_{\varepsilon\to 0+} \int_0^{+\infty} \chi_{[0,1]}(t)g(\varepsilon t)\,\mathrm{d}t = 0.$$

The desired result now follows from Lemma 3. ∎

3. PROOF OF THE MAIN THEOREM

We now begin the proof of our main result. First note that when f is radial, the Berezin transform of f can be rewritten as

$$\begin{aligned}\widetilde{f}(z) &= (1-|z|^2)^2 \int_{\mathbf{D}} \frac{f(w)\,\mathrm{d}A(w)}{|1-z\overline{w}|^4} \\ &= (1-|z|^2)^2 \int_{\mathbf{D}} f(w) \left|\sum_{n=0}^{\infty}(n+1)z^n\overline{w}^n\right|^2 \mathrm{d}A(w) \\ &= 2(1-|z|^2)^2 \sum_{n=0}^{\infty}(n+1)^2|z|^{2n} \int_0^1 f(r)\,r^{2n+1}\,\mathrm{d}r \\ &= (1-|z|^2)^2 \sum_{n=0}^{\infty}(n+1)|z|^{2n} \int_0^1 f(\sqrt{r})\,r^n\,\mathrm{d}r.\end{aligned}$$

Let $f^*(r) = f(\sqrt{r})$, $0 \leqslant r < 1$, and apply Theorem 2. We see that $\widetilde{f}(z) \to 0$ as $|z| \to 1^-$ if and only if $a_n(f^*) \to 0$ as $n \to +\infty$, where

$$a_n(f^*) = (n+1)\int_0^1 f^*(r)r^n\,\mathrm{d}r, \qquad n \geqslant 0.$$

On the other hand, the following functions constitute a natural orthonormal basis for $L^2_{\mathrm{a}}(\mathbf{D})$:

$$e_n(z) = \sqrt{n+1}\,z^n, \qquad n = 0, 1, 2, \ldots .$$

It is easy to check that T_f is a diagonal operator with respect to the basis above when f is a radial function. In fact, the diagonal elements of T_f under the basis $\{e_n\}$ are given by

$$\langle T_f e_n, e_n\rangle = (n+1)\int_0^1 f(\sqrt{r})\,r^n\,\mathrm{d}r = a_n(f^*).$$

Therefore, for a radial function f the operator T_f is compact on $L^2_{\mathrm{a}}(\mathbf{D})$ if and only if $a_n(f^*) \to 0$ as $n \to +\infty$. Combining this with the previous paragraph we have proved the equivalence of (i) and (ii) in the main theorem.

Finally, using Theorem 4 and the remark in the last paragraph we see that T_f is compact if and only if

$$\lim_{x\to 1^-} \frac{1}{1-x} \int_x^1 f^*(t)\,dt = 0.$$

Since f is bounded and

$$\frac{1}{1-x} \int_x^1 f^*(t)\,dt = \frac{2}{1+\sqrt{x}} \left[\frac{1}{1-\sqrt{x}} \int_{\sqrt{x}}^1 f(t)(t-1)\,dt + \frac{1}{1-\sqrt{x}} \int_{\sqrt{x}}^1 f(t)\,dt \right],$$

we see that the first equation in this paragraph is equivalent to

$$\lim_{x\to 1^-} \frac{1}{1-x} \int_x^1 f(t)\,dt = 0.$$

This proves the equivalence of (i) and (iii) in the main theorem.

4. CONCLUDING REMARKS

It is easy to see that our result also holds for weighted Bergman spaces on the open unit disk with radial weights like $(\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$. Our result also holds for Toeplitz operators on the Fock space (see [2]) with some obvious adjustments.

Our main theorem also suggests some natural problems. For example, we can ask whether the requirement $f \in L^\infty(\mathbf{D})$ is necessary. We can also ask the companion question for boundedness of Toeplitz operators with radial symbols. We can even ask the question of membership in the Schatten ideals for Toeplitz operators with radial symbols. The conjectures are obvious, but the techniques used in this note do not work in the case of unbounded functions.

Research supported by the National Science Foundation.

REFERENCES

1. S. AXLER, Bergman spaces and their operators, *Surveys of Some Recent Results in Operator Theory*, vol. 1, (J.B. Conway and B.B. Morrel, editors), Pitman Res. Notes Math. Ser. vol. 171, 1988, pp. 1–50.
2. C. BERGER, L. COBURN, Toeplitz operators on the Segal-Bargmann space, *Trans. Amer. Math. Soc.* **301**(1987), 813–829.

3. D. LUECKING, Trace ideal criteria for Toeplitz operators, *J. Funct. Anal.* **73**(1987), 345–368.
4. A.G. POSTNIKOV, *Tauberian Theory and its Applications*, Proc. Steklov Inst. Math, vol. 2, Amer. Math. Soc., 1980.
5. K. STROETHOFF, D. ZHENG, Toeplitz and Hankel operators on Bergman spaces, *Trans. Amer. Math. Soc.* **329**(1992), 773–794.
6. N. WIENER, Tauberian theorems, *Ann. of Math. (2)* **33**(1932), 1–100.
7. K. ZHU, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, *J. Operator Theory* **20**(1988), 329–357.
8. K. ZHU, *Operator Theory in Function Spaces*, Marcel Dekker, New York 1990.

B. KORENBLUM and K. ZHU
Department of Mathematics and Statistics
State University of New York at Albany
Albany, New York 12222
U.S.A.

Received June 24, 1994; revised October 17, 1994.