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$C_{\cdot 0}$ CONTRACTIONS: DUAL OPERATOR ALGEBRAS, JORDAN MODELS AND MULTIPLICITY

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ABSTRACT. We discuss contraction operators T in the class $C_{\cdot 0} \cap \mathbf{A}$ with defect index $d_T < \infty$, where $\mathbf{A}$ is the class of absolutely continuous contractions for which the Sz.-Nagy-Foiaş functional calculus is an isometry. We show that these form particularly nice representatives of the classes $\mathbf{A}_{n,\aleph_0}$ since their membership is completely determined by the multiplicity of either the shift piece of their Jordan model or the unitary piece of their minimal coisometric extension.

KEYWORDS: *Dual operator algebras, finite defects, Jordan model, multiplicity.*

AMS SUBJECT CLASSIFICATION: Primary 47D27; Secondary 47A45.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. Note that the ultraweak operator topology coincides with the weak* topology on $\mathcal{L}(\mathcal{H})$ (see [12]). This notion of dual algebras was introduced by S. Brown in [8], where he proved that every subnormal operator has a nontrivial invariant subspace. The theory of dual algebras has been applied to the study of invariant subspaces, dilation theory, and reflexivity (see [5]). H. Bercovici, B. Chevreau, C. Foiaş, and C. Pearcy have studied the problem of solving systems of simultaneous equations in the predual of a dual algebra (see [1], [2], [4], [5] and [11]). Central to this study have been the classes $\mathbf{A}_{m,n}$ (to be defined below) defined by Bercovici-Foiaş-Pearcy in [4]. In particular, I. Jung ([17]) proved that the classes $\mathbf{A}_{m,n}$ are distinct one from another by studying Jordan models of operators in the classes $\mathbf{A}_{n,1} \cap C_{.0}$. In another sequence of papers Chevreau-Exner-Pearcy ([10]) and Exner-Jung ([14]) gave some characterizations of the class $\mathbf{A}_{1,\aleph_0}$. By improving some results in [17] the present paper makes connection between Jordan models and this latter work on $\mathbf{A}_{1,\aleph_0}$. We show that for an operator $T \in C_{.0} \cap \mathbf{A}$ with $d_T < \infty$, membership in $\mathbf{A}_{n,\aleph_0} \backslash \mathbf{A}_{n+1,1}$ coincides with multiplicity n of either the shift part of the Jordan model of T or the unitary part of the minimal coisometric extension of T.

The notation and terminology employed herein agree with those in [3], [5], [7] and [22]. Let $\mathcal{C}_1(\mathcal{H})$ be the Banach space of trace class operators on $\mathcal{H}$ equipped with the trace norm. If $\mathcal{A}$ is a dual algebra, then it follows from [5] that $\mathcal{A}$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing $\langle T, [L]_{\mathcal{A}} \rangle = \operatorname{trace}(TL)$, $T \in \mathcal{A}$, $[L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}$. The Banach space $\mathcal{Q}_{\mathcal{A}}$ is called a *predual* of $\mathcal{A}$. We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. For x and y in $\mathcal{H}$, we define $x \otimes y$ by $(x \otimes y)(u) = (u, y)x$, for all $u \in \mathcal{H}$. The cosets $[x \otimes y]_{\mathcal{A}}$ have been essential in dual algebra work.

Suppose m and n are cardinal numbers such that $1 \leqslant m, n \leqslant \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leqslant i < m, \quad 0 \leqslant j < n, \tag{1.1}$$

where $\{[L_{ij}]\}_{\substack{0 \leqslant i < m \\ 0 \leqslant j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leqslant i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.

We write $\mathbf{D}$ for the open unit disc in the complex plane $\mathbf{C}$ and $\mathbf{T}$ for the boundary of $\mathbf{D}$. The spaces $L^p = L^p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, are the usual Lebesgue function spaces relative to normalized Lebesgue measure m on $\mathbf{T}$. The spaces $H^p = H^p(\mathbf{T})$, $1 \leqslant p \leqslant \infty$, are the usual Hardy spaces. It is well-known (cf. [13]) that the space H^∞ is the dual space of L^1/H_0^1, where $H_0^1 = \{f \in L^1 : \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}nt}\,\mathrm{d}t = 0, \text{ for } n = 0, 1, 2, \ldots\}$ and the duality is given by the pairing $\langle f, [g] \rangle = \int_{\mathbf{T}} fg\,\mathrm{d}m$, for $f \in H^\infty$, $[g] \in L^1/H_0^1$.

