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CONTINUITY OF SCHUR BLOCK-MULTIPLICATION MAPS WITH RESPECT TO VARIOUS TOPOLOGIES

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ABSTRACT. We consider maps induced on the space of bounded operator matrices via left Schur multiplication by a fixed Schur multiplier matrix. The main goal is to give a complete characterization of those multipliers that induce continuous maps when initial and final topologies are chosen from among the weak, the strong and the norm topologies.

In the process we are also able to demonstrate that multipliers inducing compact Schur multiplication maps are exactly those with an approximation property similar to one for compact operator matrices. As well we prove that diagonal truncation is *not* strong to weak continuous.

We work in general with matrices whose entries are bounded operators and we use a non-commutative extention of the usual Schur product, called "Schur block-product".

KEYWORDS: *Schur (Hadamard) block-multiplication, Schur multipliers, Block-matrices.*

AMS SUBJECT CLASSIFICATION: Primary 47B49; Secondary 47A05.

INTRODUCTION

We consider maps induced on the space of bounded operator matrices via left Schur multiplication by a fixed Schur multiplier matrix. The main goal is to give a complete characterization of those multipliers that induce continuous maps when initial and final topologies are chosen from among the weak, the strong and the norm topologies.

In the process we are also able to demonstrate that multipliers inducing compact Schur multiplication maps are exactly those with an approximation property similar to one for compact operator matrices. As well we observe that the operation of diagonal truncation is *not* strong to weak continuous.

We work in general with matrices whose entries are bounded operators and we use an extention of the usual Schur product to such matrices, introduced in [4], [5] and [7] and termed "Schur block-product". Schur block-product is a non-commutative generalization of the regular Schur product which retains many of its properties. Some distinctions, on the other hand, can be seen in the continuity properties. Block products seem to provide valuable insight into the behaviour of the original product as well as of some of its extentions.

NOTATION AND CONVENTIONS

Consider the set $M_{a\times b}$ of $a \times b$ complex matrices, where a and b are positive integers or countable infinity (denoted by "∞"). (If $a = b$ we write M_a for $M_{a\times b}$.) Some of these matrices can be regarded as bounded operators from a Banach space ℓ^p_b to a Banach space ℓ^r_a, where ℓ^q_c, with the usual norm $\|\cdot\|_q$ ($\|\cdot\|_2$ is simply written as $\|\cdot\|$), is identified as either a subspace of $M_{1\times c}$ or of $M_{c\times 1}$ depending on convenience (q, c are either positive integers or ∞). The set of elements of $M_{a\times b}$ representing bounded linear operators from ℓ^2_b to ℓ^2_a shall be denoted by $BM_{a\times b}$. $BM_{a\times b}$ is itself a Banach space and, in the case $a = b$, a C^*-algebra under the usual operator norm $\|\cdot\|$, matrix multiplication "$\circ$" and conjugate-transposition "$*$". A matrix $A \in M_{a\times b}$ represents a bounded operator from ℓ^1_b to ℓ^2_a exactly when all of its columns are in ℓ^2_a and the supremum of their ℓ^2_a-norms is finite. This supremum gives the norm of A in $B(\ell^1_b, \ell^2_a)$ and is called *the column norm of* A. Similarly, A represents a bounded operator from ℓ^2_b to ℓ^∞_a exactly when all of its rows are in ℓ^2_b and the supremum of their ℓ^2_b-norms is finite. This supremum gives the norm of A in $B(\ell^2_b, \ell^\infty_a)$ and is called *the row norm of* A.

The $i-j$-th entry of $A \in M_{a\times b}$ is denoted by $A[i,j]$, with the understanding that if $a = 1$ then $A[1,j]$ can be substituted by $A[j]$ (and similarly if $b = 1$). $M_{1\times 1}$ shall be simply identified with $\mathbf{C}$.

Recall that the *Schur product* $A \square B$ of two matrices A and B in $M_{a\times b}$ is their entrywise product, i.e. $(A \square B)[i,j] = A[i,j]B[i,j]$. (For basic information on finite and infinite dimensional Schur multiplication see [1], [3], [11], [12] and [13].)

A more general set than $M_{a\times b}$ is the set $M_{a\times b}(M_{c\times d})$ of $a \times b$ matrices with entries from $M_{c\times d}$. We shall refer to such matrices as "block-matrices". If c and d are finite then $M_{a\times b}(M_{c\times d})$ can be naturally identified with ("partitioned") $M_{ca\times db}$. If a and b are finite then $M_{a\times b}(M_{c\times d})$ can be readily identified with the tensor product $M_{c\times d} \otimes M_{a\times b}$. $M_{a\times b}(M_{1\times 1})$ and $M_{1\times 1}(M_{a\times b})$ are identified with $M_{a\times b}$.

If $A \in M_{a\times b}(M_{c\times d})$ then $A[i,j]$ ($\in M_{c\times d}$) stands for the $i-j$-th *block-entry* of A and $(A[i,j])[k,m]$ ($\in \mathbb{C}$) indicates the $k-m$-th entry of $A[i,j]$. The notation is simplified (as mentioned before) whenever one or more of a, b, c and d are 1.

We shall be mostly concerned with the subspace $M_{a\times b}(BM_{c\times d})$ of $M_{a\times b}(M_{c\times d})$. Some block-matrices in that subspace can be regarded as bounded operators from the Hilbert space $\bigoplus_{j=1}^{b} \ell_d^2$ to the Hilbert space $\bigoplus_{i=1}^{a} \ell_c^2$, in the usual way, with "$\oplus$" indicating the ℓ^2-direct sum. (Here $\bigoplus_{k=1}^{r} \ell_m^2$, with the usual ℓ^2-norm, is identified as either a subspace of $M_{r\times 1}(BM_{m\times 1})$ or of $M_{1\times r}(BM_{1\times m})$, depending on convenience.) The subspace of all such matrices in $M_{a\times b}(BM_{c\times d})$ is denoted by $BM_{a\times b}(M_{c\times d})$ and is, in the case $a = b$, a C^*-algebra under the usual operator norm, the usual block-matrix multiplication and conjugate-transposition (all denoted by the same symbols as before; note that conjugate transposition is defined on all of $M_{a\times b}(BM_{c\times d})$ by: $A^*[i,j] = (A[j,i])^*$).

To simplify notation it is convenient to make further use of the tensor product symbol "$\otimes$". If $A \in M_{c\times d}$ and $B \in M_{a\times b}$ then $A \otimes B$ shall stand for the element of $M_{a\times b}(M_{c\times d})$ specified by: $(A \otimes B)[i,j] = B[i,j] \cdot A$. In particular, if $g \in BM_{m\times 1}(\leftrightarrow \ell_m^2)$ and $h \in BM_{r\times 1}(\leftrightarrow \ell_r^2)$ then $g \otimes h \in BM_{r\times 1}(M_{m\times 1})\left(\leftrightarrow \bigoplus_{k=1}^{r} \ell_m^2\right)$.

The *Schur block-product* $A \,\square\, B$ of two block-matrices A and B in $M_{a\times b}(BM)$ is a form of entrywise product, with block-entries multiplied together as matrices, i.e. $A \,\square\, B \in M_{a\times b}(BM)$ and $(A \,\square\, B)[i,j] = A[i,j] \circ B[i,j]$. This non-commutative extention of the well-known Schur product was introduced in [4], [5] and [7], and seems to provide a natural and useful tool for further analysis.

PROPOSITION A. *Suppose A in $M_{a\times b}(BM)$ is such that $A \,\square\, T \in BM_{a\times b}(M_{c\times d})$ whenever $T \in M_{a\times b}(M_{c\times d})$. Then the linear map*

$$ {}_A\Psi : BM_{a\times b}(M_{c\times d}) \to BM_{a\times b}(M_{c\times d}) $$

defined by: ${}_A\Psi(C) = A \,\square\, C$, is bounded.

Proof. This map is clearly closed. The rest is the Closed Graph Theorem. ∎

A matrix A satisfying the hypothesis of Proposition A is called a *left Schur block-multiplier on* $BM_{a\times b}(M_{c\times d})$. The set of all such matrices in $M_{a\times b}(BM_{c\times c})$ is denoted by $SM_{a\times b}(M_{c\times d})$. Even in the familiar commutative case $b = 1$, $a = \infty$ there is no known workable set of necessary and sufficient conditions making it possible to decide which matrices belong to $SM_{a\times b}(M_{c\times d})$. If $A \in SM_{a\times b}(M_{c\times d})$

then necessarily $\sup_{i,j} ||A[i,j]|| \leqslant ||_A\Psi|| < \infty$. This condition is not sufficient. The following is a sufficient condition which is not necessary (For the proof see [6]):

PROPOSITION B. *Suppose that A in $M_{a\times b}(BM_c)$ is such that all block-rows of A are in $BM_{1\times b}(M_c)$ and the supremum of their norms is finite. Then $A \in SM_{a\times b}(M_{c\times d})$ and $||_A\Psi||$ is no larger than this supremum. In particular, $BM_{a\times b}(M_c) \subset SM_{a\times b}(M_{c\times d})$ and, for every C in $BM_{a\times b}(M_c)$, $||_C\Psi|| \leqslant ||C||$.*

The following recent material deals with the basic properties of Schur block-products in finite and infinite dimensional cases: [4], [5], [6], [7].

ADDITIONAL NOTATION AND CONVENTIONS

(1) *weak*, *str*, *norm* indicate the usual weak operator, strong operator and operator norm topologies on $BM_{a\times b}(M_{c\times d})$. In view of previous identification *weak* and *norm* can also indicate topologies on ℓ^2_r.

If no specific topology is indicated the norm topology is assumed.

(2) If $T \in BM_a(M_c)$ then $L_T : BM_{a\times b}(M_{c\times d}) \to BM_{a\times b}(M_{c\times d})$ is the bounded linear map defined by: $L_T(A) = T \circ A$. *Right multiplication* by T map R_T is defined similarly.

(3) *Block-diag* $\{A, B, C, \ldots\}$ indicates $\begin{bmatrix} A & 0 & 0 & \ldots \\ 0 & B & 0 & \ldots \\ 0 & 0 & C & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

(4) # indicates the Banach dual (space).

(5) *trn* stands for the *transpose*.

(6) If $x = (a, b, c, d, \ldots)$ in $\mathbf{C}^s$ then $\overline{x}$ stands for $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \ldots)$.

(7) A vector x in a set S is said to be a *separating vector* for the family G of functions on S if for any two distinct functions in G their values at x are distinct.

(8) If x and y are vectors in a Hilbert space H then $x\lceil\otimes\rfloor y$ denotes the rank one operator T in $B(H)$ specified by: $T(z) = \langle z, y\rangle x$.

(9) We shall use the following convention throughout this paper:

E_{mk} will stand for the $m-k$-th standard matrix unit in M_a;

F_{mk} will stand for the $m-k$-th standard matrix unit in M_b;

e_m will stand for the m-th standard basis element of ℓ^q_a, no matter what q is;

f_m will stand for the m-th standard basis element of ℓ^q_b, no matter what q is.

1. SEQUENTIAL CONTINUITY AND COMPACTNESS

This first lemma is common knowledge and it is stated here for completeness.

LEMMA 1.1. *For a sequence* $\{A_n\}_{n=1}^{\infty}$ *in* $BM_a(M_b)$

(i) *the following are equivalent to each other:*

(a) $\underset{n\to\infty}{\text{weak-lim}}\, A_n = 0$;

(b) $\sup_n \|A_n\|$ *is finite and* $\underset{n\to\infty}{\text{weak-lim}}\, A_n[i,j] = 0$ *for all* i, j;

(c) $\sup_n \|A_n\|$ *is finite and* $\lim_{n\to\infty} (A_n[i,j])[k,l] = 0$ *for all* i, j, k, l;

(ii) *as well, the following are equivalent to each other:*

(d) $\underset{n\to\infty}{\text{str-lim}}\, A_n = 0$;

(e) $\sup_n \|A_n\|$ *is finite and* $\underset{n\to\infty}{\text{str-lim}}\, A_n \circ (I_b \otimes e_i) = 0$ *for each* i;

(f) $\sup_n \|A_n\|$ *is finite and* $\lim_{n\to\infty} A_n \circ (f_j \otimes e_i) = 0$ *for each* i, j.

THEOREM 1.2. *Suppose* A *is in* $\mathrm{S}M_a(M_b)$. *Then:*

(i) ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$ *is sequentially continuous;*

(ii) ${}_A\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{str})$ *is sequentially continuous.*

Proof. It is enough to demonstrate the stated continuity properties of ${}_A\Psi$ at 0.

(i) Suppose $\{B_n\}_{n=1}^{\infty}$ is a sequence in $BM_a(M_b)$ such that $\underset{n\to\infty}{\text{str-lim}}\, B_n = 0$. Then $\underset{n\to\infty}{\text{str-lim}}\, B_n \circ (I_b \otimes e_j) = 0$, for every j, by Lemma 1.1.

$$ {}_A\Psi(B_n) \circ (I_b \otimes e_j) = \begin{bmatrix} A[1,j] & 0 & 0 & 0 \\ 0 & A[2,j] & 0 & 0 \\ 0 & 0 & A[3,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ B_n \circ (I_b \otimes e_j) $$

and therefore $\underset{n\to\infty}{\text{str-lim}}\, {}_A\Psi(B_n) \circ (I_b \otimes e_j) = 0$, for every j. So $\underset{n\to\infty}{\text{str-lim}}\, {}_A\Psi(B_n) = 0$ by Lemma 1.1, since $\sup_n \|{}_A\Psi(B_n)\| \leqslant \|{}_A\Psi\| \cdot \sup_n \|B_n\| < \infty$.

(ii) Suppose $\{B_n\}_{n=1}^{\infty}$ is a sequence in $BM_a(M_b)$ such that $\underset{n\to\infty}{\text{weak-lim}}\, B_n = 0$. Then

$$ \begin{aligned} \underset{n\to\infty}{\text{weak-lim}}({}_A\Psi(B_n))[i,j] &= \underset{n\to\infty}{\text{weak-lim}}\, A[i,j] \circ B_n[i,j] \\ &= A[i,j] \circ (\underset{n\to\infty}{\text{weak-lim}}\, B_n[i,j]) = 0, \text{ for every } i, j. \end{aligned} $$

Since $\sup_n \|{}_A\Psi(B_n)\| \leqslant \|{}_A\Psi\| \cdot \sup_n \|B_n\| < \infty$, it follows, by Lemma 1.1, that $\underset{n\to\infty}{\text{weak-lim}}\, {}_A\Psi(B_n) = 0$. ∎

Lemmas 1.3, 1.5 and 1.6 are hardly new. We present them here because they are crucial to the rest of the theorems in this section.

LEMMA 1.3. *Suppose H_1 and H_2 are Hilbert spaces of non-zero dimensions and $T \in B(H_2)$. Then T is compact if and only if $L_T : (B(H_1, H_2), \text{weak}) \to (B(H_1, H_2), \text{str})$ is sequentially continuous.*

Proof. Suppose $L_T : (B(H_1, H_2), \text{weak}) \to (B(H_1, H_2), \text{str})$ is sequentially continuous.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H_2 such that $\underset{n\to\infty}{\text{weak-lim}}\, x_n = 0$. If y is any unit vector in H_1 then $\underset{n\to\infty}{\text{weak-lim}}\, x_n \lceil \otimes \rfloor y = 0$, and consequently $\underset{n\to\infty}{\text{str-lim}}\, L_T(x_n \lceil \otimes \rfloor y) = \underset{n\to\infty}{\text{str-lim}}(T(x_n) \lceil \otimes \rfloor y) = 0$.

