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DEFORMATIONS OF VON NEUMANN ALGEBRAS

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ABSTRACT. Rieffel's deformation quantization of C^*-algebras for actions of $\mathbf{R}^d$ can also be carried out on the von Neumann algebra level.

KEYWORDS: *Deformation quantization, von Neumann algebras, noncommutative geometry.*

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Let $(A, \mathbf{R}^d, \alpha)$ be a C^*-dynamical system, meaning that A is a C^*-algebra, $\alpha : \mathbf{R}^d \to \mathrm{Aut}(A)$ is a group homomorphism, and for each $a \in A$ the function $y \mapsto \alpha_y(a)$ $(= \alpha(y)(a))$ is norm-continuous on $\mathbf{R}^d$. Rieffel ([13]) has given a prescription for constructing a "deformed" C^*-dynamical system $(A_J, \mathbf{R}^d, \alpha)$ for any skew-symmetric operator J on $\mathbf{R}^d$. The deformed C^*-algebra A_J is constructed by equipping the algebra A^∞ of elements of A which are smooth for the action with the "deformed" product

$$a \times_J b = \iint \alpha_{Ju}(a)\alpha_v(b)\mathrm{e}^{2\pi i u\cdot v}\,\mathrm{d}u\mathrm{d}v,$$

and then completing. The construction is of particular interest as a way of creating noncommutative C^*-algebras from commutative ones. Examples of deformed C^*-algebras arising in this way include the Moyal plane and the quantum torus, along with many others (see [13], Chapters 10–12).

Now let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system, meaning that M is now a von Neumann algebra and for each $a \in M$ the function $y \mapsto \alpha_y(a)$ is ultraweakly continuous. The goal of this paper is to define a deformed W^*-dynamical system $(M_J, \mathbf{R}^d, \alpha)$ (again, for J a skew-symmetric operator on $\mathbf{R}^d$) and to determine some of its basic properties. Our conclusions are perhaps relatively minor extensions of the corresponding results in the comprehensive study ([13]).

The relation of the von Neumann algebra setting to the C^*-algebra setting is as follows. Let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system. By ([11], Lemma 7.5.1) the set A of elements of M for which the action is norm-continuous, is itself an α-invariant C^*-algebra which is ultraweakly dense in M. In particular, $(A, \mathbf{R}^d, \alpha|_A)$ is a C^*-dynamical system, and it can therefore be deformed as in [13]. The deformed von Neumann algebra M_J can then be defined by taking the ultraweak closure — in some sense — of A_J (see Section 1).

Our main motivation comes from noncommutative differential geometry ([2], [3], [4], [16]), particularly the set-up introduced by Connes in [2]. Here one constructs differential geometric type data (specifically, an exterior algebra and exterior derivative) from initial data consisting of a Lie group acting on a C^*-algebra. In the present context this Lie group would simply be $\mathbf{R}^d$.

There are several reasons for pursuing this topic at the von Neumann algebra level. First observe that the measurable point of view can certainly be taken in the commutative case. Indeed, any differentiable manifold comes equipped with a canonical measure class, locally lifted from Lebesgue measure on $\mathbf{R}^d$. Then the exterior derivative can be regarded as an unbounded derivation of von Neumann algebras. We find such derivations interesting because the natural requirement of ultraweak closability of the graph has strong consequences, a point we discussed at some length in [14]. (See also [15], where we show that the weak* closable derivations of $C[0,1]$ have a particularly simple form.) The exterior derivative is in fact ultraweakly closable in the most important commutative examples, and so it is natural to work at the von Neumann algebra level in order to take advantage of this.

In particular, the present paper (especially Corollary 3.7) furnishes several new examples of noncommutative metrics along the lines discussed in [14] (i.e. C^*-derivations which extend to W^*-derivations). The Moyal plane is such an example; it is similar to the noncommutative torus as considered in ([14], Theorem 22), but harder to deal with because of the need to consider oscillatory integrals. Our results on the noncommutative torus in [14] also follow from the results given here.

In Section 1 we discuss Hilbert modules and define deformed W^*-dynamical systems. The bulk of the paper is taken up in Sections 2 and 3 in proving that the smooth elements of a deformed von Neumann algebra are the same as the smooth elements of the original algebra (Theorem 3.6); this requires a Weyl's lemma-type result for Hilbert modules (Corollary 2.3), which we develop in Section 2. Finally, in Section 4 we prove that deformation is a functor and that $(M_J)_K \cong M_{J+K}$. I wish to thank Akitaka Kishimoto for kindly allowing me to include a general result of his which greatly simplifies the proof of this last statement.

Because of our heavy reliance on the constructions and results of [13], the reader needs to be familiar with that reference.

1. HILBERT MODULES

Let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system, let A be its norm-continuous part, and let J be a skew-symmetric operator on $\mathbf{R}^d$. This notation will be maintained throughout the paper. As we indicated in the introduction, we are going to construct a deformed von Neumann algebra M_J by taking the ultraweak closure of the deformed C^*-algebra A_J. This is not quite trivial, because A_J is defined via its action on the Hilbert A-module $\mathcal{S}^A$ of A-valued Schwartz functions on $\mathbf{R}^d$. In general the algebra of all bounded adjointable operators on a Hilbert C^*-module is a C^*-algebra but not a von Neumann algebra, so A_J does not a priori sit inside of a natural von Neumann algebra. (For general background on Hilbert modules see [10] and [12].)

The best way to resolve this issue is to define an action of A_J on a self-dual Hilbert W^*-module, since the bounded operators on such a module do constitute a von Neumann algebra ([10], Proposition 3.10). That is, we want to enlarge $\mathcal{S}^A$ so that it becomes a self-dual Hilbert M-module. Algebraists call this sort of process "extension of the base ring" and perform it by tensoring the original module with the desired base ring. In the case of Hilbert modules the process has been considered in detail ([10], Section 4); here it involves taking the algebraic tensor product $\mathcal{S}^A \otimes_A M$, then factoring out null vectors and dualizing.

