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ON WEAK COMPACTNESS OF COMPOSITION OPERATORS ON BERGMAN SPACES OF SEVERAL COMPLEX VARIABLES

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ABSTRACT. Sarason proved that weak compactness of a holomorphic composition operator on H^1 of the unit disc is equivalent to norm compactness. The purpose of this paper is to obtain a result similar to Sarason's theorem for Bergman spaces over strongly pseudoconvex domains.

KEYWORDS: *Weak compactness, composition operator, Bergman spaces.*

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ be a holomorphic mapping from Ω into Ω. Then we define the *composition* operator C_φ as follows: $C_\varphi(u)(z) = u(\varphi(z))$ for all $z \in \Omega$ and functions u on Ω.

The study of composition operators in one variable has been active since the early 1970's. Most questions on composition operators were concerned with compactness, the boundedness being automatic. For the unit disk $\mathbb{D} \subset \mathbb{C}$, Shapiro and Taylor ([16]) proved that the composition operator $C_\varphi : \mathcal{H}^p(\mathbb{D}) \to \mathcal{H}^p(\mathbb{D})$ is compact for one $p \in (0, \infty)$ if and only if it is compact on $\mathcal{H}^p(\mathbb{D})$ for all $p \in (0, \infty)$. Shapiro ([15]) discussed a characterization of compactness for $C_\varphi : \mathcal{H}^p(\mathbb{D}) \to \mathcal{H}^p(\mathbb{D})$ in terms of the Nevanlinna counting function. For Bergman spaces on $\mathbb{D}$ and $\mathbb{B}_n$, see the paper by MacCluer and Shapiro ([12]). Recently, Sarason ([14]) proved that weak compactness and norm compactness for a composition operator on the Hardy space $\mathcal{H}^1(\mathbb{D})$ are equivalent.

In the case of several complex variables, many results on compactness have been provided by various authors in the last decade. For example, MacCluer ([11]) gave a characterization of compactness of $C_\varphi : \mathcal{H}^p(\mathbf{B}_n) \to \mathcal{H}^p(\mathbf{B}_n)$ in terms of a Carleson measure condition for the pullback measure $d\mu_\varphi$. And Jafari ([5], [6]) obtained the characterization for C_φ on the weighted Bergman spaces of the polydisc $\mathbf{D}^n$ and of bounded symmetric domains. More recently, corresponding to Sarason and Shapiro's work in one variable, Li and Russo ([10]) extended their results from the unit disc to strongly pseudoconvex domains in $\mathbf{C}^n$.

The purpose of this paper is to consider the analogue of Sarason's theorem for composition operators on the Bergman space $A^1(\Omega)$, where Ω is a strongly pseudoconvex domain with C^1 boundary or a bounded symmetric domain of tube type.

Our proof will use the following duality theorems. Let Ω be a strongly pseudoconvex domain or a bounded symmetric domain. There are Banach spaces $\mathcal{B}(\Omega)$, $\mathcal{B}_0(\Omega)$ of holomorphic functions on Ω such that

$$(\mathcal{B}_0)^* = A^1, \quad (A^1)^* = \mathcal{B}. \tag{1.1}$$

The space $\mathcal{B}(\Omega)$ was constructed by Krantz and Ma in [9] for strongly pseudoconvex domains and by Yan ([17]) for bounded symmetric domains. Although their definitions differ, both of these spaces are called Bloch spaces of Ω. Yan also defines a little Bloch space $\mathcal{B}_0$ which satisfies (1.1) for bounded symmetric domains of tube type. We show in this paper that for strongly pseudoconvex domains, the little Bloch space $\mathcal{B}_0$ corresponding to the definition of $\mathcal{B}$ in Krantz and Ma's paper ([9]) also satisfies (1.1).

This paper is organized in the following way. In Section 2, we introduce the properties of Bergman kernel on strongly pseudoconvex domains and present some facts for the Bergman projection on function spaces of strongly pseudoconvex domains. In Section 3 we give some proofs for duality theorems for Bergman spaces on strongly pseudoconvex domains. Then in Section 4, we prove the main theorem on compactness of composition operators on strongly pseudoconvex domains. Finally in the same section, by using Yan's duality theorems, we show that our main result is also true in the case of bounded symmetric domains of tube type.

2. PRELIMINARIES ON STRONGLY PSEUDOCONVEX DOMAINS

Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary. Let ρ be a C^∞ pluri-subharmonic defining function for Ω:

$$\Omega = \{z \in \mathbb{C}^n : \rho < 0\}.$$

Let $\mathrm{d}V$ be the Lebesgue volume measure in $\mathbb{C}^n$. For $\nu > -1$, we let

$$\mathrm{d}V_\nu(z) = |\rho(z)|^\nu \mathrm{d}V(z),$$

and

$$L^2_\nu = L^2(\Omega, \mathrm{d}V_\nu).$$

Write $H(\Omega)$ as the set of holomorphic functions on Ω. Let $A^2(\Omega)$ be the Bergman space $H(\Omega) \cap L^2(\Omega)$. P denotes the Bergman projection from $L^2(\Omega)$ onto $A^2(\Omega)$, i.e.

$$Pf(z) = \int_\Omega f(w)K(z,w)\,\mathrm{d}V(w), \quad f \in L^2(\Omega),$$

where $K(z,w)$ is the Bergman kernel. We also consider the weighted Bergman spaces $A^2_\nu(\Omega) \equiv H(\Omega) \cap L^2_\nu(\Omega)$. The orthogonal projection of L^2_ν onto A^2_ν will be denoted by P_ν,

$$P_\nu f(z) = \int_\Omega K_\nu(z,w)f(w)\,\mathrm{d}V_\nu(w), \quad f \in L^2_\nu, \tag{2.1}$$

where $K_\nu(z,w)$ is called the weighted Bergman kernel. If $\nu = 0$, then $K_0(z,w) = K(z,w)$ becomes the usual Bergman kernel.