We denote by $\mathcal{A}_T$ the dual algebra generated by T in $\mathcal{L}(\mathcal{H})$ and by $\mathcal{Q}_T$ the predual space $\mathcal{Q}_{\mathcal{A}_T}$ of $\mathcal{A}_T$. A contraction operator $T \in \mathcal{L}(\mathcal{H})$ is *absolutely continuous* if in the canonical decomposition $T = T_1 \oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0). The following is essentially ([5], Theorem 4.1).

THEOREM 1.1. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there exists a functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ for every f in H^∞. The mapping Φ_T is a norm-decreasing, weak* continuous, algebra homomorphism, and the range of Φ_T is weak* dense in $\mathcal{A}_T$. Furthermore, there exists a bounded, linear, one-to-one map φ_T of Q_T into L^1/H_0^1 such that $\Phi_T = \varphi_T^*$.*

DEFINITION 1.2. ([5]) We denote by $\mathbf{A} = \mathbf{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Furthermore, if m and n are any cardinal numbers such that $1 \leqslant m, n \leqslant \aleph_0$, we let $\mathbf{A}_{m,n} = \mathbf{A}_{m,n}(\mathcal{H})$ be the set of all T in $\mathbf{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbf{A}_{m,n})$.

Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$. We let $\mathrm{Lat}(T)$ denote the lattice of subspaces invariant for T. For a subspace $\mathcal{M} \in \mathrm{Lat}(T)$, we write $T|\mathcal{M}$ for the restriction to $\mathcal{M}$.

Throughout this paper, $\mathbf{N}$ is the set of all natural numbers. For any operator T and $n \in \mathbf{N}$, we let $T^{(n)}$ denote the operator $T^{(n)} = \underbrace{T \oplus \cdots \oplus T}_{(n)}$ (the *n-fold ampliation of* T). Let S denote the unilateral shift operator of multiplicity one. We recall that the operator $S(\theta)$ defined by $S(\theta) = (S^*|(H^2 \ominus \theta H^2))^*$, for an inner function θ, is called a *Jordan block.* Any operator of the form $S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus S^{(l)}$, where $\theta_1, \theta_2, \cdots, \theta_k$ are nonconstant (scalar valued) inner functions, each of which is a divisor of its predecessor, and $0 \leqslant k < \infty$, $0 \leqslant l \leqslant \infty$, is called a *Jordan operator.*

Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. An operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is called an *injection* if it is one-to-one. A family $\{X_\alpha\}$ of injections in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ is called *complete* if $\bigvee_\alpha X_\alpha \mathcal{H} = \mathcal{H}'$. Suppose $T \in \mathcal{L}(\mathcal{H})$ and $T' \in \mathcal{L}(\mathcal{H}')$. The operator T is said to be *injected* into T' if there is an injection $X : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $T'X = XT$, and we write $T' \succ^i T$ (or $T \prec^i T'$). The operator T is said to be *completely injected* into T' if there exists a complete family $\{X_\alpha\}$ of injections in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $T'X_\alpha = X_\alpha T$ for each α, and we write $T' \succ^{c.i.} T$ (or $T \prec^{c.i.} T'$). The operator T is said to be a *quasi-affine transform* of T' if there exists a quasi-affinity $X : \mathcal{H} \to \mathcal{H}'$ (i.e., X is one-to-one and has a dense range) such that $T'X = XT$, and we write $T' \succ T$ (or $T \prec T'$). Recall (e.g., from [22]) that the class $C_{0\cdot}$ consists of those operators T such that $\|T^n x\| \to 0$ for all $x \in \mathcal{H}$, and $C_{\cdot 0} = (C_{0\cdot})^*$. (Also, $C_{1\cdot}$ is the class those of T such that $\|T^n x\| \to 0$ only for $x = 0$, and $C_{\cdot 1} = (C_{1\cdot})^*$.) Sz.-Nagy ([21]) showed that for $T \in C_{\cdot 0}(\mathcal{H})$ with defect index $d_T < \infty$, (i.e., $d_T = \dim\overline{\{(I - T^*T)^{\frac{1}{2}}\mathcal{H}\}} < \infty$), there exists a uniquely determined Jordan operator $J_T = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(l)}$ such that $J_T \prec^{c.i.} T \prec J_T$. The operator J_T is called the *Jordan model* of T.

We will have occasion to use another model for an absolutely continuous contraction T, namely the minimal coisometric extension of T. We say that an operator T' is an *extension* of T if there exists a subspace $\mathcal{M}$ invariant for T' so that $T'|\mathcal{M}$ is unitarily equivalent to T. Let $P_{\mathcal{K}}$ denote as usual the orthogonal projection onto a subspace $\mathcal{K}$. We say that T' is a *dilation* of T (or T is a *compression* of T') if there exist subspaces $\mathcal{M}$ and $\mathcal{N}$ each invariant for T' and with $\mathcal{N} \subseteq \mathcal{M}$ so that $P_{\mathcal{M}\ominus\mathcal{N}}T'|\mathcal{M} \ominus \mathcal{N}$ is unitarily equivalent to T.

