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GROTHENDIECK TYPE NORMS FOR BILINEAR FORMS ON C^*-ALGEBRAS

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ABSTRACT. We consider three equivalent norms on the space $\mathrm{Bil}(A, B)$ of bilinear forms on a pair of C^*-algebras (A, B), namely the usual norm and two norms (here denoted $\|\cdot\|_{J^*}$ and $\|\cdot\|_{\mathrm{tb}^*}$) related to the non-commutative Grothendieck inequality and to completely bounded maps from A to B^*. The two latter norms were introduced by Blecher and the second named author some years ago. We show that the three norms are mutually distinct, and give estimates of the best constants in the comparison of these norms.

KEYWORDS: *C^*-algebra, non-commutative Grothendieck inequality, Kneser graph.*

AMS SUBJECT CLASSIFICATION: Primary 46L05; Secondary 47A30.

INTRODUCTION

In this paper, we will make comparison between the following three norms on the set of bounded bilinear forms $\mathrm{Bil}(A, B)$ on a pair of C^*-algebras (A, B):

1. The usual norm $\|V\|$ of a bilinear map $V : A \times B \to \mathbb{C}$.

2. The tracially bounded norm — here denoted $\|V\|_{\mathrm{tb}^*}$ — considered by the second named author ([9], under the name $\|V\|_{\mathrm{cb}}$) and by Blecher ([2]), i.e.

$$\|V\|_{\mathrm{tb}^*} = \sup_{n\in\mathbb{N}} \|V^{(n)}\|,$$

where $V^{(n)}$ is the bilinear form on $M_n(A) \times M_n(B)$ given by

$$V^{(n)}([a_{ij}], [b_{ij}]) = \frac{1}{n}\sum_{i,j=1}^{n} V(a_{ij}, b_{ij}).$$

3. The norm $\|V\|_{J^*}$ introduced by Blecher in [1] (under the name $\|V\|_{scb}$), i.e. $\|V\|_{J^*}$ is the smallest constant for which there exist states $\varphi_1, \varphi_2 \in S(A)$, $\psi_1, \psi_2 \in S(B)$ and numbers $s, t \in [0,1]$ such that

$$|V(a,b)| \leqslant \|V\|_{J^*}(s\varphi_1(a^*a) + (1-s)\varphi_2(aa^*))^{\frac{1}{2}}(t\psi_1(b^*b) + (1-t)\psi_2(bb^*))^{\frac{1}{2}}$$

for every pair $(a,b) \in A \times B$. Here $S(A)$ and $S(B)$ denote the state spaces of A and B respectively.

Using the non-commutative Grothendieck inequality ([7]), one has

$$\|V\| \leqslant \|V\|_{tb^*} \leqslant 2\|V\| \quad \text{and} \quad \|V\| \leqslant \|V\|_{J^*} \leqslant 2\|V\|$$

(cf. Blecher [1] and [2]), so in particular the three norms are equivalent. Also, by a simple modification of Blecher's argument in [2], Proof of Theorem 1, one gets $\|V\|_{tb^*} \leqslant \|V\|_{J^*}$ (see proof below), so altogether

$$\|V\| \leqslant \|V\|_{tb^*} \leqslant \|V\|_{J^*} \leqslant 2\|V\|. \tag{0.1}$$

Define now

$$K_{tb} = \sup_{A,B}\left(\sup\left\{\frac{\|V\|_{tb^*}}{\|V\|} \mid V \in \mathrm{Bil}(A,B),\ V \neq 0\right\}\right),$$
$$K_J = \sup_{A,B}\left(\sup\left\{\frac{\|V\|_{J^*}}{\|V\|} \mid V \in \mathrm{Bil}(A,B),\ V \neq 0\right\}\right),$$
$$K_{J,tb} = \sup_{A,B}\left(\sup\left\{\frac{\|V\|_{J^*}}{\|V\|_{tb^*}} \mid V \in \mathrm{Bil}(A,B),\ V \neq 0\right\}\right),$$

where (A,B) runs through all pairs of C^*-algebras. The constant K_J can be considered as the non-commutative analogue of the complex Grothendieck constant $K_G^{\mathbf{C}}$ (cf. [6], [8], [11]). It is clear from (0.1) that

$$1 \leqslant K_{tb} \leqslant K_J \leqslant 2 \quad \text{and} \quad 1 \leqslant K_{J,tb} \leqslant K_J.$$

Blecher proved in [2] that $K_{tb} \geqslant 4/\pi$ and he included in his thesis ([1]) an unpublished argument due to the first named author, which shows that $K_J = 2$ (cf. [1], Proposition 4.2.7). The main result of this paper is that the construction leading to $K_J = 2$ can also — after some extra work — be used to prove that $K_{tb} \geqslant \pi/2$, hence improving Blecher's lower bound from [2]. We show also that the norms $\|V\|_{tb^*}$ and $\|V\|_{J^*}$ are in general distinct, i.e. $K_{J,tb} > 1$.

The two norms $\|\cdot\|_{tb^*}$ and $\|\cdot\|_{J^*}$ on $\mathrm{Bil}(A,B)$ can be realized as the dual norms of two norms $\|\cdot\|_{tb}$ and $\|\cdot\|_J$ on the algebraic tensor product $A \otimes B$. These norms are given by (cf. [1], [2], [9]):

$$\|x\|_{\mathrm{tb}} = \inf\left\{ \sum_{k=1}^{N} \|[a_{ij}^{(k)}]\| \, \|[b_{ij}^{(k)}]\|_n \;\middle|\; x = \sum_{k=1}^{N} \left(\sum_{i,j=1}^{n} a_{ij}^{(k)} \otimes b_{ij}^{(k)} \right) \right\}$$

where $[a_{ij}^{(k)}] \in M_n(A), [b_{ij}^{(k)}] \in M_n(B), k = 1, \dots, N$ and N, n are arbitrary positive integers, and

$$\|x\|_{\mathrm{J}} = \inf\left\{ \max\left(\left\|\sum_{i=1}^{n} a_i^* a_i\right\|, \left\|\sum_{i=1}^{n} a_i a_i^*\right\| \right)^{\frac{1}{2}} \cdot \max\left(\left\|\sum_{i=1}^{n} b_i^* b_i\right\|, \left\|\sum_{i=1}^{n} b_i b_i^*\right\| \right)^{\frac{1}{2}} \right\}$$

where the infimum is over all representations $x = \sum_{i=1}^{n} a_i \otimes b_i$. Since the usual norm on $\mathrm{Bil}(A, B)$ is the dual norm to the projective tensor norm $\|x\|_\gamma$ on $A \otimes B$, the inequality (0.1) can be reformulated as

$$\|x\|_{\mathrm{J}} \leqslant \|x\|_{\mathrm{tb}} \leqslant \|x\|_\gamma \leqslant 2\|x\|_{\mathrm{J}}, \quad x \in A \otimes B. \tag{0.2}$$