For more discussions on Bergman kernel and weighted Bergman kernel, see [4], [7] and [13]. We list some useful results here for our purpose. First, let ρ be the defining function for Ω, $L_\rho(w)$ denote the Levi form. For $z, w \in \Omega$, we set

$$\sigma(z,w) = |z-w|^2 + \Big|\sum_j \frac{\partial \rho}{\partial w_j}(w)(z_j - w_j)\Big|.$$

$$R_\delta = \{(z,w) \in \overline{\Omega} \times \overline{\Omega} : |\rho(z)| + |\rho(w)| + |z-w| < \delta\}.$$

The following lemma is due to Peloso and Fefferman.

LEMMA 2.1. *Let Ω be a smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$, ν be a positive integer, then*

(i) $K_\nu(z,w) = C_\Omega|\nabla\rho(w)|^2 \det L_\rho(w) X^{-\nu}(z,w) + \widetilde{K}(z,w)$, *where* $|X(z,w)| \sim c(|\rho(z)| + |\rho(w)| + \sigma(z,w))$ *for* $(z,w) \in R_\delta$, $\widetilde{K} \in C^\infty(\overline{\Omega}\times\overline{\Omega}\setminus\Delta)$, Δ *being the diagonal of* $\partial\Omega\times\partial\Omega$ *and satisfies the estimate*

$$|\widetilde{K}(z,w)| \sim |X(z,w)|^{-(\nu+\frac{1}{2})} \cdot \big|\log|X(z,w)|\big|.$$

(ii) $K_\nu(z,z) = \Phi(z)|\rho(z)|^{-(n+1)} + \widetilde{\Phi}(z)\log|\rho(z)|$ *for z near $\partial\Omega$, where* $\Phi(z)$, $\widetilde{\Phi}(z)$ *are functions in* $C^\infty(\overline{\mathbf{D}})$, $\Phi(z)\neq 0$ *for* $z\in\partial\Omega$.

The following can be easily derived from the above:

REMARK 2.2. For any $w, z\in\Omega$,

(i) $K_\nu(z,w) = \overline{K_\nu(w,z)}$;

(ii) $K_\nu(z,w)\in C^\infty(\Omega\times\overline{\Omega})$;

(iii) $K_\nu(\cdot,w)\in L^\infty(\Omega)$.

Also we will use the Peloso's estimation, see Lemma 2.7 in [13].

LEMMA 2.3. $\int_\Omega |\nabla_z K_\nu(z,w)|\,\mathrm{d}V(w) \sim C|\rho(z)|^{-1}$, *where Ω is a smoothly bounded strongly pseudoconvex domain and ρ is a defining function.*

Notice that if $d(z)$ is the distance between the z and the boundary of the domain, then the following function is a defining function:

$$\rho(z) = \begin{cases} -d(z) & z\in\Omega \\ d(z) & z\in\mathbb{C}^n\setminus\Omega. \end{cases}$$

It is obvious that the Bergman projection P is the identity map on $A^2(\Omega)$. The following is also true.

LEMMA 2.4. *P is the identity map on $A^1(\Omega)$.*

Proof. From Beatrous paper, we know that $A^2(\Omega)$ is dense in $A^1(\Omega)$ in norm, see [1]. Let $g\in A^1(\Omega)$, there exists $\{g_n\}\in A^2(\Omega)$ such that $g_n\to g$ in $A^1(\Omega)$, since $K(\cdot,z)$ is bounded for each $z\in\Omega$, it is elementary that

$$\int_\Omega g_n(w)K(z,w)\,\mathrm{d}V(w) \to \int_\Omega g(w)K(z,w)\,\mathrm{d}V(w), \quad z\in\Omega.$$

On the other hand,

$$\int_\Omega g_n(w)K(z,w)\,\mathrm{d}V(w) = g_n(z).$$

Since g and each g_n are holomorphic functions on bounded domain Ω, the well-known inequality $|g_n - g| \leqslant C\int_\Omega |g_n - g|\,\mathrm{d}V$ implies that $g_n \to g$ pointwise. Therefore we must have

$$\int_\Omega g(w)K(z,w)\,\mathrm{d}V(w) = g(z). \quad \blacksquare$$

DEFINITION 2.5. Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$, we define the space of *Bloch functions* on Ω to be

$$\mathcal{B}(\Omega) = \Big\{f \in H(\Omega) : \sup_{z\in\Omega} |\nabla f(z)| d(z) < \infty\Big\}$$

where $d(z)$ is the distance between the z and the boundary of Ω and $\nabla f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$.

This definition is equivalent to the definition of Bloch functions in Krantz and Ma's paper, see [9]. We can also show that if the Bloch functions are equipped with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\Omega} |\nabla f(z)| d(z)$ for $f \in \mathcal{B}(\Omega)$, then $\mathcal{B}(\Omega)$ becomes a Banach space.

We also define the *little Bloch space* to be

$$\mathcal{B}_0 = \Big\{f \in \mathcal{B}(\Omega) : \lim_{z\to\partial\Omega} |\nabla f(z)| d(z) = 0\Big\}.$$

It is easy to see that the polynomials are contained in $\mathcal{B}_0$.

The following lemma is well-known and is important to the proofs of duality theorems. For completeness, we give a proof parallel to that presented in Krantz and Li's paper, see [8].

