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FINITE MEASURES ON PREDUALS AND NON-COMMUTATIVE L^p-ISOMETRIES

KEIICHI WATANABE

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ABSTRACT. We prove a linear extension theorem for continuous finite measures on preduals of von Neumann algebras. Making use of it, we determine the structure of surjective positive linear isometries between non-commutative L^p-spaces associated with arbitrary von Neumann algebras.

KEYWORDS: *von Neumann algebra, non-commutative L^p-spaces.*

AMS SUBJECT CLASSIFICATION: 46L30, 46L50.

0. INTRODUCTION

In this paper we determine the structure of surjective positive linear isometries between non-commutative L^p-spaces associated with arbitrary von Neumann algebras, where $1 < p < \infty$ and $p \neq 2$.

In the classical monograph [3], Banach states a result characterizing surjective linear isometries on ℓ^p and $L^p(0,1)$, which can be considered as L^p-spaces associated with von Neumann algebras ℓ^∞ and $L^\infty(0,1)$, respectively.

Lamperti ([18]) completed the commutative cases. Several authors had developed the theory, Broise ([4]), Russo ([20]), Arazy ([2]), Katavolos ([13], [14], [15]), Tam ([23]), and there is a complete description of isometries for the case of semifinite von Neumann algebras in Yeadon ([28]).

On the other hand, after the development of the modular theory, one can construct non-commutative L^p-spaces associated with von Neumann algebras which are not necessarily semifinite. Although there are different methods of construction, those are by Haagerup [10] (see also [24]), Araki-Masuda ([1]), Hilsum ([11]), Kosaki ([17]), Terp ([25]) etc., it is known that for a fixed von Neumann algebra those L^p-spaces are canonically isometrically isomorphic each other.

Some difficulties to deal with non-commutative L^p-spaces associated with arbitrary von Neumann algebras come from the following facts. Though one can embed the original von Neumann algebra into its L^p-spaces, no one knows which embedding is most canonical. In other words, there appear highly non-commutative obstructions such as Radon-Nikodym derivatives, which turn to be central elements in the semifinite cases. So it does not seem easy to obtain a common area between the L^p-spaces and the original von Neumann algebra, and it seems that many techniques used in semifinite case are no more available.

We work on Haagerup's L^p-spaces, since those elements are (unbounded) operators, and their polar decompositions give us informations related to the original von Neumann algebra.

As shown in [26], if there exists a surjective $*$-preserving linear L^p-isometry, then those von Neumann algebras are Jordan $*$-isomorphic. In Section 3, we will obtain the implementation for positive L^p-isometries in terms of the above Jordan $*$-isomorphism and canonical $*$-isomorphism arised from the change of states. This is an affirmative answer to [26], Remark 4.2.

The main step is to show the additivity of the map β which is defined as a transformation of Radon-Nikodym derivative. Our tool is a dual version of the linear extension theorem for probability measures on projections. This simple device will be shown in Section 2, and might be interesting itself.

1. PRELIMINARIES

We begin with some basic definitions concerning Haagerup's non-commutative L^p-spaces associated with arbitrary von Neumann algebras. For details and proofs we refer to [10] and [24]. Let φ_0 be a fixed faithful normal semifinite weight on $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. Let $\{\sigma_t^{\varphi_0}\}_{t\in\mathbf{R}}$ be the modular automorphism group with respect to φ_0. We denote by $\mathcal{N}$ the crossed product $\mathcal{M}\rtimes_{\sigma^{\varphi_0}}\mathbf{R}$, which is a von Neumann algebra generated by $\pi(x), x\in\mathcal{M}$ and $\lambda_s, s\in\mathbf{R}$, defined by

$$\begin{aligned}(\pi(x)\xi)(t) &= \sigma_{-t}^{\varphi_0}(x)\xi(t), \quad \xi\in L^2(\mathbf{R},\mathcal{H}), \quad t\in\mathbf{R},\\ (\lambda_s\xi)(t) &= \xi(t-s), \qquad \xi\in L^2(\mathbf{R},\mathcal{H}), \quad t\in\mathbf{R}.\end{aligned}$$

The dual actions, $\theta_s, s\in\mathbf{R}$, naturally extend to automorphisms on $\hat{\mathcal{N}}_+$, which is the extended positive part of $\mathcal{N}$ (cf. [8], Section 1). For each normal weight φ on $\mathcal{M}$, we denote by $\tilde{\varphi}$ the dual weight of φ on $\mathcal{N}$. It is well-known that there exists a unique faithful normal semifinite trace τ on $\mathcal{N}$ characterized by the Connes' cocycle $(D\tilde{\varphi}_0 : D\tau)_t = \lambda_t, t \in \mathbf{R}$, and τ satisfies $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbf{R}$ (cf. [9], Lemma 5.2).

