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LINEAR MAPPINGS THAT PRESERVE OPERATORS ANNIHILATED BY A POLYNOMIAL

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ABSTRACT. Let H be an infinite-dimensional Hilbert space, and let $f(x)$ be a complex polynomial with $\deg(f) \geqslant 2$. We find the general form of surjective linear mappings $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ that preserve operator roots of $f(X) = 0$ in both directions.

KEYWORDS: *Linear preserver.*

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The problem of characterizing linear operators on matrix algebras that leave invariant certain functions, subsets or relations has attracted the attention of many mathematicians in the last few decades ([18]). The first papers concerning this problem ([13], [16]) date back to the previous century. It seems that the systematic study of linear preservers begins with the paper of Marcus and Moyls ([19]). They characterized linear mappings on M_n, the algebra of all $n \times n$ matrices, that preserve the spectrum. This result has been generalized recently by Li and Pierce who obtained the general form of bijective linear operators on M_n mapping the set of matrices annihilated by a given polynomial into itself ([17], Theorem 3.3; see also [2], [14]), thus extending not only the above mentioned result due to Marcus and Moyls but also several results on linear mappings preserving nilpotents, idempotents, or r-potents.

In the recent years there has been also a considerable interest in linear preserver problems on operator algebras over infinite-dimensional spaces ([1], [3]–[10],

[15], [21]–[23], [25]–[29]). It is the aim of this paper to continue this work by studying linear mappings $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ that preserve operators annihilated by a given polynomial. Here, H is an infinite-dimensional complex Hilbert space and $\mathcal{B}(H)$ is the algebra of all bounded linear operators on H. Our approach is different from that one used in the finite-dimensional case ([2], [14], [17]).

Let us fix some notation. For any Hilbert space H we denote by $\mathcal{F}(H)$, $\mathcal{N}(H)$, $\mathcal{N}_k(H)$, and $\mathcal{N}^1(H)$ the set of all finite rank bounded linear operators, the set of all nilpotent linear bounded operators, the set of all nilpotent linear bounded operators with nilindex no greater than k, and the set of all nilpotent bounded linear operators of rank one on H, respectively. For any $x, y \in H$ we denote the inner product of these two vectors by y^*x, while xy^* denotes the rank one operator given by $(xy^*)z = (y^*z)x$. Every operator of rank one can be written in this form. The operator xy^* is nilpotent if and only if $y^*x = 0$. Let $f(x)$ be a complex polynomial. We denote by $\mathcal{V}_f$ the set of all operators $A \in \mathcal{B}(H)$ satisfying $f(A) = 0$. A mapping $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserves operators annihilated by $f(x)$ if $\varphi(\mathcal{V}_f) \subset \mathcal{V}_f$. We say that a mapping $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserves operators annihilated by $f(x)$ in both directions if for every $A \in \mathcal{B}(H)$ we have $f(\varphi(A)) = 0$ if and only if $f(A) = 0$. It should be mentioned here that some mathematicians working on linear preserver problems, e.g., Beasley and his collaborators, say that φ strongly preserves $\mathcal{V}_f$ if it preserves $\mathcal{V}_f$ in both directions.

MAIN THEOREM. *Let H be an infinite-dimensional Hilbert space, and let $f(x)$ be a complex polynomial with* $\deg(f) \geqslant 2$. *Assume that a surjective linear mapping $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserves operators annihilated by $f(x)$ in both directions and that $\varphi(I) = I$. Then φ is either an automorphism or an antiautomorphism.*

Note that the above result can be formulated more precisely. Namely, it is known that every automorphism of $\mathcal{B}(H)$ is inner ([12]), that is, it is of the form $A \mapsto TAT^{-1}$ for some invertible operator $T \in \mathcal{B}(H)$, and every antiautomorphism is of the form $A \mapsto TA^{\mathrm{tr}}T^{-1}$ for some invertible $T \in \mathcal{B}(H)$. Here, A^{tr} denotes the transpose of A relative to a fixed but arbitrary orthonormal basis. Furthermore, the assumption $\varphi(I) = I$ is only needed when $f(x) = x^2$ (cf. Theorem 2.1). For all other cases we give complete descriptions of the surjective linear mappings φ preserving $\mathcal{V}_f$ in both directions. In particular, the case $f(x) = x^k$ for some positive integer k is treated in Section 2, and the other cases are treated in Section 3.