LEMMA 2.6. *Let Ω be a strongly pseudoconvex domain, let P be the Bergman projection, then*

$$PL^\infty(\Omega) \cong \mathcal{B}(\Omega). \tag{2.2}$$

Proof. First, let $f \in L^\infty(\Omega)$, we try to show that $|\nabla Pf(z)| d(z) < \infty$. By Lemma 2.3

$$\begin{aligned} |\nabla Pf(z)| d(z) &= d(z)\Big|\int_\Omega f(w)\,\nabla_z K(z,w)\,\mathrm{d}V(w)\Big| \\ &\leqslant d(z)\|f\|_\infty\, c\cdot d(z)^{-1} \\ &= c\|f\|_\infty < \infty. \end{aligned}$$

Next, let $f \in \mathcal{B}(\Omega)$, we demonstrate that there exists $g \in L^\infty(\Omega)$ such that $f = Pg$. We can construct a partition of unity by choosing an open cover $U_0, U_1, \ldots, U_m$ of Ω so that U_0 is relatively compact in Ω and $U_i \cap \partial\Omega \neq \emptyset$ for

$i \neq 0$. There are functions χ_i, $0 \leqslant i \leqslant m$, with χ_i being characteristic function of U_i, such that $\sum_{i=0}^{m} \chi_i = 1$ on $\overline{\Omega}$. Let $\rho(z)$ be the defining function for Ω, we may choose a local holomorphic coordinates $\{z_{k_i}\}$ so that $0 < c < |\frac{\partial \rho(z)}{\partial z_{k_i}}| < C < \infty$ for all $z \in U_i$ and some $1 \leqslant k_i \leqslant n$. Then, since $\mathcal{B}(\Omega) \subset A^2(\Omega)$,

(2.3)
$$
\begin{aligned}
f(z) &= \int_\Omega f(w)K(z,w)\,\mathrm{d}V(w) \\
&= \int_\Omega \Big(\sum_{i=0}^{m} \chi_i(w)\Big) f(w)K(z,w)\,\mathrm{d}V(w) \\
&= \int_\Omega \chi_0 f(w)K(z,w)\,\mathrm{d}V(w) + \sum_{i=1}^{m} \int_\Omega \chi_i f(w)K(z,w) \frac{\partial \rho}{\partial w_{n_i}} \Big/ \frac{\partial \rho}{\partial w_{n_i}} \mathrm{d}V(w).
\end{aligned}
$$

Here, χ_0 has compact support, so $\chi_0 f$ is bounded. Let us now look at the second term of the last expression in (2.3). Understand that U_0 can be chosen so that the z in (2.3) is contained in U_0. Therefore $\chi_i(z) = 0$ for $i = 1, \dots, m$. Using integration by parts, we can get

$$
\begin{aligned}
&\sum_{i=1}^{m} \int_\Omega \chi_i f(w)K(z,w) \frac{\partial \rho/\partial w_{n_i}}{\partial \rho/\partial w_{n_i}}\,\mathrm{d}V(w) \\
&\qquad = -\sum_{i=1}^{m} \int_\Omega \frac{\partial}{\partial w_{n_i}} \left(\frac{\chi_i(w)}{\partial \rho/\partial w_{n_i}} f(w)K(z,w) \right) \rho(w)\,\mathrm{d}V(w) \\
(2.4)\qquad &\qquad = -\sum_{i=1}^{m} \int_\Omega \frac{\chi_i(w)}{\partial \rho/\partial w_{n_i}} \frac{\partial f(w)}{\partial w_{n_i}} K(z,w)\,\rho(w)\,\mathrm{d}V(w) \\
&\qquad\quad - \sum_{i=1}^{m} \int_\Omega \left(\frac{\partial}{\partial w_{n_i}} \left(\frac{\chi_i(w)}{\partial \rho/\partial w_{n_i}} \right) \right) f(w)K(z,w)\,\rho(w)\,\mathrm{d}V(w).
\end{aligned}
$$

Note that $\frac{\partial f(w)}{\partial w_{n_i}} \rho(w)$ is bounded, since f is a Bloch function. And $\frac{\chi_i(w)}{\partial \rho/\partial w_{n_i}}$ is also bounded, since $\frac{\partial \rho(z)}{\partial z_{n_i}}$ is chosen to be bounded below and above. We let

$$
(2.5) \qquad g(z) = \chi_0 f(z) - \sum_{i=1}^{m} \left(\frac{\chi_i(z) \frac{\partial f(z)}{\partial z_{n_i}} \rho(z)}{\partial \rho/\partial z_{n_i}} + \frac{\partial}{\partial z_{n_i}} \left(\frac{\chi_i(z)}{\partial \rho/\partial z_{n_i}} \right) f(z)\rho(z) \right).
$$

Then $g(z) \in L^\infty(\Omega)$, and by (2.3), $f(z) = Pg(z)$. ∎

Recall that $C_0(\Omega)$ is the set of continuous functions that vanish at the boundary $\partial\Omega$. We have an analogue for the little Bloch space. The result is well-know for the domain of the unit ball.

LEMMA 2.7. *Let Ω be a smoothly bounded strongly pseudoconvex domain, then*

$$PC_0(\Omega) = \mathcal{B}_0(\Omega),$$

where P is the Bergman projection.

Proof. We first show that $PC_0(\Omega) \subset \mathcal{B}_0(\Omega)$. Let $\varphi \in C_0(\Omega)$, we shall prove that $|\nabla(P\varphi)| \cdot d(z) \to 0$ as $z \to \partial\Omega$.