Haagerup's L^p-spaces are realized as subspaces consisting of measurable operators with respect to this trace τ. A densely defined closed operator a affiliated with $\mathcal{N}$, with its domain $\mathcal{D}(a)$, is said to be τ-measurable if there is, for each $\delta > 0$, a projection $e \in \mathcal{N}$ such that $eL^2(\mathbf{R}, \mathcal{H}) \subset \mathcal{D}(a)$ and $\tau(1-e) \leqslant \delta$. We denote by $\tilde{\mathcal{N}}$ the set of all τ-measurable operators, which becomes a complete Hausdorff topological $*$-algebra under the strong operations in the measure topology. For any subset $\mathcal{S}$ of $\tilde{\mathcal{N}}$, the set of all selfadjoint (resp. positive selfadjoint) operators in $\mathcal{S}$ shall be denoted by $\mathcal{S}_{\mathrm{sa}}$ (resp. $\mathcal{S}_+$).

Now the dual actions $\theta_s, s \in \mathbf{R}$, are extended to continuous $*$-automorphisms of $\tilde{\mathcal{N}}$. For $0 < p < \infty$, the Haagerup's L^p-space is defined by

$$L^p(\mathcal{M}; \varphi_0) = \left\{ a \in \tilde{\mathcal{N}};\ \theta_s(a) = e^{-\frac{s}{p}}\, a, s \in \mathbf{R} \right\},$$

and simply denoted by $L^p(\mathcal{M})$ whenever it is not necessary to indicate the weight φ_0. For each normal weight φ on $\mathcal{M}$, we simply denote by

$$h_\varphi = \frac{d\tilde{\varphi}}{d\tau}$$

the non-commutative Radon-Nikodym derivative of $\tilde{\varphi}$ with respect to τ. It is well-known that $\varphi \in \mathcal{M}_{*,+}$, which is the set of all normal positive linear functional on $\mathcal{M}$, if and only if h_φ is τ-measurable. The mapping $\varphi \to h_\varphi$ is extended to a linear order isomorphism from $\mathcal{M}_*$ onto $L^1(\mathcal{M})$, and so the positive linear functional tr on $L^1(\mathcal{M})$ is defined by

$$\mathrm{tr}(h_\varphi) = \varphi(1), \quad \varphi \in \mathcal{M}_*.$$

For $0 < p < \infty$, the (quasi-)norm of $L^p(\mathcal{M})$ is defined by $\|a\|_p = \mathrm{tr}(|a|^p)^{1/p}, a \in L^p(\mathcal{M})$. When $1 \leqslant p < \infty$, $L^p(\mathcal{M})$ is a Banach space, and its dual Banach space is $L^q(\mathcal{M})$ with $1/p + 1/q = 1$ by the following duality:

$$\langle a, b \rangle = \mathrm{tr}(ab) = \mathrm{tr}(ba), \quad a \in L^p(\mathcal{M}), \quad b \in L^q(\mathcal{M}).$$

Note that for any $a = u|a| \in L^p(\mathcal{M})$ with its polar decomposition, u belongs to $\mathcal{M}$ and $|a|$ belongs to $L^p(\mathcal{M})_+$. Also for any $a = a_+ - a_- \in L^p(\mathcal{M})_{\mathrm{sa}}$ with its Jordan decomposition, one has $a_+, a_- \in L^p(\mathcal{M})_+$.

We have already shown the following theorem by making use of the equality condition for the Clarkson's inequality.