The same result is also achieved by simply taking the collection of all bounded right A-module homomorphisms $\varphi : \mathcal{S}^A \to M$ (bounded with respect to the Hilbert module norm $\|f\|_2^2 = \|\langle f, f\rangle_A\|$ on $\mathcal{S}^A$). We denote this object by $(\mathcal{S}^A)'$. It is a vector space by pointwise addition of maps and twisted scalar multiplication (i.e. $(\lambda\varphi)(f) = \bar{\lambda}\varphi(f)$), and a right M-module by the action $(\varphi \cdot a)(f) = a^*\varphi(f)$. It contains $\mathcal{S}^A$ by identifying $f \in \mathcal{S}^A$ with the homomorphism

$$g \mapsto \langle f, g\rangle_A = \int f(x)^* g(x)\, \mathrm{d}x,$$

and it has an M-valued inner product $\langle \cdot, \cdot \rangle_M$ which satisfies $\langle \varphi, f\rangle_M = \varphi(f)$ for $\varphi \in (\mathcal{S}^A)'$ and $f \in \mathcal{S}^A$ (and hence $\langle f, g\rangle_M = \langle f, g\rangle_A$ for $f, g \in \mathcal{S}^A$). It is a self-dual Hilbert M-module, hence the algebra $B((\mathcal{S}^A)')$ of all bounded M-module operators on $(\mathcal{S}^A)'$ is a von Neumann algebra; the ultraweak topology is given by $T_\kappa \to T$ ultraweakly, for bounded nets $(T_\kappa) \subset B((\mathcal{S}^A)')$, if and only if $\langle T_\kappa \varphi, \psi\rangle_M \to \langle T\varphi, \psi\rangle_M$ ultraweakly in M for all $\varphi, \psi \in (\mathcal{S}^A)'$. Furthermore, the C^*-algebra $B(\mathcal{S}^A)$ of all bounded adjointable A-module operators $T : \mathcal{S}^A \to \mathcal{S}^A$ is isometrically contained in the von Neumann algebra $B((\mathcal{S}^A)')$, by defining $T\varphi = \varphi \circ T^*$ for $\varphi \in (\mathcal{S}^A)'$ and $T \in B(\mathcal{S}^A)$.

All of the preceding assertions are verified in ([10], Sections 3 and 4). (Note that Theorem 4.2 of [10] holds since A is ultraweakly dense in M and has the same unit; see [10] p. 462.) We also require the following facts.

LEMMA 1.1. (i) *Let $T \in B((\mathcal{S}^A)')$. Then $T = 0$ if and only if $Tf = 0$ for all $f \in \mathcal{S}^A$ if and only if $\langle Tf, g\rangle_M = 0$ for all $f, g \in \mathcal{S}^A$ if and only if $\langle Tf, g\rangle_M = 0$ for all scalar-valued $f, g \in \mathcal{S}^A$.*

(ii) *Let $T \in B((\mathcal{S}^A)')$ and let (T_κ) be a bounded net in $B((\mathcal{S}^A)')$. Then $T_\kappa \to T$ ultraweakly if and only if $\langle T_\kappa f, g\rangle_M \to \langle Tf, g\rangle_M$ ultraweakly in M for all $f, g \in \mathcal{S}^A$ if and only if $\langle T_\kappa f, g\rangle_M \to \langle Tf, g\rangle_M$ ultraweakly in M for all scalar-valued Schwartz functions f, g.*

Proof. (i) That $T = 0$ implies $Tf = 0$ implies $\langle Tf, g\rangle_M = 0$ for $f, g \in \mathcal{S}^A$ implies $\langle Tf, g\rangle_M = 0$ for scalar-valued f, g is clear. Conversely, suppose $\langle Tf, g\rangle_M = 0$ for all scalar-valued Schwartz functions f, g. Then $\langle T(af), bg\rangle_M = a^*\langle Tf, g\rangle_M b$ for all $a, b \in A$ implies the same for any A-linear combination of scalar-valued Schwartz functions. As the latter are clearly dense in $\mathcal{S}^A$ in Hilbert module norm, it follows that $\langle Tf, g\rangle_M = 0$ for all $f, g \in \mathcal{S}^A$.

Now for any $f \in \mathcal{S}^A$, $(Tf)(g) = \langle Tf, g\rangle_M = 0$ for all $g \in \mathcal{S}^A$ implies that $Tf = 0$. Thus, for any $\varphi \in (\mathcal{S}^A)'$

$$(T^*\varphi)(f) = \langle T^*\varphi, f\rangle_M = \langle \varphi, Tf\rangle_M = 0$$

for all $f \in \mathcal{S}^A$, hence $T^*\varphi = 0$. So $T^* = 0$, and therefore $T = 0$.

(ii) The forward directions are again vacuous. For the reverse, suppose $\langle T_\kappa f, g\rangle_M \to \langle Tf, g\rangle_M$ for all scalar-valued $f, g \in \mathcal{S}^A$ and let T' be any ultraweak cluster point of the net (T_κ). Then $\langle (T - T')f, g\rangle_M = 0$ for all scalar-valued f, g, hence $T - T' = 0$ by (i). This shows that T is the only possible ultraweak cluster point of (T_κ), so therefore $T_\kappa \to T$ ultraweakly. ∎

Since A_J is defined via its action on $\mathcal{S}^A$ and we have an embedding of $B(\mathcal{S}^A)$ into $B((\mathcal{S}^A)')$, we may consider A_J as contained in $B((\mathcal{S}^A)')$. We can then make the following definition.

DEFINITION 1.2. Let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system and let J be a skew-symmetric operator on $\mathbf{R}^d$. Let A be the norm-continuous part of M for α and define $A_J \subset B((\mathcal{S}^A)')$ as in ([13], Chapter 4) and using the above comment. Then we define the *deformed von Neumann algebra* M_J to be the ultraweak closure of A_J in $B((\mathcal{S}^A)')$.

The first thing we want to do is to show that M_J still carries an ultraweakly continuous action α of $\mathbf{R}^d$. Recall that we have a unitary action τ of $\mathbf{R}^d$ on $(\mathcal{S}^A)'$ defined by $\tau_y f(x) = f(x-y)$ for $f \in \mathcal{S}^A$ and $y \in \mathbf{R}^d$ (by the inclusion of $B(\mathcal{S}^A)$ in $B((\mathcal{S}^A)')$ it is enough to define τ_y on $\mathcal{S}^A$). Let $A^\infty \subset A$ denote the subalgebra of smooth vectors for α.