With this notation in hand, recall from [22] that an absolutely continuous contraction T has a minimal isometric dilation U_T^+, where minimality is defined in a natural way. Via the Wold decomposition one has

$$U_T^+ = S_T \oplus R_T^*, \tag{1.2}$$

where S_T acting on $\mathcal{S}_T$ is a (forward) unilateral shift of some multiplicity and R_T^* acting on $\mathcal{R}_T$ is a unitary operator (of course, either may be absent). One has easily that $(U_T^+)^* = S_T^* \oplus R_T$ is a minimal coisometric extension of T^*. In the sequel, we will usually have use for the minimal coisometric extension of T and the minimal isometric dilation of T^*; we denote this minimal coisometric extension B_T (so $B_T = (U_{T^*}^+)^*$).

The next lemma recalls some familiar facts about the two models, J_T and B_T, available for an $C_{\cdot 0}$ (hence absolutely continuous) contraction with $d_T < \infty$. The first conclusion below is from [21], and while although the second is surely not new we include a proof for the convenience of the reader. We denote by B the bilateral shift of multiplicity 1 throughout the paper.

LEMMA 1.3. *Let T be a $C_{\cdot 0}$ contraction with $d_T < \infty$. Let $n = d_{T^*} - d_T$, where we allow the possibility $n = \aleph_0$. Then*

(i) $J_T = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(n)}$ *with* $0 \leqslant k \leqslant d_T$, *and*

(ii) $B_T = S_T^* \oplus B^{(n)}$.

Proof. To prove (ii) it is convenient to use the machinery of the minimal isometric and unitary dilations of T^*. Recall from [22] (and in its notation, where we omit the subscripts T^* to ease that notation) that T^* has a minimal unitary dilation U acting on a space $\mathcal{K}$, and a restriction of U, U^+ ($= B_T^*$) acting on a space $\mathcal{K}^+$, is a minimal isometric dilation of T^*. Recall also that $\mathcal{K} = \mathrm{M}(\mathcal{L}_*) \oplus \mathcal{R}$ and $\mathcal{K}^+ = \mathrm{M}_+(\mathcal{L}_*) \oplus \mathcal{R}$, where $\mathcal{L}_*$ is some subspace wandering for U and U^+ respectively.

There is also a subspace $\mathcal{L}$ with

$$\dim(\mathcal{L}) = d_{T^*}, \tag{1.3}$$

and it is known that

$$\dim(\mathcal{L}_*) = d_T. \tag{1.4}$$

Now from ([22], page 59), since $T^* \in C_{0\cdot}$ we have $\mathcal{K} = \mathrm{M}(\mathcal{L})$ and U a bilateral shift of multiplicity d_{T^*}. Therefore,

$$\mathrm{M}(\mathcal{L}) = \mathcal{K} = \mathrm{M}(\mathcal{L}_*) \oplus \mathcal{R}. \tag{1.5}$$

Then note first that $U|\mathcal{R}$ must be a bilateral shift, and then from (1.3), (1.4), and (1.5) that the multiplicity of $U|\mathcal{R}$ must be $d_{T^*} - d_T$. Since $U|\mathcal{K}^+$ yields the minimal isometric dilation of T^*, and $\mathcal{K}^+ = \mathrm{M}_+(\mathcal{L}_*) \oplus \mathcal{R}$, the result then follows. ∎

2. JORDAN MODELS AND MULTIPLICITY

We embark upon a sequence of lemmas needed for the main theorem concerning $C_{\cdot 0}$ operators and Jordan models.

LEMMA 2.1. *If T is a contraction in $C_{\cdot 0}(\mathcal{H})$ with $d_T < \infty$ such that $J_T = S^{(n)}$, then* $\dim(\mathrm{Ker}(T^*)) = n$.

Proof. Since $J_T = S^{(n)}$, by ([23], Lemma 2.7), we have $T \in C_{10}$. According to ([18], Lemma 8), $\dim(\mathrm{Ker}(T^*)) = d_{T^*} - d_T = n$. ∎

LEMMA 2.2. *Let T be a contraction on $\mathcal{H}$ and $T \in C_{\cdot 0}$ with $d_T < \infty$. Suppose that*

$$T = \begin{pmatrix} * & * \\ 0 & \widetilde{T} \end{pmatrix}, \tag{2.1}$$

relative to a decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $J_T = S^{(r)}$ and $J_{\widetilde{T}} = S^{(n)}$. Then $r \geqslant n$.