THEOREM 1. ([26], Theorem 3.6) *Let $1 < p < \infty$ and $p \neq 2$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be σ-finite von Neumann algebras. Let φ_0 (resp. ψ_0) be a fixed faithful normal state on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). Let T be a $*$-preserving linear isometry from $L^p(\mathcal{M}_1;\varphi_0)$ onto $L^p(\mathcal{M}_2;\psi_0)$. Then there exists a Jordan $*$-isomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$ satisfying $J(s(a)) = s(T(a))$, $a \in L^p(\mathcal{M}_1;\varphi_0)_{\mathrm{sa}}$, where $s(a)$ denotes the support projection of a.*

Now there are two faithful normal states on $\mathcal{M}_2$, ψ_0 and $\varphi_0 \circ J^{-1}$. We denote the crossed product with respect to ψ_0 (resp. $\varphi_0 \circ J^{-1}$) by $\mathcal{N}_{\psi_0} = \mathcal{M}_2 \rtimes_{\sigma^{\psi_0}} \mathbf{R}$ (resp. $\mathcal{N}_2 = \mathcal{M}_2 \rtimes_{\sigma^{\varphi_0 \circ J^{-1}}} \mathbf{R}$). For each normal weight on ψ on $\mathcal{M}_2$, we denote by $\psi^{\sim\psi_0}$ (resp. $\psi^{\sim 2}$) the dual weight on $\mathcal{N}_{\psi_0}$ (resp. $\mathcal{N}_2$). Let τ_{ψ_0} (resp. τ_2) be the canonical trace on $\mathcal{N}_{\psi_0}$ (resp. $\mathcal{N}_2$). Let $\tilde{\mathcal{N}}_{\psi_0}$ (resp. $\tilde{\mathcal{N}}_2$) be the set of all τ_{ψ_0}-measurable (resp. τ_2-measurable) operators on $L^2(\mathbf{R},\mathcal{H})$.

Define a unitary operator u on $L^2(\mathbf{R};\mathcal{H})$ by

$$(u\xi)(t) = \left(D\left(\varphi_0 \circ J^{-1}\right); D\psi_0\right)_{-t} \xi(t), \quad \xi \in L^2(\mathbf{R},\mathcal{H}), \quad t \in \mathbf{R}.$$

Put $\kappa(a) = uau^*, a \in \mathcal{N}_2$. Then κ is the canonical $*$-isomorphism from $\mathcal{N}_{\psi_0}$ onto $\mathcal{N}_2$, which is related to change of states from ψ_0 to $\varphi_0 \circ J^{-1}$. Moreover, κ extends to a $*$-isomorphism $\tilde{\kappa}$ from $\tilde{\mathcal{N}}_{\psi_0}$ onto $\tilde{\mathcal{N}}_2$, and the restriction of $\tilde{\kappa}$ is a positive linear isometry from $L^p(\mathcal{M}_2;\psi_0)$ to $L^p(\mathcal{M}_2;\varphi_0 \circ J^{-1})$ (cf. [26], Lemma 2.1, Lemma 2.2).

On the other hand, since

$$\sigma^{\varphi_0 \circ J^{-1}} = J \circ \sigma^{\varphi_0} \circ J^{-1}$$

by the uniqueness of the modular automorphism group, the two W^*-dynamical systems $(\mathcal{M}_1, \mathbf{R}, \sigma^{\varphi_0})$ and $(\mathcal{M}_2, \mathbf{R}, \sigma^{\varphi_0 \circ J^{-1}})$ are covariantly Jordan $*$-isomorphic. So there exists a unique Jordan $*$-isomorphism $\tilde{J}$ from $\mathcal{N}_1$ to $\mathcal{N}_2$ extending J, where $\mathcal{N}_1 = \mathcal{M}_1 \rtimes_{\sigma^{\varphi_0}} \mathbf{R}$. Moreover, we can extend $\tilde{J}$ to a Jordan $*$-isomorphism from $\tilde{\mathcal{N}}_1$ onto $\tilde{\mathcal{N}}_2$, which is a homeomorphism with respect to their measure topologies, and the restriction of $\tilde{J}$ to $L^p(\mathcal{M}_1;\varphi_0)$ is a canonical positive linear isometry from $L^p(\mathcal{M}_1;\varphi_0)$ onto $L^p(\mathcal{M}_2;\varphi_0 \circ J^{-1})$ (cf. [26], Section 4).