PROPOSITION 1.3. *The action of α on A_J^∞ ($= A^\infty$, as sets) extends uniquely to an ultraweakly continuous action on M_J.*

Proof. First, we claim that the action β of $\mathbf{R}^d$ on $B((\mathcal{S}^A)')$ defined by conjugation with τ is ultraweakly continuous. To see this suppose $y_n \to y$ in $\mathbf{R}^d$ and note that for any $f \in \mathcal{S}^A$ we have $\tau_{y_n}(f) \to \tau_y(f)$ in Hilbert module norm. Hence

$$\langle \beta_{y_n}(T)f, g\rangle_M = \langle T\tau_{y_n}f, \tau_{y_n}g\rangle_M \to \langle T\tau_y f, \tau_y g\rangle_M = \langle \beta_y(T)f, g\rangle_M$$

for all $f, g \in \mathcal{S}^A$, where the convergence is in norm in M. Lemma 1.1 (ii) now implies that $\beta_{y_n}(T) \to \beta_y(T)$ ultraweakly, which proves the claim.

Now we want to check that the action of α on A^∞ agrees with the action of β on A_J^∞. Recall that $a \in A^\infty$ operates on $\mathcal{S}^A$ by $L_{\tilde{a}}(f) = \tilde{a} \times_J f$ where $\tilde{a}(x) = \alpha_x(a)$; that is,

$$(L_{\tilde{a}}f)(x) = \iint \alpha_{x+Ju}(a) f(x+v) e^{2\pi i u\cdot v}\, du dv.$$

Thus for any $y \in \mathbf{R}^d$ and $a \in A^\infty$

$$\begin{aligned}(L_{\alpha_y(a)^\sim} f)(x) &= \iint \alpha_{x+Ju+y}(a) f(x+v) e^{2\pi i u\cdot v} \\ &= \tau_y^{-1}\Big(\iint \alpha_{x+Ju}(a)(\tau_y f)(x+v) e^{2\pi i u\cdot v}\, du dv\Big) \\ &= (\beta_y(L_{\tilde{a}})f)(x).\end{aligned}$$

So $\alpha|_{A^\infty} = \beta|_{A_J^\infty}$, which together with the claim shows that on A_J^∞, α is the restriction of an ultraweakly continuous action on $B((\mathcal{S}^A)')$. Finally, since A_J^∞ is invariant for β and M_J is the ultraweak closure of A_J^∞, it follows that M_J is also invariant for β. Uniqueness of the action is clear by ultraweak continuity. ∎

We will henceforth use the symbol α to denote the action on M_J given by conjugation with τ. By the preceding proof, this is consistent with the original use of α.

2. SCHWARTZ FUNCTIONS

Unfortunately, we cannot assume that every element of $(\mathcal{S}^A)'$ is represented by an M-valued function on $\mathbf{R}^d$. Indeed, Manuilov ([8]) has recently shown that even completing the pre-Hilbert $L^\infty[0,1]$-module of simple $L^\infty[0,1]$-valued functions on $[0,1]$ (with the standard inner product) gives rise to elements which take values in $L^\infty[0,1]$ almost nowhere.

However, we will presently need to recognize certain elements of $(\mathcal{S}^A)'$ as A-valued functions on $\mathbf{R}^d$. What we really need is a version of Weyl's lemma which will tell us that an element of $(\mathcal{S}^A)'$ with sufficient smoothness properties actually belongs to $\mathcal{S}^A$.

The main part of the proof of this involves proving an analogous statement on the d-dimensional torus. This is accomplished by going over to $\mathbf{Z}^d$ via the Fourier transform, which does the trick because here every element of the dual module actually is an M-valued function on $\mathbf{Z}^d$ (cf. [7]).

Let $\mathbf{T}^d$ denote the d-dimensional torus and let $\mathcal{S}_0^A$ (respectively, $\mathcal{S}_0^M$) denote the A-module (resp. M-module) of continuous A-valued (resp. M-valued) functions on $\mathbf{T}^d$ which are smooth for the action τ (defined on $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ in the obvious way). Also let $(\mathcal{S}_0^A)'$ denote the M-module of right A-module homomorphisms of $\mathcal{S}_0^A$ into M.

LEMMA 2.1. *An element of $(\mathcal{S}_0^A)'$ belongs to $\mathcal{S}_0^M$ if and only if it is smooth for τ.*

Proof. The forward direction is vacuous. For the reverse direction, observe first that the functions $e_p : x \mapsto e^{2\pi i p\cdot x}$, for $p \in \mathbf{Z}^d$ and $x \in \mathbf{T}^d$, generate the M-module $(\mathcal{S}_0^A)'$ in the sense that no $\varphi \in (\mathcal{S}_0^A)'$ is orthogonal to all such functions. This is because we can approximate any continuous scalar-valued function on the torus by a finite linear combination of the e_p's, hence we can approximate any continuous A-valued function on the torus by a finite A-linear combination of the e_p's. So $\langle \varphi, e_p\rangle = \varphi(e_p) = 0$ for all $p \in \mathbf{Z}^d$ implies that $\varphi(f) = 0$ for all $f \in \mathcal{S}_0^A$, i.e. that $\varphi = 0$.

Now let $\varphi \in (\mathcal{S}_0^A)'$ be smooth for τ and define its Fourier series $\widehat{\varphi} : \mathbf{Z}^d \to M$ by $\widehat{\varphi}(p) = \langle e_p, \varphi\rangle_M$. Since φ is smooth its Fourier series decays rapidly in norm (i.e. $\|\widehat{\varphi}(p)\| \to 0$ rapidly), using the fact that differentiation on $\mathbf{T}^d$ transfers to multiplication by the coordinate functions on $\mathbf{Z}^d$ and every derivative of φ is bounded in Hilbert module norm. Therefore the series $\sum \widehat{\varphi}(p)e_p$ is uniformly summable and the sum is evidently a smooth function $\varphi' \in \mathcal{S}_0^M$ whose Fourier series is the same as that of φ. Thus $\varphi = \varphi'$ by the last paragraph, which completes the proof. ∎

There is a slight ambiguity which we need to address here. When we speak of a vector φ being smooth for τ, this means that its first derivative in the y direction,

$$\lim_{t\to 0} \frac{\tau_{ty}(\varphi)-\varphi}{t},$$

exists for all $y \in \mathbf{R}^d$, as does every subsequent derivative. The question is whether we mean that this limit exists in the uniform sense (as in the definition of $\mathcal{S}^A$ in [13]) or in the ostensibly weaker sense of Hilbert module norm. As we will be applying Lemma 2.1 and the other results of this section for vectors which are smooth in the weaker sense, we need to point out that this is all that is used in the proof. However, the argument in fact establishes smoothness in the stronger sense. Thus the two are equivalent.