It seems that without the surjectivity assumption the problem of characterizing linear mappings preserving operators annihilated by a given polynomial would become extremely difficult. Namely, even the question how to describe all (not necessarily surjective) endomorphisms of $\mathcal{B}(H)$ doesn't seem to have a simple answer. A finite-dimensional analogue of our result is given in [17]. Surprisingly, the result in the finite-dimensional case is slightly more complicated. For example, the mapping $\varphi : M_n \to M_n$ defined by $\varphi(A) = (1/2)A + (1/2n)(\mathrm{tr}A)I$, where $\mathrm{tr}(A)$ denotes the trace of A, satisfies all the assumptions of the above theorem for $f(x) = x^2$, but is neither an automorphism, nor an antiautomorphism. The reason for this appears to lie in the fact that the linear span of all square-zero operators in the finite-dimensional case is the set of all trace-zero operators, a proper subset of M_n, while it is well-known that in the infinite-dimensional case every operator can be written as a sum of five square-zero operators ([24]).

It turns out that our result is not valid for an arbitrary infinite-dimensional Banach space X. For example, for every positive integer k, $k > 2$, a surjective linear mapping $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserving operators annihilated by $f(x) = x^k$ in both directions is either an automorphism or an antiautomorphism multiplied by a nonzero constant. On the other hand, there exists an infinite-dimensional Banach space X such that the algebra $\mathcal{B}(X)$ has a non-zero multiplicative linear functional ([20], [30]). Then $\mathcal{M}$, the linear span of $\mathcal{N}_k(X)$, is a proper subspace of $\mathcal{B}(X)$. A surjective linear mapping $\varphi : \mathcal{B}(X) \to \mathcal{B}(X)$ preserving operators annihilated by $f(x) = x^k$ in both directions maps $\mathcal{M}$ onto itself. Using our methods one can determine the structure of the restriction of φ to $\mathcal{M}$, while nothing can be said about the behaviour of φ outside this subspace.

Comparing our result with the finite-dimensional one ([2], [14], [17]) we see that we need a stronger assumption on φ: it must preserve $\mathcal{V}_f$ in both directions. It would be nice to have the same result under the weaker assumption of preserving $\mathcal{V}_f$ in one direction only. It would be also interesting to obtain the result for the case $f(x) = x^2$ without the $\varphi(I) = I$ assumption.

It follows from [11], Lemma 1 and [14], p. 169 that in the finite-dimensional case every linear nonsingular mapping $\varphi : M_n \to M_n$ preserving $\mathcal{V}_f$ in one direction actually preserves $\mathcal{V}_f$ in both directions. Our methods work with minor modifications also in the finite-dimensional case. Moreover, in this special case our proof can be simplified a lot (for example, we have no problems with the continuity of φ). Therefore, our paper also gives a new simple proof (with no use of algebraic geometry) of the characterization of linear mappings on M_n preserving $\mathcal{V}_f$ for a given polynomial f.

2. LINEAR MAPPINGS THAT PRESERVE NILPOTENTS WITH BOUNDED NILINDEX

The key step in the proof of Main Theorem is to reduce the general problem to the case when $f(x) = x^k$ for some positive integer $k \geqslant 2$. This section deals with this special case. It should be mentioned that some other linear preserver problems were reduced to the problem of determining linear mappings preserving nilpotents (see [8], [17]).

We will first consider the special case that $f(x) = x^2$. This case is exceptional. First, this is the only case where we have to assume that φ maps the identity operator into itself. And second, the proof does not work in the finite-dimensional case. However, an elementary solution of the problem of characterizing linear mappings on M_n preserving square-zero matrices was given in our earlier paper ([26]).

THEOREM 2.1. *Let H be an infinite-dimensional Hilbert space, and let $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear mapping satisfying $\varphi(I) = I$. Assume that for every $A \in \mathcal{B}(H)$ we have $A^2 = 0$ if and only if $(\varphi(A))^2 = 0$. Then φ is either an automorphism or an antiautomorphism.*

