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COMPOSITION OPERATORS BETWEEN BERGMAN SPACES ON CONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. We prove a Carleson measure theorem for the Bergman spaces associated with a strictly pseudoconvex domain in $\mathbb{C}^n$. We use the theorem to study composition operators between Bergman spaces associated with a strongly convex domain in $\mathbb{C}^n$.

KEYWORDS: *Bergman space, composition operators.*

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0. INTRODUCTION

If Ω is a smoothly bounded simply connected domain in $\mathbb{C}^1$, then every holomorphic self-map of Ω induces a bounded (linear) composition operator of the associated classical (i.e., Hardy or Bergman) function spaces into themselves. A similar situation does not occur in $\mathbb{C}^n$ for $n \geqslant 2$ ([4], [15] for example). Sufficient conditions are known for a holomorphic self-map of the ball in $\mathbb{C}^n$ to induce a bounded composition operator on the associated Hardy or Bergman spaces ([13], [15]). Moreover, there are polynomial self-maps of the ball in $\mathbb{C}^2$ which induce unbounded composition operators on the associated Hardy spaces ([4]).

It was shown by MacCluer and Mercer ([14]) that a holomorphic self-map of a bounded strongly convex domain $\Omega \subset \mathbb{C}^n$ induces a bounded composition operator from the Hardy space $H^p(\Omega)$ into the Bergman space $A^p_{n-2}(\Omega)$. (See Section 1 for precise definitions.) The proof of this result uses a well-known Carleson measure theorem of Hörmander ([5]) about H^p spaces associated with a strictly pseudoconvex domain in $\mathbb{C}^n$.

In Section 3 we show that in the same situation the induced composition operator maps the Bergman space $A^p_\alpha(\Omega)$ into the Bergman space $A^p_{\alpha+n-1}(\Omega)$ boundedly — a result which recovers the known situation when $n = 1$. Our proof uses a version of Hörmander's theorem about Bergman spaces associated with a strictly pseudoconvex domain in $\mathbf{C}^n$ (see Section 2) which we believe is of independent interest; it generalizes a result of Cima and Wogen ([3]) which studied Carleson measures on the ball.

1. NOTATION AND PREPARATORY LEMMAS

Let $\Omega \Subset \mathbf{C}^n$ be a smoothly bounded strictly pseudoconvex domain, given by defining function $\rho : \mathbf{C}^n \to \mathbf{R}$. For $\xi \in \partial\Omega$, denote by T_ξ the complex tangent space to $\partial\Omega$ at ξ. As in [5], we define

$$A_\Omega(\xi, t) = \Big\{ z \in \Omega : \inf_{\substack{|w-\xi|<t^{1/2} \\ w \in T_\xi}} \operatorname{dist}(z, w) < t \Big\} \qquad (t > 0),$$

and $B_\Omega(\xi, t) = \overline{A_\Omega(\xi, t)} \cap \partial\Omega$. We write simply $A(\xi, t)$ or $B(\xi, t)$ when the context is clear.

LEMMA 1. ([5]) (i) *If $\phi : \Omega_1 \to \Omega_2$ is biholomorphic then the sets $A_{\Omega_1}(\xi, t)$ and $A_{\Omega_2}(\phi(\xi), t)$ are comparable,* i.e., *there are constants $c_1, c_2 > 0$ such that*

$$A_{\Omega_1}(\xi, c_1 t) \subset A_{\Omega_2}(\phi(\xi), t) \subset A_{\Omega_1}(\xi, c_2 t) \ \forall \xi \in \partial\Omega_1, \ t > 0 \text{ small.}$$

(ii) *We have $\sigma(B(\xi, t)) \cong t^n$, where $d\sigma$ is surface measure on $\partial\Omega$.*

For $\alpha \geqslant 0$ and $dm = dm_n$ = volume measure in $\mathbf{C}^n$, we write $d\mu_\alpha(z) = d^\alpha_\Omega(z) dm(z) = (\operatorname{dist}(z, \partial\Omega))^\alpha dm(z)$.

LEMMA 2. *For $t > 0$ small, we have $\mu_\alpha(A(\xi, t)) \cong t^{\alpha+n+1} \quad \forall \xi \in \partial\Omega$.*

Proof. For $s > 0$, let $\Omega_s = \{\rho + s < 0\} \subset \Omega$. If s is sufficiently small, the projection $\Pi : \partial\Omega_s \to \partial\Omega$ along inner normals to $\partial\Omega$ is well defined. By Lemma 1(i), and Narasimhan's Lemma ([6]), $A(\xi, t)$ is comparable to $Q(\xi, t) = \{z \in B_s(\xi, t) : 0 < s < t\}$, where $B_s(\xi, t) = B_{\Omega_s}(\Pi^{-1}(\xi) \cap \partial\Omega_s, t)$.