Moreover the constants $K_{\mathrm{tb}}, K_{\mathrm{J}}$ and $K_{\mathrm{J,tb}}$ can be expressed as

$$K_{\mathrm{tb}} = \sup_{A,B} \left(\sup\left\{ \frac{\|x\|_\gamma}{\|x\|_{\mathrm{tb}}} \,|\, x \in A \otimes B,\ x \neq 0 \right\} \right),$$
$$K_{\mathrm{J}} = \sup_{A,B} \left(\sup\left\{ \frac{\|x\|_\gamma}{\|x\|_{\mathrm{J}}} \,|\, x \in A \otimes B,\ x \neq 0 \right\} \right),$$
$$K_{\mathrm{J,tb}} = \sup_{A,B} \left(\sup\left\{ \frac{\|x\|_{\mathrm{tb}}}{\|x\|_{\mathrm{J}}} \,|\, x \in A \otimes B,\ x \neq 0 \right\} \right).$$

For further disscussion on tensor norms on C^*-algebras, see [5] and [10].

We conclude this section by giving the announced proof of $\|V\|_{\mathrm{tb}^*} \leqslant \|V\|_{\mathrm{J}^*}$: if $V \in \mathrm{Bil}(A, B)$, then for some choice of $\varphi_1, \varphi_2 \in S(A), \psi_1, \psi_2 \in S(B)$ and $s, t \in [0, 1]$,

$$|V(a, b)| \leqslant \|V\|_{\mathrm{J}^*} (s\varphi_1(a^*a) + (1-s)\varphi_2(aa^*))^{\frac{1}{2}} (t\psi_1(b^*b) + (1-t)\psi_2(bb^*))^{\frac{1}{2}}$$

for all $(a,b) \in A \times B$. Hence, for $a = [a_{ij}] \in M_n(A)$ and $b = [b_{ij}] \in M_n(B)$, we get

$$\begin{aligned}
|V^{(n)}(a,b)| &\leqslant \frac{1}{n}\sum_{i,j=1}^{n} |V(a_{ij},b_{ij})| \\
&\leqslant \frac{1}{n}\|V\|_{J^*}\sum_{i,j=1}^{n}\Big(s\varphi_1(a_{ij}^*a_{ij}) + (1-s)\varphi_2(a_{ij}a_{ij}^*)\Big)^{\frac12}\Big(t\psi_1(b_{ij}^*b_{ij}) \\
&\qquad + (1-t)\psi_2(b_{ij}b_{ij}^*)\Big)^{\frac12} \\
&\leqslant \frac{1}{n}\|V\|_{J^*}\left\{\sum_{i,j=1}^{n}(s\varphi_1(a_{ij}^*a_{ij}) + (1-s)\varphi_2(a_{ij}a_{ij}^*))\right\}^{\frac12}\left\{\sum_{i,j=1}^{n}(t\psi_1(b_{ij}^*b_{ij}) \right. \\
&\qquad \left. + (1-t)\psi_2(b_{ij}b_{ij}^*))\right\} \\
&\leqslant \frac{1}{n}\|V\|_{J^*}\left\{s\Big\|\sum_{i,j=1}^{n} a_{ij}^*a_{ij}\Big\| + (1-s)\Big\|\sum_{i,j=1}^{n} a_{ij}a_{ij}^*\Big\|\right\}^{\frac12}\left\{t\Big\|\sum_{i,j=1}^{n} b_{ij}^*b_{ij}\Big\|\right. \\
&\qquad \left. + (1-t)\Big\|\sum_{i,j=1}^{n} b_{ij}b_{ij}^*\Big\|\right\}^{\frac12}.
\end{aligned}$$

But

$$\Big\|\sum_{i,j=1}^{n} a_{ij}^*a_{ij}\Big\| = \Big\|\sum_{i=1}^{n}(a^*a)_{ii}\Big\| \leqslant n\|a^*a\| = n\|a\|^2$$

and

$$\Big\|\sum_{i,j=1}^{n} a_{ij}a_{ij}^*\Big\| = \Big\|\sum_{j=1}^{n}(aa^*)_{jj}\Big\| \leqslant n\|aa^*\| = n\|a\|^2.$$

This together with the corresponding inequalities for the b_{ij}'s gives

$$|V^{(n)}(a,b)| \leqslant \|V\|_{J^*}\|a\|\,\|b\|,$$

which proves that $\|V\|_{tb^*} \leqslant \|V\|_{J^*}$.

Note that the two other inequalities in the triple inequality (0.1) are easy: $\|V\| \leqslant \|V\|_{tb^*}$ is trivial and $\|V\|_{J^*} \leqslant 2\|V\|$ is an immediate consequence of the non-commutative Grothendieck inequality ([7]).

The rest of this paper is organized in the following way: Section 1 contains the construction leading to $K_J = 2$. Section 2 contains an explicit computation of the spectra of certain graphs. This computation is crucial for the last section (Section 3), which contains the main results.

1. THE NONCOMMUTATIVE GROTHENDIECK CONSTANT K_J

The following plays the central role throughout this paper.

THEOREM 1.1. *For any integer $n \geqslant 1$, there exist a finite dimensional Hilbert space H and $2n+1$ partial isometries $a_1, \ldots, a_{2n+1} \in B(H)$ such that*

$$\Big\| \sum_{k=1}^{2n+1} a_k \otimes a_k \Big\|_\gamma = 2n+1$$

and

$$\sum_{k=1}^{2n+1} a_k^* a_k = \sum_{k=1}^{2n+1} a_k a_k^* = (n+1) I_H.$$

Proof. Put $m = 2n+1$ and let K be an m-dimensional Hilbert space. We consider the CAR-algebra over K (cf. [4]).

Let $\widetilde{K} = \bigoplus_{j=0}^{m} K_j$ where $K_0 = \mathbf{C}, K_1 = K$ and K_j is the antisymmetric part of $\underbrace{K \otimes \cdots \otimes K}_{j}$, $(j \geqslant 2)$, so K_j is the j-particle space in the Fock representation of the CAR-relations. Note that $\dim K_j = \binom{m}{j}, (j = 0, \ldots, m)$. For $f \in K$, let $a^*(f), a(f)$ be the corresponding creation and annihilation operators. Then $f \mapsto a^*(f)$ is a linear map from K to $B(\widetilde{K})$ and $a^*(f) = a(f)^*$ for all f. Moreover,

$$a(f)a(g)^* + a(g)^* a(f) = (g|f)1 \tag{1.1}$$

$$a(f)a(g) + a(g)a(f) = 0. \tag{1.2}$$

The $a(f)$'s map K_{j+1} into K_j and the $a(f)^*$'s map K_j into K_{j+1}. Moreover,

(a) If f is a unit vector, then $a(f)$ is a partial isometry.