Thus we have a canonical positive linear isometry $\tilde{\kappa}^{-1} \circ \tilde{J}$ from $L^p(\mathcal{M}_1;\varphi_0)$ onto $L^p(\mathcal{M}_2;\psi_0)$. Our main problem turns to be the implementation $T = \tilde{\kappa}^{-1} \circ \tilde{J}$.

2. CONTINUOUS FINITE MEASURES ON THE PREDUALS OF VON NEUMANN ALGEBRAS

In this section, we consider linear extension of continuous finite measures on preduals, which is a simple tool for the next section.

DEFINITION 2. Suppose that $\mathcal{M}$ is a von Neumann algebra. A map ρ from $\mathcal{M}_{*,+}$ to $[0,\infty)$ is said to be a *continuous finite measure* on the predual if the following conditions are satisfied:

(i) $\rho(\alpha\varphi) = \alpha\rho(\varphi)$, $\alpha \geqslant 0$, $\varphi \in \mathcal{M}_{*,+}$;

(ii) $\rho(\sum \varphi_n) = \sum \rho(\varphi_n)$, whenever $\{\varphi_n\}$ is a countable family in $\mathcal{M}_{*,+}$ whose supports are orthogonal each other and the sum $\sum \varphi_n$ exists in $\mathcal{M}_{*,+}$;

(iii) $\rho(\varphi) \leqslant \|\varphi\|$, $\varphi \in \mathcal{M}_{*,+}$;

(iv) $\rho(\varphi_\lambda) \to \rho(\varphi)$, whenever $\{\varphi_\lambda\}$ is a family in $\mathcal{M}_{*,+}$ and $\|\varphi_\lambda - \varphi\| \to 0$.

Recall that a finitely additive probability measure on projections is a non-negative real-valued function μ, defined on $\mathcal{P}(\mathcal{M})$ the set of all projections in a von Neumann algebra $\mathcal{M}$, that satisfies $\mu(e+f) = \mu(e) + \mu(f)$ when $e, f \in \mathcal{P}(\mathcal{M})$ with $ef = 0$ and $\mu(1) = 1$.

THEOREM 3. (Christensen [5], Yeadon [29]) *Let $\mathcal{M}$ be a von Neumann algebra with no type I_2 summand. Let μ be a finitely additive probability measure on $\mathcal{P}(\mathcal{M})$. Then there exists a state φ on $\mathcal{M}$ whose restriction to $\mathcal{P}(\mathcal{M})$ is μ.*

The next result yields from the preceding theorem, which is also a dual version of it.

THEOREM 4. *Let ρ be a continuous finite measure on the predual of a von Neumann algebra $\mathcal{M}$. Then there exists a unique element $x \in \mathcal{M}_+$ such that $\rho(\varphi) = \varphi(x), \varphi \in \mathcal{M}_{*,+}$.*

Proof. **Step 1**: In the case that $\mathcal{M}$ is a factor of type I_n, where $n \neq 2$ is a natural number. Let τ be the canonical trace on $\mathcal{M}$ such that $\tau(1) = 1$. For $p \in \mathcal{P}(\mathcal{M})$, we define

$$\mu_\rho(p) = \rho(\tau(p\,\cdot)).$$

If $\mu_\rho(1) = 0$, then we have $\mu_\rho(p) + \mu_\rho(1-p) = 0$. It follows that $\mu_\rho = 0$, identically. Taking $x = 0$, the assertion holds. So we may assume $\mu_\rho(1) \neq 0$. Obviously, $(1/\mu_\rho(1))\mu_\rho$ is a finitely additive probability measure on $\mathcal{P}(\mathcal{M})$. By virtue of the theorem of Christensen-Yeadon, there exists a positive linear functional φ_ρ on $\mathcal{M}$ such that

$$\varphi_\rho(p) = \frac{1}{\mu_\rho(1)}\mu_\rho(p) = \frac{1}{\mu_\rho(1)}\rho(\tau(p\,\cdot)), \quad p \in \mathcal{P}(\mathcal{M}).$$