Proof. By (2.1) it is obvious that

$$T^* = \begin{pmatrix} * & 0 \\ * & \widetilde{T}^* \end{pmatrix} \tag{2.2}$$

relative to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$. Lemma 2.1 and (2.2) imply that $r = \dim(\mathrm{Ker}(T^*)) \geqslant \dim(\mathrm{Ker}(\widetilde{T}^*)) = n$. Therefore the proof is complete. ∎

LEMMA 2.3. *Let T be a $C_{\cdot 0}$ contraction on $\mathcal{H}$. Suppose that*

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix} \tag{2.3}$$

relative to a decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $T_i \in C_{10}$, $i = 1, 2$. Then $T \in C_{10}$.

Proof. To show T in $C_{1\cdot}$ it suffices to show that $\|T^n x\| \to 0$ implies $x = 0$, so suppose there is some x for which this holds. Let $x = a \oplus b \in \mathcal{H}_1 \oplus \mathcal{H}_2$. It follows from (2.3) that for $n \in \mathbf{N}$,

$$T^n = \begin{pmatrix} T_1^n & D_n \\ 0 & T_2^n \end{pmatrix}. \tag{2.4}$$

Therefore $\|T^n x\|^2 = \|T_1^n a + D_n b\|^2 + \|T_2^n b\|^2$ for any $n \in \mathbf{N}$. By assumption we have $\|T^n x\| \to 0$. This implies that $\|T_1^n a + D_n b\| \to 0$ and $\|T_2^n b\| \to 0$. Since $T_i \in C_{10}$, $i = 1, 2$, it is obvious that $b = 0 = a$ and $x = 0$, and we are done. ∎

The following theorem has as its central ingredient an improvement of ([17], Theorem 4.5).

THEOREM 2.4. *Suppose $T \in C_{\cdot 0} \cap \mathbf{A}$ satisfies $d_T < \infty$, and let n be a positive integer. Then the following are equivalent:*

(i) $T \in \mathbf{A}_{n,\aleph_0} \setminus \mathbf{A}_{n+1,1}$;
(ii) $T \in \mathbf{A}_{n,1} \setminus \mathbf{A}_{n+1,1}$;
(iii) $J_T = S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus S^{(n)}$, *with* $0 \leqslant k \leqslant d_T$;
(iv) $d_{T^*} - d_T = n$;
(v) $B_T = S_T^* \oplus B^{(n)}$.

Proof. It is immediate from the definitions that (i) $\Rightarrow$ (ii), and the equivalence of (iii), (iv), and (v) is Lemma 1.3. (Remark that, in fact, in the presence of these T is automatically in $\mathbf{A}$ in light of (v) and [2]). That (v) $\Rightarrow$ (i) is from the following result of [20] (see also [19]): let T be in $\mathbf{A}$ with R the unitary piece of its minimal coisometric extension. Suppose Σ is a Borel subset of the circle such that $m|\Sigma$ is a scalar spectral measure for R (where m denotes Lebesgue measure on the circle), and R has multiplicity at least n on Σ. If $\Sigma = \mathbf{T}$ then T is in $\mathbf{A}_{n,\aleph_0}$.

A little reflection shows that it is then enough to prove that if $T \in \mathbf{A}_{n,1}(\mathcal{H})$ for some positive integer n then there exists a positive integer r with $n \leqslant r$ and some k so that

$$J_T = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(r)}, \tag{2.5}$$

in the sense that if $k = 0$, then $J_T = S^{(r)}$. Observe that in light of (iii) $\Leftrightarrow$ (iv) we may assume that $d_{T^*} < \infty$.

Since $T \in C_{\cdot 0}$ with $d_T < \infty$, there exist nonnegative integers r and k with $0 \leqslant k \leqslant d_T$, $r = d_{T^*} - d_T$, and

$$J_T = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(r)}. \tag{2.6}$$

To simplify notation, we let $A = S^{(n)}$. Then A has an n-cyclic set and A^* has a cyclic vector (cf. [16], page 281). Since $d_{A^*} < \infty$, by ([17], Proposition 4.4), there exist $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that

$$J_{\widetilde{T}} = J_A = S^{(n)}, \quad \text{where } \widetilde{T} = T_{\mathcal{K}}, \quad \text{and } \mathcal{K} = \mathcal{M} \ominus \mathcal{N}. \tag{2.7}$$