Consider the region

$$\Omega_\delta = \{z \in \Omega : |\rho(z)| > \delta\}.$$

Then

$$\begin{aligned} |\nabla(P\varphi)| \cdot d(z) &= \left| \nabla \int_\Omega \varphi(w) K(z,w)\, \mathrm{d}V \right| \cdot d(z) \\ &= \left| d(z) \int_{\Omega_\delta} \varphi(w) \nabla K(z,w)\, \mathrm{d}V + d(z) \int_{\Omega - \Omega_\delta} \varphi(w) \nabla K(z,w)\, \mathrm{d}V \right| \\ &\leqslant d(z) \max_{w \in \Omega \setminus \Omega_\delta} |\varphi(w)| \int_\Omega |\nabla K(z,w)|\, \mathrm{d}V \\ &\quad + d(z) \left| \int_{\Omega_\delta} \varphi(w) \nabla K(z,w)\, \mathrm{d}V \right|. \end{aligned}$$

By Lemma 2.3,

$$\int_\Omega |\nabla K(z,w)|\, \mathrm{d}V \leqslant C_1 |d(z)|^{-1}.$$

Now given $\varepsilon > 0$, there exists δ such that $\max_{w \in \Omega \setminus \Omega_\delta} \varphi(w) < \varepsilon$. Then the first term of the above expression becomes $C_1\varepsilon$. Then we let z go close enough to the boundary so that z is outside of Ω_δ. Since now $\left| \int_{\Omega_\delta} \varphi(w) \nabla K(z,w)\, \mathrm{d}V \right| < C_2$ for some constant C_2 by Remark 2.2, the second term becomes less than $C_2\varepsilon$. This shows that $|\nabla(P\varphi)| \cdot d(z) \to 0$ as $z \to \partial\Omega$.

To prove that $\mathcal{B}_0(\Omega) \subset PC_0(\Omega)$, we borrow the idea in the proof for Lemma 2.6. Letting $f \in \mathcal{B}_0$, we choose a function g as in equation (2.5):

$$g(z) = \chi_0 f(z) - \sum_{i=1}^{m} \left(\frac{\chi_i(z) \frac{\partial f(z)}{\partial z_{n_i}} \rho(z)}{\partial \rho / \partial z_{n_i}} + \frac{\partial}{\partial z_{n_i}} \left(\frac{\chi_i(z)}{\partial \rho / \partial z_{n_i}} \right) f(z) \rho(z) \right).$$

By (2.3), $f(z) = Pg(z)$. We must show that $g \in C_0(\Omega)$. In fact, χ_0 has compact support in Ω, if z is very close to $\partial\Omega$, then $\chi_0(z)f(z) = 0$. Also since $f(z)$ is a little Bloch function, there exists δ, such that if $d(z) < \delta$, then

$$\Big|\sum_{i=1}^{m} \frac{\partial f(z)}{\partial z_{n_i}} \rho(z)\Big| < \varepsilon.$$

At last, since f is in $\mathcal{B}(\Omega)$, $|f(z)| < c \log \frac{1}{d(z)}$, therefore

$$|f(z)\rho(z)| \to 0 \text{ if } z \to \partial\Omega.$$

Summarize the above, we see that $g \in C_0(\Omega)$. This completes the proof of Theorem 2.7. ∎

3. DUALITY FOR BERGMAN SPACES ON STRONGLY PSEUDOCONVEX DOMAINS

In this section we present some dualities for Bergman spaces on strongly pseudoconvex domains. These results are well-known for Bergman spaces on the unit ball, but not on the strongly pseudoconvex domains.

THEOREM 3.1. *Let Ω be a strongly pseudoconvex domain, then $\mathcal{B}_0(\Omega)^* \cong A^1(\Omega)$, $A^1(\Omega)^* \cong \mathcal{B}(\Omega)$.*

Proof. The second duality is well known. For completeness, we give a proof due to Krantz and Li, see [8].

For $g \in \mathcal{B} \subset A^2(\Omega)$, by Lemma 2.6, we know that there exits $\varphi \in L^\infty$, such that $g = P\varphi$. Define a bounded functional on $A^1(\Omega)$ by

$$\ell_g(f) = \int_\Omega f\overline{\varphi}\, dV \quad f \in A^1(\Omega).$$

For $f \in A^2(\Omega)$, we have

$$\ell_g(f) = \int_\Omega f\overline{\varphi}\, dV = \int_\Omega Pf\overline{\varphi}\, dV = \int_\Omega f\overline{g}\, dV.$$

The last integral is not defined for all $f \in A^1(\Omega)$, but it dose exist for f in $A^2(\Omega)$, which is a dense subset in $A^1(\Omega)$, therefore the last integral determines a bounded linear functional on $A^1(\Omega)$. Hence $g \in \mathcal{B}$ corresponds to an element in $(A^1)^*$.

On the other hand, if $\ell \in A^1(\Omega)^*$, then by Hahn-Banach and Riesz representation theorem, there exists $h \in L^\infty(\Omega)$ such that, for each $f \in A^1(\Omega)$,

$$\ell(f) = \int_\Omega f\overline{h}\,\mathrm{d}V.$$

For $f \in A^2(\Omega)$, we have

$$\ell(f) = \int_\Omega Pf\overline{h}\,\mathrm{d}V = \int_\Omega f\overline{Ph}\,\mathrm{d}V.$$

Since A^2 is dense in A^1, the last integral actually determines a linear functional on A^1. By Lemma 2.7, Ph is in $\mathcal{B}$, therefore each ℓ in $A^1(\Omega)^*$ corresponds to an element in $\mathcal{B}(\Omega)$. This proves that $A^1(\Omega)^* \cong \mathcal{B}(\Omega)$.