Then we can take a positive element $a_\rho \in \mathcal{M}$ satisfying $\varphi_\rho = \tau(a_\rho\,\cdot)$. Fix an arbitrary positive element $b \in \mathcal{M}$. Then b can be expressed in the form $\sum \alpha_n p_n$, where $\{p_n\}$ is a finite family of orthogonal projections in $\mathcal{M}$ and $\alpha_n \geqslant 0$. Hence we have $\tau(a_\rho b) = \sum \alpha_n \tau(a_\rho p_n) = \sum \alpha_n (1/\mu_\rho(1))\rho(\tau(p_n\,\cdot)) = (1/\mu_\rho(1))\rho(\tau(b\,\cdot))$. This implies that $\rho = \mu_\rho(1)a_\rho$. Moreover, note that $\|a_\rho\| = \sup\{\varphi(a_\rho); \varphi \in \mathcal{M}_{*,+}, \|\varphi\| \leqslant 1\} \leqslant 1/\mu_\rho(1)$.

Step 2: In the case that $\mathcal{M} = B(\mathcal{H})$, where $\mathcal{H}$ is an arbitrary infinite dimensional Hilbert space. Let Tr be the canonical trace on $\mathcal{M}$. Put $\Lambda = \{p \in \mathcal{P}(\mathcal{M}); \mathrm{Tr}(p) < \infty\}$. Then Λ is an increasing net, by the parallelogram law. For $p \in \Lambda$, we define ρ_p as the restriction of ρ to $(\mathcal{M}_p)_{*,+}$, where $\mathcal{M}_p$ is the reduced von Neumann algebra. Note that each element in $(\mathcal{M}_p)_*$ is naturally considered as an element in $\mathcal{M}_*$, so that the restrictions make sense. It is easy to see that ρ_p is a continuous finite measure on $(\mathcal{M}_p)_{*,+}$. For each $p \in \Lambda$, set $\tau_p = (1/\mathrm{Tr}(p))\mathrm{Tr}(p\,\cdot)$. Applying the result in Step 1 to $(\mathcal{M}_p, \tau_p)$ and the map $(1/\rho(\tau_p))\rho_p$, we can take a unique element $a_p \in (\mathcal{M}_p)_+$ such that $(1/\rho(\tau_p))\rho_p(\varphi) = \varphi(a_p)$, $\varphi \in (\mathcal{M}_p)_{*,+}$. Now put $x_p = \rho(\tau_p)a_p$. If $p_1, p_2 \in \Lambda$ and $p_1 \leqslant p_2$, then, for any $\varphi \in (\mathcal{M}_{p_1})_{*,+}$, we have $\varphi(p_1 x_{p_2} p_1) = \rho_{p_2}(p_1\varphi p_1) = \rho(p_1\varphi p_1) = \rho_{p_1}(p_1\varphi p_1) = (p_1\varphi p_1)(x_{p_1}) = \varphi(x_{p_1})$. This implies that $p_1 x_{p_2} p_1 = x_{p_1}$. For each vector $\xi \in \bigcup_{p\in\Lambda} p\mathcal{H}$, define a function s by $s(\xi) = (x_p\xi|\xi)$, $\xi \in p\mathcal{H}$. If $\xi \in p'\mathcal{H}$ by another $p' \in \Lambda$, we have $(x_{p'}\xi|\xi) = ((p\wedge p')x_{p'}(p\wedge p')\xi|\xi) = (x_{p\wedge p'}\xi|\xi) = (x_p\xi|\xi)$, so that s is well-defined.

Moreover, we define

$$B(\xi,\eta) = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k s(\xi + \mathrm{i}^k\eta), \quad \xi,\eta \in \bigcup_{p\in\Lambda} p\mathcal{H}.$$

It is straightforward to see that B extends to a sesquilinear form on $\mathcal{H}\times\mathcal{H}$ satisfying $|B(\xi,\eta)| \leqslant \|\xi\|\,\|\eta\|$, $\xi,\eta \in \mathcal{H}$. Hence there exists a unique element $x \in B(\mathcal{H})$ such that $(x\xi|\eta) = B(\xi,\eta)$, $\xi,\eta \in \mathcal{H}$. It follows from $(x_p\xi|\eta) = (x\xi|\eta)$, $\xi,\eta \in p\mathcal{H}$ that $pxp = x_p$, $p \in \Lambda$. Since Λ increases to 1 in the strong operator topology, $pxp \to x$, weakly. In particular, $x \geqslant 0$.

