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ON POLYNOMIALLY BOUNDED WEIGHTED SHIFTS. II

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ABSTRACT. Let T be an operator-weighted shift whose weights are 2-by-2 matrices. We say that, given $\varepsilon > 0$, T is in the ε-canonical form if each weight is an upper triangular matrix (a_{ij}), with $0 \leqslant a_{11}, a_{22} \leqslant 1$ and $a_{12} \neq 0$ implies $a_{11}, a_{22} < \varepsilon$. We generalize this concept to operator-weighted shifts whose weights are n-by-n matrices and we show that every polynomially bounded weighted shift, whose weights are finite-dimensional matrices of the fixed dimension n, is similar to an operator in the ε-canonical form. This enables us to prove that every polynomially bounded weihghted shift with finite dimensional weights is similar to a contraction.

KEYWORDS: *Operator-weighted shifts, polynomially bounded operators, similarity to a contraction.*

AMS SUBJECT CLASSIFICATION: Primary 47B37; Secondary 47A65, 47A99.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *completely polynomially bounded* (notation: $T \in$(CPB)) if there exists $M \geqslant 1$ such that for every positive integer n and for every $n \times n$ matrix of polynomials (p_{ij}),

$$\|(p_{ij}(T))_{i,j=1}^{n}\| \leqslant M \sup\left\{\|(p_{ij}(\zeta))_{i,j=1}^{n}\| : |\zeta| \leqslant 1\right\}, \tag{1.1}$$

where the operator $(p_{ij}(T))$ on the left side of (1.1) is an $n \times n$ matrix with operator entries acting on the direct sum of n copies of $\mathcal{H}$ and $(p_{ij}(\zeta))$ denotes an $n \times n$ complex matrix. The infimum of all numbers M that can appear on the right hand side of (1.1) is called the *complete polynomial bound* of T and it is denoted by $M_{\text{cpb}}(T)$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* (notation: $T\in$(PB)) if there exists $M \geqslant 1$ such that (1.1) is true for $n = 1$. In this situation the infimum of all such M is called the *polynomial bound* of T and it is denoted by $M_{\text{pb}}(T)$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *power bounded* (notation: $T\in$(PW)) if there exists $M \geqslant 1$ such that (1.1) holds for $n = 1$ and for every polynomial of the special form $p(\zeta) = \zeta^m$ where m is a positive integer. Then, the infimum of all such M is called the *power bound* of T and it is denoted by $M_{\text{pw}}(T)$. Also an operator T in $\mathcal{L}(\mathcal{H})$ is said to be *similar to a contraction* (notation: $T\in$(SC)) if there exists an invertible operator S in $\mathcal{L}(\mathcal{H})$ such that $\|S^{-1}TS\| \leqslant 1$. It is easy to see that

$$\text{(SC)} \subset \text{(CPB)} \subset \text{(PB)} \subset \text{(PW)}.$$

One knows (cf. [4]) that (SC)=(CPB), but the relationship between the classes (CPB), (PB), and (PW) is not completely understood. One does know, however, that there are examples of operators in (PW) that are not in (PB) (see [1], [2], [3]).

It is the purpose of this paper to enhance our knowledge of these classes by continuing the study of power bounded operator-weighted (unilateral) shifts started in [6] and [5]. Recall that if $\{W_n\}_{n=1}^{\infty}$ is any bounded sequence from $\mathcal{L}(\mathcal{H})$, an operator T in $\mathcal{L}(\mathcal{H}^{(\aleph_0)})$ is called an *operator-weighted shift* with weight sequence $\{W_i\}_{i=1}^{\infty}$ (notation: $T = S_{\{W_i\}}$) if

$$S_{\{W_i\}}\ (k_1, k_2, \ldots, k_n, \ldots) = (0, W_1k_1, W_2k_2, \ldots, W_nk_n, \ldots),$$

for all vectors $(k_1, k_2, \ldots)$ in $\mathcal{H}^{(\aleph_0)}$. The following proposition comes from [5].

PROPOSITION 1.1. *Suppose $M \geqslant 1$ and $n \in \mathbf{N}$. Then there exists an increasing sequence $\{\omega_n(M)\}_{n=1}^{\infty}$ of positive numbers such that, for every sequence $\{W_i\}_{i=1}^{\infty}$ of mutually commuting operators in $\mathcal{L}(\mathcal{K})$, satisfying* $\dim \mathcal{K} = n$, *$S_{\{W_i\}} \in$ (PW), and $M_{\text{pw}}(S_{\{W_i\}}) \leqslant M$, we have that $S_{\{W_i\}} \in$ (SC) and $M_{\text{cpb}}(S_{\{W_i\}}) \leqslant \omega_n(M)$.*

The principal result of this paper is that we can delete the assumption that the weights are mutually commuting.

THEOREM 1.2. *Suppose $M \geqslant 1$. Then there exists an increasing sequence $\{\omega_n(M)\}_{n=1}^{\infty}$ of positive numbers such that, for every sequence $\{W_i\}_{i=1}^{\infty}$ in $\mathcal{L}(\mathcal{K})$, where* $\dim \mathcal{K} = n$, *$S_{\{W_i\}} \in$ (PW), and $M_{\text{pw}}(S_{\{W_i\}}) \leqslant M$, we have that $S_{\{W_i\}} \in$ (SC) and $M_{\text{cpb}}(S_{\{W_i\}}) \leqslant \omega_n(M)$.*

In view of [5], Theorem 4.5, we have an immediate corollary:

THEOREM 1.3. *Let $\mathcal{H}$ be a Hilbert space and let $\{W_i\}_{i=1}^{\infty}$ be a sequence of n-normal operators in $\mathcal{L}(\mathcal{H}^{(n)})$ where $W_i = (N_{jk}^{(i)})$ and $N_{jk}^{(i)}$ commutes with $N_{rs}^{(l)}$ for all positive integers i, l and for all integers j, k, r, s such that $1 \leqslant j, k, r, s \leqslant n$. Then $S_{\{W_i\}}$ is power bounded if and only if it is similar to a contraction.*

2. PRELIMINARIES

For use throughout the paper, we introduce the following notation and terminology. We write $\mathbb{C}$ for the complex plane, $\mathbb{D}$ for the open unit disc in $\mathbb{C}$, and $\mathbb{T}$ for $\partial\mathbb{D}$. As usual, we write $\mathbb{N}$ for the set of positive integers, $\mathbb{N}_0$ for the set of nonnegative integers, and $\mathbb{Z}$ for the set of all integers. If $n \in \mathbb{N} \cup \{\aleph_0\}$ and $\mathcal{K}$ is any complex Hilbert space, we write $\mathcal{K}^{(n)}$ for the (orthogonal) direct sum of n copies of $\mathcal{K}$.

If $\mathcal{K}$ is any complex Hilbert space of dimension at most $\aleph_0$ and $\{D_n\}_{n=1}^{\infty}$ is any bounded sequence from $\mathcal{L}(\mathcal{K})$, we denote by $\mathrm{Diag}(D_1, D_2, \ldots)$ the operator in $\mathcal{L}(\mathcal{K}^{(\aleph_0)})$ satisfying

$$\mathrm{Diag}(D_1, D_2, \ldots)(k_1, k_2, \ldots) = (D_1 k_1, D_2 k_2, \ldots)$$

for all vectors $(k_1, k_2, \ldots)$ in $\mathcal{K}^{(\aleph_0)}$. In the special case in which $D_n = D$ for all $n \in \mathbb{N}$, we write $\mathrm{Diag}\,(D_1, D_2, \ldots)$ simply as $\mathrm{Diag}(\{D\})$. (Of course, $\mathrm{Diag}\,(D_1, D_2, \ldots)$ is also the direct sum $\bigoplus\limits_{n=1}^{\infty} D_n$.) Furthermore, if $\{W_n\}_{n=1}^{\infty}$ is any bounded sequence from $\mathcal{L}(\mathcal{K})$, we denote by $S_{\{W_n\}}$ the operator in $\mathcal{L}(\mathcal{K}^{(\aleph_0)})$ satisfying

$$S_{\{W_n\}}\ (k_1, k_2, \ldots, k_n, \ldots) = (0, W_1 k_1, W_2 k_2, \ldots, W_n k_n, \ldots), \tag{2.1}$$

for all vectors $(k_1, k_2, \ldots)$ in $\mathcal{K}^{(\aleph_0)}$. (In other words, $S_{\{W_n\}}$ is the unilateral operator-weighted shift with weight sequence $\{W_n\}$.) If all the weights in (2.1) coincide with one weight W, we shall denote $S_{\{W_n\}}$ simply as $S_{\{W\}}$. Clearly $S_{\{W\}}$ is unitarily equivalent to the tensor product $S \otimes W$ acting on $\mathcal{H} \otimes \mathcal{K}$ where S is a unilateral shift in $\mathcal{L}(\mathcal{H})$ satisfying $Se_n = e_{n+1}$, $n \in \mathbb{N}$, for some orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$. Finally, associated with each ordered orthonormal basis $\mathfrak{X}$ of $\mathcal{K}$ are collections $\mathcal{U}(\mathcal{K}, \mathfrak{X})$ and $\mathcal{V}(\mathcal{K}, \mathfrak{X})$ of operators in $\mathcal{L}(\mathcal{K})$ defined as follows: $T \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ if the matrix of T with respect to $\mathfrak{X}$ is in upper triangular form, and $T \in \mathcal{V}(\mathcal{K}, \mathfrak{X})$ if $T \in \mathcal{U}(\mathcal{K}, \mathfrak{X})$ and its matrix (t_{ij}) with respect to the basis $\mathfrak{X}$ satisfies $0 \leqslant t_{ii} \leqslant 1$ for every i. In this paper we will be studying operator-weighted shifts whose weights are $n \times n$ matrices. Although we aim for the utmost generality, we shall confine our attention to a subclass of $n \times n$ matrices, and, therefore, to a subclass of operator-weighted shifts.