Proof. Let H be a direct sum of two closed infinite-dimensional linear subspaces H_1 and H_2 (note that we didn't assume that H_1 and H_2 are orthogonal). Let P and $Q = I - P$ be idempotents corresponding to this direct sum decomposition, that is, $\operatorname{Im} P = H_1$ and $\operatorname{Im} Q = H_2$. Assume that operators $A, B \in \mathcal{B}(H)$ satisfy $PAP = A$ and $QBQ = B$. It follows from [24], Theorem 2 that A and B can be written as sums of five operators having square zero, $A = A_1 + \cdots + A_5$, $B = B_1 + \cdots + B_5$, with $PA_iP = A_i$ and $QB_iQ = B_i$ for all $i = 1, \ldots, 5$. Clearly, $A_i + B_j$ is a square-zero operator for every pair $i, j \in \{1, \ldots, 5\}$. Consequently, we have $\varphi(A_i)\varphi(B_j) + \varphi(B_j)\varphi(A_i) = 0$, which further yields

$$\varphi(A)\varphi(B) + \varphi(B)\varphi(A) = 0. \tag{2.1}$$

Let $R \in \mathcal{B}(H)$ be an idempotent such that its image and its kernel are both infinite-dimensional. According to (2.1) we have

$$\varphi(R)\varphi(I - R) + \varphi(I - R)\varphi(R) = 0.$$

Applying the assumption $\varphi(I) = I$ we get that $\varphi(R)$ is an idempotent as well.

Let us next consider an idempotent R having a finite-dimensional image. Then we can find idempotents T_1, T_2 with infinite-dimensional images such that $RT_i = T_iR = 0$ for $i = 1, 2$, $T_1T_2 = T_2T_1 = 0$, and $I = R + T_1 + T_2$. We already know that $\varphi(T_1)$ and $\varphi(T_2)$ are idempotents. Applying (2.1) once again we see that $\varphi(T_1)\varphi(T_2) + \varphi(T_2)\varphi(T_1) = 0$. It follows that $\varphi(T_1 + T_2)$ is an idempotent as well, and consequently, $\varphi(R)$ is also an idempotent. Similarly, one can show that φ maps idempotents with finite-dimensional kernel into idempotents. Hence, φ preserves idempotents.

Next, we will prove that φ is injective. Assume on the contrary that there exists a nonzero $A \in \mathcal{B}(H)$ such that $\varphi(A) = 0$. Then A is a square-zero operator and it is easy to find a square-zero operator B such that $A+B$ is not a square-zero operator. It follows that $\varphi(B) = \varphi(A + B)$ is not a square-zero operator. This contradiction shows that φ is injective.

It was proved in [7], Theorem 1 that every bijective linear mapping on $\mathcal{B}(H)$ preserving idempotents is either an automorphism or an antiautomorphism. Applying this result we complete the proof. ∎

In order to study linear mappings preserving nilpotents of higher nilindex in both directions we will need two lemmas.

LEMMA 2.2. *Let $k > 2$ be a positive integer, H a (finite or infinite-dimensional) Hilbert space, and let $A \in \mathcal{N}_k(H)$ be a nonzero operator. Then the following conditions are equivalent.*

(i) $A \in \mathcal{N}^1(H)$.

(ii) *For every $B \in \mathcal{N}_k(H)$ satisfying $A+B \notin \mathcal{N}_k(H)$ we have $B+\alpha A \notin \mathcal{N}_k(H)$ for every nonzero complex number α.*

Proof. In order to prove that (i) implies (ii) we have to recall [27], Proposition 2.1 which states that a nonzero nilpotent $A \in \mathcal{B}(H)$ has rank one if and only if for every $B \in \mathcal{N}(H)$ satisfying $A + B \notin \mathcal{N}(H)$ we have $B + \alpha A \notin \mathcal{N}(H)$ for every nonzero complex number α.

So, assume that A is a nilpotent of rank one and that B is a nilpotent satisfying $B^k = 0$ and $(B+A)^k \neq 0$. If $B+A \notin \mathcal{N}(H)$, then by [27], Proposition 2.1 we have $B + \alpha A \notin \mathcal{N}(H)$ for every nonzero α. Obviously, this yields (ii).

It follows that we can assume from now on that $B + A$ is a nilpotent. The operator A can be written in the form $A = xz^*$ with $z^*x = 0$. Let us choose $y \in H$ satisfying $(B + A)^k y \neq 0$. It is easy to see that $V = \operatorname{span}\{x, Bx, \ldots, B^{k-1}x, y, By, \ldots, B^{k-1}y\}$ is an invariant subspace for both A and B. Let r be the smallest positive integer such that $B^r x = 0$. Let us choose a basis $\{x, Bx, \ldots, B^{r-1}x, e_1, \ldots, e_m\}$ in V. Then, with respect to the direct sum decomposition $V = \operatorname{span}\{x, Bx, \ldots, B^{r-1}x\} \oplus \operatorname{span}\{e_1, \ldots, e_m\}$, the restrictions of A and B to V have the following matrix representations

$$A_{|V} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_{|V} = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 0 & a_1 & \cdots & a_{r-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It follows from $A + B \in \mathcal{N}(H)$ that $a_1 = \cdots = a_{r-1} = 0$, and consequently, $A_{|V} B_{|V}^i A_{|V} = 0$ for every integer $i \geqslant 0$. This further implies that

$$(B_{|V} + \alpha A_{|V})^k = \alpha(A_{|V} B_{|V}^{k-1} + B_{|V} A_{|V} B_{|V}^{k-2} + \cdots + B_{|V}^{k-1} A_{|V}) = \alpha(B_{|V} + A_{|V})^k.$$