Now we claim that $\rho(\varphi) = \varphi(x)$, $\varphi \in \mathcal{M}_{*,+}$. Fix an arbitrary element $\varphi \in \mathcal{M}_{*,+}$. Note that $p\varphi p \to \varphi$ in the $\sigma(\mathcal{M}_*, \mathcal{M})$-topology. Since norm closure and weak closure coincide for a convex set in a normed space, φ can be approximated in the norm topology by a family of finite convex combinations $\sum c_k p_k \varphi p_k$, where $p_k \in \Lambda, c_k \geqslant 0$ and $\sum c_k = 1$. It follows from $\rho(\sum c_k p_k\varphi p_k) = (\sum c_k p_k \varphi p_k)(x)$ and the continuity of ρ that $\rho(\varphi) = \varphi(x)$. Thus the claim is shown.

Step 3: In the general case. Let $\mathcal{H}$ be an infinite dimensional Hilbert space which $\mathcal{M}$ is acting on. Define $\tilde{\rho}(\varphi) = \rho(\varphi|\mathcal{M})$, $\varphi \in B(\mathcal{H})_{*,+}$, where $\varphi|\mathcal{M}$ is the restriction of φ to $\mathcal{M}$. Then $\tilde{\rho}$ is a continuous finite measure on $B(\mathcal{H})_{*,+}$. Applying the result in Step 2, there exists a positive element $x \in B(\mathcal{H})$ such that $\tilde{\rho}(\varphi) = \varphi(x)$, $\varphi \in B(\mathcal{H})_{*,+}$. Moreover, for any unitary u in the commutant of $\mathcal{M}$, we have $\varphi(u^*xu) = \tilde{\rho}(u\varphi u^*) = \rho((u\varphi u^*)|\mathcal{M}) = \rho(\varphi|\mathcal{M}) = \varphi(x)$, $\varphi \in B(\mathcal{H})_{*,+}$. This implies that $x \in \mathcal{M}$ and $\rho(\varphi) = \tilde{\rho}(\varphi) = \varphi(x)$, $\varphi \in \mathcal{M}_{*,+}$. This completes the proof. ∎

3. THE STRUCTURE OF SURJECTIVE POSITIVE LINEAR L^p-ISOMETRY

In this section, we shall prove the implementation of surjective positive linear L^p-isometries.

THEOREM 5. *Let $1 < p < \infty$ and $p \neq 2$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be σ-finite von Neumann algebras. Let φ_0 (resp. ψ_0) be a fixed faithful normal state on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). Let T be a positive linear isometry from $L^p(\mathcal{M}_1;\varphi_0)$ onto $L^p(\mathcal{M}_2;\psi_0)$. Let J be the Jordan $*$- isomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$ induced by T due to Theorem 1. Let κ be the canonical isomorphism associated with the change of states ψ_0 and $\varphi_0 \circ J^{-1}$. Then T equals to the restriction of $\tilde{\kappa}^{-1} \circ \tilde{J}$ to $L^p(\mathcal{M}_1;\varphi_0)$.*

Proof. As in the proof of [26], Theorem 4.1, we define a map β from $(\mathcal{M}_1)_{*,+}$ onto $(\mathcal{M}_2)_{*,+}$ by the formula $T({h_\varphi}^{1/p}) = {h_{\beta(\varphi)}}^{1/p}$. More precisely,

$$T\left(\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\right)^{\frac{1}{p}}\right) = \left(\frac{\mathrm{d}\beta(\varphi)^{\sim\psi_0}}{\mathrm{d}\tau_{\psi_0}}\right)^{\frac{1}{p}}.$$

Then β satisfies the following conditions (cf. [16], Theorem 4.2):

(i) $\beta(\alpha\varphi) = \alpha\beta(\varphi)$, $\alpha \geqslant 0$, $\varphi \in (\mathcal{M}_1)_{*,+}$;

(ii) $\beta(\sum \varphi_n) = \sum \beta(\varphi_n)$, whenever $\{\varphi_n\}$ is a countable family in $(\mathcal{M}_1)_{*,+}$ whose supports are orthogonal each other and the sum $\sum \varphi_n$ exists in $(\mathcal{M}_1)_{*,+}$;

(iii) $\|\beta(\varphi)\| = \|\varphi\|$, $\varphi \in (\mathcal{M}_1)_{*,+}$;

(iv) $\beta(\varphi_\lambda) \to \beta(\varphi)$, whenever $\{\varphi_\lambda\}$ is a family in $(\mathcal{M}_1)_{*,+}$ and $\|\varphi_\lambda - \varphi\| \to 0$.