DEFINITION 2.1. Let $n \in \mathbf{N}$, let $\mathcal{K}$ be a Hilbert space of dimension n, and let $\mathfrak{X}$ be an ordered orthonormal basis of $\mathcal{K}$. Let $0 < \varepsilon < 1$. An operator $T \in \mathcal{L}(\mathcal{K})$ is said to be in *ε-canonical form* relative to the basis $\mathfrak{X}$ (notation: $T \in \mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$) if $T \in \mathcal{V}(\mathcal{K}, \mathfrak{X})$ and if the matrix (t_{ij}) of T with respect to the basis $\mathfrak{X}$ has the property that if $t_{ij} \neq 0$ for some $1 \leqslant i < j \leqslant n$ then $t_{ii} < \varepsilon$ and $t_{jj} < \varepsilon$. If $\{W_i\}_{i \in \mathbf{N}}$ is a bounded sequence in $\mathcal{L}(\mathcal{K})$ then the operator $S_{\{W_i\}}$, acting on $\mathcal{K}^{(\aleph_0)}$, is said to be in the *ε-canonical form* relative to the basis $\mathfrak{X}$ if for every $i \in \mathbf{N}$, W_i belongs to $\mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$, and in this case we write $S_{\{W_i\}} \in \mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$.

The significance of the subclass above comes from the following lemma.

LEMMA 2.2. *Let n, $\mathcal{K}$, and $\mathcal{H}$ be as above. Suppose $0 < \varepsilon \leqslant 1/2$, $M \geqslant 1$, and let $S_{\{W_i\}}$ be a power bounded operator-weighted shift with power bound M, that is in $\mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$. Then there exists an invertible operator $X \in \mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X^{-1} S_{\{W_i\}} X\| \leqslant 1$ and $\|X\| \leqslant 1$, $\|X^{-1}\| = (2M/\varepsilon)^{n-1}$.*

Proof. We define an operator L in $\mathcal{L}(\mathcal{K})$ as

$$L = \operatorname{Diag}\left(1, \frac{\varepsilon}{2M}, \left(\frac{\varepsilon}{2M}\right)^2, \dots, \left(\frac{\varepsilon}{2M}\right)^{n-1}\right),$$

and an operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ as $X = \bigoplus_{i \in \mathbf{N}} L_i$ where $L_i = L$ for all $i \in \mathbf{N}$. It is easy to see that $\|X\| = 1$ and $\|X^{-1}\| = (2M/\varepsilon)^{n-1}$. Furthermore, $X^{-1} S_{\{W_i\}} X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W_i} = L^{-1} W_i L$ for every $i \in \mathbf{N}$. Thus it suffices to show that $\|L^{-1} W_i L\| \leqslant 1$ for every $i \in \mathbf{N}$. We fix $i \in \mathbf{N}$, and let (a_{kj}) be the matrix of W_i in the basis $\mathfrak{X}$. Then the matrix of $L^{-1} W_i L$ is

$$\left(a_{kj} \left(\frac{\varepsilon}{2M}\right)^{j-k}\right)$$

and it is easy to see that $L^{-1} W_i L \in \mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$. Let $J_+ = \{i : 1 \leqslant i \leqslant n \text{ and } a_{ii} \geqslant \varepsilon\}$ and $J_- = \{i : 1 \leqslant i \leqslant n \text{ and } a_{ii} < \varepsilon\}$. Using the notation $\mathfrak{X} = \{x_1, \dots, x_n\}$ we see that the subspace $\mathcal{H}_+ = \bigvee_{i \in J_+} x_i$ is reducing for $L^{-1} W_i L$ and since $L^{-1} W_i L | \mathcal{H}_+$ is clearly a contraction, we concentrate on $L^{-1} W_i L | \mathcal{H}_-$, where $\mathcal{H}_- = \bigvee_{i \in J_-} x_i$. Of course, if J_- is empty there is nothing to prove, so we consider the nontrivial case when J_- is nonempty.

We make an easy observation that the norm of $L^{-1} W_i L | \mathcal{H}_-$ is the same as the norm of the matrix A obtained from $L^{-1} W_i L$ by substituting a_{ii} by 0, for

$i \in J_+$. The norm of A is clearly dominated by the sum of the norms of its diagonal and its superdiagonals. In other words

$$\|A\| \leqslant \sum_{k=1}^{n} \|A_k\|,$$

where, for every $1 \leqslant k \leqslant n$, $A_k = (\widetilde{a}_{pq})$, with $\widetilde{a}_{pq} = a_{pq}(\varepsilon/2M)^{q-p}$ if $q-p = k-1$, and $\widetilde{a}_{pq} = 0$ otherwise. Using the obvious fact that $|a_{pq}| \leqslant M$ for $p \neq q$, and $|a_{pp}| < \varepsilon$, $1 \leqslant p \leqslant n$, we have that

$$\|A_k\| \leqslant M\left(\frac{\varepsilon}{2M}\right)^{k-1}$$

for $2 \leqslant k \leqslant n$, and $\|A_1\| \leqslant \varepsilon$, so that

$$\|A\| \leqslant \varepsilon + \sum_{k=2}^{n} M\left(\frac{\varepsilon}{2M}\right)^{k-1} = \varepsilon + M\frac{\varepsilon}{2M}\,\frac{1-\left(\frac{\varepsilon}{2M}\right)^{n-1}}{1-\left(\frac{\varepsilon}{2M}\right)}$$

$$< \varepsilon + \frac{\varepsilon}{2}\,\frac{1}{1-\left(\frac{\varepsilon}{2M}\right)} < \varepsilon + \frac{\varepsilon}{2}\,\frac{1}{1-\frac{1}{2}} = 2\varepsilon \leqslant 1.$$

This completes the proof of this lemma. ∎

REMARK 2.3. We note that (with the notation above) if (a_{ij}) is the matrix of an operator $T \in \mathcal{V}(\mathcal{K}, \mathfrak{X})$ then T is in ε-canonical form if and only if for every $1 \leqslant i < j \leqslant n$ the 2×2 matrix

$$\begin{bmatrix} a_{ii} & a_{ij} \\ 0 & a_{jj} \end{bmatrix}$$

is in ε-canonical form. Accordingly, an operator-weighted shift $S_{\{W_i\}}$, where $W_i = (w_{jk}^{(i)})$ belongs to $\mathcal{V}(\mathcal{K}, \mathfrak{X})$ for every $i \in \mathbb{N}$, is in ε-canonical form if and only if, for every $1 \leqslant j < k \leqslant n$, the weighted shift with weight sequence

$$\left\{\begin{bmatrix} w_{jj}^{(i)} & w_{jk}^{(i)} \\ 0 & w_{kk}^{(i)} \end{bmatrix}\right\}_{i \in \mathbb{N}}$$

is in ε-canonical form.

REMARK 2.4. It is easy to see that if $n \in \mathbb{N}$, $\mathcal{K}$ is a Hilbert space of dimension n, $\mathfrak{X}$ is an ordered orthonormal basis of $\mathcal{K}$, and $\{W_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathcal{L}(\mathcal{K})$ such that for every $i \in \mathbb{N}$, the matrix of W_i in $\mathfrak{X}$ is $(w_{jk}^{(i)})_{j,k=1}^{n}$, then the operator $S_{\{W_i\}}$ is unitarily equivalent to an $n \times n$ operator matrix $(S_{jk})_{j,k=1}^{n}$ where, for every $1 \leqslant j, k \leqslant n$, S_{jk} is a (scalar) weighted shift with weight sequence $\{w_{jk}^{(i)}\}_{i\in\mathbb{N}}$.

Using this remark we can extend Definition 2.1 to some $n \times n$ operator matrices.

DEFINITION 2.5. If $(S_{jk})_{j,k=1}^{n}$ and $S_{\{W_i\}}$ are as in Remark 2.4, we say that $(S_{jk})_{j,k=1}^{n}$ *is in ε-canonical form*, and we write $(S_{jk}) \in \mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$, if and only if $S_{\{W_i\}}$ has the same property.

It is obvious that if $(S_{jk}) \in \mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$, then (S_{jk}) is an $n \times n$ upper triangular matrix, where, for every j, k, $1 \leqslant j, k \leqslant n$, S_{jk} is a weighted shift with weights in $[0,1]$. The following lemma is another easy consequence of the previous remarks and definitions.

LEMMA 2.6. *Let $n \in \mathbb{N}$, let $\mathcal{G}$ be a Hilbert space, and let (S_{jk}) be an upper triangular $n \times n$ operator matrix acting on $\mathcal{G}^{(n)}$, such that, for every $1 \leqslant j \leqslant k \leqslant n$, S_{jk} is a weighted shift with nonnegative weights not exceeding 1, relative to the same decomposition of $\mathcal{G}$. Then (S_{jk}) is in ε-canonical form if and only if every 2×2 operator matrix*

$$\begin{bmatrix} S_{jj} & S_{jk} \\ 0 & S_{kk} \end{bmatrix}, \quad 1 \leqslant j < k \leqslant n,$$

is in ε-canonical form.

Lemma 2.2 has shown that in order to prove Theorem 1.2 it suffices to establish a weaker conclusion that $T \in \mathcal{C}(\mathcal{K}, \mathfrak{X}, \varepsilon)$. The next lemma shows that we can also make a weaker hypothesis.