Now we want to prove that $(\mathcal{B}_0)^* \cong A^1$, under the pairing

$$\langle g, f\rangle = \int_\Omega g(z)\overline{f(z)}\,\mathrm{d}V(z),$$

where f in $A^1(\Omega)$. The above integral is not defined for all $g \in \mathcal{B}_0(\Omega)$, but it does exist for $g \in H^\infty(\Omega)$, which is a dense subset of $\mathcal{B}_0$, see [1].

To show $A^1 \subset \mathcal{B}_0{}^*$, notice that $\mathcal{B}_0 \subset \mathcal{B} \cong (A^1)^*$. Thus

$$A^1 \subset (A^1)^{**} \subset \mathcal{B}_0{}^*.$$

If $F \in (\mathcal{B}_0)^*$, we try to find $f \in A^1$ such that $F(g) = \int_\Omega g\overline{f}\,\mathrm{d}V$ for all $g \in H^\infty(\Omega)$. The following method can be found in Zhu's book for the unit disk, see [18], pp. 87–88. Consider the Hilbert space adjoint P_ν^* of P_ν as in equation (2.1), that is

$$\langle P_\nu f, g\rangle = \langle f, P_\nu^* g\rangle \quad f, g \in L^2_\nu.$$

Then

$$\int_\Omega \Big(\int_\Omega f(w)K_\nu(z,w)|\rho(w)|^\nu\,\mathrm{d}V(w)\Big)\overline{g}(z)\,\mathrm{d}V_\nu(w)$$
$$= \int_\Omega f(w)\overline{\int_\Omega K_\nu(w,z)g(z)|\rho(w)|^\nu\,\mathrm{d}V(z)}\,\mathrm{d}V_n u(w).$$

Therefore

$$P_\nu^* g(z) = |\rho(z)|^\nu \int_\Omega g(w)K_\nu(z,w)\,\mathrm{d}V(w).$$

Now, letting $V = P_{n+1}^*$, we have the following lemma.

LEMMA 3.2. *V is an embedding from $\mathcal{B}_0(\Omega)$ into $C_0(\Omega)$.*

Proof. Given $g \in \mathcal{B}_0(\Omega)$, by Theorem 2.7, there exists $\varphi \in C_0(\Omega)$, such that $g = P\varphi$. Therefore

$$\begin{aligned} Vg(z) &= |\rho(z)|^{n+1} \int_\Omega P\varphi(w) K_{n+1}(z, w)\, \mathrm{d}V(w) \\ &= |\rho(z)|^{n+1} \int_\Omega \int_\Omega \varphi(u) K(w, u)\, \mathrm{d}V(u) K_{n+1}(z, w)\, \mathrm{d}V(w) \\ &= |\rho(z)|^{n+1} \int_\Omega \varphi(u) \int_\Omega \overline{K_{n+1}(w, z) K(u, w)\, \mathrm{d}V(w)}\, \mathrm{d}V(u) \\ &= |\rho(z)|^{n+1} \int_\Omega \varphi(u) K_{n+1}(z, u)\, \mathrm{d}V(u). \end{aligned}$$

To prove Vg is in $C_0(\Omega)$, let $\Omega_\delta = \{z \in \Omega : d(z) > \delta\}$. Then

$$\begin{aligned} Vg(z) &= |\rho(z)|^{n+1} \int_\Omega \varphi(u) K_{n+1}(z, u)\, \mathrm{d}V(u) \\ &= |\rho(z)|^{n+1} \int_{\Omega_\delta} \varphi(u) K_{n+1}(z, u)\, \mathrm{d}V(u) \\ &\qquad + |\rho(z)|^{n+1} \int_{\Omega \setminus \Omega_\delta} \varphi(u) K_{n+1}(z, u)\, \mathrm{d}V(u). \end{aligned}$$

Given $\varepsilon > 0$, there exists δ such that

$$\max_{u \in \Omega \setminus \Omega_\delta} |\varphi(u)| < \varepsilon.$$

Then we choose z be in $\Omega \setminus \Omega_\delta$ and $|\rho(z)| < \varepsilon$, by Remark 2.2,

$$|\varphi(u) K_{n+1}(z, u)| < \infty, \quad u \in \Omega_\delta.$$

Hence the first term of last expression is less than $C_1\varepsilon$ for some C_1. On the other hand, by Lemma 2.1, $|K_{n+1}(z, u)|$ is comparable to $|\rho(z)|^{n+1}$ for $z, u \in \Omega \setminus \Omega_\delta$, so the second term is less than $C_2\varepsilon$ for some C_2.

To show V is an embedding, we first show that V is bounded on $\mathcal{B}_0(\Omega)$. We notice that P_{n+1} is bounded on L^1_{n+1}, so V is bounded on L^∞, therefore there exists $\varphi \in C_0(\Omega)$, such that, for $g \in \mathcal{B}_0$

$$\|Vg\|_\infty = \|V\varphi\|_\infty \leqslant C\|\varphi\|_\infty.$$

Again, we can choose ϕ such that $\|P\phi\| = \|P\varphi\|$, and $\|\phi\|_\infty \leqslant C_1\|g\|_{\mathcal{B}}$. Thus

$$\|Vg\|_\infty = \|V\varphi\|_\infty \leqslant C\|\varphi\|_\infty \leqslant C_2\|g\|_{\mathcal{B}}.$$