Therefore, $(B + \alpha A)^k y = \alpha(B_{|V} + A_{|V})^k y \neq 0$. This completes the proof of the implication (i) $\Rightarrow$ (ii).

To prove the converse statement we assume that $A \in \mathcal{N}_k(H)$ is not a rank one operator. First we will consider the case that $A^2 \neq 0$ and that $k \geqslant 4$. Then it is possible to find a direct sum decomposition of H such that the corresponding matrix representation of A is

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \tag{2.2}$$

where $A_3 \in \mathcal{N}_k(H)$ and A_1 is an operator acting on a three dimensional space with a matrix representation

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $B \in \mathcal{N}_k(H)$ have the matrix representation

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & -2A_3 \end{bmatrix}$$

with B_1 equal to

$$B_1 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

One can easily see that $A + B \notin \mathcal{N}_k(H)$, while $(B + 2A)^4 = 0$.

Next, we will consider the case that $A^2 \neq 0$ and $k = 3$. Set $H_1 = \operatorname{Ker} A$ and define H_j as the orthogonal complement of $\operatorname{Ker} A^{j-1}$ in $\operatorname{Ker} A^j$, $j = 2, 3$. Then

$$A = \begin{bmatrix} 0 & A_1 & A_3 \\ 0 & 0 & A_2 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to the direct sum decomposition $H = H_1 \oplus H_2 \oplus H_3$. Moreover, the operators $A_j : H_{j+1} \to H_j$, $j = 1, 2$, are injective. If $\dim H_1 = 1$ then H is a three dimensional space and in an appropriate basis A has the same matrix representation as A_1 in the previous case. Choosing B_1 as above we see that $A + B_1$ is not nilpotent, while $(B_1 + 2A)^3 = 0$. So, we will assume from now on that $\dim H_1 > 1$. As $A_1A_2 \neq 0$ we can find a square-zero operator $T : H_1 \to H_1$ such that $TA_1A_2 \neq 0$. We define $B \in \mathcal{B}(H)$ by

$$B = \begin{bmatrix} T & -2A_1 & -2A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $B, 2A + B \in \mathcal{N}_3(H)$, while $(A + B)^3 \neq 0$.

It remains to consider the case when A is a square-zero operator not having rank one. Then H can be decomposed into a direct sum of closed linear subspaces $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$ such that the corresponding matrix representation of A is

$$A = \begin{bmatrix} 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with A_1 and A_3 being nonzero. Hence, we can find $z \in H_3$ such that $A_1z \neq 0$. The operator A is similar to

$$\begin{bmatrix} I & M & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} A \begin{bmatrix} I & -M & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_1 & A_2 + MA_3 \\ 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, M is any bounded linear operator mapping H_2 into H_1. So, we can assume with no loss of generality that there exists $x \in H_4$ such that $A_2x = 0$ while $A_3x \neq 0$.

Assume first that $k \geqslant 4$. Let us define $T : H_1 \to H_4$, $S : H_2 \to H_3$ by $T = x(A_1z)^*$ and $S = z(A_3x)^*$. Then $A_2T = 0$, and consequently,

$$B = \begin{bmatrix} 0 & 0 & -2A_1 & -2A_2 \\ 0 & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ T & 0 & 0 & 0 \end{bmatrix}$$

and $(2A + B)$ belong to $\mathcal{N}_4(H)$, while $(A + B)^4x = -\|A_1z\|^2\,\|A_3x\|^2x$, so that $A + B \notin \mathcal{N}(H)$.