Thus

$$\mu_\alpha(A(\xi, t)) \cong \mu_\alpha(Q(\xi, t)) \cong \int_0^t \int_{B_s} d^\alpha_\Omega(z)\, d\sigma_{\Omega_s}(z)\, ds \cong \int_0^t s^\alpha \int_{B_s} d\sigma_{\Omega_s}\, ds$$

$$\cong \int_0^t s^\alpha [(1 + O(s))t]^n\, ds \qquad \text{by Lemma 1 (ii)}$$

$$\cong t^{n+\alpha+1}.$$

The constants are independent of $\xi \in \partial\Omega$. ∎

Denote by k_Ω the Kobayashi (pseudo–) distance (see [6]) on Ω. For $z \in \Omega$ and $R > 0$, let $E(z,R) = \{w \in \Omega : k_\Omega(z,w) < R\}$. We write $E(z) = E(z,1)$ and $E^2(z) = \bigcup[E(w) : E(w) \cap E(z) \neq \phi]$.

For $z \in \Omega$ near $\partial\Omega$, denote by $P_z(c_1 \mathrm{d}_\Omega(z), c_2\sqrt{\mathrm{d}_\Omega(z)})$ the polydisk centered at z with radius $c_1 \mathrm{d}_\Omega(z)$ in the complex normal direction for z, and radii $c_2\sqrt{\mathrm{d}_\Omega(z)}$ in the complex tangential directions for z.

LEMMA 3. ([7], [11]) *For $z \in \Omega$ near $\partial\Omega$, $E(z,R)$ is comparable to the polydisk $P_z(c_1 \mathrm{d}_\Omega(z), c_2\sqrt{\mathrm{d}_\Omega(z)})$, where c_1 and c_2 depend only on R. Thus $m(E(z)) \cong \mathrm{d}_\Omega^{n+1}(z)$.*

For $p > 0$ we define the (weighted) Bergman spaces associated with Ω:

$$A^p_\alpha(\Omega) = \left\{ f \in H(\Omega) : \int_\Omega |f|^p \,\mathrm{d}\mu_\alpha < +\infty \right\}.$$

2. THE MAIN RESULT

THEOREM 4. *Let μ be a positive measure on Ω. We have*

$$\mu(A(\xi,t)) \lesssim \mu_\alpha(A(\xi,t)) \iff \int_\Omega |f|^p \,\mathrm{d}\mu \lesssim \int_\Omega |f|^p \,\mathrm{d}\mu_\alpha \quad \forall f \in A^p_\alpha(\Omega).$$

Proof of sufficiency. According to [12] it is sufficient to check the following conditions:

(i) $\chi_{E(z)}(w)$ is measurable on $\Omega \times \Omega$.

(ii) $m(E^2(z)) \lesssim m(E(z))$.

(iii) $\mathrm{d}^\alpha_\Omega(u) \lesssim \mathrm{d}^\alpha_\Omega(w)$ whenever $u, w \in E(z)$.

(iv) $|f(z)|^p \lesssim (m(E(z)))^{-1} \int_{E(z)} |f|^p \,\mathrm{d}m$.

(v) $\mu(E(z)) \lesssim \mu_\alpha(E(z))$.

Condition (i) holds since k_Ω is continuous on $\Omega \times \Omega$. Also, $E^2(z) = E(z,3)$, and so (ii) holds by Lemma 3. Condition (iii) follows from the triangle inequality for k_Ω and the following estimate ([1]): Fix $z_0 \in \Omega$. There are constants $c_1, c_2 \in \mathbf{R}$ such that $c_1 - \frac{1}{2}\log \mathrm{d}_\Omega(z) \leqslant k_\Omega(z_0, z) \leqslant c_2 - \frac{1}{2}\log \mathrm{d}_\Omega(z)$. By the Cauchy formula on a polydisk, we have $|f(z)|^p \lesssim (m(P_z))^{-1} \int_{P_z} |f|^p \,\mathrm{d}m$; then (iv) follows from Lemma 3.