LEMMA 2.7. *Let $M \geqslant 1$, let $n \in \mathbb{N}$, let $\mathcal{K}$ be a Hilbert space of dimension n, let $\mathfrak{X}$ be an orthonormal basis of $\mathcal{K}$, and let $\{W_i\}_{i \in \mathbb{N}}$ be a bounded sequence in $\mathcal{L}(\mathcal{K})$ such that $S_{\{W_i\}} \in$(PW) and $M_{\mathrm{pw}}(S_{\{W_i\}}) = M$. Then there exists an invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $X^{-1} S_{\{W_i\}} X = S_{\{\widetilde{W}_i\}}$, $\|X\| \cdot \|X^{-1}\| \leqslant M$ and, for every $i \in \mathbb{N}$, $\widetilde{W}_i \in \mathcal{V}(\mathcal{K}, \mathfrak{X})$.*

Proof. One knows that each operator $T \in \mathcal{L}(\mathcal{K})$ can be uniquely decomposed as $T = QR$ where Q is a unitary operator and R has an upper triangular matrix relative to $\mathfrak{X}$. We define, by induction, a sequence $\{U_i\}_{i \in \mathbb{N}}$ of unitary operators in $\mathcal{L}(\mathcal{K})$. Let $U_1 = I$. If U_n has been selected we define U_{n+1} to be the unitary factor of $W_n U_n$ in the decomposition above. Let U be a unitary operator in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ defined as $U = \mathrm{Diag}(U_1, U_2, \ldots)$. Then $U^* S_{\{W_i\}} U = S_{\{\widetilde{W}_i\}}$ where $\widetilde{W}_i = U_{i+1}^* W_i U_i$. Since, by definition, $W_i U_i = U_{i+1} R_{i+1}$, we have that $\widetilde{W}_i = R_{i+1}$, i.e., $\widetilde{W}_i$ is represented by an upper triangular matrix, for every $i \in \mathbb{N}$. Once again using the unitary equivalence set forth in Remark 2.4, we see that $S_{\{\widetilde{W}_i\}}$ (and, hence, $S_{\{W_i\}}$) is unitarily equivalent to an $n \times n$ upper triangular operator matrix (S_{ij}). We construct the operators $\{Y_i\}_{i=1}^{n}$ in such a way that $Y_i^{-1} S_{ii} Y_i$ is a weighted shift whose weights are nonnegative numbers not greater than 1 and $\|Y_i\| \cdot \|Y_i^{-1}\| \leqslant M$ (see [7] for details). If we define $Y = \mathrm{Diag}(Y_1, \ldots, Y_n)$, we have that $\|Y\| \cdot \|Y^{-1}\| \leqslant M$. Since $Y^{-1}(S_{ij})Y$ is unitarily equivalent to an operator-weighted shift $S_{\{\widehat{W}_i\}}$, where $\widehat{W}_i \in \mathcal{V}(\mathcal{K}, \mathfrak{X})$ for every $i \in \mathbf{N}$, the proof is complete. ∎

Thus in order to prove Theorem 1.2 it suffices to exhibit a similarity $X \in \mathcal{L}(\mathcal{G}^{(n)})$ such that $X^{-1}(S_{jk})X = (\widetilde{S}_{jk})$ and $(\widetilde{S}_{jk})$ has one of the two equivalent properties stated in Lemma 2.6. More precisely, we shall prove

THEOREM 2.8. *Let $n \in \mathbf{N}$, $n \geqslant 2$, let $\mathcal{G}$ be a Hilbert space, and suppose $M \geqslant 1$. Then for every ε, $0 < \varepsilon < 1/2$, and for every δ, $1 < \delta < 1/\varepsilon$, there exists a positive number $\omega = \omega(M, \varepsilon, \delta)$ such that for every upper triangular $n \times n$ operator matrix (S_{jk}) in $\mathcal{L}(\mathcal{G}^{(n)})$, where, for every $1 \leqslant j \leqslant k \leqslant n$, S_{jk} is a weighted shift, and, for every $1 \leqslant j \leqslant n$, the weights of S_{jj} belong to $[0,1]$ (relative to the same basis of $\mathcal{G}$), there exists an invertible operator X in $\mathcal{L}(\mathcal{G}^{(n)})$ satisfying $\|X\|, \|X^{-1}\| \leqslant \omega$, $X^{-1}(S_{jk})X = (\widetilde{S}_{jk})$, and $(\widetilde{S}_{jk})$ is in ε_1-canonical form, where $\varepsilon_1 = \max\{1/\delta, \delta\varepsilon\}$.*

Clearly, Theorem 1.2 follows from Theorem 2.8, Lemma 2.2, Lemma 2.7, and the observation that, if $n = 1$, the result follows from Proposition 1.1. Thus, we concentrate on Theorem 2.8, but we shall postpone the proof of this theorem until Section 6, because our first goal is to establish a slightly stronger assertion in the special case $n = 2$.

PROPOSITION 2.9. *Let $\mathcal{K}$ be a Hilbert space of dimension 2, let $\mathfrak{X}$ be an orthonormal basis of $\mathcal{K}$, and suppose $M \geqslant 1$. Then for every ε, $0 < \varepsilon < 1/2$, and for every δ, $1 < \delta < 1/\varepsilon$, there exists a positive number $\omega = \omega(M, \varepsilon, \delta)$ such that for every sequence $\{W_i\}_{i=1}^{\infty}$ in $\mathcal{V}(\mathcal{K}, \mathfrak{X})$ satisfying $S_{\{W_i\}} \in (\mathrm{PW})$, and $M_{\mathrm{pw}}(S_{\{W_i\}}) \leqslant M$, there exists a sequence $\{X_i\}_{i \in \mathbf{N}}$ in $\mathcal{U}(\mathcal{K}, \mathfrak{X})$ such that the operator $X = \mathrm{Diag}(X_i)$ satisfies $\|X\|, \|X^{-1}\| \leqslant \omega$, $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, and, for every $i \in \mathbf{N}$, $\widetilde{W}_i$ is in the ε_1-canonical form , where $\varepsilon_1 = \max\{1/\delta, \delta\varepsilon\}$. Furthermore, with the notation $W_i = (w_{jk}^{(i)})_{j,k=1}^2$, $\widetilde{W}_i = (\widetilde{w}_{jk}^{(i)})_{j,k=1}^2$, for $i \in \mathbf{N}$, we have that $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, for every j, $1 \leqslant j \leqslant n$, and every $i \in \mathbf{N}$, such that $w_{jj}^{(i)} \leqslant \varepsilon$.*

3. PROOF OF PROPOSITION 2.9

Let ε be an arbitrary positive number less than $1/2$, let $1 < \delta < 1/\varepsilon$, and let $M \geqslant 1$. We define the following three subclasses of matrices

$$\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \tag{3.1}$$

in $\mathcal{V}(\mathcal{K}, \mathfrak{X})$. A matrix of the form (3.1) is

(1) Type I(M, ε) if $0 \leqslant \alpha \leqslant 1$, $\varepsilon < \gamma \leqslant 1$, $|\beta| \leqslant M$;

(2) Type II(M, ε) if $0 \leqslant \alpha, \gamma \leqslant \varepsilon$, $|\beta| \leqslant M$;

(3) Type III(M, ε) if $0 \leqslant \gamma \leqslant \varepsilon < \alpha \leqslant 1$, $|\beta| \leqslant M$.

In the case when it is clear from the context what values of M and ε are being used, we shall use the notation type I for type I(M, ε), and similarly for types II and III.

Let $\{W_i\}_{i \in \mathbf{N}}$ be as in the statement of the proposition and let $T = S_{\{W_i\}}$. It is easy to see that, for every $i \in \mathbf{N}$, W_i must be in one of the classes above. In order to complete the proof we shall need several technical propositions which we state here. The proofs of these assertions are postponed until Section 5.

PROPOSITION 3.1. *Let $\mathcal{K}$ be a Hilbert space of dimension 2, let $\mathfrak{X}$ be an orthonormal basis of $\mathcal{K}$, and suppose $M \geqslant 1$. Then for every ε, $0 < \varepsilon < 1/2$, and for every δ, $1 < \delta < 1/\varepsilon$, there exists a positive number $\omega = \omega(M, \varepsilon, \delta)$ such that for every sequence $\{W_i\}_{i=1}^{\infty}$ in $\mathcal{V}(\mathcal{K}, \mathfrak{X})$ that contains infinitely many operators of type I $(= \mathrm{I}(M, \varepsilon))$ and that satisfies $S_{\{W_i\}} \in (\mathrm{PW})$, and $M_{\mathrm{pw}}(S_{\{W_i\}}) \leqslant M$, there exists an operator $X \in \mathcal{L}(\mathcal{K}^{(\aleph_0)})$ such that $\|X\|, \|X^{-1}\| \leqslant \omega$, $X^{-1} S_{\{W_i\}} X = S_{\{\widetilde{W}_i\}}$, and, for every $i \in \mathbf{N}$, $\widetilde{W}_i$ is in the ε_1-canonical form , where $\varepsilon_1 = \max\{1/\delta, \delta\varepsilon\}$. Furthermore, with the notation $W_i = (w_{jk}^{(i)})_{j,k=1}^2$, $\widetilde{W}_i = (\widetilde{w}_{jk}^{(i)})_{j,k=1}^2$, for $i \in \mathbf{N}$, we have that $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, for every j, $1 \leqslant j \leqslant n$, and every $i \in \mathbf{N}$, such that $w_{jj}^{(i)} \leqslant \varepsilon$.*

An easy consequence of this proposition is that if T has infinitely many weights of type I then T satisfies the conclusions of Proposition 2.9. Thus there remains to investigate the case when T has only finitely many weights of type I, or none at all. The next proposition shows that another restriction is available.

PROPOSITION 3.2. *Let $\mathcal{K}$ be a Hilbert space of dimension 2, let $\mathfrak{X}$ be an orthonormal basis of $\mathcal{K}$, and suppose $M \geqslant 1$. Then for every ε, $0 < \varepsilon < 1/2$, and for every δ, $1 < \delta < 1/\varepsilon$, there exists a positive number $\omega = \omega(M, \varepsilon, \delta)$ such that for every sequence $\{W_i\}_{i=1}^{\infty}$ in $\mathcal{V}(\mathcal{K}, \mathfrak{X})$ that contains infinitely many operators of type II, and finitely many or no weights of type I and that satisfies $S_{\{W_i\}} \in (\mathrm{PW})$, and $M_{\mathrm{pw}}(S_{\{W_i\}}) \leqslant M$, there exists an operator $X \in \mathcal{L}(\mathcal{K}^{(\aleph_0)})$*

such that $\|X\|, \|X^{-1}\| \leqslant \omega$, $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, *and, for every* $i \in \mathbb{N}$, $\widetilde{W}_i$ *is in the* ε_1*-canonical form , where* $\varepsilon_1 = \max\{1/\delta, \delta\varepsilon\}$. *Furthermore, with the notation* $W_i = (w_{jk}^{(i)})_{j,k=1}^2$, $\widetilde{W}_i = (\widetilde{w}_{jk}^{(i)})_{j,k=1}^2$, *for* $i \in \mathbb{N}$, *we have that* $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, *for every* j, $1 \leqslant j \leqslant n$, *and every* $i \in \mathbb{N}$, *such that* $w_{jj}^{(i)} \leqslant \varepsilon$.