In particular: $\lim_{n\to\infty} (T(x_n) \lceil \otimes \rfloor y)(y) = \lim_{n\to\infty} T(x_n) = 0$. This shows that T is compact.

Conversely, suppose T is compact. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $B(H_1, H_2)$ such that $\underset{n\to\infty}{\text{weak-lim}}\, B_n = 0$. Then $0 = T(\underset{n\to\infty}{\text{weak-lim}}\, B_n(x)) = \lim_{n\to\infty} (T \circ B_n)(x)$, for every x in H_1.

In other words: $\underset{n\to\infty}{\text{str-lim}}\, T \circ B_n = 0$. So $L_T : (B(H_1, H_2), \text{weak}) \to (B(H_1, H_2), \text{str})$ is sequentially continuous. ∎

THEOREM 1.4. *For A in $SM_a(M_b)$,*

$$_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{str})$$

is sequentially continuous if and only if $A[i,j]$ is compact, for all i, j, and moreover $\lim_{i\to\infty} A[i,j] = 0$, for all j, whenever $a = \infty$.

Proof. Suppose $_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{str})$ is sequentially continuous. We shall show that

$$\begin{bmatrix} A[1,j] & 0 & 0 & 0 \\ 0 & A[2,j] & 0 & 0 \\ 0 & 0 & A[3,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

is compact, for every j; this is equivalent to the required conclusion.

By Lemma 1.3 it is enough to establish "weak-to-strong" sequential continuity of $L_{\text{block-diag}\{A[1,j],A[2,j],A[3,j],\dots\}} : BM_{a\times 1}(M_b) \to BM_{a\times 1}(M_b)$ at 0 (and therefore everywhere). Fix j. Suppose $\{D_n\}_{n=1}^{\infty}$ is a sequence in $BM_{a\times 1}(M_b)$ such that $\underset{n\to\infty}{\text{weak-lim}}\, D_n = 0$.

Let $\{B_n\}_{n=1}^{\infty}$ be the sequence in $BM_a(M_b)$ such that, for each k,

$$B_n \circ (I_b \otimes e_k) = \begin{cases} D_n & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

Then $\underset{n\to\infty}{\text{weak-lim}}\, B_n = 0$ and consequently $\underset{n\to\infty}{\text{str-lim}}\, {}_A\Psi(B_n) = 0$. Thus

$$\begin{aligned} 0 &= \underset{n\to\infty}{\text{str-lim}}\, {}_A\Psi(B_n) \circ (I_b \otimes e_j) \\ &= \underset{n\to\infty}{\text{str-lim}} \begin{bmatrix} A[1,j] & 0 & 0 & 0 \\ 0 & A[2,j] & 0 & 0 \\ 0 & 0 & A[3,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ B_n \circ (I_b \otimes e_j) \\ &= \underset{n\to\infty}{\text{str-lim}}\, L_{\text{block-diag}\{A[1,j],A[2,j],A[3,j],\ldots\}}(D_n), \text{ as required.} \end{aligned}$$

Conversely, suppose $\begin{bmatrix} A[1,j] & 0 & 0 & 0 \\ 0 & A[2,j] & 0 & 0 \\ 0 & 0 & A[3,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$ is compact, for every j. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $BM_a(M_b)$ such that $\underset{n\to\infty}{\text{weak-lim}}\, B_n = 0$. Then

$$\begin{aligned} &\underset{n\to\infty}{\text{str-lim}}\, {}_A\Psi(B_n) \circ (I_b \otimes e_j) \\ &\qquad = \underset{n\to\infty}{\text{str-lim}} \begin{bmatrix} A[1,j] & 0 & 0 & 0 \\ 0 & A[2,j] & 0 & 0 \\ 0 & 0 & A[3,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ B_n \circ (I_b \otimes e_j) \\ &\qquad = \underset{n\to\infty}{\text{str-lim}}\, L_{\text{block-diag}\{A[1,j],A[2,j],A[3,j],\ldots\}}(B_n) = 0 \end{aligned}$$

for every j, by Lemma 1.3. So $\underset{n\to\infty}{\text{str-lim}}\, {}_A\Psi(B_n) = 0$ by Lemma 1.1, since $\sup_n \|{}_A\Psi(B_n)\| \leqslant \|{}_A\Psi\| \cdot \sup_n \|B_n\| < \infty$.

This demonstrates the stated continuity of ${}_A\Psi$ at 0 and therefore everywhere. ∎

LEMMA 1.5. *Suppose H_1 and H_2 are Hilbert spaces and T is non-zero in $B(H_2)$. Then $L_T : (B(H_1, H_2), \text{str}) \to (B(H_1, H_2), \text{norm})$ is sequentially continuous if and only if H_2 is finite-dimensional.*

Proof. Suppose $L_T : (B(H_1, H_2), \text{str}) \to (B(H_1, H_2), \text{norm})$ is sequentially continuous while H_1 is infinite-dimensional.

Let x be a non-zero element of H_1 and let $\{y_n \mid n \in \mathbf{N}\}$ be an orthonormal set in H_2. Then $\operatorname{str-lim}_{n\to\infty} x\lceil\otimes\rfloor y_n = 0$ and therefore $\lim_{n\to\infty} L_T(x\lceil\otimes\rfloor y_n) = \lim_{n\to\infty} T(x)\cdot\lceil\otimes\rfloor y_n = 0$. So $T(x) = 0$ since $\|T(x)\| = \|(T(x)\lceil\otimes\rfloor y_n)(y_n)\| \leqslant \|T(x)\lceil\otimes\rfloor y_n\|$. Thus T must be the zero operator, which is a contradiction. The converse is trivial. ∎

Let S be a 0-1 matrix in M_a defined by $S[i,j] = \delta^i_{j+1}$.

LEMMA 1.6. *If $\{T_n\}_{n=1}^{\infty}$ is a sequence in $BM_a(M_b)$ such that $\sup_n \|T\|$ is finite, then*

(i) $\operatorname{str-lim}_{n\to\infty} T_n \circ (I_b \otimes (S^*)^n) = 0$;

(ii) $\operatorname{weak-lim}_{n\to\infty}(I_b \otimes S^n) \circ T_n = 0$.

Proof. (i) Clear, because $\operatorname{str-lim}_{n\to\infty}(I_b \otimes (S^*)^n) = 0$ and $\sup_n \|T_n\|$ is finite.

(ii) $\operatorname{weak-lim}_{n\to\infty}((I_b \otimes S^n) \circ T_n)[i,j] = 0$ for each i, j, because $((I_b \otimes S^n) \circ T_n)[i,j] = 0$ whenever $n > i$. So $\operatorname{weak-lim}_{n\to\infty}(I_b \otimes S^n) \circ T_n = 0$ by Lemma 1.1, since $\|(I_b \otimes S^n) \circ T_n\| \leqslant \sup_n \|T_n\| < \infty$. ∎

THEOREM 1.7. *Suppose A is a non-zero element of $\mathrm{SM}_a(M_b)$. Then ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{norm})$ is sequentially continuous if and only if b is finite, and $\lim_{n\to\infty} {}_{A-\ddot{A}_n}\Psi = 0$, whenever $a = \infty$; where $\ddot{A}_n \in \mathrm{SM}_a(M_b)$ is specified by:* $\ddot{A}_n[i,j] = \begin{cases} A[i,j] & j \leqslant n \\ 0 & \text{otherwise.} \end{cases}$

(Note that $\lim_{n\to\infty} {}_{A-\ddot{A}_n}\Psi = 0$ and $\lim_{n\to\infty} {}_{\ddot{A}_n}\Psi = {}_A\Psi$ are equivalent.)

Proof. Suppose ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{norm})$ is sequentially continuous.

Let $\{B_n\}_{n=1}^{\infty}$ be a sequence in $BM_{a\times 1}(M_b)$ such that $\operatorname{str-lim}_{n\to\infty} B_n = 0$. Let k_0 be any index such that $A \circ (I_b \otimes e_{k_0})$ is non-zero. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence in $BM_a(M_b)$ specified by:

$$D_n \circ (I_b \otimes e_j) = \begin{cases} B_n & \text{if } j = k_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\operatorname{str-lim}_{n\to\infty} D_n = 0$ and consequently $\lim_{n\to\infty} {}_A\Psi(D_n) = 0$. So

$$\lim_{n\to\infty} \begin{bmatrix} A[1,k_0] & 0 & 0 & 0 \\ 0 & A[2,k_0] & 0 & 0 \\ 0 & 0 & A[3,k_0] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ B_n = 0,$$

since

$$\begin{bmatrix} A[1,k_0] & 0 & 0 & 0 \\ 0 & A[2,k_0] & 0 & 0 \\ 0 & 0 & A[3,k_0] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ B_n = {}_A\Psi(D_n) \circ (I_b \otimes e_{k_0}).$$

This shows that $L_{\text{block-diag}\{A[1,k_0],A[2,k_0],A[3,k_0],\dots\}} : (BM_{a\times 1}(M_b), \text{str}) \to (BM_{a\times 1}(M_b), \text{norm})$ is sequentially continuous at 0 and therefore everywhere. Therefore b is finite by Lemma 1.5. If a is also finite then there is nothing left to prove in the "forward" implication.

Suppose now $a = \infty$. The first n block-columns of $A - \ddot{A}_n$ are zero. So, there exists a sequence $\{Q_n\}_{n=1}^{\infty}$ in $BM_a(M_b)$ such that $\|Q_n\| = 1$ and

$$\begin{aligned} \|{}_{A-\ddot{A}_n}\Psi(Q_n \circ (I_b \otimes (S^*)^n))\| &\geqslant \left(\|{}_{A-\ddot{A}_n}\Psi\| - \frac{1}{n}\right) \cdot \|Q_n \circ (I_b \otimes (S^*)^n)\| \\ &= \|{}_{A-\ddot{A}_n}\Psi\| - \frac{1}{n}, \text{ for every } n. \end{aligned}$$

Then $\lim_{n\to\infty} {}_A\Psi(Q_n \circ (I_b \otimes (S^*)^n)) = 0$, since $\text{str-lim}_{n\to\infty} Q_n \circ (I_b \otimes (S^*)^n) = 0$ by Lemma 1.6.

Yet ${}_A\Psi(Q_n \circ (I_b \otimes (S^*)^n)) = {}_{A-\ddot{A}_n}\Psi(Q_n \circ (I_b \otimes (S^*)^n))$. Thus $\lim_{n\to\infty} \|{}_{A-\ddot{A}_n}\Psi\| - \frac{1}{n} = 0$, i.e. $\lim_{n\to\infty} \|{}_{A-\ddot{A}_n}\Psi\| = 0$.

This completes the proof of the implication in the "forward" direction.

Conversely: if a and b are both finite then $\text{str-lim}_{n\to\infty} T_n = T$ and $\lim_{n\to\infty} T_n = T$ are equivalent, for any sequence $\{T_n\}_{n=1}^{\infty}$ in $BM_a(M_b)$. In this case the sequential continuity of ${}_A\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{norm})$ follows from Theorem 1.2.

Suppose now $a = \infty$ (b is finite) and $\lim_{n\to\infty} {}_{A-\ddot{A}_n}\Psi = 0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence in $BM_a(M_b)$ such that $\text{str-lim}_{n\to\infty} T_n = 0$. Then $\sup_n \|T_n\|$ is finite by the uniform boundedness principle. By Lemma 1.1, $\text{str-lim}_{n\to\infty} T_n \circ (I_b \otimes e_j) = 0$, for each j. Thus $\lim_{n\to\infty} T_n \circ (I_b \otimes e_j) = 0$, for each j, since b is finite and $T_n \circ (I_b \otimes e_j) \in BM_{a\times 1}(M_b)$. Let $\varepsilon > 0$ be any. There exists an index n_0 such that $\|{}_{A-\ddot{A}_n}\Psi\| < \varepsilon$, whenever $n \geqslant n_0$. There also exists an index n_1 such that $\|T_n \circ (I_b \otimes e_j)\| < \varepsilon$, for $j = 1, 2, \dots, n_0$, whenever $n \geqslant n_1$.

Let m be the larger of n_0 and n_1. If $n \geqslant m$ then

$$\begin{aligned}\|{}_A\Psi(T_n)\| &= \|{}_{A-\ddot{A}_{n_0}}\Psi(T_n) + {}_{\ddot{A}_{n_0}}\Psi(T_n)\| \\ &< \varepsilon\|T_n\| + \sum_{j=1}^{n_0} \|{}_{\ddot{A}_{n_0}}\Psi(T_n) \circ (I_b \otimes e_j)\| \\ &\leqslant \varepsilon\|T_n\| + \sum_{j=1}^{n_0} (\sup_{k,l} \|A[k,l]\|) \cdot \|T_n \circ (I_b \otimes e_j)\| \\ &\leqslant \varepsilon \cdot (\sup_n \|T_n\|) + \|{}_A\Psi\| \cdot n_0 \cdot \varepsilon = c \cdot \varepsilon,\end{aligned}$$

where c is a constant independent of n and ε.

This shows that $\lim_{n\to\infty} {}_A\Psi(T_n) = 0$. So ${}_A\Psi : (BM_a(M_b), \text{str}) \to \big(BM_a(M_b), \text{norm}\big)$ is sequentially continuous at 0, and therefore everywhere. ∎

In [11] and [13] Q. Stout described those bounded operators on a separable Hilbert space H for which no matter what basis was chosen the matrix representing the operator in the basis induced a compact Schur multiplication map on $B(H)$. These turned out to be exactly the compact operators on H. Stout also demonstrated that the larger class of those operators for which there is at least one such basis is the class of operators whose essential numerical range contains 0. The task of deciding whether a given Schur multiplier induces a compact Schur multiplication map is often not an easy one. In the remainder of this section we observe that for a Schur block-multiplication map the property of being compact is equivalent to the weak-to-norm sequentially continuity of the map. This immediately shows that compact Schur block-multiplication maps can be approximated nicely by Schur block-multiplication maps of finite rank.

LEMMA 1.8. *For all A in $SM_a(M_b)$, the image, under ${}_A\Psi$, of the closed unit ball of $BM_a(M_b)$ is weakly compact and therefore weakly closed.*

Proof. The restriction of ${}_A\Psi$ to the closed unit ball of $BM_a(M_b)$ is "weak-to-weak" continuous by Theorem 1.2. ∎

THEOREM 1.9. (Schauder [2], VI.5.2) *Suppose X and Y are Banach spaces and $T \in B(X,Y)$. Then T is compact if and only if the Banach adjoint of T is compact.*

THEOREM 1.10. ([2], VI.5.6) *Suppose X and Y are Banach spaces and $T \in B(X,Y)$. Then T is compact if and only if the Banach adjoint of T sends bounded nets that converge in the weak* topology on $Y^\#$ to nets that converge in norm on*

$X^{\#}$; (equivalently: a restriction of the Banach adjoint of T to a norm-bounded subset of $Y^{\#}$ is "weak-to-norm" continuous).*

Suppose X and Y are Banach spaces and $T \in B(X)$. If X is isomorphic to $Y^{\#}$, in such a way that T is isomorphic, under the induced isomorphism between $B(X)$ and $B(Y^{\#})$, to a Banach adjoint of an element of $B(Y)$, then T is compact if and only if a restriction of T to any norm-bounded subset of X is "weak*-to-norm" continuous (here the weak* topology is induced on X as on a dual space of Y).