(b) If $e_1, \ldots, e_m$ is an orthonormal basis for K, then the restriction of the operator $N = \sum_{k=1}^{m} a(e_k)^* a(e_k)$ to K_j is just multiplication with j (i.e. N is the number operator).

This combined with (1.1) shows that the restriction of $N' = \sum_{k=1}^{m} a(e_k) a(e_k)^*$ to K_j is multiplication with $m - j$.

Choose now a fixed orthonormal basis $e_1, \ldots, e_{2n+1}$ for K and let $a_k \in B(K_{n+1}, K_n)$ be the restriction of $a(e_k)$ to K_{n+1}. Then each a_k is a partial isometry and $a_k^* \in B(K_n, K_{n+1})$ is the restriction of $a^*(e_k)$ to K_n. Hence

$$\sum_{k=1}^{2n+1} a_k^* a_k = N_{|K_{n+1}} = (n+1) I_{K_{n+1}}$$

and

$$\sum_{k=1}^{2n+1} a_k a_k^* = N'_{|K_n} = (n+1)I_{K_n}.$$

We now compute the norm of $\sum_{k=1}^{2n+1} a_k \otimes a_k$ in $B(K_{n+1}, K_n) \otimes B(K_{n+1}, K_n)$. It is clear that

$$\left\| \sum_{k=1}^{2n+1} a_k \otimes a_k \right\|_\gamma \leqslant 2n+1.$$

To prove the converse inequality, consider the bilinear form V on $B(K_{n+1}, K_n) \times B(K_{n+1}, K_n)$ given by

$$V(x,y) = \sum_{k=1}^{2n+1} \tau(xa_k^*)\tau(ya_k^*)$$

where τ is the normalized trace on $B(K_n)$. Since the Fock representation is basis indepenent, the number

$$\tau(a(f)a(f)^*_{|K_n})$$

can only depend on $\|f\|$, and since $f \mapsto a(f)^*$ is linear, we get

$$\tau(a(f)a(f)^*_{|K_n}) = C\|f\|^2$$

where $C \geqslant 0$ is a constant. By the polarization identity, we have

$$\tau(a(f)a(g)^*_{|K_n}) = C(g|f).$$

In particular, $\tau(a_k a_\ell^*) = C\delta_{k\ell}$.

Using $\sum_{k=1}^{2n+1} a_k a_k^* = (n+1)I_{K_n}$, we get $C = \frac{n+1}{2n+1}$. Therefore, we have

$$\sum_{k=1}^{2n+1} V(a_k, a_k) = \sum_{k,\ell=1}^{2n+1} \tau(a_k a_\ell^*)\tau(a_k a_\ell^*) = \sum_{k=1}^{2n+1} \tau(a_k a_k^*)^2 = (2n+1)C^2.$$

If we can show that $\|V\| \leqslant C^2$ (as a bilinear form on $B(K_{n+1}, K_n) \times B(K_{n+1}, K_n)$) then it will follow that $\left\| \sum_{k=1}^{2n+1} a_k \otimes a_k \right\|_\gamma \geqslant 2n+1$, so actually $\left\| \sum_{k=1}^{2n+1} a_k \otimes a_k \right\|_\gamma = 2n+1$.

Observe first that the definition of V is invariant under change of the basis: i.e. if $[g_{\ell k}]_{\ell,k=1}^{2n+1}$ is a unitary matrix and $b_k = \sum_{\ell=1}^{2n+1} g_{\ell k} a_\ell$, $(k = 1, \ldots, 2n+1)$, then also

$$V(x,y) = \sum_{k=1}^{2n+1} \tau(xb_k^*)\tau(yb_k^*).$$

Let $x \in B(K_{n+1}, K_n)$ be fixed, and let x_0 be the orthogonal projection onto $\operatorname{span}\{a_1, \ldots, a_{2n+1}\}$ with respect to the inner product $\tau(z_1 z_2^*)$ on $B(K_{n+1}, K_n)$. Since $a_1, \ldots, a_{2n+1}$ are pairwise orthogonal and $\|a_1\|_2 = \cdots = \|a_{2n+1}\|_2 = \sqrt{C}$, one can choose $b_1, \ldots, b_{2n+1}$ as above such that b_1 and x_0 are proportional.

Hence

$$\tau(xb_k^*) = 0, \quad \text{for } k = 2, \ldots, 2n+1.$$

Therefore

$$V(x,y) = \tau(xb_1^*)\tau(yb_1^*).$$

Since the map $f \mapsto a(f)_{|K_{n+1}}$ is conjugate linear, b_1 is of the form $b_1 = a(f_0)_{|K_n}$ for some unit vector $f_0 \in K$. In particular, b_1 is a partial isometry and $\tau(b_1 b_1^*) = C$. Hence we also have $\tau(|b_1^*|) = C$. Thus $|\tau(xb_1^*)| \leqslant C\|x\|$ and $\tau(yb_1^*) \leqslant C\|y\|$, which imply that $|V(x,y)| \leqslant C^2 \|x\|\,\|y\|$.

Since this holds for all $x, y \in B(K_{n+1}, K_n)$, we have $\|V\| \leqslant C^2$ and so

$$\Big\| \sum_{k=1}^{2n+1} a_k \otimes a_k \Big\|_\gamma \geqslant 2n+1.$$

Since $\binom{2n+1}{n} = \binom{2n+1}{n+1}$, we have $\dim K_n = \dim K_{n+1}$. Thus it is possible to realize the a_k's as operators from a Hilbert space H into itself, where $\dim H = \binom{2n+1}{n}$. ∎

COROLLARY 1.2.

$$K_J = 2.$$

Proof. Since $\Big\| \sum_{k=1}^{2n+1} a_k \otimes a_k \Big\|_\gamma = 2n+1$ and $\Big\| \sum_{k=1}^{2n+1} a_k \otimes a_k \Big\|_J \leqslant n+1$, we have

$$\frac{\Big\| \sum_{k=1}^{2n+1} a_k \otimes a_k \Big\|_\gamma}{\Big\| \sum_{k=1}^{2n+1} a_k \otimes a_k \Big\|_J} \geqslant \frac{2n+1}{n+1} \longrightarrow 2 \qquad (n \to \infty). \quad ∎$$

REMARK 1.3. By the same method, one can prove that if n_1, n_2 are positive integers, then there exist two Hilbert spaces H_1, H_2 with $\dim H_1 = \binom{n_1+n_2+1}{n_1}$,