We now want to prove an analogous result for A-valued functions on $\mathbf{R}^d$. Supposing $\varphi \in (\mathcal{S}^A)'$ is smooth for τ, we say φ *decays rapidly* if $f\psi \in (\mathcal{S}^A)'$ for any scalar polynomial f on $\mathbf{R}^d$ and any derivative ψ of φ. Here $f\psi$ is defined by $f\psi(g) = \psi(\bar{f}g)$, and the issue is whether this defines a bounded function of g. Let $\mathcal{S}^M$ denote the M-module of those continuous M-valued functions in $(\mathcal{S}^A)'$ which are smooth for τ and decay rapidly.

THEOREM 2.2. *A rapidly decaying element of $(\mathcal{S}^A)'$ belongs to $\mathcal{S}^M$ if and only if it is smooth for τ.*

Proof. The forward direction is vacuous. For the converse, suppose $\varphi \in (\mathcal{S}^A)'$ is rapidly decaying and smooth. We claim that $f\varphi$ is also smooth, for any compactly supported scalar-valued C^∞ function f on $\mathbf{R}^d$. To see this observe first that the map $g \mapsto fg$ belongs to $B(\mathcal{S}^A)$ and has norm $\|f\|_\infty$; thus by the isometric embedding of $B(\mathcal{S}^A)$ in $B((\mathcal{S}^A)')$ it follows that $f\psi \in (\mathcal{S}^A)'$ and $\|f\psi\|_2 \leqslant \|f\|_\infty \|\psi\|_2$ for all $\psi \in (\mathcal{S}^A)'$. Now let $y \in \mathbf{R}^d$ and observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tau_{ty}(f\varphi)\Big|_{t=0} &= \lim \frac{\tau_{ty}(f\varphi)-f\varphi}{t}\\ &= \lim \frac{(\tau_{ty}f-f)(\tau_{ty}\varphi)}{t} + \lim \frac{f(\tau_{ty}(\varphi)-\varphi)}{t}.\end{aligned}$$

The second term converges to $f\varphi'$ since $(\tau_{ty}(\varphi)-\varphi)/t$ converges to φ', and this also implies that $\tau_{ty}(\varphi) \to \varphi$; together with the convergence of $(\tau_{ty}f - f)/t$ to f', this then shows that the first term converges to $f'\varphi$. Thus we have verified the Leibniz rule for derivatives and this implies the claim.

The point of the claim is that by taking f compactly supported but equal to 1 on some compact subset X of $\mathbf{R}^d$, we get a smooth function $f\varphi$ which agrees with φ on X (in the sense that $f\varphi(g) = \varphi(g)$ for all $g \in \mathcal{S}^A$ supported on X)

and is compactly supported (in the sense that $f\varphi(g) = 0$ for all $g \in \mathcal{S}^A$ supported away from f). If we can show that $f\varphi$ is a continuous M-valued function on $\mathbf{R}^d$, the same will then be true of φ. In other words we have reduced to the case of compactly supported smooth functions, or equivalently, smooth functions on the torus. By the lemma such a function is in $\mathcal{S}^M$, so we are done. ∎

We already have one action, τ, of $\mathbf{R}^d$ on $(\mathcal{S}^A)'$. Now define another action σ by $(\sigma_y f)(x) = \alpha_{-y}(f(x))$ for $f \in \mathcal{S}^A$. This is not unitary but instead satisfies $\langle \sigma_y \varphi, \sigma_y \psi \rangle_M = \alpha_{-y}(\langle \varphi, \psi \rangle_M)$.

COROLLARY 2.3. *A rapidly decaying element of $(\mathcal{S}^A)'$ belongs to $\mathcal{S}^A$ if and only if it is smooth for τ and norm-continuous for σ.*

Proof. Suppose $f \in \mathcal{S}^A$; we must show that it is norm-continuous for σ. For any $\varepsilon > 0$ fix a compact subset $X \subset \mathbf{R}^d$ such that

$$\int_{\mathbf{R}^d - X} \|f(x)\|^2 \, dx \leqslant \varepsilon.$$

By continuity $f(X)$ is a norm-compact subset of A, and since α is norm-continuous on A there exists $\delta > 0$ such that $|y| \leqslant \delta$ implies $\|\alpha_y(a) - a\| \leqslant \sqrt{\varepsilon/\mu(X)}$ for all $a \in f(X)$, where μ denotes Lebesgue measure. Thus, $|y| \leqslant \delta$ implies

$$\begin{aligned}
\|\sigma_y f - f\|^2 &= \left\| \int (\sigma_y f(x) - f(x))^* (\sigma_y f(x) - f(x)) \, dx \right\| \\
&\leqslant \int \|\sigma_y f(x) - f(x)\|^2 \, dx \\
&\leqslant \int_{\mathbf{R}^d - X} \left(\|\alpha_{-y}(f(x))\| + \|f(x)\|\right)^2 dx + \int_X \|\alpha_{-y}(f(x)) - f(x)\|^2 \, dx \\
&\leqslant \int_{\mathbf{R}^d - X} \left(2\|f(x)\|\right)^2 dx + \int_X \left(\frac{\varepsilon}{\mu(X)}\right) dx \\
&\leqslant 5\varepsilon.
\end{aligned}$$

Thus f is norm-continuous for σ, as desired.

For the reverse direction, suppose $\varphi \in (\mathcal{S}^A)'$ decays rapidly and is smooth for τ and norm-continuous for σ. Then $\varphi \in \mathcal{S}^M$ by the theorem. To see that φ is actually an A-valued function, set $g_n(x) = n^d h(nx)$ for some positive $h \in L^1(\mathbf{R}^d)$ with $\int h(x) \, dx = 1$ (so that (g_n) is a convolution approximate unit in $L^1(\mathbf{R}^d)$). Observe that

$$\alpha_{-y}(\langle \varphi, \tau_x(g_n) \rangle_M) = \langle \sigma_y(\varphi), \sigma_y(\tau_x(g_n)) \rangle_M = \langle \sigma_y(\varphi), \tau_x(g_n) \rangle_M,$$

hence for any $x \in \mathbf{R}$, the element $\langle \varphi, \tau_x(g_n) \rangle_M \in M$ is norm-continuous for α and therefore in A. But since φ is continuous on $\mathbf{R}^d$, $\langle \varphi, \tau_x(g_n) \rangle_M$ converges in norm to $\varphi(-x)$, so we conclude that $\varphi(-x) \in A$ and hence that φ is an A-valued function. This completes the proof. ∎

3. SMOOTH VECTORS

By Proposition 1.3 there is an action α of $\mathbf{R}^d$ on M_J, which on A_J^∞ agrees with the original action α (hence our use of the same name for it). According to Theorem 7.1 of [13], A_J^∞ is precisely the part of A_J which is smooth for α. We now wish to prove the same thing for M_J, that A_J^∞ is precisely the part of M_J which is smooth for α.