THEOREM 1.11. *Suppose A is an element of $SM_a(M_b)$. Then the following are equivalent:*

(i) ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{norm})$ *is sequentially continuous;*

(ii) *A restriction of ${}_A\Psi$ to a norm-bounded subset of $BM_a(M_b)$ is "weak-to-norm" continuous;*

(iii) *The image, under ${}_A\Psi$, of the closed unit ball of $BM_a(M_b)$ is compact;*

(iv) *${}_A\Psi$ is compact;*

(v) *${}_A\Psi$ maps every bounded set to a precompact set.*

Proof. That (a) $\Leftrightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) is standard and clear, whereas "(d) $\Leftrightarrow$ (e)" is a definition of compactness of ${}_A\Psi$.

We show that (d) $\Rightarrow$ (b):

In [6] it is shown that ${}_A\Psi$ is isomorphic to a Banach adjoint of a bounded operator on the Banach space of trace-class elements of $BM_a(M_b)$. Suppose ${}_A\Psi$ is compact. Then the restriction of ${}_A\Psi$ to any norm-bounded subset of $BM_a(M_b)$ is "ultraweak-to-norm" continuous by Theorem 1.10.

The ultraweak and the weak topologies coincide on norm-bounded subsets of $BM_a(M_b)$. Therefore (b) follows. ∎

LEMMA 1.12. *Suppose A is a non-zero element of $SM_a(M_b)$. Then ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{norm})$ is sequentially continuous if and only if b is finite and $\lim_{n\to\infty} {}_{A_n}\Psi = {}_A\Psi$ (equivalently: $\lim_{n\to\infty} {}_{A-A_n}\Psi = 0$), whenever $a = \infty$; where $A_n \in SM_a(M_b)$ is specified by: $A_n[i,j] = \begin{cases} A[i,j] & \text{if } i,j \leqslant n \\ 0 & \text{otherwise.} \end{cases}$*

Proof. Let $\ddot{A}_n$ be as in Theorem 1.7, and let $\dot{A}_n \in SM_a(M_b)$ be specified, for each n, by:

$$\ddot{A}_n[i,j] = \begin{cases} \dot{A}[i,j] & \text{whenever } i \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

First observe that $\lim_{n\to\infty} {}_{A-A_n}\Psi = 0$ if and only if $\lim_{n\to\infty} {}_{A-\ddot{A}_n}\Psi = 0$ and $\lim_{n\to\infty} {}_{A-\dot{A}_n}\Psi = 0$. [Both $\|{}_{A-\dot{A}_n}\Psi\|$ and $\|{}_{A-\ddot{A}_n}\Psi\|$ are dominated by $\|{}_{A-A_n}\Psi\|$. This demonstrates the implication in the "forward" direction. The converse follows froom the relations $\|{}_{A-A_n}\Psi\| = \|{}_{A-\ddot{A}_n}\Psi + {}_{\ddot{A}_n-A_n}\Psi\| \leqslant \|{}_{A-\ddot{A}_n}\Psi\| + \|{}_{\ddot{A}_n-A_n}\Psi\| \leqslant \|{}_{A-\ddot{A}_n}\Psi\| + \|{}_{A-\dot{A}_n}\Psi\|$.]

Suppose ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{norm})$ is sequentially continuous. Then ${}_A\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{norm})$ is sequentially continuous and consequently b is finite and $\lim_{n\to\infty} {}_{A-\ddot{A}_n}\Psi = 0$, whenever $a = \infty$, all by Theorem 1.7. To demonstrate the implication in the "forward" direction it remains to show that if $a = \infty$ then $\lim_{n\to\infty} {}_{A-\dot{A}_n}\Psi = 0$.

The first n block-rows of $A - \dot{A}_n$ are zero. So, there exists a sequence $\{Q_n\}_{n=1}^{\infty}$ in $BM_a(M_b)$ such that $\|Q_n\| = 1$ and

$$\begin{aligned} \|{}_{A-\dot{A}_n}\Psi((I_b \otimes S^n) \circ Q_n)\| &\geqslant \left(\|{}_{A-\dot{A}_n}\Psi\| - \frac{1}{n}\right) \cdot \|(I_b \otimes S^n) \circ Q_n\| \\ &= \|{}_{A-\dot{A}_n}\Psi\| - \frac{1}{n}, \text{ for every } n. \end{aligned}$$

Then $\lim_{n\to\infty} {}_A\Psi((I_b \otimes S^n) \circ Q_n) = 0$, since $\text{weak-}\lim_{n\to\infty}(I_b \otimes S^n) \circ Q_n = 0$ by Lemma 1.6. Yet ${}_A\Psi((I_b \otimes S^n) \circ Q_n) = {}_{A-\dot{A}_n}\Psi((I_b \otimes S^n) \circ Q_n)$. Thus $\lim_{n\to\infty} \|{}_{A-\dot{A}_n}\Psi\| - \frac{1}{n} = 0$, i.e. $\lim_{n\to\infty} \|{}_{A-\dot{A}_n}\Psi\| = 0$. This completes the proof of the implication in the "forward" direction.

Conversely: If a and b are both finite then $\text{weak-}\lim_{n\to\infty} T_n = T$ and $\lim_{n\to\infty} T_n = T$ are equivalent, for any sequence $\{T_n\}_{n=1}^{\infty}$ in $BM_a(M_b)$. In this case the sequential continuity of ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{norm})$ follows from Theorem 1.2.

Suppose now $a = \infty$ (b is finite). ${}_A\Psi$ is compact since it is a limit in norm of a sequence of operators of finite rank. The required continuity of ${}_A\Psi$ now follows from Theorem 1.11. ∎

2. FULL CONTINUITY

THEOREM 2.1. *Suppose A is a non-zero element of* $SM_a(M_b)$. *Then* ${}_A\Psi$: $(BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{norm})$ *is continuous if and only if b is finite and $A[i, j]$ is non-zero for only finitely many i, j.*

Proof. Suppose ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{norm})$ is continuous. That b is finite follows from Theorem 1.8. The pre-image, under ${}_A\Psi$, of the open unit ball of $BM_a(M_b)$ is weakly open and contains 0. Therefore there exist $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$ in $\bigoplus_{i=1}^{a} \ell_b^2$ and an $\varepsilon > 0$ such that $\|{}_A\Psi(T)\| < 1$ whenever $|\langle T(x_k), y_k\rangle| < \varepsilon$, for every k.

Let $\mathcal{F}$ stand for the set $\{T \in BM_a(M_b) \mid |\langle T(x_k), y_k\rangle| = 0, \text{ for every } k\}$. Then $\mathcal{F}$ is a linear space and every element R of $\mathcal{F}$ satisfies $\|{}_A\Psi(R)\| < 1$. This can only happen if ${}_A\Psi(R) = 0$, for every element R of $\mathcal{F}$, in other words: if $\mathcal{F}$ is a subset of the kernel of ${}_A\Psi$. Since $\mathcal{F}$ is clearly a space of finite linear co-dimension in $BM_a(M_b)$, so is the kernel of ${}_A\Psi$.

The linear co-dimension in $BM_a(M_b)$ of the kernel of ${}_A\Psi$ is at least as large as the number of non-zero block-entries of A. Therefore A has finitely many non-zero block-entries.

This completes the proof of the implication in the "forward" direction.

Conversely, suppose b is finite and $A[i, j]$ is non-zero for only finitely many i, j. Consider any index n such that $A[i, j] = 0$ whenever $i > n$ or $j > n$. Then ${}_A\Psi = \sum_{i,j=1}^{n} {}_{A_{ij}}\Psi$, where A_{ij} in $BM_a(M_b)$ is the matrix specified by:

$$A_{ij}[k, l] = \begin{cases} A[i, j] & \text{if } (k, l) = (i, j) \\ 0 & \text{otherwise.} \end{cases}$$

That each ${}_{A_{ij}}\Psi$ is "weak-to-norm" continuous is immediate because $L_{A[i,j]}$: $(BM_b, \text{weak}) \to (BM_b, \text{norm})$ is continous for each i, j, whenever b is finite. ∎

THEOREM 2.2. *Suppose A is a non-zero element of* $SM_a(M_b)$. *Then* ${}_A\Psi$: $(BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{norm})$ *is continuous if and only if b is finite and A has only finitely-many non-zero block-columns.*

Proof. Suppose ${}_A\Psi$ has the stated continuity property. Then b is finite by Theorem 1.7. If a is finite then there is nothing left to prove; therefore assume $a = \infty$. The pre-image, under ${}_A\Psi$, of the open unit ball of $BM_\infty(M_b)$ is a strongly open set containing 0.

Therefore there exist $x_1, x_2, \ldots, x_n$ in $\bigoplus_{i=1}^{\infty} \ell_b^2$ and an $\varepsilon > 0$ such that $\|_A\Psi(T)\| < 1$ whenever $\|T(x_k)\| < \varepsilon$, for every k. In particular, ${}_A\Psi(T) = 0$ whenever $T(x_k) = 0$, for every k.

CLAIM: Suppose D and G are two permutation matrices in BM_∞. Define $\tilde{D}$ and $\tilde{G}$ by:

$$\tilde{D} = I_b \otimes D, \quad \tilde{G} = I_b \otimes G, \quad (\tilde{D}, \tilde{G} \in BM_\infty(M_b)).$$

For $i = 1, 2, 3, \ldots, n$ let $y_i = \tilde{D}^{-1}(x_i)$, $\left(y_i \in \bigoplus_{i=1}^{\infty} \ell_b^2\right)$. Then $\tilde{G} \circ A \circ \tilde{D}$ is a well-defined element of $SM_\infty(M_b)$.

Moreover:

(1) $\|_{\tilde{G}\circ A\circ\tilde{D}}\Psi(T)\| < 1$ whenever $\|T(y_i)\| < \varepsilon$, for $i = 1, 2, \ldots, n$.

(2) ${}_{\tilde{G}\circ A\circ\tilde{D}}\Psi(T) = 0$ whenever $T(y_i) = 0$, for $i = 1, 2, \ldots, n$.

Proof of the claim. That $\tilde{G} \circ A \circ \tilde{D}$ is a well-defined element of $SM_\infty(M_b)$ is clear, since $(\tilde{G} \circ A \circ \tilde{D}) \,\square\, T = \tilde{G} \circ (A \,\square\, (\tilde{G}^{-1} \circ T \circ \tilde{D}^{-1})) \circ \tilde{D}$ for any T in $BM_\infty(M_b)$.

Suppose now that T is an element of $BM_\infty(M_b)$ and $\|T(y_i)\| < \varepsilon$, for $i = 1, 2, \ldots, n$. Then $\|(\tilde{G}^{-1} \circ T \circ \tilde{D}^{-1})(x_i)\| = \|(\tilde{G}^{-1} \circ T)(y_i)\| = \|T(y_i)\| < \varepsilon$, for $i = 1, 2, \ldots, n$. Consequently $\|_A\Psi(\tilde{G}^{-1} \circ T \circ \tilde{D}^{-1})\| < 1$. Yet $\|_A\Psi(\tilde{G}^{-1} \circ T \circ \tilde{D}^{-1})\| = \|A \,\square\, (\tilde{G}^{-1} \circ T \circ \tilde{D}^{-1})\| = \|\tilde{G}^{-1} \circ ((\tilde{G} \circ A \circ \tilde{D}) \,\square\, T) \circ \tilde{D}^{-1}\| = \|(\tilde{G} \circ A \circ \tilde{D}) \,\square\, T\| = \|_{\tilde{G}\circ A\circ\tilde{D}}\Psi(T)\|$, so that (1) is verified. That (2) follows from (1) is apparent. The proof of the claim is complete.

Let m be an integer such that $mb > n$. Suppose A has infinitely many non-zero block-columns. Then there exist permutation matrices D and G in BM_∞ such that the element $\widehat{A}_m$ of $BM_m(M_b)$ that is the north-west m-by-m block-corner of $(I_b \otimes G) \circ A \circ (I_b \otimes D)$ has no zero block-columns. We now make use of the claim to assume, without loss of generality, that D and G are both identity matrices. That is: we take it that the north-west m-by-m block-corner of A has no zero block-columns. (We pass to $(I_b \otimes G) \circ A \circ (I_b \otimes D)$ in place of A, and to y_i's in place of x_i's. The numbers of non-zero block-columns in $(I_b \otimes G) \circ A \circ (I_b \otimes D)$ and in A are the same.) For each $i = 1, 2, \ldots, n$ let z_i be the cannonical projection of x_i onto the direct sum $\bigoplus_{i=1}^{m} \ell_b^2$, of the first m summands of $\bigoplus_{i=1}^{\infty} \ell_b^2$. For each j we let g_j and f_j stand for the j-th standard basis elements of ℓ_m^2 and of ℓ_b^2 respectively. Then $\{f_t \otimes g_r \mid 1 \leqslant t \leqslant b,\ 1 \leqslant r \leqslant m\}$ is a basis of $\bigoplus_{i=1}^{m} \ell_b^2$. Since the number of elements in the basis is mb, which is strictly larger than n, there must be an element of the basis lying outside of the span of $z_1, z_2, \ldots, z_n$.

Let $f_{t_0} \otimes g_{r_0}$ be such an element. There exists an index q_0 in $\{1, 2, \ldots, m\}$ such that $A[q_0, r_0]$ is non-zero.

Let w be a non-zero element of $\ker^{\perp}(A[q_0, r_0])$ in ℓ^2_b. From basic linear algebra there exists a matrix B_m in $M_m(M_b)$ such that

$$B_m(z_i) = 0, \quad i = 1, 2, \ldots, n \quad \text{and} \quad B_m(f_{t_0} \otimes g_{r_0}) = w \otimes g_{q_0}.$$

In particular, then w is in the range of $B_m[q_0, r_0]$, so that $A[q_0, r_0] \circ B_m[q_0, r_0] \neq 0$. If B is the matrix in $BM_\infty(M_b)$ specified by:

$$B[i,j] = \begin{cases} B_m[i,j] & \text{if } 1 \leqslant i, j \leqslant m \\ 0 & \text{otherwise,} \end{cases}$$

then $B(x_i) = 0$, for $i = 1, 2, \ldots, n$, and ${}_A\Psi(B) \neq 0$ (since $A[q_0, r_0] \circ B[q_0, r_0] \neq 0$). This is a contradiction.

The proof of the implication in the "forward" direction is complete.

For the converse it is enough to demonstrate that if b is finite and if A has only one non-zero block-column, which, without loss of generality is its first block-column, then ${}_A\Psi : (BM_\infty(M_b), \text{str}) \to (BM_\infty(M_b), \text{norm})$ is continuous. Yet, under this hypothesis ${}_A\Psi = L_{\text{block-diag}\{A[1,1], A[2,1], A[3,1], \ldots\}} \circ R_{\text{block-diag}\{I_b, 0_b, 0_b, \ldots\}}$. Clearly $R_{\text{block-diag}\{I_b, 0_b, 0_b, \ldots\}} : (BM_\infty(M_b), \text{str}) \to (BM_\infty(M_b), \text{norm})$ is continuous because b is finite. The result follows since $L_{\text{block-diag}\{A[1,1], A[2,1], A[3,1], \ldots\}}$ is *norm-to-norm* continuous. ∎

This next result is certainly common knowledge. It is nevertheless relevant enough to this paper to be stated with proof.