$\dim H_2 = \binom{n_1+n_2+1}{n_2}$ and partial isometries $a_1, \dots, a_{n_1+n_2+1}$ in $B(H_1, H_2)$ such that

$$\left\| \sum_{k=1}^{n_1+n_2+1} a_k \otimes a_k \right\|_\gamma = n_1 + n_2 + 1$$

and

$$\sum_{k=1}^{n_1+n_2+1} a_k^* a_k = (n_1+1) I_{H_1}, \qquad \sum_{k=1}^{n_1+n_2+1} a_k a_k^* = (n_2+1) I_{H_2}.$$

By tensoring H_1 and H_2 with ℓ^2, one can actually obtain that the a_k's live on the same infinite dimensional Hilbert space. This shows that the dual version of the non-commutative Grothendieck inequality ([7])

$$\left\|\sum_{i=1}^n x_i \otimes y_i\right\|_\gamma \leqslant \left(\left\|\sum_{i=1}^n x_i^* x_i\right\| + \left\|\sum_{i=1}^n x_i x_i^*\right\|\right)^{\frac{1}{2}} \left(\left\|\sum_{i=1}^n y_i^* y_i\right\| + \left\|\sum_{i=1}^n y_i y_i^*\right\|\right)^{\frac{1}{2}}$$

cannot be improved, in the sense that the smallest function $f(\cdot,\cdot)$ for which

$$\left\|\sum_{i=1}^n x_i \otimes y_i\right\|_\gamma \leqslant f\left(\left\|\sum_{i=1}^n x_i^* x_i\right\|, \left\|\sum_{i=1}^n x_i x_i^*\right\|\right)^{\frac{1}{2}} f\left(\left\|\sum_{i=1}^n y_i^* y_i\right\|, \left\|\sum_{i=1}^n y_i y_i^*\right\|\right)^{\frac{1}{2}}$$

holds for arbitrary C^*-algebras is $f(s,t) = s + t$.

REMARK 1.4. It is straightforward to check that

$$\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_J = n+1, \quad \|V\|_{J^*} = \frac{n+1}{2n+1} \quad \text{and } \|V\| = \left(\tfrac{n+1}{2n+1}\right)^2$$

with $a_k, k = 1, 2, \dots, 2n+1$ and V as in the proof of Theorem 1.1.

2. SPECTRA OF THE KNESER GRAPH $K_{2r+k}^{(r)}$ FOR $k = 1$

Put $T = \{1, \dots, 2r+1\}$ and let $T^{(r)}$ be the set of r-subsets of T. The Kneser graph $K_{2r+1}^{(r)}$ (cf. [3]) consists of the vertices $T^{(r)}$ where two r-sets are joint if and only if they are disjoint, so the adjacency matrix A for the Kneser graph $K_{2r+1}^{(r)}$ is, for $M, N \in T^{(r)}$,

$$A(M,N) = \begin{cases} 1 & \text{if } M \cap N = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Each vertex in $K_{2r+1}^{(r)}$ is joint to $r+1$ vertices, so each row and each column in A contains $(r+1)$ 1's.

THEOREM 2.1. *Let A be the adjacency matrix of the Kneser graph $K^{(r)}_{2r+1}$. Then*

$$\mathrm{sp}(A) = \{r+1, -r, r-1, -(r-2), \ldots, (-1)^r\}$$

and the dimension for the eigenspace of A belonging to $(-1)^{r+1-k}k$, $k = 1, 2, \ldots, r+1$ is

$$\binom{2r+1}{r+1-k} - \binom{2r+1}{r-k}.$$

Proof. First, we compute the spectrum of A. Define $(0,1)$-matrices $E_0, E_1, \ldots, E_r$ by

$$E_i(M,N) = \begin{cases} 1 & \text{if } |M \cap N| = r - i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_0 = I$ and $E_r = A$. Elementary combinatrial arguments give that the E_i's are mutually commuting matrices and that

$$\begin{aligned} E_r E_0 &= E_r, \\ E_r E_1 &= rE_r + 2E_{r-1}, \\ E_r E_2 &= (r-1)E_{r-1} + 3E_{r-2}, \\ E_r E_3 &= (r-2)E_{r-2} + 4E_{r-3}, \\ &\vdots \\ E_r E_r &= E_1 + (r+1)E_0. \end{aligned} \tag{2.1}$$

From this, it follows that $\mathrm{alg}(A, I)$ the algebra generated by A and I is equal to

$$\mathrm{span}\{E_0, E_1, \ldots, E_r\}$$

which has dimension $r+1$. Since A is selfadjoint, it follows that A has precisely $r+1$ distinct eigenvalues $\lambda_1, \ldots, \lambda_{r+1}$. The projection P_λ of the eigenspace corresponding to $\lambda \in \{\lambda_1, \ldots, \lambda_{r+1}\}$ is contained in $\mathrm{alg}(A, I)$: i.e.

$$P_\lambda = \sum_{i=0}^{r} c_{\lambda,i} E_i, \quad c_{\lambda,i} \in \mathbb{R}.$$

Then $AP_\lambda = \lambda P_\lambda = \sum_{i=0}^{r} \lambda c_{\lambda,i} E_i$.

If $(\alpha_0, \ldots, \alpha_r) \in \mathbb{R}^{r+1}$, it follows from (2.1) that

$$A\left(\sum_{i=0}^{r} \alpha_i E_i\right) = \sum_{i=0}^{r} \beta_i E_i$$

where $(\beta_0, \ldots, \beta_r) \in \mathbf{R}^{r+1}$ is given by

$$\begin{pmatrix} \beta_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \beta_r \end{pmatrix} = \begin{pmatrix} & & & & & r+1 \\ & & & & r & 1 \\ & & & r-1 & 2 & \\ & & \iddots & \iddots & & \\ & 3 & \iddots & & & \\ 2 & r-1 & & & & \\ 1 \quad r & & & & & \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix}$$

Hence if $\lambda \in \mathrm{sp}(B)$, then $\lambda \in \mathrm{sp}(A)$, where

$$B = \begin{pmatrix} & & & & & r+1 \\ & & & & r & 1 \\ & & & r-1 & 2 & \\ & & \iddots & \iddots & & \\ & 3 & \iddots & & & \\ 2 & r-1 & & & & \\ 1 \quad r & & & & & \end{pmatrix}$$

So if B has $r+1$ distinct eigenvalues, we will get that $\mathrm{sp}(A) = \mathrm{sp}(B)$. Put

$$\begin{aligned} x_0 &= (1, 1, \ldots, 1) & y_0 &= (1, 1, \ldots, 1) \\ x_1 &= (r, r-1, \ldots, 0) & y_1 &= (0, \ldots, r-1, r) \\ x_2 &= \left(\binom{r}{2}, \binom{r-1}{2}, \ldots, 0\right) & y_2 &= \left(0, \ldots, \binom{r-1}{2}, \binom{r}{2}\right) \\ &\vdots & &\vdots \\ x_r &= (1, 0, \ldots, 0) & y_r &= (0, \ldots, 0, 1). \end{aligned}$$