In the remaining case that $k = 3$ one can find a bounded linear operator $T : H_2 \to H_3$ such that $A_1 T A_3 \neq 0$. Obviously, the operator

$$B = \begin{bmatrix} 0 & 0 & -2A_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfies $(A+B)^3 \neq 0$ and $(2A+B)^3 = 0$. This completes the proof. ∎

LEMMA 2.3. *Let H be an infinite-dimensional Hilbert space, k an integer no smaller than 3, and let $A \in \mathcal{B}(H)$ be a nonzero square-zero operator. Assume also that A is not a rank one operator. Let B be any operator from $\mathcal{B}(H)$. Suppose that for every finite rank nilpotent operator $T \in \mathcal{B}(H)$ the operator $A + T$ belongs to $\mathcal{N}_k(H)$ if and only if $B + T \in \mathcal{N}_k(H)$. Then $A = B$.*

Proof. Our assumptions imply that $\dim \operatorname{Ker} A = \infty$ and $\dim \overline{\operatorname{Im} A^*} \geqslant 2$. The operators A and B have the following matrix representations

$$A = \begin{bmatrix} 0 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

with respect to the direct sum decomposition $H = H_1 \oplus H_2$ where $H_1 = \operatorname{Ker} A$ and $H_2 = \overline{\operatorname{Im} A^*}$.

Let us first prove that $B_4 = 0$. Assume on the contrary that there exists $x \in H_2$ such that $B_4 x = y \neq 0$. Choose a rank one operator $C \in \mathcal{B}(H_2, H_1)$ satisfying $Cx = -B_2 x$. We will first consider the case that $y = \lambda x$ for some nonzero complex number λ. If we define a nilpotent finite rank operator T by

$$T = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$$

then obviously $(A+T)^2 = 0$ and $(B+T)x = \lambda x$ which contradicts our assumptions. So, it remains to consider the case that x and y are linearly independent. Then we can find a rank one square-zero operator $D \in \mathcal{B}(H_2)$ satisfying $Dx = x - y$. If

$$T = \begin{bmatrix} 0 & C \\ 0 & D \end{bmatrix}$$

then $(A+T)^3 = 0$ while $(B+T)x = x$. This contradiction shows that $B_4 = 0$.

In our next step we will prove that $B_1 = \lambda I$ for some complex number $\lambda \in \mathbf{C}$. Assume on the contrary that there exists $x \in H_1$ such that x and $B_1 x = y$ are linearly independent. Then we can find a square-zero rank one operator $D \in \mathcal{B}(H_1)$

such that $Dx = x - y$. We can also find a rank one operator $C \in \mathcal{B}(H_2, H_1)$ satisfying $CB_3x = -B_2B_3x$. Then the operator

$$T = \begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix}$$

satisfies $(A+T)^3 = 0$ and $(B+T)(x+B_3x) = x + B_3x$. This contradiction shows that $B_1 = \lambda I$ for some complex number λ.

Let us now prove that $B_3 = 0$. Assume that this is not true, so that there exists $x \in H_1$ such that $B_3x = z \neq 0$. We choose $y \in H_2$ such that y and z are linearly independent. Then we can find a rank one square-zero operator $D \in \mathcal{B}(H_2)$ satisfying $Dy = y - z$. We choose a rank one operator $C \in \mathcal{B}(H_2, H_1)$ such that $Cy = x - \lambda x - B_2y$ and define

$$T = \begin{bmatrix} 0 & C \\ 0 & D \end{bmatrix}.$$

Once again we have $(A+T)^3 = 0$ and $(B+T)(x+y) = x+y$. So, we have $B_3 = 0$ and since $B^k = 0$ we have also $\lambda = 0$.

Let y be any nonzero vector from H_1. Then we can find a nonzero $x \in H_1$ with $y^*x = 0$ and a finite rank operator $C \in \mathcal{N}_k(H_1)$ such that $C^{k-1} = xy^*$. Let $D \in \mathcal{B}(H_2, H_1)$ be any rank one operator satisfying $D^*y = -A_2^*y$. If

$$T = \begin{bmatrix} C & D \\ 0 & 0 \end{bmatrix}$$

then $(A + T)^k = 0$, and consequently, $(B + T)^k = 0$. This is equivalent to $(B_2 + D)^*y = 0$. As y was an arbitrary nonzero vector from H_1 we have $A_2 = B_2$ which completes the proof. ∎

THEOREM 2.4. *Let H be an infinite-dimensional Hilbert space, k a positive integer no smaller than 3, and let $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear mapping. Assume that for every $A \in \mathcal{B}(H)$ we have $A^k = 0$ if and only if $\varphi(A)^k = 0$. Then $\varphi = c\theta$ where θ is either an automorphism or an antiautomorphism and c is a nonzero complex number.*

Proof. As in the proof of Theorem 2.1 we show that φ is bijective. It follows from Lemma 2.2 that φ preserves nilpotent operators of rank one in both directions. It was proved in [27], Proof of the Main Theorem that if $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a bijective linear mapping preserving nilpotents of rank one in both directions then there exist a nonzero complex number c and a bounded bijective linear operator $S : H \to H$ such that either

(i) $\varphi(T) = cSTS^{-1}$ for every nilpotent $T \in \mathcal{F}(H)$; or

(ii) $\varphi(T) = cST^{\mathrm{tr}}S^{-1}$ for every nilpotent $T \in \mathcal{F}(H)$. Here, T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis.