For condition (v), let $z \in \Omega$ near $\partial\Omega$. Pick $\xi \in \partial\Omega$ so that $|\xi - z| = d_\Omega(z)$. By Lemma 3, $E(z) \subset A(\xi, t)$, where t is proportional to $d_\Omega(z)$. By hypothesis we have $\mu(E(z)) \leqslant \mu(A(\xi,t)) \lesssim \mu_\alpha(A(\xi,t))$. Now by Lemma 2 and 3, we have $m(A(\xi,t)) \cong t^{n+1} \cong d_\Omega^{n+1}(z) \cong m(E(z))$ and thus $\mu_\alpha(A(\xi,t)) \lesssim \mu_\alpha(E(z))$ and we are done.

We remark that [11] uses Luecking's Theorem ([12]) in a similar fashion.

Proof of necessity. We first prove the assertion in case Ω is in addition strongly convex; our proof is motivated by [5]. Fix $\xi \in \partial\Omega$. By Lemma 1 and [8], [9] we may assume that $\xi = (1, 0, 0, \ldots, 0)$ and that in a neighborhood U of ξ, Ω is given by $x_1 > \varphi(x_2, \ldots, x_N) - 1$ ($x_1 = \operatorname{Re} z_1$, $x_2 = \operatorname{Im} z_1, \ldots, x_N = \operatorname{Im} z_n$ and $N = 2n$), where φ is a positive definite quadratic form in $\widehat{x} = (x_2, \ldots, x_N)$, plus terms which are $o(|\widehat{x}|^2)$. Moreover, we have

$$(*) \qquad \xi \in \overline{\Omega} \text{ and } \xi_1 = 1 \Leftrightarrow \xi = (1, 0, \ldots, 0)$$

(cf. Lempert's Theorem 5 in [14]).

For $t > 0$ the function $f_t(z) = (1 + t - z_1)^{-k}$ is holomorphic on Ω; k is a positive integer to be chosen. We assume that t is small, so that $A(\xi, t) \subset U$. Here we have $A = A(\xi, t) = \{z \in \Omega : |1 - z_1| < t\}$. Therefore

$$t^{-kp}\mu(A(\xi,t)) = t^{-kp}\int_A d\mu \lesssim \int_A |f_t|^p\, d\mu \lesssim \int_\Omega |f_t|^p\, d\mu_\alpha \text{ by hypothesis.}$$

Now

$$\begin{aligned}
\int_\Omega |f_t|^p\, d\mu_\alpha &\lesssim C + \int_A |1 + t - z_1|^{-pk}\, d\mu_\alpha(z) \\
&\lesssim C + \int_0^t s^\alpha \int_{B_s} \left(|x_2| + t + |x_1 + 1|\right)^{-kp} d\sigma_{\Omega_s}(x)\, ds \\
&\lesssim C + \int_0^t s^\alpha \int_{B_s} \left(|x_2| + t + |\beta(s)\widehat{x}|^2\right)^{-kp} d\sigma_{\Omega_s}(x)\, ds \\
&\lesssim C + \int_0^t s^\alpha \int \left(t + |\beta(s)\widehat{\widehat{x}}|^2\right)^{1-kp} d\widehat{\widehat{x}}\, ds \\
&\lesssim C + t^{n-kp}\int_0^t s^\alpha (1 + O(s))\, ds \\
&\lesssim C + t^{n-kp} t^{\alpha+1}
\end{aligned}$$

where $\beta(s) = 1 + O(s)$, $\hat{\hat{x}} = (x_3, \ldots, x_N)$, and $2(kp-1) > 2n-2$, i.e., $kp > n$. Then by Lemma 2 we have $\mu(A(\xi,t)) \lesssim t^{\alpha+n+1} \cong \mu_\alpha(A(\xi,t))$. The constants depend continuously on $\xi \in \partial\Omega$.

In case Ω is merely strictly pseudoconvex, we may still assume (by Narasimhan's Lemma) that Ω has the special form above in a neighborhood U of a boundary point — except that $(*)$ may no longer hold. So from here we proceed as in [5]: we can define functions f_t holomorphic on Ω analogous to those above, using a solution to the $\overline{\partial}$ equation on a strictly pseudoconvex domain $\tilde{\Omega} \supset \Omega$. The well-known estimate ([6]): $u \in C^1(\tilde{\Omega}) \Rightarrow$

$$\sup_{\Omega} |u| \lesssim \|u\|_{L^2(\tilde{\Omega})} + \|\overline{\partial} u\|_{L^\infty(\tilde{\Omega})}$$

plays an important role. The rest of the argument is the same as in the strongly convex case. ∎