Clearly, these two propositions imply that, unless T has only finitely many weights of type I and of type II (or none at all), T satisfies the conclusions of Proposition 2.9.

So we may suppose that there exists $N \in \mathbb{N}$ such that W_i is of type III, for every $i > N$. Again, we note that $S_{\{W_i\}}$ is unitarily equivalent to a 2×2 operator matrix

$$\begin{bmatrix} S_{\{\alpha_i\}} & S_{\{\beta_i\}} \\ 0 & S_{\{\gamma_i\}} \end{bmatrix}, \tag{3.2}$$

acting on an orthogonal direct sum of two copies of a Hilbert space $\mathcal{G}$, with $0 \leqslant \gamma_i < \varepsilon < \alpha_i \leqslant 1$ and $|\beta_i| \leqslant M$ for every $i > N$. One knows (cf. [5], Lemma 3.3 and the proof of Proposition 2.7) that there exists an operator $X \in \mathcal{L}(\mathcal{G})$ whose matrix relative to the same basis of $\mathcal{G}$ is diagonal and which satisfies $\|X\| \leqslant M$, $\|X^{-1}\| \leqslant 1/\varepsilon$, and $X^{-1}S_{\{\alpha_i\}}X = S_{\{\widetilde{\alpha}_i\}}$ where, for $i > N$, $\widetilde{\alpha}_i = 1$ or $\widetilde{\alpha}_i \leqslant \varepsilon$ and $\widetilde{\alpha}_i \leqslant \alpha_i$ whenever $\alpha_i \leqslant \varepsilon$. Then the operator $\widetilde{X} \in \mathcal{L}(\mathcal{G} \oplus \mathcal{G})$ defined as $\widetilde{X} = \mathrm{Diag}(X, I)$ has the property that

$$\widetilde{T} = \widetilde{X}^{-1} \begin{bmatrix} S_{\{\alpha_i\}} & S_{\{\beta_i\}} \\ 0 & S_{\{\gamma_i\}} \end{bmatrix} \widetilde{X} = \begin{bmatrix} S_{\{\widetilde{\alpha}_i\}} & S_{\{\widetilde{\beta}_i\}} \\ 0 & S_{\{\gamma_i\}} \end{bmatrix},$$

where $S_{\{\widetilde{\beta}_i\}} = X^{-1}S_{\{\beta_i\}}$, and it is obvious that for every $i > N$, $\beta_i = 0$ if and only if $\widetilde{\beta}_i = 0$. Of course, $\widetilde{T}$ is unitarily equivalent to $S_{\{\widetilde{W}_i\}}$ and $\widetilde{M} = M_{\mathrm{pw}}(\widetilde{T}) \leqslant M^2(1/\varepsilon)$. Furthermore, it suffices to show that $\widetilde{T}$ satisfies the conclusions of the proposition. Finally, we note that for every $i > N$, $\widetilde{W}_i$ is type III$(\widetilde{M}, \varepsilon)$ (with $\widetilde{\alpha}_i = 1$). Thus, we concentrate on the operator-weighted shift $\widetilde{T}$ with the property that, for any $i > N$, $\widetilde{W}_i$ is of type III of the form

$$\begin{bmatrix} 1 & \beta_i \\ 0 & \gamma_i \end{bmatrix} \tag{3.3}$$

with $0 \leqslant \gamma_i \leqslant \varepsilon$ and $|\beta_i| \leqslant \widetilde{M}$. The following proposition shows how to handle this situation.

PROPOSITION 3.3. *Let $\mathcal{K}$ be a Hilbert space of dimension 2, let $\mathfrak{X}$ be an orthonormal basis of $\mathcal{K}$, and suppose $M \geqslant 1$. Then for every ε, $0 < \varepsilon < 1/2$, and for every δ, $1 < \delta < 1/\varepsilon$, there exists a positive number $\omega_1 = \omega_1(M, \varepsilon, \delta)$ such that for every sequence $\{W_i\}_{i=1}^{\infty}$ in $\mathcal{V}(\mathcal{K}, \mathfrak{X})$ for which there exists a positive integer N such that, for $i \geqslant N$, W_i is type* III *of the form* (3.3), *and which satisfies $S_{\{W_i\}} \in$ (PW), and $M_{\mathrm{pw}}(S_{\{W_i\}}) \leqslant M$, there exists an operator $X \in \mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant \omega_1$, $X^{-1} S_{\{W_i\}} X = S_{\{\widetilde{W}_i\}}$, and, for every $i \in \mathbf{N}$, $\widetilde{W}_i$ is in the ε_1-canonical form , where $\varepsilon_1 = \max\{1/\delta, \delta\varepsilon\}$. Furthermore, with the notation $W_i = (w_{jk}^{(i)})_{j,k=1}^2$, $\widetilde{W}_i = (\widetilde{w}_{jk}^{(i)})_{j,k=1}^2$, for $i \in \mathbf{N}$, we have that $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, for every j, $1 \leqslant j \leqslant n$, and every $i \in \mathbf{N}$, such that $w_{jj}^{(i)} \leqslant \varepsilon$.*

Clearly, it follows from Proposition 3.3 that $\widetilde{T}$ satisfies the conclusions of Proposition 2.9 with $\omega_1(\widetilde{M})$ instead of $\omega(M)$. But that, in its turn, implies that T satisfies the conclusions of Proposition 2.9 with $\omega(M) = M(1/\varepsilon)\omega_1(\widetilde{M})$. This completes the proof of Proposition 2.9.

4. SOME USEFUL SIMILARITIES

Before we can prove Propositions 3.1–3.3, we need several lemmas. It will be helpful to introduce some notation and terminology in order to simplify the statements of these lemmas. The letter J will stand for either $\mathbf{N}$ or any initial finite segment of $\mathbf{N}$. The letter Λ (with or without an index) will denote a finite subset of $\mathbf{N}$, μ (with or without an index) will be the smallest, and ν (with or without an index) the largest element of Λ. The letter Ω (with or without an index) will denote the set of all positive integers between μ and ν that do not belong to Λ, and, finally, the symbol Ξ stands for the set of positive integers that are not in Λ or Ω. Also, we shall employ the notation $A < B$ for nonempty subsets of $\mathbf{N}$, if $\sup A < \inf B$. In the case when $A = \{a\}$ we shall write $a < B$ for $A < B$, and $B < a$ for $B < A$. Since the mentioned lemmas are needed to point out some relationship between weights of types I–III, we fix $M \geqslant 1$, $0 < \varepsilon \leqslant 1/2$, and $1 < \delta < 1/\varepsilon$, and we shall work under a hypothesis that $S_{\{W_i\}}$ is power bounded with power bound M, acting on $\mathcal{K}^{(\aleph_0)}$, where $\dim \mathcal{K} = 2$, and that there exists a fixed orthonormal basis of $\mathcal{K}$ in which every weight W_i is represented by an upper triangular matrix of one of the types I–III. We shall be employing the following terminology.

DEFINITION 4.1. Let $(\Lambda_i)_{i\in J}$ be a family of subsequences of $\mathbf{N}$. We say that (Λ_i) is *increasing* if for any $i, j \in J$, $i < j$, we have that $\Lambda_i < \Lambda_j$. If $(\Lambda_i')_{i\in J'}$ and $(\Lambda_i'')_{i\in J''}$ are two (finite or infinite) *increasing families*, we say that (Λ_i') *interlaces* (Λ_j'') if for any $i, j \in J'$ there exists $k \in J''$ such that $\Lambda_i' < \Lambda_k'' < \Lambda_j'$. If each of the two families interlaces the other one, we say that they are *interlaced*. In this situation, we extend the definition of Ω so that it includes also the set of all $k \in \mathbb{N}$ such that $\Lambda'_i < k < \Lambda''_j$ or $\Lambda''_j < k < \Lambda'_i$ for some $i \in J', j \in J''$; and $k \notin \left(\bigcup_i \Lambda'_i\right) \cup \left(\bigcup_j \Lambda''_j\right)$.

We start with a very simple assertion.

LEMMA 4.2. *Let $\Lambda = \{\mu, \nu\}$, and let W_μ, W_ν be operators of type* III *such that $\forall k \in \Omega$, W_k is of type* II. *Then there exists an invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant 1 + M$, $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_i = W_i$ for $i \in \Xi$, $\widetilde{\alpha}_i = \alpha_i$, $\widetilde{\gamma}_i = \gamma_i$, $|\widetilde{\beta}_i| \leqslant M(M+1)^2$ for every $i \in \Lambda \cup \Omega$, and $\widetilde{W}_\nu$ is diagonal.*

Proof. Let $X = \mathrm{Diag}(X_i)$ where $X_i = I$ for $i \in \Xi \cup \{\mu\}$, and

$$X_i = \begin{bmatrix} 1 & -\beta_\nu \\ 0 & 1 \end{bmatrix}, \quad \text{for } i \in \Omega \cup \{\nu\}.$$

Clearly, X is a bounded invertible operator satisfying $\|X\|, \|X^{-1}\| \leqslant 1 + M$, and the weight sequence $\{\widetilde{W}_i\}$ has desired properties. ∎

Lemma 4.2 easily generalizes to the case when Λ has more than 2 elements.