Then one easily gets

$$\begin{aligned} Bx_0 &= (r+1)y_0 \\ Bx_1 &= ry_1 \\ Bx_2 &= (r-1)y_2 \\ &\vdots \\ Bx_r &= y_r. \end{aligned}$$

However $y_i \in \operatorname{span}\{x_0, \ldots, x_r\}$ for $i = 0, \ldots, r$. Explicitly

$$\begin{aligned}
y_0 &= x_0 \\
y_1 &= r x_0 - x_1 \\
y_2 &= \binom{r}{2} x_0 - \binom{r-1}{1} x_1 + x_2 \\
y_3 &= \binom{r}{3} x_0 - \binom{r-1}{2} x_1 + \binom{r-2}{1} - x_3 \\
&\vdots \\
y_r &= x_0 - x_1 + \cdots + (-1)^r x_r .
\end{aligned}$$

Hence the matrix of B with respect to the basis $x_0, x_1, \ldots, x_r$ is an upper triangular matrix with diagonal elements $r+1, -r, r-1, -(r-2), \ldots, (-1)^r$. This proves that

$$\operatorname{sp}(A) = \operatorname{sp}(B) = \{r+1, -r, r-1, -(r-2), \ldots, (-1)^r\}.$$

Next, we compute the dimensions of the eigenspaces. Let d_k, $(k = 1, \ldots, r+1)$ denote the dimension of the eigenspace belonging to the eigenvalue $\lambda_k = (-1)^{r+1-k} k$. Then

$$\operatorname{Tr}(A^m) = \sum_{k=1}^{r+1} d_k \lambda_k^m ,$$

where Tr is the usual trace with $\operatorname{Tr}(E_0) = |T^{(r)}| = \binom{2r+1}{r}$. On the other hand,

$$A^m = A^m E_0 = \sum_{i=0}^{r} c_i^{(m)} E_i ,$$

where

$$\begin{pmatrix} c_0^{(m)} \\ \vdots \\ \vdots \\ c_r^{(m)} \end{pmatrix} = B^m \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Since $\operatorname{Tr}(E_i) = 0$ for $i = 1, \ldots, r$, we have

$$\operatorname{Tr}(A^m) = \binom{2r+1}{r} c_0^{(m)} = \binom{2r+1}{r} (B^m)_{oo} ,$$

where $(B^m)_{oo}$ denotes the element of B^m in the upper left corner. Clearly B^2 is of the form

$$B^2 = \begin{pmatrix} * & * & & & \\ * & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & * \\ & & & * & * \end{pmatrix} .$$

By induction, one gets that the matrix elements of B^{2n+1} is zero unless $r-n+2 \leqslant i+j \leqslant r+n+3$. In particular, $(B^{2n+1})_{oo} = 0$ for $n = 0, 1, \ldots, r-1$: i.e.

$$\operatorname{Tr}(A^{2n+1}) = 0 \quad \text{for } n = 0, 1, \ldots, r-1$$

or equivalently

$$\sum_{k=1}^{r+1} d_k \lambda_k^{2n+1} = 0 \quad \text{for } n = 0, 1, \ldots, r-1.$$

Since $|\lambda_k| = k$ and $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ alternate in sign, we get

$$\sum_{k=1}^{r+1} (-1)^k d_k k^{2n+1} = 0 \quad \text{for } n = 0, 1, \ldots, r-1. \tag{2.2}$$

These r equations together with the obvious equation

$$\sum_{k=1}^{r+1} d_k = \binom{2r+1}{r} \tag{2.3}$$

determines the multiplicities d_k.

Put

$$g_k = \frac{(-1)^k}{k} d_k \quad \text{for } k = 1, 2, \ldots, r+1$$
$$g_{-k} = g_k \quad \text{for } k = 1, 2, \ldots, r+1$$

and choose g_0 such that

$$\sum_{k=-r-1}^{r+1} g_k = 0.$$

Then (2.2) is equivalent to

$$\sum_{k=-r-1}^{r+1} g_k k^{2n+2} = 0 \quad \text{for } n = 0, 1, \ldots, r-1.$$

Hence

$$\sum_{k=-r-1}^{r+1} g_k k^m = 0 \quad \text{for } m = 0, 1, \ldots, 2r+1. \tag{2.4}$$

(The sum is 0 when m is odd, because $g_k = g_{-k}$.) The polynomials $1, k, k^2, \ldots, k^{2r+2}$ form a basis for the $(2r+3)$-dimensinal vector space of functions on $\{-r-1, \ldots, 0, \ldots, r+1\}$. Therefore any two non-zero solutions to the equations (2.4) are proportional.

We claim that

$$g'_k = (-1)^k \binom{2r+2}{r+1-k} \quad \text{for } k = -r-1, \dots, r+1$$

is a solution.

Let X be the set of polynomials : $\mathbb{Z} \to \mathbb{R}$ and define a linear map $T : X \to X$ by

$$T(P)(k) = -P(k+1) + 2P(k) - P(k-1).$$

Then it is easy to see that

$$\operatorname{degree} T(P) \leqslant \operatorname{degree} P - 2.$$

Thus $T^{r+1}(P) = 0$ for any polynomial of degree $\leqslant 2r+1$. However

$$T^{r+1}(P)(0) = \sum_{k=-r-1}^{r+1} (-1)^k \binom{2r+2}{r+1-k} P(k).$$

This shows that

$$\text{(2.5)} \qquad \sum_{k=-r-1}^{r+1} g'_k k^m = 0 \quad \text{for } m = 0, 1, \dots, 2r+1.$$

Hence we have proved that

$$d_k = (-1)^k k g_k = ck \binom{2r+2}{r+1-k} \quad \text{for } k = 1, 2, \dots, r+1,$$

where c is a constant. But

$$k\binom{2r+2}{r+1-k} = (r+1)\left\{\binom{2r+1}{r+1-k} - \binom{2r+1}{r-k}\right\}.$$

Thus

$$\sum_{k=1}^{r+1} d_k = c(r+1) \sum_{k=1}^{r+1} \left\{\binom{2r+1}{r+1-k} - \binom{2r+1}{r-k}\right\} = c(r+1)\binom{2r+1}{r}.$$

By (2.3), $c = \frac{1}{r+1}$ and we get

$$d_k = \frac{k}{r+1}\binom{2r+2}{r+1-k} = \binom{2r+1}{r+1-k} - \binom{2r+1}{r-k}. \quad \blacksquare$$