This result is suggested in a comment on p. 39 of [13]; see also Theorem 1.2 of [6] for a related result involving the Heisenberg group and Theorem 22 of [14] for the special case of the noncommutative torus. Its execution requires a fair amount of preliminaries. We preserve the notation of the preceding sections, and throughout this section let D denote the subalgebra of M_J which is smooth for α, and which we will eventually prove equals A_J^∞.

LEMMA 3.1. *For any $y \in \mathbf{R}^d$ and $T \in M_J$ we have $\alpha_y(T) = \tau_y^{-1} T \tau_y = \sigma_y^{-1} T \sigma_y$.*

Proof. Recall that $\alpha_y(T) = \tau_y^{-1} T \tau_y$ by definition (see the comment following Proposition 1.3). So the second equality is the one at issue.

Let $y \in \mathbf{R}^d$, $a \in A^\infty$, and $f \in \mathcal{S}^A$. Then

$$\begin{aligned}
(\sigma_y^{-1} L_{\tilde{a}} \sigma_y) f(x) &= \alpha_y\Big(\iint \alpha_{x+Ju}(a)(\sigma_y f)(x+v) e^{2\pi i u \cdot v}\, du dv\Big) \\
&= \alpha_y\Big(\iint \alpha_{x+Ju}(a)\alpha_{-y}(f(x+v)) e^{2\pi i u \cdot v}\, du dv\Big) \\
&= \iint \alpha_{x+Ju+y}(a) f(x+v) e^{2\pi i u \cdot v}\, du dv \\
&= \widetilde{\alpha_y(a)} \times_J f(x) \\
&= (\tau_y^{-1} L_{\tilde{a}} \tau_y) f(x).
\end{aligned}$$

Thus $\tau_y^{-1} L_{\tilde{a}} \tau_y = \sigma_y^{-1} L_{\tilde{a}} \sigma_y$ holds for all $a \in A^\infty$. As A_J^∞ is ultraweakly dense in M_J, it follows that the same equation holds for all $T \in M_J$. ∎

LEMMA 3.2. *Let $T \in D$ and $f \in S^A$. Then for any coordinate function x_j on $\mathbf{R}^d$ we have $x_j(Tf) \in (S^A)'$ and*

$$x_j(Tf) = T(x_j f) + (2\pi i)^{-1} \sum_k J_{kj} (\partial_k T) f,$$

where $\partial_k T$ denotes $\mathrm{d}(\alpha_{tv_k}(T))/\mathrm{d}t\big|_{t=0}$, $v_k \in \mathbf{R}^d$ being the kth basis vector.

Proof. For $T \in A_J^\infty$ this is [13], Proposition 3.2. Thus for any $T \in M_J$ we can find a net $(T_\kappa) \subset A_J^\infty$ which converges ultraweakly to T and satisfies the desired equation. For any $g \in S^A$ we then have

$$\langle x_j(T_\kappa f), g\rangle_M = \langle T_\kappa(x_j f), g\rangle_M - (2\pi i)^{-1} \sum_k \overline{J}_{kj} \langle (\partial_k T_\kappa) f, g\rangle_M.$$

Substituting tv_k for y, dividing by t, and taking the limit of

$$\tau_y(T_\kappa f) - T_\kappa f = (\tau_y T_\kappa \tau_y^{-1})(\tau_y(f)) - T_\kappa f = (\alpha_y(T_\kappa) - T_\kappa)(\tau_y(f)) + T_\kappa(\tau_y(f) - f)$$

shows that $(\partial_k T_\kappa) f = \partial_k(T_\kappa f) - T_\kappa(\partial_k f)$, both partials on the right being differentiation with respect to τ. Thus we can rewrite the preceding equation as

$$\langle T_\kappa f, x_j g\rangle_M = \langle T_\kappa(x_j f), g\rangle_M + (2\pi i)^{-1} \sum_k \overline{J}_{kj} \Big[\langle T_\kappa f, \partial_k g\rangle_M + \langle T_\kappa(\partial_k f), g\rangle_M\Big].$$

(It is clear that if $\partial_k \varphi$ exists then it agrees with the distributional derivative $\langle \partial_k \varphi, g\rangle = -\langle \varphi, \partial_k g\rangle$, hence the first term in the sum.) We can now take the ultraweak limit and replace T_κ with T in the above equation. Moving the partial derivative back onto T (we can do this since $T \in D$) gives

$$\langle Tf, x_j g\rangle_M = \langle T(x_j f), g\rangle_M - (2\pi i)^{-1} \sum_k \overline{J}_{kj} \langle (\partial_k T) f, g\rangle_M,$$

which implies that $x_j(Tf) \in (S^A)'$ and yields the desired equation. ∎

LEMMA 3.3. *If $f \in S^A$ and $T \in D$ then $Tf \in S^A$.*

Proof. Suppose $f \in S^A$ and $T \in D$. We wish to apply Corollary 2.3, and therefore must show that Tf is smooth for τ, is norm-continuous for σ, and decays rapidly.

Differentiability of Tf for τ was indicated in the proof of Lemma 3.2, and an easy induction then implies that Tf is smooth for τ. Similarly

$$\sigma_y(Tf) - Tf = (\alpha_y(T) - T)(\sigma_y(f)) + T(\sigma_y(f) - f);$$

as the right side goes to zero in Hilbert module norm as $y \to 0$, this implies that Tf is norm-continuous for σ.