LEMMA 2.3. *Let A be an s-by-s matrix of complex numbers. Then the linear span of columns of A, taken as elements of $\mathbb{C}^s$, is finite-dimensional if and only if the linear span of the rows of A is finite-dimensional. (The two dimensions are then, in fact, equal.)*

Proof. Suppose the linear span of columns of A is finite-dimensional and let $\{f_1, f_2, \ldots, f_n\}$ be a basis of that span in $\mathbb{C}^s$. For each index j there exist scalars $c(1,j), c(2,j), \ldots, c(n,j)$ such that the j-th column of A equals $c(1,j)f_1 + c(2,j)f_2 + \cdots + c(n,j)f_n$. For each index $i = 1, 2, \ldots, n$, let B_i be the matrix in M_s such that k-th column of B_i is $c(i,k)f_i$, for every k. Then $A = B_1 + B_2 + \cdots + B_n$. The span of the rows of B_i is a subset of $\text{span}\{(c(i,1), c(i,2), \ldots, c(i,s)) \mid 1 \leqslant i \leqslant n\}$ and is therefore one-dimensional. Thus the dimension of the span of the rows of A is at most n. The converse implication and the equality of the two dimensions follow when the above argument is applied to the transpose of A. ∎

It is important to observe that Lemma 2.3 can be trivially extended (with essentially the same proof) to cover the case of block-matrices: If A is a matrix in $M_a(M_b)$ then $\operatorname{span}\{A(f_j \otimes e_i) \,|\, 1 \leqslant i < a+1,\ 1 \leqslant j < b+1\}$ is finite-dimensional if and only if $\operatorname{span}\{A^{\mathrm{trn}}(f_j \otimes e_i) \,|\, 1 \leqslant i < a+1,\ 1 \leqslant j < b+1\}$ is finite-dimensional. Here A^{trn} is the transpose of A; that is $(A^{\mathrm{trn}}[i,j])[k,l] = (A[j,i])[l,k]$ for all i,j,k,l. (Note that $A(f_j \otimes e_i)$ and $A^{\mathrm{trn}}(f_j \otimes e_i)$ are well-defined elements of $(\mathbf{C}^b)^a$ for all i,j.)

LEMMA 2.4. *Suppose A is an element of $SM_a(M_b)$. Suppose $\{T_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ are norm-bounded sequences in BM_b such that*

$$\begin{bmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \text{ and } \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

are both invertible in $BM_a(M_b)$. Let

$$B = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ A \circ \begin{bmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}.$$

Then B is in $SM_a(M_b)$; moreover, ${}_A\Psi : (BM_a(M_b), \mathrm{weak}) \to (BM_a(M_b), \mathrm{str})$ is continuous if and only if ${}_B\Psi : (BM_a(M_b), \mathrm{weak}) \to (BM_a(M_b), \mathrm{str})$ is continuous.

Proof. That B is in $SM_a(M_b)$ and ${}_B\Psi = L_{\text{block-diag}\{Q_1,Q_2,Q_3,\dots\}} \circ {}_A\Psi \circ L_{\text{block-diag}\{T_1,T_2,T_3,\dots\}}$ is clear. The rest follows since $L_D : BM_a(M_b) \to BM_a(M_b)$ is *weak-to-weak* and *strong-to-strong* continuous for every D in $BM_a(M_b)$. ∎

THEOREM 2.5. *Suppose A is an element of $SM_a(M_b)$. Then*

$${}_A\Psi : (BM_a(M_b), \mathrm{weak}) \to (BM_a(M_b), \mathrm{str})$$

is continuous if and only if A has finitely many non-zero block-rows and both $\operatorname{span}_{i,j}(\operatorname{Ran} A[i,j])$ and $\operatorname{span}_{i,j}(\operatorname{Ker}^{\perp} A[i,j])$ are finite-dimensional.

Proof. Suppose ${}_A\Psi$ has the stated continuity property. Fix an element u in $\bigoplus_{i=1}^{a} \ell_b^2$ such that $\langle u, f_j \otimes e_i\rangle \neq 0$ for all i and j. Then there exist $x_1, x_2, x_3, \dots, x_n$ and $y_1, y_2, y_3, \dots, y_n$ in $\bigoplus_{i=1}^{a} \ell_b^2$ and a $\delta > 0$ such that $\|({}_A\Psi(T))(u)\| < 1$ for those T in $BM_a(M_b)$ that satisfy $\langle T(x_k), y_k\rangle < \delta$, for all k.

Step 1. [Here we show that A has finitely many non-zero block-rows and that $\operatorname{span}_{i,j}(\operatorname{Ker}^{\perp} A[i,j])$ is finite-dimensional].

First of all $\|({}_A\Psi(T))(u)\| < 1$ for any T in $BM_a(M_b)$ such that $T^*(y_k) = 0$, for all k; (since such T satisfies $\langle T(x_k), y_k\rangle = 0 < \delta$, for all k). Then $({}_A\Psi(T))(u) = 0$, because the above also holds true for every scalar multiple of such T.

Let z be any element of $\{y_1, y_2, y_3, \ldots, y_n\}^{\perp}$ in $\bigoplus_{i=1}^{a} \ell_b^2$ and let i_0 and j_0 be any positive integers not exceeding a and b respectively. Denote by T_0 the operator in $BM_a(M_b)$ specified by

$$T_0(f_j \otimes e_i) = \begin{cases} z & \text{if } i = i_0 \text{ and } j = j_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $T_0^*(y_k) = 0$, for all k, and therefore $({}_A\Psi(T_0))(u) = 0$; that is

$\sum_{m=1}^{a} (A[i,m] \cdot T_0[i,m])(u[m]) = 0$, for all i;

i.e. $(A[i,i_0] \cdot T_0[i,i_0])(u[i_0]) = 0$, for all i;

i.e. $A[i,i_0]((u[i_0])[j_0] \cdot z[i]) = 0$, for all i;

i.e. $A[i,i_0](z[i]) = 0$, for all i, (because $(u[i_0])[j_0] \neq 0$);

i.e. $z[i] \in \operatorname{Ker}(A[i,i_0])$, for all i;

i.e. $z \in \operatorname{Ker}(A \circ (I_b \otimes E_{i_0 i_0}))$.

We can therefore conclude that $\{y_1, y_2, y_3, \ldots, y_n\}^{\perp} \subset \operatorname{Ker}(A \circ (I_b \otimes E_{jj}))$, for all j; that is $\operatorname{Ker}^{\perp}(A \circ (I_b \otimes E_{jj})) \subset \operatorname{span}\{y_1, y_2, y_3, \ldots, y_n\}$, for all j. Thus $\{w \otimes e_j \mid w \in \operatorname{Ker}^{\perp} A[i,j]\} = \operatorname{Ker}^{\perp}(A[i,j] \otimes E_{ij}) = \operatorname{Ker}^{\perp}((I_b \otimes E_{ii}) \circ A \circ (I_b \otimes E_{jj})) \subset \operatorname{Ker}^{\perp}(A \circ (I_b \otimes E_{jj})) \subset \operatorname{span}\{y_1, y_2, y_3, \ldots, y_n\}$, for all i, j. This shows that $\operatorname{span}_{i,j}\{w \otimes e_j \mid w \in \ker^{\perp} A[i,j]\}$ is at most n-dimensional.

Consequently A has no more than n non-zero block rows and the dimension of $\operatorname{span}_{i,j}(\operatorname{Ker}^{\perp} A[i,j])$ is at most n.

Step 2. [Here we show that $\operatorname{span}_{i,j}(A[i,j])(f_t)$ is finite-dimensional, for every t.]

First of all, $({}_A\Psi(T))(u) = 0$ for any T in $BM_a(M_b)$, such that $T(x_k) = 0$ for all k; (because such T satisfies: $\langle T(x_k), y_k\rangle = 0 < \delta$, for all k).

Let w be any element of $\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}^{\perp}$ in $\bigoplus_{i=1}^{a} \ell_b^2$ and let i_0 and j_0 be any positive integers not exceeding a and b respectively. Denote by T_1 the operator in $BM_a(M_b)$ specified by

$$T_1^*(f_j \otimes e_i) = \begin{cases} \bar{w} & \text{if } i = i_0 \text{ and } j = j_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $T_1(x_k) = 0$, for all k, and therefore $({}_A\Psi(T_1))(u) = 0$; that is

$$\sum_{m=1}^{a} (A[i,m] \cdot T_1[i,m])(u[m]) = 0, \quad \text{for all } i;$$

$$\text{i.e. } \sum_{m=1}^{a} (A[i_0,m] \cdot T_1[i_0,m])(u[i_0]) = 0;$$

$$\text{i.e. } \sum_{m=1}^{a} (A[i_0,m])(\langle w[m], \bar{u}[m]\rangle f_{j_0}) = 0;$$

$$\text{i.e. } \sum_{m=1}^{a} \langle w[m], \bar{u}[m]\rangle \cdot (A[i_0,m])(f_{j_0}) = 0;$$

$$\text{i.e. } \sum_{m=1}^{a} ((A[i_0,m])(f_{j_0})\lceil \otimes \rfloor \bar{u}[m])(w[m]) = 0.$$

Denote by Q the operator in $BM_{1\times a}(M_b)$ specified by $Q[j] = (A[i_0,j])(f_{j_0})\cdot \lceil\otimes\rfloor \bar{u}[j]$, for all j. (Such Q is bounded because $\sum_{j=1}^{a} \|u[j]\|^2 < \infty$ and $\|(A[i_0,j])(f_{j_0})\| \leqslant \|A[i_0,j]\| \leqslant \|{}_A\Psi\| < \infty$.) What we have shown so far is that $\operatorname{Ker}(Q)$ contains w. We can therefore conclude that $\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}^\perp \subset \operatorname{Ker}(Q)$; that is $\operatorname{Ran}(Q^*) \subset \operatorname{span}\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}$; i.e. $\operatorname{span}_r(Q^*(f_r)) \subset \operatorname{span}\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}$. Now: $Q^*[j] = \bar{u}[j]\lceil\otimes\rfloor(A[i_0,j])(f_{j_0})$, so that $(Q^*[j])(f_r) = \langle f_r, (A[i_0,j])(f_{j_0})\rangle \bar{u}[j]$, for all j.

Thus

$$Q^*(f_r) = \begin{bmatrix} \langle f_r, A[i_0,1](f_{j_0})\rangle I_b & 0 & 0 & 0 \\ 0 & \langle f_r, A[i_0,2](f_{j_0})\rangle I_b & 0 & 0 \\ 0 & 0 & \langle f_r, A[i_0,3](f_{j_0})\rangle I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} (\bar{u})$$
$$\subset \operatorname{span}\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\},$$

for all r. From the start u was chosen in such a way that $\bar{u}$ is a separating vector for the set

$$\left\{ \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 \\ 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \in BM_a(M_b) \,\middle|\, D_m \text{ is diagonal, for all } m \right\}.$$

So

$$\operatorname*{span}_r \left(\begin{bmatrix} \langle f_r, A[i_0,1](f_{j_0})\rangle I_b & 0 & 0 & 0 \\ 0 & \langle f_r, A[i_0,2](f_{j_0})\rangle I_b & 0 & 0 \\ 0 & 0 & \langle f_r, A[i_0,3](f_{j_0})\rangle I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \right)$$

is finite-dimensional and consequently

$$\operatorname*{span}_r\left(\begin{bmatrix} \langle A[i_0,1](f_{j_0}),f_r\rangle & 0 & 0 & 0 \\ 0 & \langle A[i_0,2](f_{j_0}),f_r\rangle & 0 & 0 \\ 0 & 0 & \langle A[i_0,3](f_{j_0}),f_r\rangle & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}\right)$$

is finite-dimensional.

Let G be the matrix in $M_{a\times b}$ such that m-th row of G equals $(A[i_0,m](f_{j_0}))^{\text{trn}}$, for each m. The last argument has shown that the span of the columns of G is finite-dimensional and therefore the span of the rows of G is finite-dimensional by Lemma 2.3. In other words $\operatorname*{span}_m(A[i_0,m](f_{j_0}))^{\text{trn}}$ is finite-dimensional, or equivalently $\operatorname*{span}_m A[i_0,m](f_{j_0})$ is finite-dimensional. It now follows that $\operatorname*{span}_{i,m} A[i,m](f_{j_0})$ is finite-dimensional, since i_0 was arbitrary and A has finitely many non-zero block-rows. Because the choice of j_0 was arbitrary, it follows that $\operatorname*{span}_{i,m} A[i,m](f_t)$ is finite-dimensional, for every t.

Step 3. [Here we show that $\operatorname*{span}_{i,j}(\operatorname{Ran} A[i,j])$ is finite-dimensional].

We have established in *Step 1* that $\operatorname*{span}_{i,j}(\operatorname{Ker}^{\perp} A[i,j])$ is finite-dimensional. Therefore there exists a unitary matrix V in BM_b and a positive integer m_0 such that all non-zero entries of $A[i,j]\circ V$ lie in first m_0 columns, for all i and j. For each i and j the (at most m_0-dimensional) span of the first m_0 columns of $A[i,j]\circ V$ coincides with the closed span of the columns of $A[i,j]$, so that the latter is finite-dimensional and is simply the range of $A[i,j]$. Write

$$D = A\circ\begin{bmatrix} V & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}.$$

Then $D\in SM_a(M_b)$ and ${}_D\Psi : (BM_a(M_b),\text{weak}) \to (BM_a(M_b),\text{str})$ is continuous by Lemma 2.4. Thus the conclusions of *Steps* 1 and 2 apply to D.

In particular, $\operatorname*{span}_{i,j}(D[i,j])(f_t)$ is finite-dimensional, for every t. This shows that $\operatorname*{span}_{i,j,t}(D[i,j])(f_t)$ is finite-dimensional, since $(D[i,j])(f_t)=0$ for all i,j, whenever $t>m_0$. Yet $\operatorname*{span}_{i,j}(\operatorname{Ran} A[i,j]) = \operatorname*{span}_{i,j}(\operatorname{Ran} D[i,j]) = \operatorname*{span}_{i,j}(\operatorname*{span}_t(D[i,j])(f_t))$ $= \operatorname*{span}_{i,j,t}(D[i,j])(f_t)$. So $\operatorname*{span}_{i,j}(\operatorname{Ran} A[i,j])$ is finite-dimensional.

This is the end of Step 3 and thus of the proof of the implication in the "forward" direction.

To establish the converse it is enough to demonstrate that if A has only one non-zero block-row, (which from now on, without loss of generality, we will assume to be its first block-row), and if both $\operatorname{span}_{i,j}(\operatorname{Ran} A[i,j])$ and $\operatorname{span}_{i,j}(\operatorname{Ker} A[i,j])$ are finite-dimensional then ${}_A\Psi$ has the stated continuity property.

Suppose A satisfies the hypothesis and denote by k_0 is the larger of the two finite dimensions: $\dim(\operatorname{span}_j(\operatorname{Ran} A[1,j]))$ and $\dim(\operatorname{span}_j(\operatorname{Ker}^{\perp} A[1,j]))$. There exist unitary matrices W_1 and W_2 in BM_b such that W_1^* and W_2 map $\operatorname{span}_j(\operatorname{Ker}^{\perp} A[1,j])$ and $\operatorname{span}_j(\operatorname{Ran} A[1,j])$, respectively, into the span of $\{e_1, e_2, \ldots, e_{k_0}\}$. For each j, all non-zero entries of $W_2 \circ A[1,j] \circ W_1$ lie within k_0-by-k_0 north-west corner. Because of Lemma 2.4, it is sufficient to demonstrate that ${}_Q\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{str})$ is continuous, where

$$Q = \begin{bmatrix} W_2 & 0 & 0 & 0 \\ 0 & W_2 & 0 & 0 \\ 0 & 0 & W_2 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \circ A \circ \begin{bmatrix} W_1 & 0 & 0 & 0 \\ 0 & W_1 & 0 & 0 \\ 0 & 0 & W_1 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}.$$

Therefore we may assume, without loss of generality, that all non-zero entries of $A[1,j]$ already lie within k_0-by-k_0 north-west corner, for each j; (that is, treat Q as a *new* A).