LEMMA 4.3. *Let $N \in \mathbb{N}$, let $\Lambda = \{n_i\}_{i=1}^N$, and let W_i be of type* III *for every $i \in \Lambda$. Moreover assume that W_i is of type* II *for every $i \in \Omega$. Then there exists an invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$, $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$ where $\widetilde{W}_i = W_i$ for $i \in \Xi$, $\widetilde{\alpha}_i = \alpha_i$, $\widetilde{\gamma}_i = \gamma_i$, $|\widetilde{\beta}_i| \leqslant M[1 + M/(1-\varepsilon)]^2$ for every $i \in \Lambda \cup \Omega$, and $\widetilde{W}_i$ is diagonal for $i \in \Lambda \setminus \{\mu\}$.*

Proof. Let $a_k = \sum_{j=k}^{N} \beta_j \prod_{s=k}^{j-1} \gamma_s$, for $1 \leqslant k \leqslant N$, with the understanding that $\prod_{s=k}^{k-1} \gamma_s = 1$. Let

$$A_k = \begin{bmatrix} 1 & -a_k \\ 0 & 1 \end{bmatrix}.$$

Define $X = \mathrm{Diag}(X_i)$ where $X_i = I$ for $i \in \Xi \cup \mu$, and $X_i = A_{k+1}$ for $n_k < i \leqslant n_{k+1}$, $k = 1, \ldots, N-1$. It is easy to see that $|a_k| \leqslant M/(1-\varepsilon)$, so X is a bounded invertible operator with $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$. Verification of the properties of the sequence $\{\widetilde{W}_i\}$ is left to the reader. ∎

Our next step is an easy generalization of Lemma 4.3.

LEMMA 4.4. *Let $(\Lambda_i)_{i\in J}$ be an increasing family of finite sequences in* $\mathbf{N}$. *Suppose that for any $i,j \in J$, $i < j$, and any $x \in \Lambda_i$, $y \in \Lambda_j$ we have $x < y$. Suppose that $\forall\, i \in J$, $\forall\, k \in \Lambda_i$, W_k is of type* III, *and that $\forall\, i \in J$, $\forall\, k \in \Omega_i$, W_k is of type* II. *Then there exists a bounded invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$ and such that $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$ where $\widetilde{W}_i = W_i$ for any $i \in \Xi$, $\widetilde{\alpha}_i = \alpha_i$, $\widetilde{\gamma}_i = \gamma_i$, $|\widetilde{\beta}_i| \leqslant M[1 + M/(1-\varepsilon)]^2$ for every $i \notin \Xi$, and $\widetilde{W}_i$ is diagonal for any $j \in \mathbf{N}$ and any $i \in \Lambda_j \setminus \{\mu_j\}$.*

Our next task is to establish the analogues of lemmas 4.2–4.4 for the operators of type I. Again we start with the simplest case.

LEMMA 4.5. *Let $\Lambda = \{\mu, \nu\}$, and let W_μ, W_ν be operators of type* I *such that $\forall\, k \in \Omega$, W_k is of type* II. *Then there exists an invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant 1 + M$, $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_i = W_i$ for $i \in \Xi$, $\widetilde{\alpha}_i = \alpha_i$, $\widetilde{\gamma}_i = \gamma_i$, $|\widetilde{\beta}_i| \leqslant M(1+M)^2$ for every $i \in \Lambda \cup \Omega$, and $\widetilde{W}_\mu$ is diagonal.*

Proof. Let $X = \mathrm{Diag}(X_i)$ where $X_i = I$ for $i \in \Xi \cup \{\mu\}$, and

$$X_i = \begin{bmatrix} 1 & \beta_\mu \\ 0 & 1 \end{bmatrix}, \quad \text{for } i \in \Omega \cup \{\nu\}.$$

It is easy to verify that X is invertible with $\|X\|, \|X^{-1}\| \leqslant 1 + M$ and that the weight sequence $\{\widetilde{W}_i\}$ has desired properties. ∎

The case when Λ has more than two elements is slightly different from the situation in Lemma 4.3.

LEMMA 4.6. *Let $\Lambda = \{n_k\}_{k\in J}$, and let W_i be of type* I *for every $i \in \Lambda$. Moreover, assume that W_i is of type* II *for every $i \in \Omega$. Then there exists an invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1+\varepsilon)$ and such that $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_i = W_i$ for $i \in \Xi$, $\widetilde{\alpha}_i = \alpha_i$, $\widetilde{\gamma}_i = \gamma_i$, $|\widetilde{\beta}_i| \leqslant M[1 + M/(1-\varepsilon)]^2$ for every $i \in \Lambda \cup \Omega$, and $\widetilde{W}_i$ is diagonal for $i \in \Lambda$ except for $i = \nu$ in the case when J is finite.*

Proof. Let $b_k = \sum\limits_{j=1}^{k}\Big(\prod\limits_{s=j+1}^{k}\alpha_s\Big)\beta_j$, for $1 \leqslant k < N+1$, with the understanding that $\prod\limits_{s=k+1}^{k}\alpha_s = 1$. Let

$$B_k = \begin{bmatrix} 1 & b_k \\ 0 & 1 \end{bmatrix}.$$

Define $X = \mathrm{Diag}(X_i)$ where $X_i = I$ for $i \in \Xi \cup \{\mu\}$, and $X_i = B_k$ for $n_k < i \leqslant n_{k+1}$, $k < N$. It is easy to see that $|b_k| \leqslant M/(1-\varepsilon)$, so X is a bounded invertible operator with $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$. Verification of the properties of the sequence $\{\widetilde{W}_i\}$ is left to the reader. ∎

As before, the case when there is more than one sequence Λ is an easy extension of the previous lemmas.

LEMMA 4.7. *Let* $(\Lambda_i)_{i\in J}$ *be an increasing family of finite sequences in* $\mathbb{N}$. *Suppose that* $\forall\, i \in \mathbb{N}$, $\forall\, k \in \Lambda_i$, W_k *is of type* I, *and that* $\forall\, i \in J$, $\forall\, k \in \Omega_i$, W_k *is of type* II. *Then there exists a bounded invertible operator* X *in* $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ *such that* $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$ *and such that* $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, *where* $\widetilde{W}_i = W_i$ *for any* $i \in \Xi$, $\widetilde{\alpha}_i = \alpha_i$, $\widetilde{\gamma}_i = \gamma_i$, $|\widetilde{\beta}_i| \leqslant M[1 + M/(1-\varepsilon)]^2$ *for every* $i \notin \Xi$, *and* $\widetilde{W}_i$ *is diagonal for any* $j \in J$ *and any* $i \in \Lambda_j \setminus \{\nu_j\}$.

In the previous lemmas we were concerned with operators of type III or type I. Now we can work on both types simultaneously.

LEMMA 4.8. *Let* $\Lambda_1 = \{n_i\}_{i\in J}$ *and* $\Lambda_2 = \{m_i\}_{i\in J}$ *be increasing sequences of positive integers such that for any* $i \in J$,

$$n_i < m_i, \tag{4.1}$$

and if $i + 1 \in J$ *then*

$$m_i < n_{i+1}. \tag{4.2}$$

Let W_i *be of type* I *(resp. type* III*) for every* $i \in \Lambda_1$ *(resp.* $i \in \Lambda_2$*), and suppose that* $\forall\, i \in J$, *and* $\forall\, k$, $n_i < k < m_i$, W_k *is of type* II. *Let* δ *be a positive number,* $1 < \delta < 1/\varepsilon$. *Then there exists an invertible operator* X *in* $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ *such that* $\|X\|, \|X^{-1}\| \leqslant \delta$ *and such that* $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, *where* $\widetilde{W}_k = W_k$ *whenever* $m_i < k < n_{i+1}$, $\widetilde{\alpha}_k = \alpha_k$, $\widetilde{\gamma}_k = \gamma_k$, $|\widetilde{\beta}_k| \leqslant \delta^2 M$ *whenever* $n_i < k < m_i$, *and* $\widetilde{W}_k$ *is of type* $\mathrm{II}(\delta^2 M, \max(1/\delta, \delta\varepsilon))$ *with* $\widetilde{w}_{jj}^{(k)} \leqslant \delta w_{jj}^{(k)}$, $1 \leqslant j \leqslant 2$, *for any* $k \in \Lambda_1 \cup \Lambda_2$.

Proof. Let $X = \mathrm{Diag}(X_k)$ where $X_k = I$ for $k \leqslant n_1$ and whenever $m_i < k \leqslant n_{i+1}$, and

$$X_k = \begin{bmatrix} \frac{1}{\delta} & 0 \\ 0 & \delta \end{bmatrix}$$

otherwise. Once again, it is easy to see that X is a bounded invertible operator on $\mathcal{K}^{(\aleph_0)}$ satisfying $\|X\|, \|X^{-1}\| \leqslant \delta$, and that the sequence $\{\widetilde{W}_k\}$ has the required properties. ∎

In Lemma 4.8 we have assumed that the first weight that is not of type II must be of type I. However, we can easily dispense of this assumption.

LEMMA 4.9. *Lemma 4.8 remains valid if the conditions (4.1) and (4.2) are replaced by the conditions*

$$m_i < n_i, \tag{4.3}$$

and

$$n_i < m_{i+1}. \tag{4.4}$$

Proof. If we define the operator T' as the operator-weighted shift with weight sequence $\{W_i\}_{i=0}^{\infty}$, where

$$W_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.5}$$

then, since W_0 is of type I, T' satisfies the hypotheses of Lemma 4.8. In other words, there exists an invertible operator X' such that $X'^{-1}TX'$ is a weighted shift whose weights $\{\widetilde{W}_i\}_{i=0}^{\infty}$ satisfy the conclusions of Lemma 4.8. Moreover, it follows from the proof of Lemma 4.8 that X' is a diagonal operator, with some sequence $\{X_i\}_{i=0}^{\infty}$ on the diagonal. It is easy to see that if X is a diagonal operator with the sequence $\{X_i\}_{i\in\mathbf{N}}$ on the diagonal, then $X^{-1}TX$ is an operator-weighted shift satisfying the conclusions of Lemma 4.8. ∎

The following lemma is an easy consequence of the preceding results.