We shall prove that β is also additive. For an arbitrary fixed element $y \in (\mathcal{M}_2)_+$ such that $\|y\| \leqslant 1$, we put $\rho_y(\varphi) = \beta(\varphi)(y)$, $\varphi \in (\mathcal{M}_1)_{*,+}$. Then obviously ρ_y is a continuous finite measure on $(\mathcal{M}_1)_{*,+}$. Therefore, by Theorem 4, there exists a unique element $x = x_{\rho_y} \in \mathcal{M}_1$ such that $\rho_y(\varphi) = \varphi(x)$, $\varphi \in (\mathcal{M}_1)_{*,+}$. Now for each pair $\varphi, \psi \in (\mathcal{M}_1)_{*,+}$, we have $\beta(\varphi+\psi)(y) = \rho_y(\varphi+\psi) = (\varphi+\psi)(x) = \varphi(x) + \psi(x) = \rho_y(\varphi) + \rho_y(\psi) = \beta(\varphi)(y) + \beta(\psi)(y) = (\beta(\varphi)+\beta(\psi))(y)$. Since $y \in (\mathcal{M}_2)_+$, $\|y\| \leqslant 1$ is arbitrary, this shows the desired formula

$$\beta(\varphi+\psi) = \beta(\varphi) + \beta(\psi), \quad \varphi, \psi \in (\mathcal{M}_1)_{*,+}.$$

Then the map β extends to a surjective positive linear isometry from $(\mathcal{M}_1)_*$ to $(\mathcal{M}_2)_*$. The transposed map ${}^t\beta$ of β is a surjective positive linear isometry from $\mathcal{M}_2$ to $\mathcal{M}_1$. It follows from [12], Theorem 7, that ${}^t\beta(y) = {}^t\beta(1)J_0(y)$, $y \in \mathcal{M}_2$, where ${}^t\beta(1)$ is a unitary and J_0 is a Jordan $*$-isomorphism from $\mathcal{M}_2$ onto $\mathcal{M}_1$. Since ${}^t\beta(1) \geqslant 0$, we have ${}^t\beta(1) = 1$. Thus ${}^t\beta$ itself is a Jordan $*$-isomorphism, so that

$$J(s(\varphi)) = s(\beta(\varphi)) = s(\varphi \circ {}^t\beta) = ({}^t\beta)^{-1}(s(\varphi)), \quad \varphi \in (\mathcal{M}_1)_{*,+}.$$

Since $s(\varphi)$, $\varphi \in (\mathcal{M}_1)_{*,+}$ runs over the whole $\mathcal{P}(\mathcal{M}_1)$, this implies that $J = ({}^t\beta)^{-1}$.

Finally, for any $\varphi \in (\mathcal{M}_1)_{*,+}$, we compute the Radon-Nikodym derivative:

$$T\left(h_\varphi{}^{\frac{1}{p}}\right)^p = T\left(\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\right)^{\frac{1}{p}}\right)^p = \frac{\mathrm{d}\beta(\varphi)^{\sim\psi_0}}{\mathrm{d}\tau_{\psi_0}} = \frac{\mathrm{d}(\varphi \circ J^{-1})^{\sim\psi_0}}{\mathrm{d}\tau_{\psi_0}}$$

$$= \tilde{\kappa}^{-1}\left(\frac{\mathrm{d}(\varphi \circ J^{-1})^{\sim 2}}{\mathrm{d}\tau_2}\right) = \tilde{\kappa}^{-1} \circ \tilde{J}\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\right) = \tilde{\kappa}^{-1} \circ \tilde{J}(h_\varphi).$$

This implies $T = \tilde{\kappa}^{-1} \circ \tilde{J}$ on $L^p(\mathcal{M}_1; \varphi_0)$. This completes the proof. ∎