The fact that $(x_1^2 + \cdots + x_d^2)^n (Tf)$ is bounded follows inductively from Lemma 3.2. The corresponding statement for derivatives of Tf then follows inductively using the formula $\partial_k(Tf) = (\partial_k T) f + T(\partial_k f)$. ∎

As in [13], p. 4, let $\{e_p : p \in \mathbb{Z}^d\}$ be a collection of positive compactly-supported scalar-valued C^∞ functions on $\mathbb{R}^d$ with the property that $e_p = \tau_p e_0$ and $\sum_p e_p = 1$. Also, for $n \in \mathbb{N}$ define

$$f_n = \sum_{|p| \leqslant n} e_p.$$

As in [13] let $\mathcal{B}^A$ denote the space of bounded, uniformly continuous A-valued functions on $\mathbb{R}^d$ which are smooth for τ, and for $F \in \mathcal{B}^A$ define $L_F \in B(\mathcal{S}^A) \subset B((\mathcal{S}^A)')$ by $L_F(g) = F \times_J g$.

LEMMA 3.4. *Let $T \in D$ and $g, h \in \mathcal{S}^A$. Then*

$$\langle L_{Tf_n} g, h\rangle_A \to \langle Tg, h\rangle_A$$

in the norm of A.

Proof. Note first that

$$(L_{\tilde{a}} f) \times_J g = \tilde{a} \times_J f \times_J g = L_{\tilde{a}}(f \times_J g)$$

for all $f, g \in \mathcal{S}^A$ and $a \in A^\infty$, i.e. every element of A_J^∞ commutes with right multiplication by g. Therefore the same is true of any operator in M_J.

Now

$$\begin{aligned}\langle L_{Tf_n} g, h\rangle_A &= \langle (Tf_n) \times_J g, h\rangle_A \\ &= \langle T(f_n \times_J g), h\rangle_A \\ &= \langle f_n \times_J g, T^* h\rangle_A.\end{aligned}$$

By [13], Proposition 3.4, $f_n \times_J g \to 1 \times_J g = g$ uniformly on compact subsets of $\mathbb{R}^d$; and since $T^*h \in \mathcal{S}^A$ by Lemma 3.3, it follows that $\langle f_n \times_J g, T^*h\rangle_A \to \langle g, T^*h\rangle_A$ in the norm of A. Therefore

$$\langle L_{Tf_n} g, h\rangle_A \to \langle g, T^*h\rangle_A = \langle Tg, h\rangle_A,$$

as desired. ∎

Observe now that for any $T \in D$ and $p \in \mathbb{Z}^d$, we have

$$Te_p = T\tau_p e_0 = \tau_p(\tau_p^{-1} T \tau_p) e_0 = \tau_p(\sigma_p^{-1} T \sigma_p) e_0 = \tau_p \sigma_p^{-1}(Te_0).$$

Therefore $\|Te_p(x)\| = \|Te_0(x-p)\|$. Since $Te_0 \in \mathcal{S}^A$ by Lemma 3.3, it follows that $\sum_p Te_p$ converges uniformly on compact subsets of $\mathbb{R}^d$ to a bounded, uniformly continuous, smooth A-valued function, i.e. an element of $\mathcal{B}^A$. Denote this function by F.

LEMMA 3.5. *Let $T \in D$ and define $F \in \mathcal{B}^A$ as above. Then for any $g, h \in \mathcal{S}^A$ we have*

$$\langle L_{Tf_n} g, h \rangle_A \to \langle L_F g, h \rangle_A$$

in the norm of A.

Proof. Let $g, h \in \mathcal{S}^A$. Then $(Tf_n) \times_J g \to F \times_J g$ uniformly on compact subsets by [13], Lemma 3.8. Therefore

$$\begin{aligned} \langle L_{Tf_n} g, h \rangle_A &= \langle (Tf_n) \times_J g, h \rangle_A \\ &\to \langle F \times_J g, h \rangle_A \\ &= \langle L_F g, h \rangle_A \end{aligned}$$

with convergence in the norm of A. ∎

THEOREM 3.6. *Let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system, A^∞ its subalgebra of smooth vectors, and J a skew-symmetric operator on $\mathbf{R}^d$. Then an operator $T \in M_J$ is smooth for α if and only if $T \in A_J^\infty$.*

Proof. Every element of A_J^∞ is smooth for α by [13], comment p. 50. Conversely, suppose $T \in M_J$ is smooth for α. Define f_n and F as above. Then by Lemmas 3.4 and 3.5

$$\langle Tg, h \rangle_A = \langle L_F g, h \rangle_A$$

for all $g, h \in \mathcal{S}^A$, hence $T = L_F$ by Lemma 1.1 (i).

For $y \in \mathbf{R}^d$ define a function $F_y \in \mathcal{B}^A$ by $F_y(x) = \alpha_y(F(x - y))$. Then $T = \tau_y \sigma_y^{-1} T \sigma_y \tau_y^{-1}$ implies

$$L_F = \tau_y \sigma_y^{-1} L_F \sigma_y \tau_y^{-1} = L_{F_y}$$

i.e. $L_{F - F_y} = 0$. Consideration of the formula

$$(F' \times_J g)(x) = \int F'(x + Ju)\widehat{g}(u)e^{-2\pi i u \cdot x} du$$

([13], Proposition 3.1) with $F' = F - F_y$ and $\widehat{g}$ supported on a small neighborhood of the origin, shows that this implies $F = F_y$. Thus $F(y) = F_y(y) = \alpha_y(F(0))$, so that $F = \tilde{a}$ with $a = F(0) \in A$. Then the fact that $\tilde{a} \in \mathcal{B}^A$ implies that actually $a \in A^\infty$. We conclude that $T = L_{\tilde{a}} \in A_J^\infty$. ∎

COROLLARY 3.7. *The norm-continuous part of M_J for the action α is A_J.*

Proof. By [1], Proposition 3.1.6, the smooth part is norm-dense in the norm-continuous part. Thus the norm-continuous part is $\overline{A_J^\infty} = A_J$. ∎

4. FURTHER PROPERTIES OF M_J

Let $(M, \mathbf{R}^d, \alpha)$ and $(N, \mathbf{R}^d, \beta)$ be two W^*-dynamical systems and let A and B be their respective subalgebras of norm-continuous vectors. Then any equivariant normal homomorphism $\theta : M \to N$ restricts to a homomorphism from A^∞ into B^∞, and according to [13], Theorem 5.12, this restriction is also a homomorphism from A_J^∞ to B_J^∞ which extends to an equivariant homomorphism $\theta_J : A_J \to B_J$. Our first goal in this section is to show that θ_J extends to an equivariant normal homomorphism from M_J into N_J. The proof of this is fairly straightforward, if slightly tedious.