Consequently, it is sufficient to establish the required converse implication for every matrix A with the property that the only non-zero block-row of A is its first block-row, and such that there are two numbers r and s with the property that for each j the only entry of $A[1,j]$ that is possibly non-zero is $(A[1,j])[r,s]$. Without loss of generality we may assume both r and s to be 1. For such an A:

$${}_A\Psi = L_{\text{block-diag}\{F_{11},0,0,\ldots\}} \circ R_{\text{block-diag}\{(A[1,1])[1,1]I_b,(A[1,2])[1,1]I_b,(A[1,3])[1,1]I_b,\ldots\}}.$$

Yet

$$L_{\text{block-diag}\{F_{11},0,0,\ldots\}} : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{str})$$

and

$$R_{\text{block-diag}\{(A[1,1])[1,1]I_b,(A[1,2])[1,1]I_b,\ldots\}} : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{str})$$

are both continuous.

This clearly demonstrates the required continuity of ${}_A\Psi$. ∎

Here we get the first glimpse of the unusual behaviour of diagonal truncation on BM_∞. For

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

the *diagonal truncation* of A, denoted by diag(A), is

$$\operatorname{diag}(A) = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}.$$

In other words (if $A \in BM_\infty$) $\operatorname{diag}(A) = {}_I\Psi(A)$. From Theorem 1.4 the diagonal truncation is weak to strong sequentially continuous. Yet according to Theorem 2.5 it is *not* weak to strong continuous.

We shall improve on this claim after we derive the characterizations of those multipliers that induce weak to weak, strong to weak and strong to strong Schur multiplication maps. These are the most dfficult results of the paper and they require a number of lemmas.

LEMMA 2.6. *Suppose A in $SM_a(M_b)$ is such that ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{str})$ is continuous. Then*

$$\operatorname{span}\left\{ \begin{pmatrix} (A[i,1])[j,k] \\ (A[i,2])[j,k] \\ (A[i,3])[j,k] \\ \vdots \end{pmatrix} \,\middle|\, 1 \leqslant i < a+1,\ 1 \leqslant j,\ k < b+1 \right\}$$

is finite-dimensional.

Proof. Suppose A satisfies the hypotheses. Fix an element u in $\bigoplus_{i=1}^{a} \ell_b^2$ such that $\langle u, f_j \otimes e_i \rangle \neq 0$ for all i and j. Then there exist $x_1, x_2, x_3, \ldots, x_n$ in $\bigoplus_{i=1}^{a} \ell_b^2$ and an $\varepsilon > 0$ such that $\|({}_A\Psi(T))(u)\| < 1$ for those T in $BM_a(M_b)$ that satisfy $\|T(x_m)\| < \varepsilon$, for all m.

Let i and j be any integers such that $0 < i < a+1$, $0 < j < b+1$. Let v be any element of $\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}^\perp$ and let T be the matrix in $BM_a(M_b)$ specified by

$$T^{\mathrm{trn}}(f_k \otimes e_m) = \begin{cases} v & \text{if } (k,m) = (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|({}_A\Psi(T))(u)\| < 1$, since $T(x_m) = 0$, for all m. Thus $({}_A\Psi(T))(u) = 0$ because the above also holds true for every scalar multiple of T. That is:

$$0 = \sum_{k=1}^{a}(A[i,k] \circ T[i,k])(u[k]) = \sum_{k=1}^{a}(A[i,k])(\langle v[k], \bar{u}[k]\rangle(f_j)) \\ = \sum_{k=1}^{a}\langle v[k], \bar{u}[k]\rangle(A[i,k])(f_j),$$

i.e., for each r,

$$0 = \sum_{k=1}^{a}\langle v[k], \bar{u}[k]\rangle(A[i,k])[r,j] = \sum_{k=1}^{a}\langle v[k], \overline{((A[i,k])[r,j])(u[k])}\,\rangle = \langle v, w_r\rangle;$$

where, for each r, w_r is the element of $\bigoplus_{i=1}^{a} \ell_b^2$ specified by $w_r[k] = \overline{((A[i,k])[r,j])(u[k])}$, for each k. Thus $\bar{w}_r \in \operatorname{span}\{x_1, x_2, x_3, \ldots, x_n\}$, for each r, since $v \in \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n\}^{\perp}$. On the other hand:

$$\bar{w}_r = \begin{bmatrix} (A[i,1])[r,j]I_b & 0 & 0 & 0 \\ 0 & (A[i,2])[r,j]I_b & 0 & 0 \\ 0 & 0 & (A[i,3])[r,j]I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}(u)$$
$$= \left(I_b \otimes \begin{bmatrix} (A[i,1])[r,j] & 0 & 0 & 0 \\ 0 & (A[i,2])[r,j] & 0 & 0 \\ 0 & 0 & (A[i,3])[r,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \right)(u).$$

Observe that u was chosen in such a way that it is a separating vector for the subspace

$$\left\{ I_b \otimes \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \;\middle|\; \{c_1, c_2, c_3, \ldots\} \subseteq \ell_a^\infty \right\} \text{ of } BM_a(M_b).$$

Therefore

$$\operatorname{span}\left\{ \left(I_b \otimes \begin{bmatrix} (A[i,1])[r,j] & 0 & 0 & 0 \\ 0 & (A[i,2])[r,j] & 0 & 0 \\ 0 & 0 & (A[i,3])[r,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \right)(u) \;\middle|\right.$$

$$\Big| 0 < i < a+1, 0 < j, r < b+1 \Big\}$$

is a subspace of $\operatorname{span}\{x_1, x_2, x_3, \ldots, x_n\}$, and thus is finite-dimensional.

Consequently

$$\operatorname{span}\Big\{ I_b \otimes \begin{bmatrix} (A[i,1])[r,j] & 0 & 0 & 0 \\ 0 & (A[i,2])[r,j] & 0 & 0 \\ 0 & 0 & (A[i,3])[r,j] & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \Big|$$

$$\Big| 0 < i < a+1, 0 < j, r < b+1 \Big\}$$

is finite-dimensional and the claim of the lemma follows immediately. ∎

LEMMA 2.7. *Suppose A is in $M_{a\times r}(M_{b\times s})$. Then*

$$\operatorname{span}\left\{ \begin{pmatrix} (A[i,1])[j,k] \\ (A[i,2])[j,k] \\ (A[i,3])[j,k] \\ \vdots \end{pmatrix} \Bigg| \, 1 \leqslant i < a+1, \ 1 \leqslant j < b+1, \ 1 \leqslant k < s+1 \right\}$$

is finite-dimensional if and only if the span of block-columns of A, taken as elements of $M_{a\times 1}(M_{b\times s})$, is finite-dimensional in $M_{a\times 1}(M_{b\times s})$.

Proof. Let $\{g_m \mid 1 \leqslant m < r+1\}$ and $\{h_k \mid 1 \leqslant k < s+1\}$ be the standard basis of ℓ^2_r and ℓ^2_s respectively.

Let Q be the matrix in $M_{a\times r}(M_{s\times 1}(M_{b\times 1}))$ specified by $((Q[i,m])[k])[j] = (A[i,m])[j,k]$ for all i, j, k, m. In simpler terms: for each i, m, $Q[i,m]$ is the *vertical stack* of columns of $A[i,m]$. Then

$$Q^{\mathrm{trn}}(f_j \otimes (h_k \otimes e_i)) = \begin{pmatrix} ((Q[i,1])[k])[j] \\ ((Q[i,2])[k])[j] \\ ((Q[i,3])[k])[j] \\ \vdots \end{pmatrix} = \begin{pmatrix} (A[i,1])[j,k] \\ (A[i,2])[j,k] \\ (A[i,3])[j,k] \\ \vdots \end{pmatrix}.$$

Therefore

$$\operatorname{span}\left\{ \begin{pmatrix} (A[i,1])[j,k] \\ (A[i,2])[j,k] \\ (A[i,3])[j,k] \\ \vdots \end{pmatrix} \Bigg| \, 1 \leqslant i < a+1, \ 1 \leqslant j < b+1, \ 1 \leqslant k < s+1 \right\}$$

is finite-dimensional if and only if $\operatorname{span}\{Q^{\text{trn}}(f_j \otimes (h_k \otimes e_i)) \,|\, 0 < i < a+1, 0 < k < s+1, 0 < j < b+1\}$ is finite-dimensional.

By Lemma 2.3 the latter span is finite-dimensional if and only if $\operatorname{span}\{Q(g_m) \,|\, 0 < m < r+1\}$ is finite-dimensional; note that $Q(g_m) = \begin{pmatrix} Q[1,m] \\ Q[2,m] \\ Q[3,m] \\ \vdots \end{pmatrix}$.

A minute's consideration should now convince the reader that $\operatorname{span}\{Q(g_m) \,|\, 0 < m < r+1\}$ is finite-dimensional if and only if the span of the block-columns of A is finite-dimensional in $M_{a\times 1}(M_{b\times s})$. ∎

LEMMA 2.8. *Suppose A is an element of $SM_a(M_b)$. If the span of the block-columns of A is finite-dimensional in $M_{a\times 1}(M_b)$, then both ${}_A\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{str})$ and ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$ are continuous.*

Proof. For non-triviality assume A is non-zero. Let $G_1, G_2, G_3, \ldots, G_n$ be block-columns of A forming a basis for the span of the block-columns of A; ($G_i \in M_{a\times 1}(BM_b)$ and $\sup_j \|G_j[j]\| \leqslant \|{}_A\Psi\| < \infty$, for each i).

There exists another basis $F_1, F_2, F_3, \ldots, F_n$ for the span of the block-columns of A in $M_{a\times 1}(BM_b)$ such that $\sup_{i,j} \|F_i[j]\|$ is finite and, for each $i = 1, 2, \ldots, n$, there exist j_i, k_i and m_i with the property that

$$(F_t[j_i])[k_i, m_i] = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The latter condition simply states that each F_i has an *indicator entry* which sets it apart from all other basis elements. (The basis $F_1, F_2, F_3, \ldots, F_n$ is obtained from $G_1, G_2, G_3, \ldots, G_n$ by a Gauss-Jordan process.) For each r, let $c_1(r), c_2(r), c_3(r), \ldots, c_n(r)$ be complex numbers such that

$$(r\text{-th block-column of } A) = c_1(r)F_1 + c_2(r)F_2 + c_3(r)F_3 + \cdots + c_n(r)F_n.$$

Then

$$(A[j_i, r])[k_i, m_i] = \sum_{p=1}^{n} c_p(r)(F_p[j_i])[k_i, m_i] = c_i(r),$$

for each i and r.

Consequently, $\sup_{i,j} |c_i(j)| \leqslant \sup_{i,j} \|A[i,j]\| \leqslant \|{}_A\Psi\| < \infty$.

Let u in $\bigoplus_{i=1}^{a} \ell_b^2$ and $\varepsilon > 0$ be any. Then $\{T \in BM_a(M_b) \mid \|T(u)\| < \varepsilon\}$ is a *generic* subbasic neighbourhood of 0 in the strong topology.

We show that the pre-image, under ${}_A\Psi$, of this set contains a strongly open neighbourhood of 0. This is enough to establish the continuity of ${}_A\Psi$: $(BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{str})$ at 0, and therefore everywhere.

Suppose T is an element of $BM_a(M_b)$. Then:

$$\begin{aligned}
\|({}_A\Psi(T))(u)\|^2 &= \sum_{i=1}^{a} \left\| \sum_{j=1}^{a} (A[i,j] \circ T[i,j])(u[j]) \right\|^2 \\
&= \sum_{i=1}^{a} \left\| \sum_{j=1}^{a} \sum_{k=1}^{n} c_k(j)(F_k[i] \circ T[i,j])(u[j]) \right\|^2 \\
&= \sum_{i=1}^{a} \left\| \sum_{j=1}^{a} \sum_{k=1}^{n} (F_k[i] \circ T[i,j])(c_k(j)u[j]) \right\|^2 .
\end{aligned}$$

For each k, let x_k be the element of $\bigoplus_{i=1}^{a} \ell_b^2$ specified by: $x_k[j] = c_k(j)u[j]$, for all j. (Clearly $\sum_{j=1}^{a} \|c_k(j)u[j]\|^2 = \sum_{j=1}^{a} |c_k(j)|^2 \|u[j]\|^2 \leqslant \|{}_A\Psi\|^2 \sum_{j=1}^{a} \|u[j]\|^2 = \|{}_A\Psi\|^2 \|u\|^2 < \infty$.) Then

$$\begin{aligned}
\sum_{i=1}^{a} \left\| \sum_{j=1}^{a} (F_k[i] \circ T[i,j])(c_k(j)u[j]) \right\|^2 &= \sum_{i=1}^{a} \|F_k[i]((T[i,1], T[i,2], T[i,3], \ldots)(x_k))\|^2 \\
&\leqslant \sup_i \|F_k[i]\|^2 \sum_{i=1}^{a} \|(T[i,1], T[i,2], T[i,3], \ldots)(x_k)\|^2 \\
&= (\sup_i \|F_k[i]\|^2) \|T(x_k)\|^2 < \infty, \text{ for each } k.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|({}_A\Psi(T))(u)\|^2 &= \sum_{i=1}^{a} \left\| \sum_{k=1}^{n} \sum_{j=1}^{a} (F_k[i] \circ T[i,j])(c_k(j)u[j]) \right\|^2 \\
&\leqslant \sum_{i=1}^{a} \left(\sum_{k=1}^{n} \left\| \sum_{j=1}^{a} (F_k[i] \circ T[i,j])(c_k(j)u[j]) \right\| \right)^2
\end{aligned}$$

$$\leqslant \sum_{i=1}^{a}\left(n\sum_{k=1}^{n}\left\|\sum_{j=1}^{a}(F_k[i]\circ T[i,j])(c_k(j)u[j])\right\|^2\right)$$
$$= n\sum_{k=1}^{n}\sum_{i=1}^{a}\left\|\sum_{j=1}^{a}(F_k[i]\circ T[i,j])(c_k(j)u[j])\right\|^2$$
$$\leqslant n\sum_{k=1}^{n}(\sup_i \|F_k[i]\|^2)\|T(x_k)\|^2$$
$$\leqslant n\left(\sup_{i,k}\|F_k[i]\|^2\right)\sum_{k=1}^{n}\|T(x_k)\|^2.$$

(In the above calculation we have made use of the following well-known inequality:

$$\left(\sum_{t=1}^{n} d_t\right)^2 \leqslant n\sum_{t=1}^{n} d_t^2, \text{ for all real } d_1, d_2, d_3, \ldots, d_n;\ n\in\mathbb{N}.$$

We state the proof for completeness:

$$\left(\sum_{t=1}^{n} d_t\right)^2 = (d_1+d_2+d_3+\cdots+d_n)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} d_i d_j$$
$$\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{d_i^2+d_j^2}{2} = \sum_{i=1}^{n}\left(n\frac{d_i^2}{2}+\frac{1}{2}\sum_{j=1}^{n} d_j^2\right)$$
$$= \frac{n}{2}\sum_{i=1}^{n} d_i^2 + \frac{n}{2}\sum_{j=1}^{n} d_j^2 = n\sum_{t=1}^{n} d_t^2.\ )$$

It is now clear that the set $\left\{T \in BM_a(M_b)\,\middle|\, \|T(x_k)\| < \varepsilon/(n\cdot\sup_{i,k}\|F_k[i]\|),\ k = 1,2,\ldots,n\right\}$, which is a basic strongly open neighbourhood of 0, is contained in the pre-image, under ${}_A\Psi$, of the set $\{T \in BM_a(M_b) \mid \|T(u)\| < \varepsilon\}$. Thus the continuity of ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{str})$ is established.