Hence by ([23], Lemma 2.7) and (2.7), we have $\widetilde{T} \in C_{10}$. Furthermore, we can write

$$T = \begin{pmatrix} * & * & * \\ 0 & \widetilde{T} & * \\ 0 & 0 & * \end{pmatrix} \tag{2.8}$$

relative to a decomposition $\mathcal{N} \oplus \mathcal{K} \oplus \mathcal{M}^{\perp}$. Put

$$T_1 = T|(\mathcal{N} \oplus \mathcal{K}) = \begin{pmatrix} A' & * \\ 0 & \widetilde{T} \end{pmatrix} \tag{2.9}$$

relative to the decomposition $\mathcal{N} \oplus \mathcal{K}$. Then it is easy to show that $T_1 \prec^i T$. It follows from ([21], Theorem 4) that

$$J_{T_1} = S(\theta'_1) \oplus \cdots \oplus S(\theta'_{k'}) \oplus S^{(r')}, \tag{2.10}$$

where $0 \leqslant k' \leqslant k$, $0 \leqslant r' \leqslant r$. Let

$$A' = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \tag{2.11}$$

be its unique triangular form of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$. Then we have

$$T_1 = \begin{pmatrix} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & \widetilde{T} \end{pmatrix} \tag{2.12}$$

relative to some decomposition. Let

$$\widetilde{T'} = \begin{pmatrix} A_2 & * \\ 0 & \widetilde{T} \end{pmatrix}. \tag{2.13}$$

By Lemma 2.3, $\widetilde{T'} \in C_{10}$. According to ([23], Lemma 2.7) and (2.10),

$$J_{A_1} = S(\theta'_1) \oplus \cdots \oplus S(\theta'_{k'}) \quad \text{and} \quad J_{\widetilde{T'}} = S^{(r')}. \tag{2.14}$$

By Lemma 2.2, (2.7), (2.10), (2.13) and (2.14), we have $n \leqslant r' \leqslant r$. The proof is complete. (Note that via the theorem as a whole in fact $r = n$.) ∎

There is also a theorem corresponding, with necessary modifications, to the case $n = \aleph_0$ in the one above. Its proof is easy from that above and the well known fact that $\mathbf{A}_{\aleph_0,\aleph_0} = \bigcap_{n=1}^{\infty} \mathbf{A}_{n,n}$ (see [5], Theorem 6.3), so we omit it.

THEOREM 2.5. *Suppose $T \in C_{\cdot 0} \cap \mathbf{A}$ satisfies $d_T < \infty$. Then the following are equivalent:*

(i) $T \in \mathbf{A}_{\aleph_0,\aleph_0}$;

(ii) $T \in \mathbf{A}_{\aleph_0,1}$;

(iii) $J_T = S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus S^{(\aleph_0)}$, *with* $0 \leqslant k \leqslant d_T$;

(iv) d_{T^*} *is infinite;*

(v) $B_T = S_T^* \oplus B^{(\aleph_0)}$.

We should remark that the result (ii) $\Leftrightarrow$ (iii) in Theorem 2.4, in which information about multiplicity is deduced from information about membership in some $\mathbf{A}_{m,n}$, is of a type fairly rare in dual algebra theory. There are a number of results of the general form "T has (some sort of) multiplicity k implies $T \in \mathbf{A}_{k',k''}$ where k' and k'' are related to k in some way" (see, for example, [15], [20], and [19]). Other than the results in this paper, we know of similar results in the reverse direction only in the context of normal operators (cf. [15]).

3. CONSEQUENCES

The following corollary captures the fact that operators in $\mathbf{A} \cap C_{\cdot 0}$ are completely well behaved with respect to ampliation, direct sum, and "factoring out" as regards their class membership. The results follow easily from Theorem 2.4.

COROLLARY 3.1. *Suppose T and T' are operators in $\mathbf{A} \cap C_{\cdot 0}$ with $d_T < \infty$ and $d_{T'} < \infty$, and suppose $m, n \in \mathbf{N}$. Then*

(i) $T \in \mathbf{A}_{m,\aleph_0} \setminus \mathbf{A}_{m+1,1}$ *implies* $T^{(n)} \in \mathbf{A}_{n\cdot m,\aleph_0} \setminus \mathbf{A}_{n\cdot m+1,1}$,

(ii) $T \in \mathbf{A}_{m,\aleph_0} \setminus \mathbf{A}_{m+1,1}$ *and* $T' \in \mathbf{A}_{n,\aleph_0} \setminus \mathbf{A}_{n+1,1}$ *implies* $T \oplus T' \in \mathbf{A}_{m+n,\aleph_0} \setminus \mathbf{A}_{m+n+1,1}$, *and*

(iii) $T \in \mathbf{A}_{m,\aleph_0} \setminus \mathbf{A}_{m+1,1}$ *and* $T \oplus T' \in \mathbf{A}_{m+n,\aleph_0} \setminus \mathbf{A}_{m+n+1,1}$ *implies* $T' \in \mathbf{A}_{n,\aleph_0} \setminus \mathbf{A}_{n+1,1}$.

It turns out as well that class membership is nicely behaved with respect to formation of an upper triangular operator with $C_{\cdot 0}$ operators of finite defect on the diagonals. Recall that for a contraction T we denote by R_T the unitary piece of the minimal coisometric extension of T. We begin with Lemma 1.4 from [6].