LEMMA 4.10. *Let $(\Lambda_i')_{i\in J'}$ and $(\Lambda_i'')_{i\in J''}$ be two interlaced families of subsequences of $\mathbf{N}$. Suppose that for every $i \in J'$ and every $k \in \Lambda_i'$, W_k is of type* I, *that for any $i \in J''$ and any $k \in \Lambda_i''$, W_k is type* III, *that for any $k \in \Omega$, W_k is type* II. *Then there exists a bounded invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that*

$$\|X\|, \|X^{-1}\| \leqslant \left[1 + \frac{M}{1-\varepsilon}\right]\left[1 + \frac{M}{1-\varepsilon}\left(1 + \frac{M}{1-\varepsilon}\right)^2\right]\delta, \tag{4.6}$$

and such that $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_i = W_i$ for any $i \in \Xi$, and $\widetilde{W}_i$ is either a diagonal contraction or it is of type $\mathrm{II}(\delta^2 M, \max(1/\delta, \delta\varepsilon))$ *otherwise. Furthermore, $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbf{N}$.*

Proof. In the notation of Lemma 4.4, there exists a similarity X such that $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$ and such that $T_1 = X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$ where each weight of type II becomes type $\mathrm{II}(M[1 + M/(1-\varepsilon)]^2, \varepsilon)$, each weight of type I remains unchanged and each weight of type III is diagonal (hence a contraction) except for W_{μ_i}, $i \in \mathbf{N}$, which is of type $\mathrm{III}(M[1+M/(1-\varepsilon)]^2, \varepsilon)$. Also $\widetilde{w}_{jj}^{(i)} = w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbf{N}$. Obviously, $M_1 = M_{\mathrm{pw}}(T_1) \leqslant M[1+M/(1-\varepsilon)]^2$. To this operator and its power bound M_1 we apply Lemma 4.7, thereby obtaining a similarity Y such that $\|Y\|, \|Y^{-1}\| \leqslant 1 + M_1/(1-\varepsilon)$ and such that $T_2 = Y^{-1} S_{\{\widetilde{W}_i\}} Y = S_{\{\widehat{W}_i\}}$ where each weight of type II becomes type II$(M_1[1+M_1/(1-\varepsilon)]^2, \varepsilon)$, each weight of type III remained unchanged, and each weight of type I is diagonal (hence a contraction), except for W_{ν_i}, $i \in \mathbb{N}$. Furthermore, $\widehat{w}_{jj}^{(i)} = \widetilde{w}_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbb{N}$. Now Lemma 4.8 or Lemma 4.9 yields another similarity Z such that $\|Z\|, \|Z^{-1}\| \leqslant \delta$ and such that $T_3 = Z^{-1} T_2 Z = S_{\{\check{W}_i\}}$ is an operator-weighted shift whose each weight is either a diagonal contraction or of type II$(\delta^2 M_2, \max(1/\delta, \delta\varepsilon))$ with $\check{w}_{jj}^{(i)} \leqslant \delta \widehat{w}_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbb{N}$. Thus $(XYZ)^{-1}T(XYZ)$ is a weighted shift with the required properties and, clearly,

$$\begin{aligned}\|XYZ\|, \|(XYZ)^{-1}\| &\leqslant \left[1 + \frac{M}{1-\varepsilon}\right] \left[1 + \frac{M_1}{1-\varepsilon}\right] \delta \\ &\leqslant \left[1 + \frac{M}{1-\varepsilon}\right] \left[1 + \frac{M}{1-\varepsilon} \left(1 + \frac{M}{1-\varepsilon}\right)^2\right] \delta,\end{aligned}$$

which completes the proof of this lemma. ∎

In the previous lemmas we have studied the interplay between weights of types III and I. Now we turn our attention to weights of types II and III.

LEMMA 4.11. *Let $\Lambda_1 = \{n_i\}_{i \in J}$ and $\Lambda_2 = \{m_i\}_{i \in J}$ be two interlaced families of one-element subsets of* $\mathbb{N}$. *Let W_i be of type* II *(resp. type* III*) for every $i \in \Lambda_1$ (resp. $i \in \Lambda_2$), and suppose that $\forall\, k \in \Omega$, W_k is either diagonal or of type* II. *Then there exists an invertible operator X in $\mathcal{L}(\mathcal{K}^{(\aleph_0)})$ such that $\|X\| = 1$, $\|X^{-1}\| = \delta$ and such that $X^{-1} S_{\{W_i\}} X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_k = W_k$ whenever $k \in \mathfrak{X}i$, and $\widetilde{W}_k$ is either diagonal or of type* II$(\delta M, \max(1/\delta, \delta\varepsilon))$ *with $\widetilde{w}_{jj}^{(k)} \leqslant \delta w_{jj}^{(k)}$, $1 \leqslant j \leqslant 2$, for any $k \notin \Xi$.*

Proof. Let $X = \operatorname{Diag}(X_k)$ where $X_k = I$ for $k \leqslant n_1$ and whenever $m_i < k \leqslant n_{i+1}$, and

$$X_k = \begin{bmatrix} 1/\delta & 0 \\ 0 & 1 \end{bmatrix}$$

otherwise. Once again, we leave the details of the proof to the reader. ∎

Again, the following lemma is an easy corollary of the previous results.

LEMMA 4.12. *Let $(\Lambda_i')_{i \in J'}$ and $(\Lambda_i'')_{i \in J''}$ be two interlaced families of subsequences of* $\mathbb{N}$ *such that Ω is empty. Suppose that for every $i \in J'$ and every*

$k \in \Lambda_i'$, W_k is of type III, *and that for any $i \in J''$ and any $k \in \Lambda_i''$, W_k is of type* II. *Then there exists a bounded invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that*

$$\|X\|, \|X^{-1}\| \leqslant \left(1 + \frac{M}{1-\varepsilon}\right)\delta,$$

and such that $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_i = W_i$ for any $i \in \Xi$, and $\widetilde{W}_i$ is either a diagonal operator of type III *(hence a contraction), or it is of type* II$(\delta M, \varepsilon_1)$, *otherwise. Furthermore, $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbf{N}$.*

Proof. We apply Lemma 4.4 to the family $(\Lambda_i')_{i \in J'}$ to obtain a bounded invertible operator X in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|X\|, \|X^{-1}\| \leqslant 1 + M/(1-\varepsilon)$ and such that $X^{-1}S_{\{W_i\}}X = S_{\{\widetilde{W}_i\}}$, where $\widetilde{W}_i = W_i$ for any $i \notin \bigcup_{j \in J'} \Lambda_j$, $\widetilde{W}_i$ is a diagonal operator of type III for any $j \in J'$ and any $i \in \Lambda_j \setminus \{\mu_j\}$, and for every $j \in J'$, $\widetilde{W}_{\mu_j}$ is of type III$(M[1 + M/(1-\varepsilon)]^2, \varepsilon)$. Also $\widetilde{w}_{kk}^{(i)} = w_{kk}^{(i)}$, $1 \leqslant k \leqslant 2$, $i \in \mathbf{N}$. Now the families $\Lambda_1 = \{\mu_j\}_{j \in J'}$, $\Lambda_2 = \{\nu_j\}_{j \in J''}$ satisfy the hypotheses of Lemma 4.11, hence there exists a bounded invertible operator Y in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ such that $\|Y\| = 1$, $\|Y^{-1}\| = \delta$, and such that $X^{-1}S_{\{\widetilde{W}_i\}}X = S_{\{\widehat{W}_i\}}$ where the sequence $\{\widehat{W}_i\}_{i \in \mathbf{N}}$ has the required properties. Clearly

$$\|XY\|, \|(XY)^{-1}\| \leqslant \left(1 + \frac{M}{1-\varepsilon}\right)\delta,$$

and the lemma is proved. ∎

5. PROOFS OF PROPOSITIONS

In this section we will prove the propositions 3.1–3.3, thereby completing the proof of Proposition 2.9.

5.1. *Proof of Proposition* 3.1. Let ε and δ be arbitrary real numbers such that $0 < \varepsilon < 1/2$ and $1 < \delta < 1/\varepsilon$. We set

$$\omega = \left[1 + \frac{M}{1-\varepsilon}\right]\left[1 + \frac{M}{1-\varepsilon}\left(1 + \frac{M}{1-\varepsilon}\right)^2\right]\delta.$$

Using all weights of type III and type I, we obtain two interlaced families $(\Lambda_i')_{i \in J'}$, $(\Lambda_i'')_{i \in J''}$ which satisfy the conditions of Lemma 4.10. Of course, if there are no weights of type III, then J'' is the empty set and $J' = \{1\}$. Thus Lemma 4.10 shows that there exists an invertible operator X satisfying (4.6) and such that $T_1 = X^{-1}TX$ is a weighted shift $S_{\{\widetilde{W}_i\}}$ all of whose weights are either diagonal contractions or they are of type II$(\delta^2 M, \varepsilon_1)$, where $\varepsilon_1 = \max\{1/\delta, \delta\varepsilon\}$, and $\widetilde{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbf{N}$. In other words, T_1 is in the ε_1-canonical form, and $\|X\|, \|X^{-1}\| \leqslant \omega$. This completes the proof of this proposition. ∎

5.2. *Proof of Proposition* 3.2. Let ε and δ be arbitrary real numbers such that $0 < \varepsilon < 1/2$ and $1 < \delta < 1/\varepsilon$. We set

$$\omega_0 = \left[1 + \frac{M}{1-\varepsilon}\right]\left[1 + \frac{M}{1-\varepsilon}\left(1 + \frac{M}{1-\varepsilon}\right)^2\right]\delta,$$

and

$$\omega = \omega_0\left[1 + \frac{M{\omega_0}^2}{1-\varepsilon_1}\right]\delta.$$

First we consider the case when $\{W_i\}_{i\in\mathbb{N}}$ contains finitely many weights of type III or none at all. Then all weights of type III and of type I form two interlaced families $(\Lambda'_i)_{i\in J'}$ and $(\Lambda''_j)_{j\in J''}$ where both J' and J'' are either finite or empty. Therefore, Lemma 4.10 implies that there exists an invertible operator X satisfying (4.6) and such that $T_1 \equiv X^{-1}TX$ is an operator-weighted shift $S_{\{\widetilde{W}_i\}}$ whose all weights are of type II$(\delta^2 M, \varepsilon_1)$, or diagonal operators of type III or type I (hence contractions) with $\widetilde{w}_{jj}^{(i)} = w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbb{N}$. Since $\omega_0 < \omega$, this completes the proof in the case when there are finitely many weights of type III.