Next we consider the question of continuity of ${}_A\Psi : (BM_a(M_b), \mathrm{weak}) \to (BM_a(M_b), \mathrm{weak})$.

Let u, v in $\bigoplus_{i=1}^{a} \ell_b^2$ and $\varepsilon > 0$ be any. Then $\{T \in BM_a(M_b) \mid |\langle T(u), v| < \varepsilon\}$ is a *generic* subbasic neighbourhood of 0 in the weak topology. We show that the pre-image under ${}_A\Psi$ of this set contains a weakly open neighbourhood of 0. This is enough to establish the continuity of ${}_A\Psi : (BM_a(M_b), \mathrm{weak}) \to (BM_a(M_b), \mathrm{weak})$

at 0 and therefore everywhere. Suppose T is an element of $BM_a(M_b)$. Then:

$$\begin{aligned}\langle({}_A\Psi(T))(u), v\rangle &= \sum_{i=1}^{a}\left\langle\sum_{j=1}^{a}(A[i,j]\circ T[i,j])(u[j]), v[i]\right\rangle\\ &= \sum_{i=1}^{a}\left\langle\sum_{j=1}^{a}\sum_{k=1}^{n}c_k(j)(F_k[i]\circ T[i,j])(u[j]), v[i]\right\rangle\\ &= \sum_{i=1}^{a}\sum_{j=1}^{a}\sum_{k=1}^{n}\langle(T[i,j])(c_k(j)u[j]), (F_k[i])^*(v[i])\rangle.\end{aligned}$$

For each k, let x_k and y_k be elements of $\bigoplus_{i=1}^{a}\ell_b^2$ specified by: $x_k[j] = c_k(j)u[j]$ and $y_k[j] = (F_k[j])^*(v[j])$, for all j. (Clearly $\sum_{j=1}^{a}\|c_k(j)u[j]\|^2 = \sum_{j=1}^{a}|c_k(j)|^2\|u[j]\|^2 \leqslant \|{}_A\Psi\|^2\sum_{j=1}^{a}\|u[j]\|^2 = \|{}_A\Psi\|^2\|u\|^2 < \infty$, and

$$\begin{aligned}\sum_{j=1}^{a}\|(F_k[j])^*(v[j])\|^2 \leqslant \sum_{j=1}^{a}\|F_k[j]\|^2\|v[j]\|^2 &\leqslant \left(\sup_{k,j}\|F_k[j]\|\right)^2\sum_{j=1}^{a}\|v[j]\|^2\\ &= \left(\sup_{k,j}\|F_k[j]\|\right)^2\|v\|^2 < \infty.\ )\end{aligned}$$

Then

$$\begin{aligned}&\sum_{i=1}^{a}\sum_{j=1}^{a}\langle(T[i,j])(c_k(j)u[j]), (F_k[i])^*(v[i])\rangle\\ &\qquad= \sum_{i=1}^{a}\langle(T[i,1], T[i,2], T[i,3], \ldots)(x_k), (F_k[i])^*(v[i])\rangle\\ &\qquad= \langle T(x_k), y_k\rangle,\end{aligned}$$

for each k. Consequently,

$$\langle({}_A\Psi(T))(u), v\rangle = \sum_{k=1}^{n}\sum_{i=1}^{a}\sum_{j=1}^{a}\langle(T[i,j])(c_k(j)u[j]), (F_k[i])^*(v[i])\rangle = \sum_{k=1}^{n}\langle T(x_k), y_k\rangle.$$

This shows that the set $\{T\in BM_a(M_b)\,|\,|\langle T(x_k), y_k\rangle| < \varepsilon/n, k = 1, 2, \ldots, n\}$, which is a basic weakly open neighbourhood of 0, is contained in the pre-image, under ${}_A\Psi$, of the set $\{T\in BM_a(M_b)\,|\,|\langle T(u), v\rangle| < \varepsilon\}$. So the continuity of ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$ is established. ∎

THEOREM 2.9. *Suppose A is an element of $SM_a(M_b)$. Then*

$$_A\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{str})$$

is continuous if and only if the span of block-columns of A is finite-dimensional in $M_{a\times 1}(M_b)$.

Proof. Combine the results of Lemmas 2.6, 2.7, 2.8. ∎

LEMMA 2.10. *Suppose that A in $SM_a(M_b)$ is such that only finitely many block-rows of A are non-zero. For each i, j, let B_{ij} be the matrix in $M_{b\times a}$ such that, for every k, the k-th column of B_{ij} equals the j-th column of $A[i,k]$, (i.e. $B_{ij}(e_k) = (A[i,k])(f_j)$). It follows $(B_{ij})^{\text{trn}}$ represents a bounded linear transformation from ℓ_b^2 to ℓ_a^∞.*

Suppose that, for every x in ℓ_b^2 and for each index i, $\text{span}\{(B_{ij})^{\text{trn}}(x) \mid 1 \leqslant j < b+1\}$ is finite-dimensional in ℓ_a^∞. Then

$$_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$$

is continuous.

Proof. First of all $\|B_{ij}(e_k)\| = \|(A[i,k])(f_j)\| \leqslant \|A[i,k]\| \leqslant \|_A\Psi\| < \infty$, for each k, and therefore $(B_{ij})^{\text{trn}}$ does indeed represent a bounded linear transformation from ℓ_b^2 to ℓ_a^∞. (Recall that a matrix represents a bounded operator from ℓ_b^2 to ℓ_a^∞ exactly when all of its rows are in ℓ_b^2 and their norms are uniformly bounded.) It is enough to establish the result for A that only has one non-zero block-row. Therefore assume that A is in $SM_{1\times a}(M_b)$, rather than in $SM_a(M_b)$. To this end it is sufficient to prove that if u and v are chosen arbitrarily from $\bigoplus_{i=1}^{a} \ell_b^2$ and ℓ_b^2 respectively, and if $\varepsilon > 0$ is any, then the pre-image, under $_A\Psi$, of $\{S \in BM_{1\times a}(M_b) \mid |\langle S(u), v\rangle| < \varepsilon\}$ contains a weakly open neighbourhood of 0.

Now $\text{span}_j\, B_{1,j}^{\text{trn}}(\bar{v})$ is finite-dimensional in ℓ_a^∞. There exists a basis $g_1, g_2, g_3, \ldots, g_n$ of this span, such that, for each k in $\{1, 2, \ldots, n\}$, there is an i_k with the property that

$$g_m[i_k] = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{otherwise,} \end{cases}$$

for all m. (This states that each g_t has an *indicator* entry which sets it apart from all other basis elements.)

Such a basis $g_1, g_2, g_3, \ldots, g_n$ can be obtained from any given basis by a Gauss-Jordan process. Let $\{c_k(j) \mid 1 \leqslant k \leqslant n, 1 \leqslant j \leqslant b+1\}$ be the set of complex numbers such that $B_{1,j}^{\text{trn}}(\bar{v}) = c_1(j)g_1 + c_2(j)g_2 + c_3(j)g_3 + \cdots + c_n(j)g_n$, for each j. Then $c_k(j) = (B_{1j}^{\text{trn}}(\bar{v}))[i_k] = \langle B_{1j}(e_{i_k}), v\rangle = \langle (A[1,i_k])(f_j), v\rangle$, for all j, k. Therefore: $\sum_{j=1}^{b} |c_k(j)|^2 = \sum_{j=1}^{b} |\langle f_j, (A[1,i_k])^*(v)\rangle|^2 = \|(A[1,i_k])^*(v)\|^2 \leqslant \|A[1,i_k]\|^2\|v\|^2 \leqslant \|{}_A\Psi\|^2\|v\|^2$, for all k. Therefore $\sup_k \sum_{j=1}^{b} |c_k(j)|^2 \leqslant \|{}_A\Psi\|^2\|v\|^2$.

Let T in $BM_{1\times a}(M_b)$ be any. Then

$$\begin{aligned}
\langle ({}_A\Psi(T))(u), v\rangle &= \Big\langle \sum_{m=1}^{a} (A[1,m] \circ T[1,m])(u[m]), v\Big\rangle \\
&= \Big\langle \sum_{m=1}^{a} (A[1,m])\Big(\sum_{j=1}^{b} \langle (T[1,m])^{\mathrm{trn}}(f_j), \overline{u[m]}\rangle f_j\Big), v\Big\rangle \\
&= \Big\langle \sum_{m=1}^{a}\sum_{j=1}^{b} \langle (T[1,m])^{\mathrm{trn}}(f_j), \overline{u[m]}\rangle (A[1,m])(f_j), v\Big\rangle \\
&= \sum_{m=1}^{a}\sum_{j=1}^{b} \langle (T[1,m])^{\mathrm{trn}}(f_j), \overline{u[m]}\rangle \cdot \langle (A[1,m])(f_j), v\rangle \\
&= \sum_{m=1}^{a}\sum_{j=1}^{b} \langle (T[1,m])^{\mathrm{trn}}(f_j), \overline{u[m]}\rangle \cdot (B_{1j}^{\mathrm{trn}}(\overline{v}))[m] \\
&= \sum_{m=1}^{a}\sum_{j=1}^{b} \left(\langle (T[1,m])^{\mathrm{trn}}(f_j), \overline{u[m]}\rangle \cdot \Big(\sum_{q=1}^{n} c_q(j) g_q[m]\Big)\right) \\
&= \sum_{m=1}^{a}\sum_{j=1}^{b}\sum_{q=1}^{n} \langle (T[1,m])^{\mathrm{trn}}(c_q(j)f_j), \overline{g_q[m]} \cdot \overline{u[m]}\rangle.
\end{aligned}$$

For each q, let w_q is the element of ℓ_b^2 specified by $w_q[j] = c_q(j)$ for all j; and let z_q be the element of $\bigoplus_{i=1}^{a} \ell_b^2$ specified by $z_q[m] = g_q[m]u[m]$, for all m. (We have already demonstrated that $\sup_k \sum_{j=1}^{b} |c_k(j)|^2$ is finite. As well, note that $|g_q[m]| \leqslant \|g\|_{\ell_a^\infty} < \infty$, for all m). Then

$$\begin{aligned}
\sum_{m=1}^{a}\sum_{j=1}^{b} \langle (T[1,m])^{\mathrm{trn}}(c_q(j)f_j), \overline{g_q[m]} \cdot \overline{u[m]}\rangle &= \sum_{m=1}^{b} \langle (T[1,m])^{\mathrm{trn}}(w_q), \overline{z_q[m]}\rangle \\
&= \sum_{m=1}^{a} \langle (T[1,m])(z_q[m]), \overline{w}_q\rangle \\
&= \langle T(z_q), \overline{w}_q\rangle,
\end{aligned}$$

for each q. Thus we have obtained: $\langle ({}_A\Psi(T))(u), v\rangle = \sum_{q=1}^{n} \langle T(z_q), \overline{w}_q\rangle$. From here it is apparent that the pre-image, under ${}_A\Psi$, of $\{S \in BM_{1\times a}(M_a) \mid |\langle S(u), v\rangle| < \varepsilon\}$

contains $\{T \in BM_{1\times a}(M_b) \,|\, |\langle T(z_q), \overline{w}_q\rangle| < \varepsilon/n,\ q = 1,2,\ldots,n\}$, which is a weakly open neighbourhood of 0. The proof is now complete. ∎

LEMMA 2.11. *Suppose A in $\mathrm{SM}_a(M_b)$ is such that ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{weak})$ is continuous. Then there exist A_1 and A_2 in $\mathrm{SM}_a(M_b)$ with that same property satisfying:*

(i) *$A_1 + A_2 = A$;*

(ii) *A_1 has finitely many non-zero block-rows;*

(iii) *For each i, j, let $B_{ij}^{(1)}$ be the matrix in $M_{b\times a}$ constructed from A_1 as in the statement of Lemma 2.10. Then $\mathrm{span}\Big\{\big(B_{ij}^{(1)}\big)^{\mathrm{trn}}(x) \,\big|\, 1 \leqslant i < a+1,\ 1 \leqslant j < b+1\Big\}$ is finite-dimensional in ℓ_a^∞, for each x in ℓ_b^2;*

(iv) $\mathrm{span}\left\{\begin{pmatrix}(A_2[i,1])[j,k]\\(A_2[i,2])[j,k]\\(A_2[i,3])[j,k]\\ \vdots\end{pmatrix} \,\middle|\, 1 \leqslant i < a+1, 1 \leqslant j,k < b+1\right\}$ *is finite-dimensional.*

Proof. If a is finite, simply take $A_1 = A$ and $A_2 = 0$. Thus from now on we assume that $a = \infty$. Suppose ${}_A\Psi$ has the stated continuity property.

Step 1. [Here we show that $\mathrm{span}_{i,j}\begin{pmatrix}\langle (A[i,1])(f_j), z_i\rangle\\ \langle (A[i,2])(f_j), z_i\rangle\\ \langle (A[i,3])(f_j), z_i\rangle\\ \vdots\end{pmatrix}$ is finite-dimensional, for every sequence $\{z_m\}_{m=1}^\infty$ of elements of ℓ_b^2.]

It is clearly sufficient to demonstrate the claim for sequences $\{z_m\}_{m=1}^\infty$ such that $\sum_{m=1}^\infty \|z_m\|^2 < \infty$. Given such a sequence, let z be the vector in $\bigoplus_{i=1}^\infty \ell_b^2$ specified by $z[m] = z_m$, for all m.

Let w in $\bigoplus_{i=1}^\infty \ell_b^2$ be any. Since ${}_A\Psi : (BM_\infty(M_b), \mathrm{str}) \to (BM_\infty(M_b), \mathrm{weak})$ is continuous, there exist $x_1, x_2, x_3, \ldots, x_n$ in $\bigoplus_{i=1}^\infty \ell_b^2$ and a $\delta > 0$ such that $|\langle({}_A\Psi(T))(w), z\rangle| < 1$ whenever T in $BM_\infty(M_b)$ satisfies $\|T(x_t)\| < \delta$, for all t. If $T(x_t) = 0$, for all t, then $\langle({}_A\Psi(T))(w), z\rangle = 0$ since all complex multiples of T annihilate $x_1, x_2, x_3, \ldots, x_n$.