3. COMPARISON OF THE NORMS $\|\cdot\|_J$, $\|\cdot\|_{tb}$ and $\|\cdot\|_\gamma$

Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis of a Hilbert space K such that $\dim K = 2n+1$ as in Section 1. For a base $e_{m_1} \wedge \cdots \wedge e_{m_n} \in K_n$ and a base $e_{m'_1} \wedge \cdots \wedge e_{m'_{n+1}} \in K_{n+1}$, the exterior product satisfies that

$$(e_{m'_1} \wedge \cdots \wedge e_{m'_{n+1}}) \wedge (e_{m_1} \wedge \cdots \wedge e_{m_n}) = \begin{cases} \quad e_1 \wedge \cdots \wedge e_{2n+1} \\ \text{or} \quad -e_1 \wedge \cdots \wedge e_{2n+1} \\ \text{or} \quad 0 \end{cases}$$

after alternating. Then we may regard

$$(e_{m'_1} \wedge \cdots \wedge e_{m'_{n+1}}) \wedge (e_{m_1} \wedge \cdots \wedge e_{m_n})$$

as a number, explicitly $1, -1$ or 0. Let M be a set $\{m_1, \ldots, m_n\}$ ordered by $m_1 < \cdots < m_n$ where $m_k \in \{1, \ldots, 2n+1\}$ and denote $e_{m_1} \wedge \cdots \wedge e_{m_n}$ by e_M. We choose $\{m'_1, \ldots, m'_{n+1}\} \subset \{1, \ldots, 2n+1\}$ such that

$$(e_{m'_1} \wedge \cdots \wedge e_{m'_{n+1}}) \wedge e_M = 1$$

and denote $e_{m'_1} \wedge \cdots \wedge e_{m'_{n+1}}$ by e_{M^c}.

Note that the inner products on the basis of K_n and K_{n+1} are respectively given by

$$(e_M | e_N) = e_{M^c} \wedge e_N \quad \text{for } e_M, e_N \in K_n,$$
$$(e_{M^c} | e_{N^c}) = e_{M^c} \wedge e_N \quad \text{for } e_{M^c}, e_{N^c} \in K_{n+1}.$$

In this setting, it is clear that

$$(a_k \otimes a_k e_{M^c} \otimes e_{N^c} | e_S \otimes e_T) = (a_k^* \otimes a_k^* e_M \otimes e_N | e_{S^c} \otimes e_{T^c}).$$

So if we identify K_n with K_{n+1} by letting e_M correspond to e_{M^c}, then $a_k \otimes a_k$ is selfadjoint on $H \otimes H$ in Theorem 1.1.

From now on, we shall use this identification.

Let v be the selfadjoint unitary operator on $H \otimes H$ given by

$$v(\xi \otimes \eta) = \eta \otimes \xi \quad \text{for } \xi, \eta \in H.$$

Since

$$\begin{aligned}(a_k \otimes a_k v e_M \otimes e_N | e_S \otimes e_T) &= e_k \wedge e_N \wedge e_S \cdot e_k \wedge e_M \wedge e_T \\ &= (v a_k \otimes a_k e_M \otimes e_N | e_S \otimes e_T),\end{aligned}$$

v commutes with $a_k \otimes a_k$. It follows that $\sum_{k=1}^{2n+1} a_k \otimes a_k v$ is selfadjoint.

Let C and D be subsets in $\{1,\dots,2n+1\}$ such that

$$|C| = |D| = n-i \quad (0 \leqslant i \leqslant n) \quad \text{and} \quad C \cap D = \emptyset$$

and e_{QQ} the one-dimensional projection : $H \to \mathbb{C}e_Q$, where e_Q is a base in H as above. For fixed C, D and i $(0 \leqslant i \leqslant n)$, we put

$$P_{i,C,D} = \sum_{\substack{Q,R \\ |Q\cap R|=i \\ Q\setminus R=C \\ R\setminus Q=D}} e_{QQ} \otimes e_{RR}.$$

Note that $\{P_{i,C,D}\}_{i,C,D}$ are mutually orthogonal projections such that

$$\bigoplus_i \bigoplus_{C,D} P_{i,C,D} = I$$

on $H \otimes H$.

LEMMA 3.1.

(i) *$P_{i,C,D}$ commutes with* $\sum_{k=1}^{2n+1} a_k \otimes a_k v$.

(ii) $P_{i,C,D}\left(\sum_{k=1}^{2n+1} a_k \otimes a_k\right)v$ *has the matrix representation* $(-1)^{n-i}A^{(i)}$, *where* $A^{(i)}$ *is the adjacency matrix of the Kneser graph* $K_{2i+1}^{(i)}$.

Proof. Put $F_{i,C,D} = \{(Q,R) \mid |Q\cap R| = i, Q\setminus R = C, R\setminus Q = D\}$. For $e_M, e_N, e_S, e_T \in H$, we have

$$\begin{aligned}
&(P_{i,C,D}a_k \otimes a_k v e_M \otimes e_N | e_S \otimes e_T) \\
&= \sum_{(Q,R)\in F_{i,C,D}} ((e_k \wedge e_N) \otimes (e_k \wedge e_M) | e_S \otimes e_T)\delta_{Q,S}\delta_{R,T} \\
&= \sum_{(Q,R)\in F_{i,C,D}} e_k \wedge e_N \wedge e_S \cdot e_k \wedge e_M \wedge e_T \delta_{Q,S}\delta_{R,T} \qquad (3.1)\\
&= \begin{cases} e_k \wedge e_N \wedge e_S \cdot e_k \wedge e_M \wedge e_T & \text{if } (M,N),(S,T) \in F_{i,C,D} \text{ and} \\ & \quad |\{k\}\cup N\cup S| = |\{k\}\cup M \cup T| = 2n+1, \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

By the same way, it is easy to see that $P_{i,C,D}$ commutes with $a_k \otimes a_k v$. In the upper case of (3.1), we note that

$$e_k \wedge e_N \wedge e_S = e_k \wedge e_M \wedge e_T \cdot \operatorname{sgn}\begin{pmatrix} c_1 & \dots & c_{n-i} & d_1 & \dots & d_{n-i} \\ d_1 & \dots & d_{n-i} & c_1 & \dots & c_{n-i}\end{pmatrix}$$

where $C = \{c_1, \dots, c_{n-i}\}$ and $D = \{d_1, \dots, d_{n-i}\}$. Thus it follows that

$$\begin{aligned} e_k \wedge e_N \wedge e_S \cdot e_k \wedge e_M \wedge e_T \\ &= \operatorname{sgn}\begin{pmatrix} 1 & \dots & n-i & n-i+1 & \dots & 2(n-i) \\ n-i+1 & \dots & 2(n-i) & 1 & \dots & n-i \end{pmatrix} \\ &= (-1)^{n-i}. \end{aligned}$$

This implies that the number $(P_{i,C,D}a_k \otimes a_k v e_M \otimes e_N | e_S \otimes e_T)$ does not depend on the choise of k, that is $(-1)^{n-i}$ or 0.