LEMMA 4.1. *Let $(M, \mathbf{R}^d, \alpha)$ and $(N, \mathbf{R}^d, \beta)$ be W^*-dynamical systems and let $(M \oplus N, \mathbf{R}^d, \alpha \oplus \beta)$ be the direct sum. Then for any skew-symmetric J we have a canonical isomorphism $M_J \oplus N_J \cong (M \oplus N)_J$.*

Proof. Let A and B be the norm-continuous parts of the two systems; it is easy to see that $A \oplus B$ is the norm-continuous part of the direct sum system. Now $M_J \oplus N_J$ acts on $(\mathcal{S}^A)' \oplus (\mathcal{S}^B)'$ while $(M \oplus N)_J$ acts on $(\mathcal{S}^{A\oplus B})'$, both of which are Hilbert $(M \oplus N)$-modules. It will suffice to show that $(\mathcal{S}^A)' \oplus (\mathcal{S}^B)'$ is canonically isomorphic to $(\mathcal{S}^{A\oplus B})'$; having done this it is easy to check that the actions of $A^\infty \oplus B^\infty$ agree, hence their ultraweak closures $M_J \oplus N_J$ and $(M \oplus N)_J$ are the same.

Isomorphism of the Hilbert modules is achieved by checking

$$\mathcal{S}^{A\oplus B} \cong \mathcal{S}^A \oplus \mathcal{S}^B \quad \text{and} \quad (\mathcal{S}^A \oplus \mathcal{S}^B)' \cong (\mathcal{S}^A)' \oplus (\mathcal{S}^B)',$$

both of which are more or less trivial. ∎

LEMMA 4.2. *Let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system and let I be an ultraweakly closed ideal of M which is invariant for α. Then $(M/I, \mathbf{R}^d, \alpha)$ is also a W^*-dynamical system and for any skew-symmetric J we have a canonical isomorphism of M_J/I_J with $(M/I)_J$.*

Proof. First we must observe that the quotient action $\alpha_y(a + I) = \alpha_y(a) + I$ is ultraweakly continuous on M/I. This is clear because the quotient map $M \to M/I$ is ultraweakly continuous, hence $y_n \to y$ (in $\mathbf{R}^d$) implies $\alpha_{y_n}(a) \to \alpha_y(a)$ (ultraweakly in M) implies $\alpha_{y_n}(a) + I \to \alpha_y(a) + I$ (ultraweakly in M/I). So $(M/I, \mathbf{R}^d, \alpha)$ is indeed a W^*-dynamical system.

Now M is canonically isomorphic to $I \oplus (M/I)$, so by Lemma 4.1, M_J is canonically isomorphic to $I_J \oplus (M/I)_J$. Thus M_J/I_J is canonically isomorphic to $(M/I)_J$, which completes the proof. ∎

LEMMA 4.3. *Let $(M, \mathbf{R}^d, \alpha)$ and $(N, \mathbf{R}^d, \beta)$ be W^*-dynamical systems and suppose $M \subset N$ and $\alpha = \beta|_M$. Let J be a skew-symmetric operator on $\mathbf{R}^d$. Then M_J is canonically isomorphic to the ultraweak closure of A^∞ (the smooth part of M) in N_J.*

Proof. Let A and B be the norm-continuous parts of the two systems. Then we must compare the ultraweak closure M_J of A^∞ in its action on $(\mathcal{S}^A)'$ with the ultraweak closure M'_J of A^∞ in its action on $(\mathcal{S}^B)'$.

Let (a_κ) be a net in A^∞ which is bounded in M_J; by [13], Proposition 5.4, the norm of an element of $A^\infty \subset M_J$ is the same as its norm in M'_J, so (a_κ) is also bounded in M'_J. Now the last part of Lemma 1.1 (ii) implies that a_k goes to zero ultraweakly in M_J if and only if it goes to zero ultraweakly in M'_J. From this it follows that the identity map from A^∞ into itself extends to an isomorphism between M_J and M'_J. ∎

THEOREM 4.4. *Let $(M, \mathbf{R}^d, \alpha)$ and $(N; \mathbf{R}^d, \beta)$ be W^*-dynamical systems and let $\theta : M \to N$ be an equivariant normal homomorphism. Let J be skew-symmetric on $\mathbf{R}^d$. Then there is a unique equivariant normal homomorphism $\theta_J : M_J \to N_J$ which agrees with θ on smooth vectors.*

Proof. Since θ is automatically nonexpansive, it is clear that it takes the norm-continuous part of M into the norm-continuous part of N, and hence also the smooth part of M into the smooth part of N.

Let I be the kernel of θ and factor θ into $\theta^1 : M \to M/I$ and $\theta^2 : M/I \to N$. Then by Lemmas 4.2 and 4.3 we have canonical normal homomorphisms θ^1_J : $M_J \to (M/I)_J$ and $\theta^2_J : (M/I)_J \to N_J$, and it is clear that these agree with θ^1 and θ^2 on smooth vectors. Then $\theta_J = \theta^2_J \circ \theta^1_J$ is a normal homomorphism from M_J into N_J which agrees with θ on smooth vectors. Equivariance follows from the obvious equivariance of θ^1_J and θ^2_J; uniqueness is clear by ultraweak continuity. ∎

Now we move to our final topic; in analogy with [13], Theorem 6.4, we wish to show that $(M_J)_K$ is naturally isomorphic to M_{J+K}, for any skew-symmetric operators J and K on $\mathbf{R}^d$. One way to prove this, along the lines of [13], Theorem 6.4, is to show directly that $(M_J)_K \cong M_{K\oplus J} \cong M_{J+K}$ where $K \oplus J$ is a skew-symmetric operator on $\mathbf{R}^d \oplus \mathbf{R}^d$ and the action is $(\alpha \oplus \alpha)_{(t,x)}(a) = \alpha_{t+x}(a)$. However, it is easier to deduce the von Neumann algebra result from the C^*-algebra result, using the following theorem of Akitaka Kishimoto ([5]). It generalizes [9], Proposition 6.1, which dealt with the case of ergodic actions.