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REFERENCES

1. H. ARAKI, T. MASUDA, Positive cones and L^p-spaces for von Neumann algebras, *Publ. Res. Inst. Math. Sci.* **18**(1982), 339–411.
2. J. ARAZY, The isometries of C_p, *Israel J. Math.* **22**(1975), 247–256.
3. S. BANACH, Théorie des Operations Linéaries, Warsaw 1932.
4. M.M. BROISE, Sur les isomorphismes de certaines algèbres de von Neumann, *Ann. Sci. École Norm. Sup. (4)* **83**(1966), 91–111.
5. E. CHRISTENSEN, Measures on projections and physical states, *Comm. Math. Phys.* **86**(1982), 529–538.
6. J. DIXMIER, Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. France* **81**(1953), 9–39.
7. H.A. DYE, On the geometry of projections in certain operator algebras, *Ann. of Math. (2)* **66**(1955), 73–88.
8. U. HAAGERUP, Operator valued weight in von Neumann algebras I, *J. Funct. Anal.* **32**(1979), 175–206.
9. U. HAAGERUP, Operator valued weight in von Neumann algebras II, *J. Funct. Anal.* **33**(1979), 339–361.
10. U. HAAGERUP, L^p-spaces associated with an arbitrary von Neumann algebra, *Colloq. Internat. CNRS* **274**(1979), 175–184.
11. M. HILSUM, Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale, *J. Funct. Anal.* **40**(1981), 151–169.
12. R.V. KADISON, Isometries of operator algebras, *Ann. of Math. (2)* **54**(1951), 325–338.

13. A. KATAVOLOS, Isometries of non-commutative L^p-spaces, *Canad. J. Math.* **28** (1976), 1180–1186.
14. A. KATAVOLOS, Are non-commutative L^p-spaces really non-commutative?, *Canad. J. Math.* **33**(1981), 1319–1327.
15. A. KATAVOLOS, Non-commutative L^p-spaces II, *Canad. J. Math.* **34**(1982), 1208–1214.
16. H. KOSAKI, Applications of uniform convexity of non-commutative L^p-spaces, *Trans. Amer. Math. Soc.* **283**(1984), 265–282.
17. H. KOSAKI, Applications of the complex interpolation method to a von Neumann algebra: Non-commutative L^p-spaces, *J. Funct. Anal.* **56**(1984), 29–78.
18. J. LAMPERTI, On the isometries of certain function spaces, *Pacific J. Math.* **8**(1958), 459–466.
19. E. NELSON, Note on non-commutative integration, *J. Funct. Anal.* **15**(1974), 103–116.
20. B. RUSSO, Isometries of L^p-spaces associated with finite von Neumann algebras, *Bull. Amer. Math. Soc.* **74**(1968), 228–232.
21. I.E. SEGAL, A non-commutative extension of abstract integration, *Ann. of Math. (2)* **57**(1953), 401–457.
22. Ş. STRĂTILĂ, *Modular Theory in Operator Algebras*, Abacus Press, Tunbridge Wells, England 1981.
23. P.K. TAM, Isometries of L^p-spaces associated with semifinite von Neumann algebras, *Trans. Amer. Math. Soc.* **254**(1979), 339–354.
24. M. TERP, L^p-spaces associated with arbitrary von Neumann algebras, info notes, Copenhagen University, 1981.
25. M. TERP, Interpolation spaces between a von Neumann algebra and its predual, *J. Operator Theory* **8**(1982), 327–360.
26. K. WATANABE, On isometries between non-commutative L^p-spaces associated with arbitrary von Neumann algebras, *J. Operator Theory* **28**(1992), 267–279.
27. F.J. YEADON, Non-commutative L^p-spaces, *Math. Proc. Cambridge Philos. Soc.* **77**(1975), 91–102.
28. F.J. YEADON, Isometries of non-commutative L^p-spaces, *Math. Proc. Cambridge Philos. Soc.* **90**(1981), 41–50.
29. F.J. YEADON, Finitely additive measures on projections in finite W^*-algebras, *Bull. London Math. Soc.* **16**(1984), 145–150.

KEIICHI WATANABE
Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-21
JAPAN

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