On the other hand, we have

$$\begin{aligned} &\{((M,N),(S,T)) \,|\, (P_{i,C,D} \sum_{k=1}^{2n+1} a_k \otimes a_k v e_M \otimes e_N | e_S \otimes e_T) \neq 0\} \\ &\quad = \{((M,N),(S,T)) \,|\, (M,N),(S,T) \in F_{i,C,D}, N \cap S = M \cap T = \emptyset\}. \end{aligned} \tag{3.2}$$

Moreover the set in (3.2) is bijective to

$$\{(X,Y) \,|\, X, Y \subset \{1, \dots, 2n+1\} \setminus (C \cup D), X \cap Y = \emptyset, |X| = |Y| = i\} \tag{3.3}$$

by

$$((M,N),(S,T)) \mapsto (M \cap N, S \cap T) \quad \text{and} \quad (X,Y) \mapsto ((X \cup C, X \cup D), (Y \cup C, Y \cup D)).$$

Since $|\{1, \dots, 2n+1\} \setminus (C \cup D)| = 2i+1$, the set in (3.3) is the indices which the Kneser graph $K_{2i+1}^{(i)}$ is joint. Hence the matrix corresponding to $P_{i,C,D} \sum_{k=1}^{2n+1} a_k \otimes a_k v$ is $(-1)^{n-i} A^{(i)}$. ∎

For $x \in B(H) \otimes B(H)$, we set $\|x\|_1 = \tau \otimes \tau(|x|)$ where τ is the normalized trace on $B(H)$.

LEMMA 3.2.

$$\left\| \sum_{k=1}^{2n+1} a_k \otimes a_k \right\|_1 = \frac{\binom{4n+1}{2n}}{\binom{2n+1}{n}^2}.$$

Proof. We may identify $P_{i,C,D} \sum_{k=1}^{2n+1} a_k \otimes a_k v$ with $(-1)^{n-i} A^{(i)}$. Then we can respect that

$$\begin{aligned} \sum_{k=1}^{2n+1} a_k \otimes a_k v &= \bigoplus_i \bigoplus_{C,D} P_{i,C,D} \sum_{k=1}^{2n+1} a_k \otimes a_k v \\ &= \bigoplus_i \bigoplus_{C,D} (-1)^{n-i} A^{(i)}. \end{aligned}$$

For fixed i, the multiplicity of $(-1)^{n-i}A^{(i)}$ is $|\{(C,D)\,|\,C\cap D=\emptyset, |C|=|D|=n-i\}|$, that is $\frac{(2n+1)!}{(n-i)!^2(2i+1)!}$.

Since $A^{(i)}$ has the eigenvalues $(-1)^{i+1-j}j, (j=1,\ldots,i+1)$, then $P_{i,C,D}\sum_{k=1}^{2n+1} a_k \otimes a_k v$ has the eigenvalues $\lambda_j=(-1)^{n+1-j}j, (j=1,\ldots,i+1)$ and the dimension of the eigenspace of $P_{i,C,D}\sum_{k=1}^{2n+1} a_k\otimes a_k v$ belonging to λ_j is

$$\binom{2i+1}{i+1-j}-\binom{2i+1}{i-j}.$$

Let E_j be the eigenspace of $\sum_{k=1}^{2n+1} a_k\otimes a_k v$ belonging to λ_j, then we have

$$\begin{aligned}\dim E_j &= \sum_{i=0}^{n}\left\{\binom{2i+1}{i+1-j}-\binom{2i+1}{i-j}\right\}\frac{(2n+1)!}{(n-i)!^2(2i+1)!}\\ &= \binom{2n+1}{n+1-j}^2-\binom{2n+1}{n-j}^2.\end{aligned}$$

Since $|\lambda_j|=j$, we get

$$\begin{aligned}\dim H\otimes H\left\|\sum_{k=1}^{2n+1} a_k\otimes a_k\right\|_1 &= \dim H\otimes H\left\|\sum_{k=1}^{2n+1} a_k\otimes a_k v\right\|_1\\ &= \sum_{k=0}^{n}\binom{2n+1}{k}^2=\frac{1}{2}\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2\\ &= \frac{1}{2}\binom{4n+2}{2n+1}=\binom{4n+1}{2n}.\end{aligned}$$

Since $\dim H=\binom{2n+1}{n}$, we obtain

$$\left\|\sum_{k=1}^{2n+1} a_k\otimes a_k\right\|_1=\frac{\binom{4n+1}{2n}}{\binom{2n+1}{n}^2}. \quad ∎$$

Lemma 3.3.

$$\left\|\sum_{k=1}^{2n+1} a_k\otimes a_k\right\|_2=\frac{n+1}{\sqrt{2n+1}}.$$

Proof. As in the proof of Theorem 1.1, $\|a_k\|_2^2=\frac{n+1}{2n+1}$ for all k. Then we have

$$\Big\|\sum_{k=1}^{2n+1} a_k\otimes a_k\Big\|_2^2 = \tau\otimes\tau\left(\Big(\sum_{k=1}^{2n+1} a_k\otimes a_k\Big)^*\Big(\sum_{k=1}^{2n+1} a_k\otimes a_k\Big)\right)$$
$$= \sum_{k,\ell=1}^{2n+1}\tau(a_k^*a_\ell)\tau(a_k^*a_\ell) = \sum_{k=1}^{2n+1}\tau(a_k^*a_k)^2$$
$$= \sum_{k=1}^{2n+1}\|a_k\|_2^4 = \frac{(n+1)^2}{2n+1}. \quad \blacksquare$$

Let $\overline{H}$ be the "conjugate" Hilbert space whose underlying real vector space is the same as H and the identity map $\xi\mapsto\overline{\xi}$ is conjugate linear with the inner product $(\overline{\xi}|\overline{\eta}) = \overline{(\xi|\eta)}$. If A is a set in $B(H)$ and $x\in A$, then there exists a unique $\overline{x}\in B(\overline{H})$ such that $\overline{x}\overline{\xi} = \overline{x\xi}$ for $\overline{\xi}\in\overline{H}$. We denote $\{\overline{x}\in B(\overline{H})\,|\,x\in A\}$ by $\overline{A}$.