THEOREM 4.5. (Kishimoto) *Let M and N be von Neumann algebras and α (resp. β) be an action of a locally compact group G on M (resp. N). Let A (resp. B) denote the norm continuous part of M under α (resp. N under β). Suppose $\varphi : A \to B$ is a surjective isomorphism, equivariant for the actions of α and β. Then φ extends to an isomorphism $\overline{\varphi}$ from M onto N.*

Proof. Let $x \in M$. For $f \in C_c(G)$ let

$$\alpha_f(x) = \int f(t)\alpha_t(x)\,\mathrm{d}t.$$

Then $\alpha_f(x) \in A$. If $x \in A$ then one has

$$\varphi \circ \alpha_f(x) = \beta_f \circ \varphi(x).$$

For $f, g \in C_c(G)$ and $x \in M$

$$\varphi \circ \alpha_{f*g}(x) = \varphi \circ \alpha_f \circ \alpha_g(x) = \beta_f \circ \varphi \circ \alpha_g(x).$$

Let $\{g_i\}$ be an approximate unit for $C_c(G)$, i.e., $g_i \geqslant 0$, $\int g_i(t)\mathrm{d}t = 1$, and for any $f \in C_c(G)$

$$\|g_i * f - f\|_{L^1} \to 0.$$

Let y be a weak*-limit point of $\varphi \circ \alpha_{g_i}(x)$. Then

$$\varphi \circ \alpha_f(x) = \beta_f(y) \tag{4.1}$$

since $\|\alpha_{f*g_i}(x) - \alpha_f(x)\| \leqslant \|f * g_i - f\|_{L^1}\|x\| \to 0$. If z is another weak*-limit point of $\varphi \circ \alpha_{g_i}(x)$, then

$$(\varphi \circ \alpha_f(x) =) \quad \beta_f(y) = \beta_f(z).$$

Since this is true for any $f \in C_c(G)$, one obtains that $y = z$. We define $\overline{\varphi}$ by

$$\overline{\varphi}(x) = y.$$

If $x \in A$, then $\|\varphi \circ \alpha_{g_i}(x) - \varphi(x)\| = \|\alpha_{g_i}(x) - x\| \to 0$ and so $\overline{\varphi}(x) = \varphi(x)$. Thus $\overline{\varphi}$ extends φ. It is obvious that $\overline{\varphi}$ is linear and $\|\overline{\varphi}\| \leqslant 1$.

By the above argument, one can define $\overline{\varphi^{-1}} : N \to M$. Note for $y \in N$, $f \in C_c(G)$

$$\varphi^{-1} \circ \beta_f(y) = \alpha_f \circ \overline{\varphi^{-1}}(y) \quad \text{or} \quad \varphi \circ \alpha_f(\overline{\varphi^{-1}}(y)) = \beta_f(y).$$

Comparing with (4.1) one obtains that $\overline{\varphi} \circ \overline{\varphi^{-1}}(y) = y$. In this way one obtains that $\overline{\varphi}$ is an isometric linear bijection and $(\overline{\varphi})^{-1} = \overline{\varphi^{-1}}$.

If $x_1 \geqslant x_2 \geqslant 0$, then $\overline{\varphi}(x_1) \geqslant \overline{\varphi}(x_2) \geqslant 0$, because $\varphi \circ \alpha_{g_i}(x_1) \geqslant \varphi \circ \alpha_{g_i}(x_2)$. Let $\{x_\mu\}$ be a bounded increasing net in M. Then it follows that $\lim \overline{\varphi}(x_\mu) = \overline{\varphi}(\lim x_\mu)$. Hence if ω is a normal state on N then $\omega \circ \overline{\varphi}$ is a normal state on M. Conversely, if ω' is a normal state on M then $\omega' \circ \overline{\varphi}^{-1}$ is a normal state on N. This shows that $\overline{\varphi}^* : N^* \to M^*$ maps N_* onto M_* or φ is weak*-continuous. Hence $\overline{\varphi}$ is an isomorphism of M onto N. ∎

COROLLARY 4.6. *Let $(M, \mathbf{R}^d, \alpha)$ be a W^*-dynamical system and let J and K be skew-symmetric operators on $\mathbf{R}^d$. Then $(M_J)_K \cong M_{J+K}$.*

Proof. Let A be the norm-continuous part of M. By [13], Theorem 6.5 $(A_J)_K \cong A_{J+K}$, and according to Corollary 3.7 these are the norm-continuous parts of $(M_J)_K$ and M_{J+K}, respectively. The result now follows from Theorem 4.5. ∎

It is perhaps worth noting that the generalization of Theorem 4.5 to homomorphisms is false: an equivariant homomorphism from A to B does not necessarily extend to a homomorphism from M to N. A counterexample can be constructed as follows. Let $S = \mathbf{T} \times \mathbf{N}$, equipped with Lebesgue measure cross counting measure, and define an action of $\mathbf{T}$ on $L^\infty(S)$ by translation, $\alpha_s(f)(t,n) = f(t-s,n)$. Weak*-continuity of α follows easily from weak*-continuity of the corresponding action on $\mathbf{T}$. Also, the norm-continuous part of $L^\infty(S)$ is the set $C_{\mathrm{be}}(S)$ of functions $f(t,n)$ with the property that the sequence of functions $f_n = f(\cdot, n) \in L^\infty(\mathbf{T})$ is uniformly bounded and equicontinuous.

Let Ω be a free ultrafilter on $\mathbf{N}$, and define a homomorphism $\pi : C_{\mathrm{be}}(S) \to C_{\mathrm{be}}(S)$ by

$$(\pi f)(t,n) = \begin{cases} f(t,n) & \text{if } n \geqslant 1; \\ \lim\limits_{k \in \Omega} f(t,k) & \text{if } n = 0. \end{cases}$$

Since the family (f_n) is uniformly bounded and equicontinuous it follows that the limit function $\lim\limits_{k\in\Omega} f(\cdot, k)$ is bounded and continuous, so $\pi(f) \in C_{\mathrm{be}}(S)$.

It is clear that π is equivariant for the action α. However, it does not extend to a normal homomorphism from $L^\infty(S)$ into itself. To see this consider the sequence of functions f_k defined by

$$f_k(n,t) = \begin{cases} 1 & \text{if } n \geqslant k; \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_k \to 0$ weak*, but $(\pi f)(\cdot, 0) = 1$.

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