LEMMA 3.2. *Let T be a completely non-unitary contraction on $\mathcal{H}$ and let $\mathcal{H}_1$ be a subspace invariant for T. Write T in its two by two decomposition with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$,*

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}. \tag{3.1}$$

Then R_T is unitarily equivalent to $R_{T_1} \oplus R_{T_2}$.

We also have need for some tools from [9]. For T an absolutely continuous contraction that paper defines a set $X_T \subseteq \mathbf{T}$ which captures that portion of the unit circle on which T "wants" to be in $\mathbf{A}_{\aleph_0,\aleph_0}$ (so, for example, $X_T = \mathbf{T}$ if and only if $T \in \mathbf{A}_{\aleph_0,\aleph_0}$). Denote the class of operators T in $C_{\cdot 0}$ with $d_T < \infty$ by $C^{\mathrm{d}}_{\cdot 0}$. In this language, it makes sense to ask about X_T for T in $(C^{\mathrm{d}}_{\cdot 0} \cap \mathbf{A}) \backslash \mathbf{A}_{\aleph_0,\aleph_0}$. As one might expect from a little work with Theorem 2.4 and Corollary 3.1 (and Corollary 3.5 below), it turns out that $X_T = \emptyset$. To see this, it is elementary from [9] that if T is a compression of T' then $X_T \subseteq X_{T'}$. Recall next that every absolutely continuous contraction with both defect indices finite is the compression of a bilateral shift of multiplicity $d_T + d_{T^*}$ (see [22], Theorem 7.4). Finally, such a bilateral shift is not in $\mathbf{A}_{\aleph_0,\aleph_0}$ from, for example, [15], and a little work with Möbius transforms shows that in fact its X set must be empty.

For convenience of notation, let $\mathbf{A}_{0,k}$ denote the class of absolutely continuous contractions for any k. Also, as usual we abbreviate $\mathbf{A}_{\aleph_0,\aleph_0}$ to $\mathbf{A}_{\aleph_0}$. Note that the following improves (ii) and (iii) of Corollary 3.1.

COROLLARY 3.3. *Suppose T is an absolutely continuous contraction in $C_{\cdot 0}$ satisfying $d_T < \infty$, and suppose T has some upper triangular decomposition*

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}. \tag{3.2}$$

Then $T \in \mathbf{A}_{\aleph_0}$ if and only if $T_1 \in \mathbf{A}_{\aleph_0}$ or $T_2 \in \mathbf{A}_{\aleph_0}$. If not, then for any non-negative integers m and n we have:

(i) *Suppose $T_1 \in \mathbf{A}_{n,\aleph_0} \backslash \mathbf{A}_{n+1,1}$. Then $T \in \mathbf{A}_{m+n,\aleph_0} \backslash \mathbf{A}_{m+n+1,1}$ if and only if $T_2 \in \mathbf{A}_{m,\aleph_0} \backslash \mathbf{A}_{m+1,1}$, and*

(ii) *Suppose $T_2 \in \mathbf{A}_{m,\aleph_0} \backslash \mathbf{A}_{m+1,1}$. Then $T \in \mathbf{A}_{m+n,\aleph_0} \backslash \mathbf{A}_{m+n+1,1}$ if and only if $T_1 \in \mathbf{A}_{n,\aleph_0} \backslash \mathbf{A}_{n+1,1}$.*

Proof. An easy computation shows that if T has the triangularization above then T_1 and T_2 are indeed in the class $C_{\cdot 0}$. Assume first that $T \in \mathbf{A}_{\aleph_0}$. From ([22], Proposition 3.6), we have $d_{T_1} \leqslant d_T$, and thus either $T_1 \in \mathbf{A}_{\aleph_0}$ or $d_{T_1^*}$ is finite using Theorems 2.4 and 2.5. If $d_{T_1^*}$ is finite we noted above that $X_{T_1} = \emptyset$; it then follows from a result in [9] (using $T_1 \in C_{\cdot 0}$ in the triangular factorization) that $X_{T_2} = X_T$ must be the whole circle, and thereby that $T_2 \in \mathbf{A}_{\aleph_0}$. Of course, if either of T_1 or T_2 is in $\mathbf{A}_{\aleph_0}$ it is well known that T must be also.

If none of T, T_1, and T_2 are in $\mathbf{A}_{\aleph_0}$ we have immediately that d_{T^*} is finite; it then follows from ([22], Proposition 3.6), that d_{T_1}, d_{T_2}, $d_{T_1^*}$, and $d_{T_2^*}$ are all finite. The lemma above and Theorems 2.4 and 2.5 then give the result in the finite cases. ∎

With the aid of a proposition we see that class membership is well behaved under powers as well. The next result, and its proof, are from [9].