Now we try to show that $\|g\|_{\mathcal{B}} \leqslant C\|Vg\|_\infty$.
We first show $PVg(z) = g(z)$ for $g \in \mathcal{B}_0$ and $z \in \Omega$,

$$\begin{aligned} PVg(z) &= \int_\Omega K(z,w)|\rho(w)|^{n+1}\Big(\int_\Omega K_{n+1}(w,u)g(u)\,\mathrm{d}V(u)\Big)\,\mathrm{d}V(w) \\ &= \int_\Omega g(u)\int_\Omega K(z,w)|\rho(w)|^{n+1}K_{n+1}(w,u)\,\mathrm{d}V(w)\,\mathrm{d}V(u) \\ &= \int_\Omega g(u)\overline{K(u,z)}\,\mathrm{d}V(u) \\ &= g(z). \end{aligned}$$

Then $\|g\|_{\mathcal{B}} \leqslant C\|Vg\|_\infty$ by boundedness of the Bergman projection P. ∎

Since V is an embedding, $X = V\mathcal{B}_0$ is a closed subspace of $C_0(\Omega)$. $F \cdot V^{-1}$ is a bounded functional on X. By Hahn-Banach extension theorem, there exists a bounded complex measure μ, such that

$$F \cdot V^{-1}(h) = \int_\Omega h\,\mathrm{d}\mu(z), \quad h \in C_0(\Omega).$$

Now let $g \in \mathcal{B}_0$. Then

$$\begin{aligned} F(g) &= F \cdot V^{-1} \cdot V(g) \\ &= \int_\Omega Vg(z)\,\mathrm{d}\mu(z) \\ &= \int_\Omega |\rho(z)|^{n+1}\int_\Omega g(w)K_{n+1}(z,w)\,\mathrm{d}V(w)\,\mathrm{d}\mu(z) \\ &= \int_\Omega g(w)\int_\Omega |\rho(z)|^{n+1}K_{n+1}(z,w)\,\mathrm{d}\mu(z)\,\mathrm{d}V(w). \end{aligned}$$

If we let $f(z) = \int_\Omega |\rho(w)|^{n+1}K_{n+1}(z,w)\,\mathrm{d}\overline{\mu}(w)$, then

$$F(g) = \int_\Omega g(z)\overline{f(z)}\,\mathrm{d}V(z).$$

f is analytic since $K_{n+1}(z,w)$ is for each $w \in \Omega$. f is also in $L^1(\Omega)$ since

$$\begin{aligned}\int_\Omega |f(z)|\,\mathrm{d}V(z) &\leqslant \int_\Omega\int_\Omega |\rho(w)|^{n+1}|K_{n+1}(z,w)|\,\mathrm{d}|\mu(w)|\,\mathrm{d}V(z)\\ &= \int_\Omega\int_\Omega |\rho(w)|^{n+1}|K_{n+1}(z,w)|\,\mathrm{d}V(z)\,\mathrm{d}|\mu(w)|\\ &\leqslant C\int_\Omega \mathrm{d}|\mu(w)| < \infty.\end{aligned}$$

This show that $(\mathcal{B}_0)^* \cong A^1$ under the pairing $\langle g, f\rangle = \int_\Omega g(z)\overline{f(z)}\,\mathrm{d}V(z)$. ∎

4. MAIN RESULT

Now we are ready to state our main result.

THEOREM 4.1. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $\varphi : \Omega \to \Omega$ be a holomorphic mapping. Then the following statements are equivalent:*

(i) *$C_\varphi : A^1(\Omega) \to A^1(\Omega)$ is compact;*

(ii) *$C_\varphi : A^1(\Omega) \to A^1(\Omega)$ is weakly compact;*

(iii) *$C_\varphi^* : \mathcal{B}(\Omega) \to \mathcal{B}_0(\Omega)$ is bounded.*

Proof. It is obvious that (i) implies (ii). Now we want to show that (ii) implies (iii). First we need a theorem in [2], which states that a linear operator T on a Banach space X is weakly compact if and only if either T^* is weakly compact on X^* or $T^{**} : X^{**} \to X$ is bounded. We also need the following lemma.

LEMMA 4.2. *Let C_φ be a composition operator on $A^1(\Omega)$, then*

$$C_\varphi^*(\mathcal{B}_0(\Omega)) \subset \mathcal{B}_0(\Omega).$$

Proof. Let $u \in \mathcal{B}_0(\Omega)$. Then $C_\varphi^* u$ is in $\mathcal{B}(\Omega)$, hence in $A^2(\Omega)$, and can be written as:

$$\begin{aligned}C_\varphi^* u(z) &= \int_\Omega C_\varphi^* u(w) k(z,w)\,\mathrm{d}V(w)\\ &= \int_\Omega u(w)\overline{C_\varphi K(w,z)}\,\mathrm{d}V(w), \quad z \in \Omega.\end{aligned}$$

Since $\mathcal{B}_0(\Omega) = PC_0(\Omega)$, there exists $u_0 \in C_0(\Omega)$ such that $u = Pu_0$. The above equation becomes

$$\begin{aligned} C_\varphi^* u(z) &= \int_\Omega Pu_0(w)\overline{C_\varphi K(w,z)}\,\mathrm{d}V(w) \\ &= \int_\Omega u_0(w)\overline{PC_\varphi K(w,z)}\,\mathrm{d}V(w) \\ &= \int_\Omega u_0(w)\overline{C_\varphi K(w,z)}\,\mathrm{d}V(w). \end{aligned}$$