Let v be any element of $\{\overline{x}_1, \overline{x}_2, \overline{x}_3, \ldots, \overline{x}_n\}^\perp$ in $\bigoplus_{i=1}^\infty \ell_b^2$, and let i_0 and j_0 be any positive integers, with j_0 not exceeding b. Denote by T_0 the operator in

$BM_\infty(M_b)$ specified by

$$(T_0)^{\mathrm{trn}}(f_j \otimes e_i) = \begin{cases} v & \text{if } (i,j) = (i_0, j_0) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $T(x_t) = 0$, for all t. Therefore:

$$\begin{aligned} 0 = \langle ({}_A\Psi(T))(w), z\rangle &= \Big\langle \Big(\sum_{m=1}^{\infty} A[i_0, m] \circ T[i_0, m]\Big)(w[m]), z[i_0]\Big\rangle \\ &= \sum_{m=1}^{\infty} \langle (A[i_0, m])(\langle v[m], \overline{w[m]}\rangle f_{j_0}), z[i_0]\rangle \\ &= \sum_{m=1}^{\infty} \langle v[m], \overline{w[m]}\rangle \langle (A[i_0, m])(f_{j_0}), z[i_0]\rangle \\ &= \sum_{m=1}^{\infty} \langle v[m], \langle z[i_0], (A[i_0, m])(f_{j_0})\rangle \overline{w[m]}\rangle. \end{aligned}$$

Let u stand for

$$\begin{bmatrix} \langle z_{i_0}, (A[i_0,1])(f_{j_0})\rangle I_b & 0 & 0 & 0 \\ 0 & \langle z_{i_0}, (A[i_0,2])(f_{j_0})\rangle I_b & 0 & 0 \\ 0 & 0 & \langle z_{i_0}, (A[i_0,3])(f_{j_0})\rangle I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} (\overline{w}).$$

Then u is in $\bigoplus_{i=1}^{\infty} \ell_b^2$ and $0 = \langle ({}_A\Psi(T))(w), z\rangle = \langle v, u\rangle$, so that $u \in \operatorname{span}\{\overline{x}_1, \overline{x}_2, \overline{x}_3, \ldots, \overline{x}_n\}$. Therefore

$$\begin{bmatrix} \langle (A[i_0,1])(f_{j_0}), z_{i_0}\rangle I_b & 0 & 0 & 0 \\ 0 & \langle (A[i_0,2])(f_{j_0}), z_{i_0}\rangle I_b & 0 & 0 \\ 0 & 0 & \langle (A[i_0,3])(f_{j_0}), z_{i_0}\rangle I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} (w)$$

is an element of the $\operatorname{span}\{x_1, x_2, x_3, \ldots, x_n\}$. We can now conclude that

$$\operatorname*{span}_{i,j} \begin{bmatrix} \langle (A[i,1])(f_j), z_i\rangle I_b & 0 & 0 & 0 \\ 0 & \langle (A[i,2])(f_j), z_i\rangle I_b & 0 & 0 \\ 0 & 0 & \langle (A[i,3])(f_j), z_i\rangle I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} (w)$$

is a subspace of the $\operatorname{span}\{x_1, x_2, x_3, \ldots, x_n\}$, for every w in $\bigoplus_{i=1}^{\infty} \ell_b^2$. Since w can be selected to be a separating vector for the set

$$\left\{ \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 \\ 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix} \,\middle|\, D_m \text{ is a diagonal element of } B(M_b) \right\},$$

(there are many vectors in $\bigoplus_{i=1}^{\infty} \ell_b^2$ that separate this set), it must be true that

$$\operatorname*{span}_{i,j} \begin{bmatrix} \langle (A[i,1])(f_j), z_i\rangle I_b & 0 & 0 & 0 \\ 0 & \langle (A[i,2])(f_j), z_i\rangle I_b & 0 & 0 \\ 0 & 0 & \langle (A[i,3])(f_j), z_i\rangle I_b & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

is finite-dimensional (of dimension at most n). Thus

$$\operatorname*{span}_{i,j} \begin{bmatrix} \langle (A[i,1])(f_j), z_i\rangle & 0 & 0 & 0 \\ 0 & \langle (A[i,2])(f_j), z_i\rangle & 0 & 0 \\ 0 & 0 & \langle (A[i,3])(f_j), z_i\rangle & 0 \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

is finite-dimensional. This clearly provides the desired conclusion.

Step 2. For each i, j, let C_{ij} be the matrix in $M_{\infty\times b}$ such that, for every k, k-th column of $(C_{ij})^{\mathrm{trn}}$ equals the j-th column of $A[i,k]$. Then $\|(C_{ij})^{\mathrm{trn}}(e_k)\| = \|(A[i,k])(f_j)\| \leqslant \|A[i,k]\| = \|_A\Psi\| < \infty$, for each k, and therefore C_{ij} represents a bounded linear transformation from ℓ_b^2 to ℓ_a^∞. If $y \in \ell_b^2$ then $(C_{ij}(y))[k] = \langle (C_{ij})^{\mathrm{trn}}(e_k), \overline{y}\rangle = \langle (A[i,k])(f_j), \overline{y}\rangle$, for every k. With this in mind we apply the result obtained in Step 1 to conclude that if $\{y_t\}_{t=1}^{\infty}$ is any sequence of elements of ℓ_b^2 then $\operatorname*{span}_{i,j} C_{ij}(y_i)$ is finite-dimensional in ℓ_a^∞.

Step 3. [Here we show that there exists a positive integer i_0 such that $\operatorname*{span}_{i>i_0, j}(\operatorname{Ran} C_{ij})$ is finite-dimensional in ℓ_a^∞.]

We start by demonstrating that there exists a positive integer i_{00} such that $\operatorname*{span}_{j}(\operatorname{Ran} C_{ij})$ is finite-dimensional whenever $i \geqslant i_{00}$.

Suppose no such number exists. Then there is an unbounded sequence $\{i_t\}_{t=1}^{\infty}$ of positive integers such that $\operatorname*{span}_{j}(\operatorname{Ran} C_{i_t j})$ is infinite-dimensional, for every t.

Consequently there is a sequence $\{y_t\}_{t=1}^{\infty}$ of elements of ℓ_b^2 and a sequence $\{j_t\}_{t=1}^{\infty}$ of positive integers such that, for every q, $C_{i_q j_q}(y_q)$ does not belong to $\operatorname{span}\{C_{i_t j_t}(y_t) \mid 1 \leqslant t < q\}$; (in other words: $\{C_{i_t j_t}(y_t) \mid t \in \mathbb{N}\}$ is linearly independent).

This contradicts conclusions of Step 2 and therefore proves the existence of the number i_{00} as claimed.

Now we use a "proof by contradiction" once again, to demonstrate the existence of i_0. Suppose that $\operatorname*{span}_{i>i_0, j}(\operatorname{Ran} C_{ij})$ is infinite-dimensional (in ℓ_a^∞) for every positive integer i_0. Then there exists a sequence $\{(\widehat{i}_t, \widehat{j}_t)\}_{t=1}^{\infty}$ of pairs of positive integers such that $\widehat{i}_t \to \infty$ as $t \to \infty$, and, for every q, $\operatorname{Ran}(C_{\widehat{i}_q \widehat{j}_q}) \not\subset \operatorname{span}\{\operatorname{Ran}(C_{\widehat{i}_t \widehat{j}_t}) \mid 1 \leqslant t < q\}$; (recall that for every i exceeding i_{00}, $\operatorname{span}_j(\operatorname{Ran} C_{ij})$ is finite-dimensional). Therefore, for every q, there exists an element y_q of ℓ_b^2 such that $C_{\widehat{i}_q \widehat{j}_q}(y_q) \notin \operatorname{span}\{\operatorname{Ran}(C_{\widehat{i}_t \widehat{j}_t}) \mid 1 \leqslant t < q\}$.

Consequently, $\{C_{\widehat{i}_t \widehat{j}_t}(y_t) \mid t \in \mathbb{N}\}$ is linearly independent and this again contradicts the results of Step 2. Thus the claim in this step is verified.

Step 4. [Here we complete the proof of Lemma 2.11.]

Let $A_1 = \left(I_b \otimes \left(\sum_{r=1}^{i_0} E_{rr}\right)\right) \circ A$ and $A_2 = A - A_1$. Then A_1 has finitely many non-zero block-rows. Also clearly $A_1, A_2 \in SM_\infty(M_b)$ and

$$ {}_{A_1}\Psi = L_{I_b \otimes \left(\sum_{r=1}^{i_0} E_{rr}\right)} \circ {}_A\Psi. $$

Since

$$ L_{I_b \otimes \left(\sum_{r=1}^{i_0} E_{rr}\right)} : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak}) $$

is clearly continuous, the continuity of ${}_{A_1}\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{weak})$ follows. The continuity of ${}_{A_2}\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{weak})$ now follows trivially. (So far we have satisfied all the requirements up to (ii); we now proceed to (iii)). For each $r = 1, 2$ and $1 \leqslant i < \infty$ and $1 \leqslant j < b+1$, let $B_{ij}^{(r)}$ be the matrix in $M_{\infty \times b}$ such that, for every k, the k-th column of $B_{ij}^{(r)}$ equals the j-th column of $A_r[i,k]$. Then

$$ B_{ij}^{(1)} = \begin{cases} (C_{ij})^{\text{trn}} & \text{if } 1 \leqslant i \leqslant i_0 \\ 0 & \text{otherwise} \end{cases} $$

and

$$ B_{ij}^{(2)} = \begin{cases} (C_{ij})^{\text{trn}} & \text{if } i_0 < i \\ 0 & \text{otherwise.} \end{cases} $$

That the requirement (iii) is now satisfied follows immediately from the result of Step 2. It remains only to consider the requirement (iv). The $\operatorname{span}_{i,j} \operatorname{Ran}\left((B_{ij}^{(2)})^{\text{trn}}\right)$ is finite-dimensional by the result of Step 3. Therefore

$$ \operatorname*{span}_{\substack{i>i_0\\ j,m}} \begin{pmatrix} (A[i,1])[m,j] \\ (A[i,2])[m,j] \\ (A[i,3])[m,j] \\ \vdots \end{pmatrix} = \operatorname*{span}_{\substack{i>i_0\\ j,m}} C_{ij}(f_m) = \operatorname*{span}_{i,j,m} \left(B_{ij}^{(2)}\right)^{\text{trn}} (f_m) $$

$$ = \operatorname*{span}_{i,j} \left(\operatorname*{span}_{m} \left(B_{ij}^{(2)}\right)^{\text{trn}} (f_m)\right) = \operatorname*{span}_{i,j} \operatorname{Ran} \left(B_{ij}^{(2)}\right)^{\text{trn}} $$

$$ = \text{finite-dimensional.} \quad \blacksquare $$

LEMMA 2.12. *Suppose A is an element of $M_{1\times a}(BM_b)$. For each j, let B_j be the matrix in $M_{b\times a}$ such that, for each k, the k-th column of B_j equals the j-th column of $A[k]$. Then*

(i) $\operatorname{span}\{B_j \mid 1 \leqslant j < b+1\}$ *is finite-dimensional in $M_{b\times a}$ if and only if* $\operatorname{span}\{(A[m])^{\mathrm{trn}}(f_i) \mid 1 \leqslant i < b+1,\ 1 \leqslant m < a+1\}$ *is finite-dimensional in ℓ_b^2 (that is the total span of all rows of all block-entries of A is finite-dimensional).*

(ii) $\operatorname{span}\{(B_j)^{\mathrm{trn}}(f_k) \mid 1 \leqslant j,k < b+1\}$ *is finite-dimensional in ℓ_a^∞ if and only if* $\operatorname{span}\{A[m] \mid 1 \leqslant m < a+1\}$ *is finite-dimensional in BM_b.*

Proof. (i) Let $\widehat{A}$ denote the matrix in $M_{a\times 1}(BM_b)$ specified by $\widehat{A}[i] = A[i]$, for all i. Then clearly $\operatorname{span}\{(A[m])^{\mathrm{trn}}(f_j) \mid 1 \leqslant i,m < b+1\} = \operatorname{span}\{(\widehat{A}[m])^{\mathrm{trn}}(f_i) \mid 1 \leqslant i,m < b+1\} = \operatorname{span}\{(i\text{-th row of } \widehat{A}[m])^{\mathrm{trn}} \mid 1 \leqslant i,m < b+1\}$ in ℓ_b^2. By Lemma 2.3 the latter span is finite-dimensional if and only if the span of *scalar columns* of $\widehat{A}$ is finite-dimensional in $\ell_b^2 \times \ell_b^2 \times \ell_b^2 \times \ldots$ (a times); that is, if and only if $\operatorname{span}\{\widehat{A}(f_i) \mid 1 \leqslant i \leqslant b+1\}$ is finite-dimensional in $\ell_b^2 \times \ell_b^2 \times \ell_b^2 \times \ldots$ (a times). It takes a minute's consideration to see that the span of *scalar columns* of $\widehat{A}$ is finite-dimensional exactly when the $\operatorname{span}\{B_j \mid 1 \leqslant j < b+1\}$ is finite-dimensional.

(ii) It has already been shown in Lemma 2.10 that each $(B_j)^{\mathrm{trn}}$ represents a bounded linear transformation from ℓ_b^2 to ℓ_a^∞.

Let $\widehat{B}$ denote the matrix in $M_{a\times 1}(M_{b\times a})$ specified by $\widehat{B}[j] = B_j$, for all j. Then $\operatorname{span}\{(B_j)^{\mathrm{trn}}(f_k) \mid 1 \leqslant j,k < b+1\} = \operatorname{span}\{(\widehat{B}[j])^{\mathrm{trn}}(f_k) \mid 1 \leqslant j,k < b+1\} = \operatorname{span}\{(k\text{-th row of } \widehat{B}[j])^{\mathrm{trn}} \mid 1 \leqslant j,k < b+1\}$ in ℓ_a^∞. By Lemma 2.3 the latter span is finite-dimensional if and only if the span of scalar columns of $\widehat{B}$ is finite-dimensional in $\ell_b^2 \times \ell_b^2 \times \ell_b^2 \times \ldots$; that is, if and only if $\operatorname{span}\{\widehat{B}(e_k) \mid 1 \leqslant k < b+1\}$ is finite-dimensional in $\ell_b^2 \times \ell_b^2 \times \ell_b^2 \times \ldots$. It is easy to see that the span of scalar columns of $\widehat{B}$ is finite-dimensional exactly when the $\operatorname{span}\{A[m] \mid 1 \leqslant m < a+1\}$ is finite-dimensional in BM_b. ∎

DEFINITION. ([8]) Let V and W be a vector space and let $L(V,W)$ stand for the space of linear maps from V to W. We say that a subset S of $L(V,W)$ is *locally finite-dimensional* if $\operatorname{span}\{T(x) \mid T \in S\}$ is finite-dimensional for every x in V.

THEOREM 2.13. ([8]) *Let V and W be Banach spaces. A subset S of $B(V,W)$ is locally finite-dimensional if and only if there exists a finite-dimensional subspace R of $B(V,W)$, containing no operators of finite rank, and a finite-dimensional subspace N of W, such that S is contained in the vector sum $R + \{T \in B(V,W) \mid \operatorname{Range}(T) \subset N\}$. Moreover if M is a closed subspace complementary to N in W then R can be taken to be a subspace of $\{T \in B(V,W) \mid \operatorname{Ran}(T) \subset M\}$.*

LEMMA 2.14. *Suppose* $\operatorname{span}\{u_i \mid i = 1, 2, 3, \ldots\}$ *is finite-dimensional in* ℓ_a^∞ *and* $\sum_{i=1}^{\infty} |u_i[j]|^2 < \infty$, *for every* j. *Then* $\sum_{i=1}^{\infty} \|u_i\|_{\ell_a^\infty}^2 < \infty$.