PROPOSITION 3.4. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and let j be a positive integer. Let $B_T = S_T^* \oplus R_T$ acting on $\mathcal{K} = \mathcal{S} \oplus \mathcal{R}$ denote as usual the minimal co-isometric extension of T, and denote similarly the minimal co-isometric extension of T^j. Then $R_{(T^j)}$ is unitarily equivalent to $(R_T)^j$.*

Proof. We give the proof for the case $j = 2$, leaving other cases to the reader. Since it is clear that $B_T^2 = (S_T^*)^2 \oplus (R_T)^2$ acting on $\mathcal{K}$ is some co-isometric extension of T^2, we may assume that there exists $\mathcal{K}_1 \subseteq \mathcal{K}$, reducing for B_T^2, such that $B_T^2|\mathcal{K}_1$ is the minimal co-isometric extension of T^2 (see [11]). It suffices to show that $\mathcal{R} \subseteq \mathcal{K}_1$ or equivalently that $\mathcal{K}_1^\perp \subseteq \mathcal{S}$. Since B_T is a co-isometry, this is in turn equivalent to

$$\mathcal{K}_1^\perp \subseteq \bigcup_{n=1}^{\infty} \operatorname{Ker}(B_T^{2n}). \tag{3.3}$$

To prove (3.3) we recall some facts about the minimal coisometric extension (again from [22]). From the minimality of B_T, we may deduce that $\bigvee_{n\geqslant 0} (B_T^*)^n \mathcal{H} = \mathcal{K}$. Further, there is a subspace $\mathcal{L}$ of $\mathcal{K}$ defined by

$$\mathcal{L} = \overline{\{(B_T^* - T^*)h : h \in \mathcal{H}\}} \tag{3.4}$$

such that

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{L} \oplus B_T^*\mathcal{L} \oplus \cdots \oplus (B_T^*)^n \mathcal{L} \oplus \cdots \tag{3.5}$$

Note that since $\mathcal{K}_1$ is reducing for B_T^2, we have

$$(B_{T^2})^* = ((B_T^2)|\mathcal{K}_1)^* = ((B_T^2)^*|\mathcal{K}_1) = (B_T^*)^2|\mathcal{K}_1. \tag{3.6}$$

Let $\mathcal{L}_2$ be the space analogous to $\mathcal{L}$, but for B_{T^2}, defined by

$$\mathcal{L}_2 = \overline{\{(B_{T^2}^* - (T^2)^*)h : h \in \mathcal{H}\}}. \tag{3.7}$$

Then it is clear from (3.6) that

$$\mathcal{L}_2 = \overline{\{((B_T^*)^2 - T^{*2})h : h \in \mathcal{H}\}}. \tag{3.8}$$

And again using (3.6) we have the decomposition

$$\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{L}_2 \oplus (B_T^*)^2\mathcal{L}_2 \oplus \cdots \oplus (B_T^*)^{2n}\mathcal{L}_2 \oplus \cdots \ . \tag{3.9}$$

One may then compute easily that $\mathcal{L}_2 \subseteq \mathcal{L} \oplus B_T^*\mathcal{L}$ and that for any u and v in $\mathcal{L}$ such that $B_T^*u \oplus v \perp \mathcal{L}_2$, we have $B_T^*u \oplus v \in \operatorname{Ker}(B_T^2)$. It is easy to perform similar computations to yield $((B_T^*)^2\mathcal{L} \oplus (B_T^*)^3\mathcal{L}) \ominus (B_T^*)^2\mathcal{L}_2 \subseteq \operatorname{Ker}(B_T^4)$, and so on. The result (3.3) then follows from the decompositions given by (3.5) and (3.9). ∎

A general analysis in [9] shows that if $T \in \mathbf{A}_{1,\aleph_0}$ then $T^n \in \mathbf{A}_{n,\aleph_0}$. With the aid of the proposition above, and in our special situation, we can do better.

COROLLARY 3.5. *Suppose $T \in \mathbf{A} \cap C_{\cdot 0}$ satisfies $d_T < \infty$, and let $m, n \in \mathbf{N}$. Then*

(i) *$T \in \mathbf{A}_{n,1} \backslash \mathbf{A}_{n+1,1}$ implies $T^m \in \mathbf{A}_{m \cdot n,1} \backslash \mathbf{A}_{m \cdot n+1,1}$, and*
(ii) *$T^m \in \mathbf{A}_{m \cdot n+1,1}$ implies $T \in \mathbf{A}_{n+1,1}$.*

Proof. It suffices to use Proposition 3.4 and count multiplicities. ∎

We may gain some information about Jordan models as well; the following is immediate if we use the proof from ([17], Lemma 5.1), along with Theorem 2.4.