Now let $\Omega_\delta = \{z \in \Omega : |d(z)| > \delta\}$ and consider the following:

$$\begin{aligned} |\nabla C_\varphi^* u(z)|\,|d(z)| &= |d(z)|\left|\nabla \int_\Omega u_0(w)\overline{C_\varphi K(w,z)}\,\mathrm{d}V(w)\right| \\ &\leqslant |d(z)|\int_\Omega |u_0(w)|\,|\nabla_{\bar z} C_\varphi K(w,z)|\,\mathrm{d}V(w) \\ &= |d(z)|\int_{\Omega_\delta} |u_0(w)|\,|\nabla_{\bar z} C_\varphi K(w,z)|\,\mathrm{d}V(w) \\ &\quad + |d(z)|\int_{\Omega\setminus\Omega_\delta} |u_0(w)|\,|\nabla_{\bar z} C_\varphi K(w,z)|\,\mathrm{d}V(w) \\ &\leqslant |d(z)|\int_{\Omega_\delta} |u_0(w)|\,|\nabla_{\bar z} C_\varphi K(w,z)|\,\mathrm{d}V(w) \\ &\quad + |d(z)|\max_{w\in\Omega\setminus\Omega_\delta}(|u_0(w)|)\int_{\Omega\setminus\Omega_\delta} |\nabla_{\bar z} C_\varphi K(w,z)|\,\mathrm{d}V(w). \end{aligned} \tag{4.1}$$

Given $\varepsilon > 0$, there exists δ such that

$$\max_{w\in\Omega\setminus\Omega_\delta} |u_0(w)| < \varepsilon.$$

Then we choose z close to $\partial\Omega$ so that $d(z) < \varepsilon$ and $z \in \Omega \setminus \varphi(\Omega_\delta)$. Since

$$K(w,z) \in C^\infty(\Omega \times \overline{\Omega}),$$

therefore $\nabla_{\bar z} K(\varphi(w), z)$ is bounded. Hence the first term goes to zero. For the second term, consider

$$\begin{aligned} \int_\Omega |\nabla_{\bar z} C_\varphi K(w,z)|\,\mathrm{d}V(w) &= \int_\Omega |C_\varphi \nabla_{\bar z} K(w,z)|\,\mathrm{d}V(w) \\ &\leqslant \|C_\varphi\| \int_\Omega |\nabla_{\bar z} K(w,z)|\,\mathrm{d}V(w) \\ &\leqslant C|d(z)|^{-1}. \end{aligned}$$

Hence the second term is less than $C\varepsilon$. This shows that $|\nabla C_\varphi^* u(z)|\,|d(z)| \to 0$ as $z \to \partial\Omega$, therefore $C_\varphi^* u$ is in $\mathcal{B}_0$. ∎

To show (ii) implies (iii), let $C_\varphi : A^1(\Omega) \to A^1(\Omega)$ be weakly compact. We want to write C_φ as a dual of some linear operator T_φ. By Remark 4.2, C_φ^* maps $\mathcal{B}_0(\Omega)$ into $\mathcal{B}_0(\Omega)$, we can define T_φ to be $C_\varphi^* | \mathcal{B}_0(\Omega)$. And we can show that $(T_\varphi)^* = C_\varphi$. In fact, let h be in the lowest level, i.e., the $\mathcal{B}_0$. Every functional in $(\mathcal{B}_0)^*$ is of the form $\ell_g(h) = \int_\Omega h\bar{g}\,\mathrm{d}V$ for some g in A^1. Now we want to show that $(C_\varphi^*)^*$ acting on ℓ_g is represented by C_φ acting on g, i.e.

$$((C_\varphi^*)^*\ell_g)(h) = \ell_{C_\varphi g}(h).$$

Consider the following

$$((C_\varphi^*)^*\ell_g)(h) = \int_\Omega (C_\varphi)^* h\bar{g}\,\mathrm{d}V = \overline{\int_\Omega C_\varphi g\bar{h}\,\mathrm{d}V} = \int_\Omega h\overline{C_\varphi g}\,\mathrm{d}V = \ell_{C_\varphi g}(h).$$

This shows that $(T_\varphi)^* = C_\varphi$.

We know that $A^1(\Omega) = (\mathcal{B}_0(\Omega))^*$ and $\mathcal{B}(\Omega) = (A^1(\Omega))^*$, so the weak compactness of C_φ implies the weak compactness of T_φ. Therefore C_φ^*, which is $(T_\varphi)^{**}$, maps $\mathcal{B}(\Omega)$ into $\mathcal{B}_0(\Omega)$.

Now we complete the proof by showing that (iii) implies (i). Assume $C_\varphi^* : \mathcal{B}(\Omega) \to \mathcal{B}_0(\Omega)$. To prove C_φ is compact, it suffices to show that for each bounded sequence $\{u_n\}$ in $A^1(\Omega)$ which converges to 0 in w*-topology, $C_\varphi(u_n)$ converges to 0 in $A^1(\Omega)$.

Now let $\{u_n\}$ be a bounded sequence in $A^1(\Omega)$ that converges to 0 in weak star topology, i.e.

$$\langle u_n, f\rangle \to 0, \quad \text{for every} \quad f \in \mathcal{B}_0(\Omega).$$

Since $K_z(\cdot) = K(\cdot, z) \in C^\infty(\overline{\Omega})$ for $z \in \Omega$, therefore $|\nabla K_z(w)|\,|\rho(w)| \to 0$ as $w \to 0$, i.e. K_z is in $\mathcal{B}_0$. Hence

$$u_n(z) = \langle u_n, K_z\rangle \to 0$$

which means $u_n \to 0$ for every $z \in \Omega$. So we get

$$C_\varphi(u_n)(z) = u_n(\varphi(z)) \to 0, \quad z \in \Omega. \tag{4.2}$$

That is, $C_\varphi(u_n)$ converges to 0 point-wise. Therefore $C_\varphi(u_n)$ converges to 0 in V-measure.