Next, we consider the case when $\{W_i\}_{i\in\mathbb{N}}$ contains infinitely many weights of type III. Let i_0 be the largest positive integer such that W_{i_0} is of type I, let j_0 be the smallest positive integer greater then i_0 such that W_{j_0} is of type III, and let k_0 be the smallest positive integer greater than j_0 such that W_{k_0} is of type II. Again, all the weights of type I and the weights W_i of type III for $i < k_0$ are indexed by two interlaced families $(\Lambda'_i)_{i\in J'}$ and $(\Lambda''_j)_{j\in J''}$ where both J' and J'' are either finite or empty. Using Lemma 4.10 we obtain an invertible operator X satisfying (4.6), and such that $T_1 \equiv X^{-1}TX$ is a power bounded operator-weighted shift with power bound $M_1 \equiv M_{\mathrm{pw}}(T_1) \leqslant M{\omega_0}^2$, whose all weights $\widetilde{W}_i$, for $i < k_0$, are either diagonal operators of type III or type I (hence contractions), or they are of type II$(\delta^2 M, \varepsilon_1)$. Also $\widetilde{w}_{jj}^{(i)} = w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbb{N}$. Now we concentrate on weights $\widetilde{W}_i = W_i$ for $i \geqslant k_0$, and we notice that the operators of type II and III form two interlaced families that satisfy the hypotheses of Lemma 4.12. Thus, there exists a similarity Y satisfying $\|Y\|, \|Y^{-1}\| \leqslant [1 + M_1/(1-\varepsilon_1)]\delta$, and such that $T_2 = Y^{-1}T_1Y$ is an operator-weighted shift whose all weights $\widehat{W}_i$, for $i \geqslant k_0$, are either diagonal contractions or of type II$(\delta M, \varepsilon_1)$, and $\widehat{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbb{N}$. Since, for $i < k_0$, all the weights $\widehat{W}_i$ of T_2 coincide with the corresponding weights $\widetilde{W}_i$ of T_1, we conclude that the operator T_2 is in the ε_1-canonical form. Finally, we note that $T_2 = (YX)^{-1}T(YX)$ and

$$\|(YX)^{-1}\|, \|YX\| \leqslant \omega_0\left[1 + \frac{M_1}{1-\varepsilon}\right]\delta = \omega.$$

This completes the proof of this proposition. ∎

5.3. *Proof of Proposition* 3.3. Let ε and δ be arbitrary real numbers such that $0 < \varepsilon < 1/2$ and $1 < \delta < 1/\varepsilon$. We set

$$\omega_1 = \left[1 + \frac{M_1}{1-\varepsilon}\right]\left[1 + \frac{M_1}{1-\varepsilon}\left(1 + \frac{M_1}{1-\varepsilon}\right)^2\right]\delta\left(1 + \frac{M}{1-\varepsilon}\right),$$

and we consider a sequence $\{W_i\}_{i\in\mathbf{N}}$ such that for $i \geqslant N$, where N is some positive integer, W_i is type III of form (3.1). Let k be a positive integer greater than N. Consider the infinite series

$$\sum_{i=k}^{\infty} \beta_i \prod_{j=k}^{i-1} \gamma_j,$$

with the understanding that $\prod_{j=k}^{k-1} \gamma_j = 1$. We will show that this series converges. Indeed, let $\{\beta_{k_i}\}_{i\in\mathbf{N}}$ be the nonzero members of $\{\beta_i\}_{i\geqslant k}$. Then the series above can be rewritten as

$$\sum_{i=1}^{\infty} \beta_{k_i} \prod_{j=k}^{k_i-1} \gamma_j.$$

We note that if $\beta_i \neq 0$ for some $i > N$, then W_i is of type III, so $\gamma_i < \varepsilon$. Since for every $i \in \mathbf{N}$, $\gamma_i \leqslant 1$ we have that

$$\left|\sum_{i=1}^{\infty} \beta_{k_i} \prod_{j=k}^{k_i-1} \gamma_j\right| \leqslant \sum_{i=1}^{\infty} |\beta_{k_i}| \prod_{j=k}^{k_i-1} \gamma_j \leqslant \sum_{i=1}^{\infty} |\beta_{k_i}| \prod_{j=1}^{i-1} \gamma_{k_j} \leqslant \sum_{i=1}^{\infty} M\varepsilon^{i-1} = M\frac{1}{1-\varepsilon}.$$

Thus, the series converges for every $k > N$. We denote its sum by s_k, and note that $|s_k| \leqslant M/(1-\varepsilon)$. Let

$$Y_k = \begin{bmatrix} 1 & -s_k \\ 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathcal{K}), \quad k > N,$$

let $Y_k = I$ for $k \leqslant N$, and let $Y = \operatorname{Diag}(Y_k) \in \mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$. Then $\|Y^{-1}\|, \|Y\| \leqslant 1 + M/(1-\varepsilon)$, $T_1 \equiv Y^{-1}S_{\{W_i\}}Y = S_{\{\widetilde{W}_i\}}$ is a power bounded operator-weighted shift with power bound $M_1 \equiv M_{\text{pw}}(T_1) \leqslant (1 + M/(1-\varepsilon))^2 M$, and

$$\widetilde{W}_i = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_i \end{bmatrix}, \quad \text{for any } i > N,$$

with $0 \leqslant \gamma \leqslant 1$, and $\widetilde{W}_i = W_i$ for $i < N$. Clearly $\widetilde{w}_{jj}^{(i)} = w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbf{N}$.

Next we split all the weights of type III and type I (among the first N) into two interlaced families satisfying the conditions of Lemma 4.10. Thus, there exists a bounded invertible operator $\widetilde{X}$ in $\mathcal{L}\left(\mathcal{K}^{(\aleph_0)}\right)$ satisfying (4.6) with M_1 instead of M, and such that $T_2 \equiv \widetilde{X}^{-1}T_1\widetilde{X}$ is an operator-weighted shift with weight sequence $\{\widehat{W}_i\}_{i=0}^{\infty}$, where, for any $i \leqslant N$, $\widehat{W}_i$ is either a diagonal contraction or it is of type II$(\delta^2 M_1, \varepsilon_1)$ with $\widehat{w}_{jj}^{(i)} \leqslant \delta w_{jj}^{(i)}$, $1 \leqslant j \leqslant 2$, $i \in \mathbf{N}$, and $\widehat{W}_i = \widetilde{W}_i$ for $i > N$. Thus T_2 is in the ε_1-canonical form and, since $T_2 = (Y\widetilde{X})^{-1}T(Y\widetilde{X})$ and $\|(Y\widetilde{X})^{-1}\|, \|Y\widetilde{X}\| \leqslant \omega_1$, the proposition is proved. ∎

6. PROOF OF THEOREM 2.8

We introduce an ordering in the set $\{(i,j) : 1 \leqslant i < j \leqslant n\}$. We say that (i_1, j_1) *precedes* (i_2, j_2), and we write $(i_1, j_1) \prec (i_2, j_2)$ if $j_1 - i_1 < j_2 - i_2$ or if $j_1 - i_1 = j_2 - i_2$ and $i_1 < i_2$ (and, hence, $j_1 < j_2$). In other words we have a sequence

$$(1,2),(2,3),\ldots,(n-1,n),(1,3),\ldots,(n-2,n),\ldots,(1,n-1),(2,n),(1,n). \tag{6.1}$$

For any p, q, such that $1 \leqslant p < q \leqslant n$, Proposition 2.9 yields diagonal operators A, B, C in $\mathcal{L}(\mathcal{G})$ such that A and C are both invertible and

$$\begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} S_{pp} & S_{pq} \\ 0 & S_{qq} \end{bmatrix} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \\ 0 & K_3 \end{bmatrix} \tag{6.2}$$

where K_1, K_2, K_3 are weighted shifts (with weights belonging to $[0,1]$) relative to the same basis of $\mathcal{G}$, and the matrix on the right side of (6.2) is in the ε_1-canonical form. Furthermore, if any weight of S_{pp} (resp. S_{qq}) is at most ε, the corresponding weight of K_1 (resp. K_3) is at most $\delta\varepsilon$, hence does not exceed ε_1. Applied to the $n \times n$ matrix (S_{ij}) this implies that, if we define $X = (x_{ij})$ as

$$x_{ii} = \begin{cases} A & \text{if } i = p, \\ C & \text{if } i = q, \\ I & \text{otherwise;} \end{cases}$$

and, for $i \neq j$

$$x_{ij} = \begin{cases} B & \text{if } i = p,\ j = q, \\ 0 & \text{otherwise;} \end{cases}$$

then $X^{-1}(S_{ij})X = (\widetilde{S}_{ij})$ has the submatrix

$$\begin{bmatrix} \widetilde{S}_{pp} & \widetilde{S}_{pq} \\ 0 & \widetilde{S}_{qq} \end{bmatrix}$$

in the ε_1-canonical form. The key observation, which we shall prove now, is that if $(r,t) \prec (p,q)$ and if

$$\begin{bmatrix} S_{rr} & S_{rt} \\ 0 & S_{tt} \end{bmatrix} \tag{6.3}$$

is in the ε-canonical form, then

$$\begin{bmatrix} \widetilde{S}_{rr} & \widetilde{S}_{rt} \\ 0 & \widetilde{S}_{tt} \end{bmatrix} \tag{6.4}$$

is in ε_1-canonical form. In other words, going along the sequence (6.1), we can transform (one by one) the 2×2 submatrices into ε-canonical form without losing this property for the preceding submatrices. Therefore, we concentrate on the statement above, which we state as

LEMMA 6.1. *Let $n \in \mathbf{N}$, let $\mathcal{G}$ be a Hilbert space, and let (S_{ij}) be an upper triangular $n \times n$ operator matrix acting on $\mathcal{G}^{(n)}$, such that S_{ij} is a weighted shift (relative to the same basis of $\mathcal{G}$). Let p and q be positive integers satisfying $1 \leqslant p < q \leqslant n$, and suppose that, for every $(r,t) \prec (p,q)$, the matrix (6.3) is in the ε-canonical form. Then there exists an invertible operator $X = X(p,q)$ on $\mathcal{G}^{(n)}$ such that $\|X\|, \|X^{-1}\| \leqslant \omega(M)$ and such that $X^{-1}(S_{ij})X = (\widetilde{S}_{ij})$ is an $n \times n$ upper triangular operator matrix, $\widetilde{S}_{ij}$ is a weighted shift for every $1 \leqslant i \leqslant j \leqslant n$, and the 2×2 matrix*

$$\begin{bmatrix} \widetilde{S}_{pp} & \widetilde{S}_{pq} \\ 0 & \widetilde{S}_{qq} \end{bmatrix} \tag{6.5}$$

is in ε_1-canonical form. Moreover, for every $(r,t) \prec (p,q)$, the matrix (6.4) is in the ε_1-canonical form.