COROLLARY 3.6. *Suppose $T \in \mathbf{A}_{n,1} \cap C_{\cdot 0}(\mathcal{H})$ and $d_T < \infty$. Then $T \in C_{10}$ if and only if there exists $r \in \mathbf{N}$ with $r \geqslant n$ such that $J_T = S^{(r)}$.*

A completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be of *class* C_0 if there exists a non-zero function $u \in H^\infty(\mathbf{T})$ such that the functional calculus $u(T) = 0$ (cf. [3]). Recall that if $C \in C_0$ with $d_{C^*} < \infty$, then $J_C = S(\theta_1) \oplus \cdots \oplus S(\theta_k)$ for some $0 \leqslant k < \infty$ (cf. [3]). Recall also that for such a C, we have $d_{C^*} = d_C$. The following is then immediate.

THEOREM 3.7. *Let $C \in C_0$ with $d_{C^*} < \infty$ and let $T \in C_{\cdot 0}(\mathcal{H})$ with $d_T < \infty$. If $T \oplus C \in \mathbf{A}_{n,1}$, for some $n \in \mathbf{N}$, then $d_{T^*} - d_T \geqslant n$.*

We also recapture the following theorem from [17]; recall that S denotes the unilateral shift of multiplicity 1.

COROLLARY 3.8. *If $C \in C_0$ with $d_{C^*} < \infty$, then $S^{(n)} \oplus C \in \mathbf{A}_{n,\aleph_0} \setminus \mathbf{A}_{n+1,1}$ for any $n \in \mathbf{N}$.*

4. REMARKS AND QUESTIONS

Observe that since $(C_{\cdot 0})^* = C_{0\cdot}$ and $\mathbf{A}^*_{m,n} = \mathbf{A}_{n,m}$ the results above have implications for a theory of operators in the class $C_{0\cdot}$ with some defect index finite; we leave these corollaries to the interested reader. Recall that we denote the class of operators T in $C_{\cdot 0}$ with $d_T < \infty$ by $C^{\mathrm{d}}_{\cdot 0}$. Similarly straightforward is the following: one way to view Theorem 2.4 is as the assertion that one has, for any n,

$$C^{\mathrm{d}}_{\cdot 0} \cap \mathbf{A}_{n,1} = C^{\mathrm{d}}_{\cdot 0} \cap \mathbf{A}_{n,m} = C^{\mathrm{d}}_{\cdot 0} \cap \mathbf{A}_{n,\aleph_0}, \quad m \in \mathbf{N}. \tag{4.1}$$

This generalizes in a special case the result in [14] that

$$C_{\cdot 0} \cap \mathbf{A}_{1,1} = C_{\cdot 0} \cap \mathbf{A}_{1,m} = C_{\cdot 0} \cap \mathbf{A}_{1,\aleph_0}, \quad m \in \mathbf{N},$$

and leads to the obvious question as to whether (4.1) generalizes to arbitrary elements of $C_{\cdot 0}$.

A similar natural question is whether (or to what extent) the classes $\mathbf{A}_{n,\aleph_0}$ are determined by multiplicity of the unitary part of the minimal coisometric extension (the Jordan model no longer available). Since it is easy to construct a normal operator in $C_{00} \cap \mathbf{A}_{\aleph_0,\aleph_0}$, whose minimal coisometric extension therefore has *no* unitary part, the question must necessarily be sharpened. It is well known that for T in $\mathbf{A}_{1,\aleph_0}$ one may produce a restriction of T to an invariant subspace or a compression of T to a semi-invariant subspace in $C_{\cdot 0} \cap \mathbf{A}_{1,\aleph_0}$ (see [14] or [10]). Since the restriction or compression of an operator in C_{00} is again in C_{00}, the normal operator example above shows that there is no hope of capturing the class in the multiplicity of the unitary piece of some restriction or compression. Can one do business at least in the case $T \notin \mathbf{A}_{\aleph_0,\aleph_0}$, or under some other special assumption?

Finally, we remark on the annoying fact that while for T in $C^{\mathrm{d}}_{\cdot 0} \cap \mathbf{A}$ we are well able to factor out powers of the unilateral shift as summands (Corollary 3.1), for bilateral shift summands we are unable to improve on an old result of [15] that, for any absolutely continuous contraction T, if $T \oplus B^{(n)} \in \mathbf{A}_{n+1,n+1}$ then T is in $\mathbf{A} = \mathbf{A}_{1,1}$. We make here the obvious conjecture that for T in $C^{\mathrm{d}}_{\cdot 0} \cap \mathbf{A}$, $T \oplus B^{(n)} \in \mathbf{A}_{n+m,n+m}$ implies T in $\mathbf{A}_{m,m}$.

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