Secondly, for every $g \in \mathcal{B}(\Omega)$,

$$\langle C_\varphi(u_n), g\rangle = \langle u_n, C_\varphi^* g\rangle = \langle u_n, f\rangle \to 0,$$

where $f = C_\varphi^* g$ is in $\mathcal{B}_0$. This says that

$$\{C_\varphi(u_n)\} \to 0, \quad \text{weakly}. \tag{4.3}$$

By a theorem in [2] on page 295, (4.2) and (4.3) imply that $C_\varphi(u_n) \to 0$ in A^1 norm. This shows that C_φ is compact, which completes the proof for the main theorem. ∎

If Ω is a bounded symmetric domain of tube type in $\mathbb{C}^n$, the main result is still true provided the Bloch and little Bloch spaces are replaced by Yan's "Bloch" and "little Bloch" spaces. First, let us state Yan's result, see [17].

THEOREM 4.3. [Yan] *Let r be the rank of Ω, $p = 2n/r$ and s be a positive integer. Let*

(i) $$\widetilde{\mathcal{B}}^s(\Omega) = \{f \in A^2(\Omega) : \sup_{z\in\Omega} K(z,z)^{-\frac{s}{p}} |(D^s f)(z)| < \infty\}$$

and

(ii) $$\widetilde{\mathcal{B}}_0^s(\Omega) = \{f \in A^2(\Omega) : \lim_{z\to\partial\Omega} K(z,z)^{-\frac{s}{p}} |(D^s f)(z)| = 0\}$$

where D^s is a differential operator on Ω such that

$$D_z^s K(z,w) = c_s K(z,w)^{1+\frac{s}{p}},$$

then, for all $s > (n/r) - 1$,

$$A^1(\Omega)^* \cong \widetilde{\mathcal{B}}^s(\Omega), \quad \widetilde{\mathcal{B}}_0^s(\Omega)^* = A^1(\Omega).$$

Furthermore, if P is the Bergman projection, then

$$P : L^\infty(\Omega) \to \widetilde{\mathcal{B}}^s(\Omega), \quad P : C_0(\Omega) \to \widetilde{\mathcal{B}}_0^s(\Omega)$$

are both bounded and onto.

Using Yan's theorem, we are able to use similar proofs to obtain a result similar to our main theorem. We provide a proof for the analogue of Lemma 4.2 for bounded symmetric domains of tube type.

LEMMA 4.4. *If Ω is a bounded symmetric domain of tube type and C_φ is a composition operator on $A^1(\Omega)$, then*

$$C_\varphi^*(\widetilde{\mathcal{B}}_0^s(\Omega)) \subset \widetilde{\mathcal{B}}_0^s(\Omega).$$

Proof. Let $u \in \widetilde{\mathcal{B}}_0^s(\Omega)$, then $C_\varphi^* u$ is in $A^2(\Omega)$ and can be written as:

$$\begin{aligned} C_\varphi^* u(z) &= \int_\Omega C_\varphi^* u(w) k(z,w)\, dV(w) \\ &= \int_\Omega u(w)\overline{C_\varphi K(w,z)}\, dV(w), \quad z \in \Omega. \end{aligned}$$

Since $\widetilde{\mathcal{B}}_0^s(\Omega) = PC_0(\Omega)$, there exists $u_0 \in C_0(\Omega)$ such that $u = Pu_0$. The above equation becomes

$$C_\varphi^* u(z) = \int_\Omega Pu_0(w)\overline{C_\varphi K(w,z)}\, dV(w) = \int_\Omega u_0(w)\overline{K(\varphi(w),z)}\, dV(w).$$

Now consider

$$\begin{aligned} h(z,z)^s(D^s C_\varphi^* u)(z) &= c_s h(z,z)^s \int_\Omega D^s(u_0 C_\varphi K(z,w))\, dV(w) \\ &= c_s h(z,z)^s \int_\Omega u_0(w) D_z^s K(z,\varphi(w))\, dV(w) \\ &= c_s h(z,z)^s \int_{\Omega_\delta} u_0(w) C_\varphi^* h(z,w)^{-(p+s)}\, dV(w) \\ &\quad + c_s h(z,z)^s \int_{\Omega\setminus\Omega_\delta} u_0(w) C_\varphi^* h(z,w)^{-(p+s)}\, dV(w) \end{aligned}$$

where $\Omega_\delta = \{z \in \Omega : d(z) < \delta\}$. Use a similar argument to the last part of the proof for Lemma 4.2 and by Theorem 4.1 in Faraut and Koranyi's paper [3], we see that $h(z,z)^s(D^s C_\varphi^* u)(z)$ is in $C_0(\Omega)$. Hence $C_\varphi^* u$ is in $\widetilde{\mathcal{B}}_0^s(\Omega)$. ∎

Now we state our main result in the case of bounded symmetric domains.

THEOREM 4.5. *Let Ω be a bounded symmetric domain of tube type in $\mathbb{C}^n$. Let $\varphi : \Omega \to \Omega$ be a holomorphic mapping. Then the following statements are equivalent:*

(i) $C_\varphi : A^1(\Omega) \to A^1(\Omega)$ *is compact;*

(ii) $C_\varphi : A^1(\Omega) \to A^1(\Omega)$ *is weakly compact;*

(iii) $C_\varphi^* : \widetilde{B}^s(\Omega) \to \widetilde{B}_0^s(\Omega)$ *is bounded.*

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