Proof. Let n be the dimension of the span of $\{u_i \mid i = 1, 2, 3, \ldots\}$. Without loss of generality we may assume that this span must have a basis $\{v_1, v_2, \ldots, v_n\}$ such that

$$v_i[j] = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leqslant i,\ j \leqslant n$. Now $u_i = a_1^{(i)} v_1 + a_2^{(i)} v_2 + \cdots + a_n^{(i)} v_n$, where $a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}$ stand for the appropriate scalars. Clearly

$$\sum_{i=1}^{\infty} |u_i[j]|^2 = \sum_{i=1}^{\infty} |a_j^{(i)}|^2, \text{ so that } \sum_{i=1}^{\infty} |a_j^{(i)}|^2 < \infty,$$

for every j. Yet

$$\|u_i\|_{\ell_a^\infty} \leqslant |a_1^{(i)}|\, \|v_1\|_{\ell_a^\infty} + |a_2^{(i)}|\, \|v_2\|_{\ell_a^\infty} + \cdots + |a_n^{(i)}|\, \|v_n\|_{\ell_a^\infty},$$

for each i, and consequently

$$\|u_i\|_{\ell_a^\infty}^2 \leqslant n\left(|a_1^{(i)}|^2 \|v_1\|_{\ell_a^\infty}^2 + |a_2^{(i)}|^2 \|v_2\|_{\ell_a^\infty}^2 + \cdots + |a_n^{(i)}|^2 \|v_n\|_{\ell_a^\infty}^2\right),$$

for each i, by the inequality stated in the proof of Lemma 2.8. Therefore we have

$$\begin{aligned}\sum_{i=1}^{\infty} \|u_i\|_{\ell_a^\infty}^2 &\leqslant n\Big(\|v_1\|_{\ell_a^\infty}^2 \sum_{i=1}^{\infty} |a_1^{(i)}|^2 + \|v_2\|_{\ell_a^\infty}^2 \sum_{i=1}^{\infty} |a_2^{(i)}|^2 + \cdots + \|v_n\|_{\ell_a^\infty}^2 \sum_{i=1}^{\infty} |a_n^{(i)}|^2\Big) \\ &< \infty. \quad \blacksquare\end{aligned}$$

LEMMA 2.15. *Let* A *in* $M_a(BM_b)$ *be such that only finitely many block-rows of* A *are non-zero. If* $\sup_j \|A[i,j]\|$ *is finite and* $\operatorname{span}_j A[i,j]$ *is finite-dimensional in* BM_b, *for every* i, *then* A *is in* $SM_a(M_b)$.

Proof. It is enough to consider the case when only the first block-row of A is non-zero. Let n be the dimension of $\operatorname{span}_j A[1,j]$. There exists a basis $\{T_1, T_2, \ldots, T_n\}$ of this span such that for each $k = 1, 2, \ldots, n$ there exist i_k and j_k such that

$$T_m[i_k, j_k] = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{otherwise.} \end{cases}$$

$A[1,j] = a_1^{(j)}T_1 + a_2^{(j)}T_2 + \cdots + a_n^{(j)}T_n$, where $a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}$ stand for the appropriate scalars. Then $a_k^{(j)} = (A[1,j])[i_k, j_k]$, for all j, k, so that $\sup_{j,k} |a_k^{(j)}| \leqslant \sup_j \|A[1,j]\| < \infty$. The first block-row of A equals $\sum_{k=1}^{n} \boxed{a_k^{(1)}T_k \; a_k^{(2)}T_k \; a_k^{(3)}T_k \ldots}$ and it is clear that each of the terms $\boxed{a_k^{(1)}T_k \; a_k^{(2)}T_k \; a_k^{(3)}T_k \ldots}$ is in $SM_{1\times a}(M_b)$, because $\sup_{j,k} |a_k^{(j)}| < \infty$. Thus it follows that $A \in SM_a(M_b)$. ∎

LEMMA 2.16. *For A in $SM_a(M_b)$ the following are equivalent:*

(i) ${}_A\Psi : (BM_a(M_b), \text{str}) \to (BM_a(M_b), \text{weak})$ *is continuous.*

(ii) ${}_A\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$ *is continuous.*

(iii) *There exist A_1 and A_2 in $SM_a(M_b)$ satisfying:*

 (a) $A_1 + A_2 = A$;

 (b) *The span of the block-columns of A_1 is finite-dimensional in $M_{a\times 1}(M_b)$;*

 (c) *A_2 has finitely many non-zero block rows;*

 (d) *For each i, j, let $B_{ij}^{(2)}$ be the matrix in $M_{b\times a}$ constructed from A_2 as in the statement of Lemma 2.10.*
 Then $\text{span}\left\{ \left(B_{ij}^{(2)}\right)^{\text{trn}}(x) \,\middle|\, 1 \leqslant i < a+1,\ 1 \leqslant j < b+1 \right\}$ *is finite-dimensional in ℓ_a^∞, for each x in ℓ_b^2.*

(iv) *There exist A_1 and A_2 in $SM_a(M_b)$ satisfying:*

 (a) $A_1 + A_2 = A$;

 (b) *A_2 has finitely many non-zero block rows;*

 (c) *The span of the block-columns of A_1 is finite-dimensional in $M_{a\times 1}(M_b)$;*

 (d) *There exist matrices S_1 and S_2 in $SM_a(M_b)$, each with finitely many non-zero block-rows, and such that:*

  (d1) $S_1 + S_2 = A_2$.

  (d2) *The* $\text{span}_{i,j}\, S_1[i,j]$ *is finite-dimensional in BM_b;*

  (d3) *The* $\text{span}_{i,j}(\ker^\perp S_2[i,j])$ *is finite-dimensional in ℓ_b^2;*

Proof. That (ii) implies (i) is trivial. That (i) implies (iii) follows directly from Lemma 2.11 and Lemma 2.7. Suppose now that condition (iii) holds. Then ${}_{A_1}\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$ is continuous by Lemma 2.8, and ${}_{A_2}\Psi : (BM_a(M_b), \text{weak}) \to (BM_a(M_b), \text{weak})$ is continuous by Theorem 2.10. Thus (ii) holds since ${}_A\Psi = {}_{A_1}\Psi + {}_{A_2}\Psi$. Therefore (iii) implies (ii). We complete the proof by showing that (iii) is equivalent to (iv). Suppose (iii) holds. For each i,j,k : $\|B_{ij}^{(2)}(e_k)\| = \|(A_2[i,k])(f_j)\| \leqslant \|A_2[i,k]\| \leqslant \|_{A_2}\Psi\| < \infty$. Therefore $\left(B_{ij}^{(2)}\right)^{\text{trn}}$ represents a bounded linear transformation from to ℓ_b^2 to ℓ_a^∞.

The subset $\left\{\left(B_{ij}^{(2)}\right)^{\text{trn}} \mid 1 \leqslant i < a+1, 1 \leqslant j < b+1\right\}$ of the space of elements of $M_{a\times b}$ representing operators in $B(\ell_b^2, \ell_a^\infty)$, satisfies the hypothesis of the Theorem 2.13. Thus, there is a finite-dimensional subspace M of ℓ_a^∞ and a finite-dimensional subspace F of the space of elements of $M_{a\times b}$ representing operators in $B(\ell_b^2, \ell_a^\infty)$, such that $\left\{\left(B_{ij}^{(2)}\right)^{\text{trn}} \mid 1 \leqslant i < a+1, 1 \leqslant j < b+1\right\} \subset F + \{Q \in B(\ell_b^2, \ell_a^\infty) \mid \text{Ran}(Q) \subset M\}$, where the sum is a vector space sum.

Moreover the ranges of all operators in F can be assumed to lie within a closed compliment of M in ℓ_a^∞. Therefore exist two families $\{G_{ij}^{(2)} \mid 1 \leqslant i < a+1, 1 \leqslant j < b+1\}$ and $\{D_{ij}^{(2)} \mid 1 \leqslant i < a+1, 1 \leqslant j < b+1\}$ of matrices in $M_{a\times b}$, all representing bounded linear transformation from ℓ_b^2 to ℓ_a^∞, such that, for each i,j, $\left(B_{ij}^{(2)}\right)^{\text{trn}} = G_{ij}^{(2)} + D_{ij}^{(2)}$, $\operatorname{span}_{i,j} G_{ij}^{(2)}$ and $\operatorname{span}_{i,j,k} D_{ij}^{(2)}(f_k)$ are both finite-dimensional, and there exists a constant K such that $\max(\|G_{ij}^{(2)}(x)\|_{\ell_a^\infty}, \|D_{ij}^{(2)}(x)\|_{\ell_a^\infty}) \leqslant K\left\|\left(B_{ij}^{(2)}\right)^{\text{trn}}(x)\right\|_{\ell_a^\infty}$, for all x in ℓ_b^2 and all i,j.

Yet $\left(\left(B_{ij}^{(2)}\right)^{\text{trn}}(x)\right)[k] = ((A_2[i,k])^{\text{trn}}(x))[j]$, from the definition of $B_{ij}^{(2)}$. Thus $\sum_{j=1}^{b}\left|\left(\left(B_{ij}^{(2)}\right)^{\text{trn}}(x)\right)[k]\right|^2 = \sum_{j=1}^{b}|((A_2[i,k])^{\text{trn}}(x))[j]|^2 = \|(A_2[i,k])^{\text{trn}}(x)\|^2 \leqslant \|A_2[i,k]\|^2\|x\|^2 \leqslant \|_{A_2}\Psi\|^2\|x\|^2$, for all x in ℓ_b^2 and all i,k.

Apply Lemma 2.14 to conclude that $\sum_{j=1}^{b}\left\|\left(B_{ij}^{(2)}\right)^{\text{trn}}(x)\right\|^2 < \infty$, for all x in ℓ_b^2 and all i. Therefore $\sum_{j=1}^{b}\left\|G_{ij}^{(2)}(x)\right\|^2 < \infty$ and $\sum_{j=1}^{b}\left\|D_{ij}^{(2)}(x)\right\|^2 < \infty$, for all x in ℓ_b^2 and all i.

Let S_1 be the matrix in $M_a(M_b)$ specified by $(S_1[i,k])(f_j) = \left(D_{ij}^{(2)}\right)^{\text{trn}}(e_k)$, for each i,j,k, and S_2 be the matrix in $M_a(M_b)$ specified by $(S_2[i,k])(f_j) = \left(G_{ij}^{(2)}\right)^{\text{trn}}(e_k)$, for each i,j,k. Both S_1 and S_2 have finitely many non-zero rows, since A_2 does. It is apparent that $S_1 + S_2 = A_2$, because $\left(B_{ij}^{(2)}\right)^{\text{trn}} = G_{ij}^{(2)} + D_{ij}^{(2)}$, for each i,j. Now $((S_2[i,k])^{\text{trn}}(x))[j] = (G_{ij}^{(2)}(x))[k]$ and $((S_1[i,k])^{\text{trn}}(x))[j] = (D_{ij}^{(2)}(x))[k]$, for all x in ℓ_b^2 and all i,j,k. Therefore $\sum_{j=1}^{b}|((S_2[i,k])^{\text{trn}}(x))[j]|^2 = \sum_{j=1}^{b}|(G_{ij}^{(2)}(x))[k]|^2 \leqslant \sum_{j=1}^{b}\|G_{ij}^{(2)}(x)\|^2 < \infty$, and consequently $(S_2[i,k])^{\text{trn}}(x) \in \ell_b^2$

with $\|(S_2[i,k])^{\mathrm{trn}}(x)\|^2 \leqslant \sum_{j=1}^{b} \|G_{ij}^{(2)}(x)\|^2 < \infty$, for all x in ℓ_b^2 and all i,j,k. Thus $(S_2[i,k])^{\mathrm{trn}} \in BM_b$, for all i,k, and, for each i, $\{(S_2[i,k])^{\mathrm{trn}} \mid 1 \leqslant k \leqslant a+1\}$ is a pointwise bounded set of operator matrices (in BM_b). The latter set is therefore uniformly bounded by the uniform boundedness principle.

Consequently, the set $\{S_2[i,k] \mid 1 \leqslant k \leqslant a+1\}$ is also uniformly bounded in BM_b, for every i. The same can be said about the set $\{S_1[i,k] \mid 1 \leqslant k \leqslant a+1\}$.

It remains to observe that, for each i, Lemma 2.12, part (ii), demonstrates, (with A being the i-th block-row of S_1 and $B_j = \left(D_{ij}^{(2)}\right)^{\mathrm{trn}}$), that $\operatorname{span}_k(S_1[i,k])$ (and also $\operatorname{span}_{i,k}(S_1[i,k])$) is finite-dimensional in BM_b, so that Lemma 2.15 can be applied to conclude that $S_1 \in SM_a(M_b)$. That S_2 is also in $SM_a(M_b)$ follows immediately since $S_1 + S_2 = A_2$. That $\operatorname{span}_{i,k}(\ker^{\perp} S_2[i,j])$ is finite-dimensional in ℓ_b^2 follows easily from Lemma 2.12, part (i).

We have now demonstrated that (iv) holds true. It is simple to see that the proof demonstrating that (iii) implies (iv) can be traversed in the opposite direction to show that (iv) implies (iii). ∎

THEOREM 2.17. *For A in $SM_a(M_b)$ the following are equivalent:*

(i) ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{weak})$ *is continuous.*

(ii) ${}_A\Psi : (BM_a(M_b), \mathrm{weak}) \to (BM_a(M_b), \mathrm{weak})$ *is continuous.*

(iii) *There exist A_1 and A_2 in $SM_a(M_b)$ satisfying:*

(a) $A_1 + A_2 = A$;

(b) *A_2 has finitely many non-zero block-rows and $\operatorname{span}_{i,j}(\ker^{\perp} A_2[i,j])$ is finite-dimensional in ℓ_b^2;*

(c) ${}_{A_1}\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{str})$ *is continuous.*

Proof. Let A_1 in the present theorem stand for the sum of A_1 and S_1 of part (iv) (d) in Lemma 2.16. Then Theorem 2.9 and Lemma 2.16 provide the necessary conclusions. ∎

It is now easy to provide an example of a Schur block-multiplier which induces a map that is weak-to-weak and strong-to-weak but not strong-to-strong continuous. Indeed, let A be the block-matrix in $BM_\infty(M_\infty)$ with one non-zero block-row which is:

$$\left[\begin{array}{ccccc|ccccc|ccccc|c}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}\right].$$

By Theorem 2.9 ${}_A\Psi : (BM_a(M_b), \mathrm{str}) \to (BM_a(M_b), \mathrm{str})$ is not continuous since the span of the block-columns of A is not finite-dimensional. Yet ${}_A\Psi : (BM_a(M_b), \text{weak or str}) \to (BM_a(M_b), \mathrm{weak})$ is clearly continuous by Theorem 2.17 (with $A_2 = A$).

It is apparent that Theorem 2.17 indicates that no such example exists if b is finite, since in that case ${}_{A_2}\Psi$ is also strong-to-strong continuous. We restate this in the case $b = 1$, i.e. in the case of ordinary Schur product.

COROLLARY 2.18. *For A in SM_a the following are equivalent:*

(i) ${}_A\Psi : (BM_a, \mathrm{weak}) \to (BM_a, \mathrm{weak})$ *is continuous;*

(ii) ${}_A\Psi : (BM_a, \mathrm{str}) \to (B, \mathrm{weak})$ *is continuous;*

(iii) ${}_A\Psi : (BM_a, \mathrm{str}) \to (BM_a, \mathrm{str})$ *is continuous;*

(iv) *A is a matrix with of finite column (row) rank.*

That the identity matrix in BM_∞ does not induce a Schur multiplication map that is strong to weak continuous follows immediately from Corollary 2.18. Yet by Theorem 1.2, ${}_I\Psi$ is strong-to-strong sequentially continuous. In other words, even though for every sequence $\{T_n\}_{n=1}^{\infty}$ converging strongly to zero (in BM_∞), $\{\mathrm{diag}(T_n)\}_{n=1}^{\infty}$ also converges to zero strongly, *there exists a net $\{T_\alpha \,|\, \alpha \in \Im\}$ converging strongly to zero in BM_∞ such that $\{\mathrm{diag}(T_\alpha) \,|\, \alpha \in \Im\}$ doesn't even converge to zero weakly!*

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