Define the bilinear form W on $B(H)\times B(\overline{H})$ by

$$W(x,\overline{y}) = \frac{2n+1}{n+1}V(x,y^*) \qquad \text{for } x\in B(H) \text{ and } \overline{y}\in B(\overline{H})$$

and let P be the orthogonal projection: $B(H)\to \operatorname{span}\{a_1,\dots,a_{2n+1}\}$ with respect to the inner product $(x|y)_\tau$ on $B(H)$: i.e.

$$P(x) = \frac{2n+1}{n+1}\sum_{k=1}^{2n+1}(x|a_k)_\tau a_k.$$

We note that $W(x,\overline{y}) = (P(x)|P(y))_\tau$.

Let $W^{(k)}$ be the bilinear form on $M_k(B(H))\times M_k(B(\overline{H}))$, such that

$$W^{(k)}([x_{i,j}],[\overline{y}_{i,j}]) = \frac{1}{k}\sum_{i,j=1}^{k} W(x_{ij},\overline{y}_{ij})$$

for $[x_{ij}]\in M_k(B(H))$ and $[\overline{y}_{ij}]\in M_k(B(\overline{H}))$. Then

$$\|W\|_{\mathrm{tb}^*} = \sup_k\|W^{(k)}\|.$$

THEOREM 3.4.

$$K_{\mathrm{tb}} \geqslant \frac{\pi}{2}.$$

Proof. It is easy to see that

$$W^{(k)}([x_{ij}],[\overline{y}_{ij}]) = \Big(P\otimes I_k[x_{ij}]\,|\,P\otimes I_k[y_{ij}]\Big)_{\tau\otimes\tau_k}$$

where τ_k is the normalized trace on $M_k(\mathbb{C})$.

Let $u\left|\sum_{k=1}^{2n+1} a_k \otimes a_k\right|$ be the polar decomposition of $\sum_{k=1}^{2n+1} a_k \otimes a_k$. Then we have

$$\|W\|_{\mathrm{tb}^*} \geqslant \|(P\otimes I)u\|_2^2 \geqslant \frac{\left|\left((P\otimes I)u \mid \sum_{k=1}^{2n+1} a_k \otimes a_k\right)_{\tau\otimes\tau}\right|^2}{\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_2^2}$$
$$= \frac{\left|\left(u \mid \sum_{k=1}^{2n+1} a_k \otimes a_k\right)_{\tau\otimes\tau}\right|^2}{\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_2^2} = \frac{\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_1^2}{\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_2^2}.$$

Since $\|W\| = \frac{n+1}{2n+1}$ by Remark 1.4 and the definition, and $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ by the Stirling formula, we obtain that

$$\frac{\|W\|_{\mathrm{tb}^*}}{\|W\|} \geqslant \frac{(2n+1)\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_1^2}{(n+1)\left\|\sum_{k=1}^{2n+1} a_k \otimes a_k\right\|_2^2}$$
$$= \frac{(2n+1)^2\binom{4n+2}{2n}^2}{(n+1)^3\binom{2n+1}{n}^4}$$
$$\to \frac{\pi}{2} \qquad (n\to\infty). \quad ∎$$

THEOREM 3.5.

$$K_{\mathrm{J,tb}} > 1.$$

Proof. Put $A = M_3(\mathbb{C})$ with the normalized trace τ and let $W(x,\overline{y}) = (P(x)|P(y))_\tau$ for $x \in A$ and $\overline{y} \in \overline{A}$ as in Theorem 3.4. Since the unit ball in a unital C^*-algebra is the closed convex hull of the unitary elements and the unit ball in $M_n(A)$ is compact, we can choose unitary elements $u_n \in M_n(A)$ and $\overline{v}_n \in M_n(\overline{A})$ for each n such that

$$W^{(n)}(u_n, \overline{v}_n) = \|W^{(n)}\|.$$

It is easy to see that

$$|W^{(n)}(x,\overline{y})| \leq W^{(n)}(x,\overline{x})^{\frac{1}{2}} W^{(n)}(y,\overline{y})^{\frac{1}{2}} \quad \text{for all } x \in M_n(A) \text{ and all } \overline{y} \in M_n(\overline{A}).$$

Thus we may assume that

$$W^{(n)}(u_n, \overline{u}_n) = \|W^{(n)}\|.$$

Let $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ be a free ultrafilter and τ_n the normalized trace on $M_n(\mathbf{C})$. We set that

$$M = \bigoplus_{n=1}^{\infty} \frac{M_n(A)}{I_\omega}$$

where

$$I_\omega = \Big\{(x_n) \in \bigoplus_{n=1}^{\infty} M_n(A) \,|\, \lim_{n\to\omega} \tau \otimes \tau_n(x_n^* x_n) = 0\Big\}.$$

Then M is a factor of type II_1 with the trace

$$\tau \otimes \tau_\omega(x) = \lim_{n\to\omega} \tau \otimes \tau_n(x_n)$$

where (x_n) are representing sequences for x. It is clear that M is $*$-isomorphic to $A \otimes N$ where

$$N = \bigoplus_{n=1}^{\infty} \frac{M_n(\mathbf{C})}{J_\omega}, \quad J_\omega = \Big\{(x_n) \in \bigoplus_{n=1}^{\infty} M_n(\mathbf{C}) \,|\, \lim_{n\to\omega} \tau_n(x_n^* x_n) = 0\Big\}.$$

We define the bilinear form $\widetilde{W}$ on $M \times \overline{M}$ by

$$\widetilde{W}(x, \overline{y}) = \lim_{n\to\omega} W^{(n)}(x_n, \overline{y}_n),$$

where $\overline{M}$ is the "conjugate" algebra for M as in Theorem 3.4. Let u be the unitary in M with representing sequence (u_n). Then clearly

$$\widetilde{W}(u, \overline{u}) = \lim_{n\to\omega} W^{(n)}(u_n, \overline{u}_n) = \|W\|_{\mathrm{tb}^*}.$$

Assume that

$$\|W\|_{\mathrm{tb}^*} = \|W\|_{\mathrm{J}^*} \quad (\text{i.e. } \|W\|_{\mathrm{tb}^*} = 1 \text{ by Remark 1.4}).$$

Then

$$\|P \otimes I_N(u)\|_2^2 = \widetilde{W}(u, \overline{u}) = \|W\|_{\mathrm{tb}^*} = \|u\|_2^2.$$

This implies that u is in the range of $P \otimes I_N$. Thus there exist $x, y, z \in N$ such that

$$u = a_1 \otimes x + a_2 \otimes y + a_3 \otimes z :$$

i.e.

$$u = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$

Clearly it follows that

$$zz^* + yy^* = z^*z + y^*y = 1$$
$$zz^* + xx^* = x^*x + z^*z = 1$$
$$xx^* + yy^* = y^*y + x^*x = 1$$

and

$$0z - z^*0 - y^*x = 0.$$

Hence we get that $xx^* = x^*x = yy^* = y^*y = \frac{1}{2}$ and $y^*x = 0$.

This is a contradiction. ∎

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