3. APPLICATION TO COMPOSITION OPERATORS

Throughout this section $\Omega \Subset \mathbb{C}^n$ is smoothly bounded and strongly convex. Fix $z_0 \in \Omega$. For each $x \in \partial\Omega$ there is a unique extremal map (with respect to k_Ω) $\varphi_x : \overline{\Delta} \to \overline{\Omega}$ such that $\varphi_x(0) = z_0$, $\varphi_x(1) = x$ ([8], [2]). There is also a map $\Psi : \overline{\Omega} \to \overline{\mathbb{B}}$ (with $\Psi(z_0) = 0$) called the spherical representation ([10]) which maps extremal disks $\varphi_x(\Delta)$ onto slices through the origin. Ψ is a diffeomorphism away from z_0. The spherical representation and the estimate ([8], [1]): $\mathrm{d}_\Omega(\varphi(\lambda)) \cong 1 - |\lambda|$ for *any* extremal map $\varphi : \Delta \to \Omega$ with $\varphi(0) = z_0$ yield the following (cf. [14], Lemma 1):

LEMMA 5. *Let $f \in L^1(\Omega)$, with support away from a neighborhood of z_0. There is an $\varepsilon > 0$ such that*

$$\int_\Omega f \, \mathrm{d}\mu_\alpha \cong \int_\varepsilon^1 r^{2n-1} \int_{\partial\Omega} \int_0^{2\pi} f \circ \varphi_x(re^{i\theta}) \, \mathrm{d}\theta \, \mathrm{d}\sigma(x) (1-r)^\alpha \, \mathrm{d}r.$$

Let $\phi : \Omega \to \Omega$ be holomorphic. The composition operator $C_\phi : H(\Omega) \to H(\Omega)$ induced by ϕ is given by $C_\phi f = f \circ \phi$. Theorem 7 of [14] asserts that $C_\phi : H^p(\Omega) \to A^p_{n-2}(\Omega)$ boundedly. We prove the following companion result.

THEOREM 6. *$C_\phi : A^p_\alpha(\Omega) \to A^p_{\alpha+n-1}(\Omega)$ boundedly.*

Proof. The argument is similar to that in [14]; we omit some of the details. Let $\phi(z_0) = w_0$, and $\gamma = \alpha + n - 1$. By Theorem 4, Lemma 2, and change of variables, it suffices to show that $\mu_\gamma \circ \phi^{-1} A(\xi, t) \lesssim t^{\alpha+n+1}$ $\forall \xi \in \partial\Omega$, $t > 0$. By Lemma 5 we have

$$\mu_\gamma \circ \phi^{-1} A(\xi,t) \lesssim \int_0^1 r^{2n-1} \int_{\partial\Omega} \int_0^{2\pi} \chi_{\phi^{-1}(A)} \circ \varphi_x(re^{i\theta})\, d\theta\, d\sigma(x)(1-r)^\gamma\, dr$$

$$\lesssim \int_{\partial\Omega} \int_{\Delta} \chi_{\phi^{-1}(A)} \circ \varphi_x(\lambda)(1-|\lambda|)^\gamma\, dm_1(\lambda)\, d\sigma(x),$$

where Δ is the unit disk in $\mathbf{C}^1$. Now by [14], Lemma 6, the sets $A(\xi, t)$ are comparable to the sets $S(\xi,t) = \{z \in \Omega : |1 - \varphi_\xi^{-1} \circ p_\xi(z)| < t\}$, where $\varphi_\xi(0) = w_0$ and $p_\xi : \Omega \to \Omega$ is the associated holomorphic retraction, i.e., $p_\xi \circ p_\xi = p_\xi$ and $p_\xi \circ \varphi_\xi(\lambda) = \varphi_\xi(\lambda)$ $\forall \lambda \in \Delta$ ([8], [9]). Thus the above integral is nonvanishing if and only if

$$\phi \circ \varphi_x(\lambda) \in S(\xi,t) \Leftrightarrow |1 - \varphi_\xi^{-1} \circ p_\xi \circ \phi \circ \varphi_x(\lambda)| < t \Leftrightarrow \tau(\lambda) \in A_\Delta(1,t) = S_\Delta(1,t),$$

where $\tau = \varphi_\xi^{-1} \circ p_\xi \circ \phi \circ \varphi_x : \Delta \to \Delta$, $\tau(0) = 0$. Now C_τ is bounded on $A^p_\alpha(\Delta)$, from which it follows that $\mu_\alpha \circ \tau^{-1} S_\Delta(1,t) \lesssim t^{\alpha+2}$; the constant depending only on $\tau(0)$ (see [14], Lemma 3). Thus we have $\mu_\gamma \circ \phi^{-1} A(\xi,t) \lesssim t^{\gamma+2} = t^{\alpha+n+1}$ and we are done. ∎

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