Proof. We define $X = (x_{ij})$ as above. Clearly, $\widetilde{S}_{ij}$ is an upper triangular matrix, and the estimates on $\|X\|$ and $\|X^{-1}\|$ follow from Proposition 2.9. In order to prove the remaining assertions of this lemma we introduce the notation $(a_{ij}) = X^{-1} - I_{\mathcal{G}^{(n)}}$, $(c_{ij}) = X - I_{\mathcal{G}^{(n)}}$. Then, an easy calculation shows that $(\widetilde{S}_{ij}) = X^{-1}(S_{ij})X$ is a matrix whose (i,j) entry $\widetilde{S}_{ij}$ is

$$S_{ij} + S_{ip}c_{pj} + S_{iq}c_{qj} + a_{ip}S_{pj} + a_{iq}S_{qj} + a_{ip}S_{pp}c_{pj} + a_{ip}S_{pq}c_{qj} + a_{iq}S_{qq}c_{qj}. \tag{6.6}$$

Since both (a_{ij}) and (c_{ij}) have the only nonzero entries if $i \in \{p,q\}$ or $j \in \{p,q\}$, it is obvious that $i \notin \{p,q\}$ and $j \notin \{p,q\}$ implies $\widetilde{S}_{ij} = S_{ij}$. Therefore, it suffices to establish the assertion in the case when at least one of i, j belongs to $\{p,q\}$.

First we consider the case when $i = p$. Then (6.6) reduces to

$$S_{pj} + S_{pp}c_{pj} + S_{pq}c_{qj} + a_{pp}S_{pj} + a_{pq}S_{qj} + a_{pp}S_{pp}c_{pj} + a_{pp}S_{pq}c_{qj} + a_{pq}S_{qq}c_{qj}. \tag{6.7}$$

If $j = p$ then (6.7) is

$$S_{pp} + S_{pp}c_{pp} + a_{pp}S_{pp} + a_{pp}S_{pp}c_{pp} = (1+a_{pp})S_{pp}(1+c_{pp}) = A^{-1}S_{pp}A = K_1. \tag{6.8}$$

If $j = q$ then (6.7) is

$$\begin{aligned} &S_{pq} + S_{pp}c_{pq} + S_{pq}c_{qq} + a_{pp}S_{pq} + a_{pq}S_{qq} + a_{pp}S_{pp}c_{pq} + a_{pp}S_{pq}c_{qq} + a_{pq}S_{qq}c_{qq} \\ &\quad = (1+a_{pp})S_{pq}(1+c_{qq}) + a_{pq}S_{qq}(1+c_{qq}) + (1+a_{pp})S_{pp}c_{pq} \\ &\quad = A^{-1}S_{pq}C + (-A^{-1}BC^{-1})S_{qq}C + A^{-1}S_{pp}B = K_2. \end{aligned} \tag{6.9}$$

If $j \neq p$ and $j \neq q$, (6.7) becomes

$$\begin{aligned} S_{pj} + a_{pp}S_{pj} + a_{pq}S_{qj} &= (1+a_{pp})S_{pj} + a_{pq}S_{qj} \\ &= A^{-1}S_{pj} + (-A^{-1}BC^{-1})S_{qj}. \end{aligned} \tag{6.10}$$

It is easy to see that if $j < p$, $\widetilde{S}_{ij} = 0$, and if $p < j < q$, (6.10) becomes $A^{-1}S_{pj}$. This shows that if $i = p$, $\widetilde{S}_{ij} = 0$ for $j < p$ and $\widetilde{S}_{ij}$ is a weighted shift for $j \geqslant p$.

Next we turn our attention to the case $i = q$. In this situation, (6.6) reduces to

$$\begin{aligned} S_{qj} + S_{qq}c_{qj} + a_{qq}S_{qj} + a_{qq}S_{qq}c_{qj} &= (1 + a_{qq})(S_{qj} + S_{qq}c_{qj}) \\ &= C^{-1}(S_{qj} + S_{qq}c_{qj}). \end{aligned} \tag{6.11}$$

If $j < q$, then $\widetilde{S}_{ij} = 0$ and it is clear that $\widetilde{S}_{qj}$ is a weighted shift for every $j \geqslant q$. Notice that, if $j = q$, then (6.11) is

$$(1 + a_{qq})S_{qq}(1 + c_{qq}) = C^{-1}S_{qq}C = K_3. \tag{6.12}$$

Finally, if $i \neq p$ and $i \neq q$, then, as we noticed before, we consider two possibilities: $j = p$ or $j = q$. If $j = p$, we can write (6.6) as

$$S_{ip} + S_{ip}c_{pp} + a_{ip}S_{pp} + a_{ip}S_{pp}c_{pp}. \tag{6.13}$$

Of course, $i > p$ implies $\widetilde{S}_{ij} = 0$ and if $i < p$, (6.13) reduces to

$$S_{ip}(1 + c_{pp}) = S_{ip}A, \tag{6.14}$$

and it follows that, for $j = p$, $\widetilde{S}_{ij}$ is a weighted shift. If $j = q$, (6.6) becomes

$$S_{iq} + S_{ip}c_{pq} + S_{iq}c_{qq} + a_{ip}S_{pq} + a_{iq}S_{qq} + a_{ip}S_{pp}c_{pq} + a_{ip}S_{pq}c_{qq} + a_{iq}S_{qq}c_{qq}. \tag{6.15}$$

It is easy to see that $i > q$ implies $\widetilde{S}_{ij} = 0$, and if $i < q$ (and $i \neq p$), then (6.15) reduces to

$$S_{iq} + S_{ip}c_{pq} + S_{iq}c_{qq} = S_{ip}B + S_{iq}C, \tag{6.16}$$

which shows that, once again, $\widetilde{S}_{ij}$ is a weighted shift.

Next we investigate the properties of 2×2 submatrices. We notice that (6.8), (6.9), and (6.12), together with the fact that the right side of (6.2) is in the ε_1-canonical form, show that the matrix (6.5) has the same property. It remains to be proved that, if $(r, t) \prec (p, q)$, the matrix (6.4) is in the ε_1-canonical form. Since, for $i \notin \{p, q\}$, $\widetilde{S}_{ii} = S_{ii}$ we see, using (6.8) and (6.12), that both $\widetilde{S}_{rr}$ and $\widetilde{S}_{tt}$ have nonnegative weights not greater than 1. Since $(r, t) \prec (p, q)$, $\widetilde{S}_{rt}$ is given by one of the formulae (6.9), (6.11), (6.12), or (6.13). In each case, for any nonzero weight of $\widetilde{S}_{rt}$, the corresponding weight of S_{rt} is also different from zero which implies that the corresponding weights of S_{rr} and S_{tt} are at most ε. For $r \notin \{p, q\}$, $\widetilde{S}_{rr} = S_{rr}$ so its appropriate weight is at most ε, and similarly for $\widetilde{S}_{tt}$. Finally, if $r = p$ or $r = q$, then $\widetilde{S}_{rr}$ is given by (6.8) or by (6.12), and the similar statement can be made for $\widetilde{S}_{tt}$. But, we have already seen that if a weight of S_{pp} or S_{pq} is at most ε, then the corresponding weight of K_1 or K_3 is at most ε_1. This completes the proof of this lemma. ∎

Now we can finish the proof of Theorem 2.8. If $n = 2$, the assertion follows from Proposition 2.9. If $n \geqslant 3$, we define

$$\varepsilon = 2^{-2^{\frac{n(n-1)}{2}}},$$

and we define the sequence $\varepsilon_1 = \varepsilon$, $\varepsilon_k = \sqrt{\varepsilon_{k-1}}$, $2 \leqslant k \leqslant n(n-1)/2$, and the sequence $\{\delta_k\}$ as $\delta_k = \varepsilon_k^{-1/2}$, $1 \leqslant k \leqslant n(n-1)/2$. Notice that for every k, $\max(1/\delta_k, \delta_k\varepsilon_k) = \max({\varepsilon_k}^{1/2}, {\varepsilon_k}^{1/2}) = {\varepsilon_k}^{1/2} = \varepsilon_{k+1}$ so Lemma 6.1 provides the inductive step to get from first k submatrices in ε_k-canonical form to first $k+1$ submatrices in ε_{k+1}-canonical form. Thus, after repeating the procedure $n(n-1)/2$ times, we obtain a matrix $(\widehat{S}_{ij})$ that is in $\varepsilon_{n(n-1)/2}$-canonical form. Since an easy calculation shows that $\varepsilon_{n(n-1)/2} < 1/2$, the theorem is proved. ∎

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