In the first case we define a new mapping $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\phi(A) = c^{-1}S^{-1}\varphi(A)S.$$

Obviously, ϕ is a bijective linear mapping preserving nilpotents with nilindex no greater than k in both directions. Moreover, $\phi(T) = T$ for every finite rank nilpotent operator. It follows from Lemma 2.3 that ϕ maps every square-zero operator into itself. As every operator from $\mathcal{B}(H)$ is a sum of five square-zero operators ([24]) we have $\phi(A) = A$ for every $A \in \mathcal{B}(H)$. This completes the proof in the first case. In almost the same way we get in case (ii) that $\varphi = c\theta$ where θ is an antiautomorphism of $\mathcal{B}(H)$. ∎

3. PROOF OF THE MAIN RESULT: THE GENERAL CASE

Let $f(x) = (x - x_1)\cdots(x - x_k)$ be a complex polynomial with $\deg(f) = k \geqslant 2$. Here, $x_1, \ldots, x_k$ are possibly repeated complex numbers. Let $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear mapping that preserve operators annihilated by $f(x)$ in both directions. First we will show that φ preserves $\mathcal{N}_k(H)$ in both directions. Let $N \in \mathcal{B}(H)$ be a nilpotent of nilindex $r \leqslant k$. Then there exists a direct sum decomposition of H into closed subspaces $H = H_1 \oplus \cdots \oplus H_r$ such that

$$N = \begin{bmatrix} 0 & N_{1,2} & N_{1,3} & \ldots & N_{1,r} \\ 0 & 0 & N_{2,3} & \ldots & N_{2,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N_{r-1,r} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

with respect to this decomposition. To see that this is true one can take $H_1 = \operatorname{Ker} N$ and define H_j as the orthogonal complement of $\operatorname{Ker} N^{j-1}$ in $\operatorname{Ker} N^j$, $j = 2, \ldots, r$. If

$$A = \begin{bmatrix} x_1 I & 0 & \ldots & 0 \\ 0 & x_2 I & \ldots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_r I \end{bmatrix}$$

then $f(A + \alpha N) = 0$ for every complex number α. It follows that $f(\varphi(A) + \alpha\varphi(N)) = 0$ for every α. All coefficients in this operator polynomial must be zero. In particular, the coefficient at α^k must be zero, and hence, $\varphi(N)^k = 0$.

Assume now that $\varphi(M)^k = 0$ for some operator $M \in \mathcal{B}(H)$. As before we find $C \in \mathcal{B}(H)$ such that $f(C + \alpha\varphi(M)) = 0$, $\alpha \in \mathbf{C}$. The surjectivity of φ yields the existence of $D \in \mathcal{B}(H)$ such that $\varphi(D) = C$. It follows that $f(D + \alpha M) = 0$ for every complex number α, and consequently $M^k = 0$.

We are now ready to prove our main result using the above considerations and the results from the previous section. We have to distinguish several cases. Let us first assume that $f(x) = (x - a)^2$ for some nonzero complex number a.

PROPOSITION 3.1. *Let H be an infinite-dimensional Hilbert space, a a nonzero complex number, and let $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear mapping. Assume that for every $A \in \mathcal{B}(H)$ we have $(A - aI)^2 = 0$ if and only if $(\varphi(A) - aI)^2 = 0$. Then φ is either an automorphism or an antiautomorphism.*

Proof. We have already proved that φ preserves square-zero operators in both directions. All we have to do is to show that $\varphi(I) = I$ and then the result follows directly from Theorem 2.1.

Let N be any square-zero operator. Then we have $f(aI + \alpha N) = 0$ for every complex number α, and consequently, $(\varphi(aI) + \alpha\varphi(N) - aI)^2 = 0$. It follows that $XA + AX = 0$ where $X = \varphi(N)$ and $A = \varphi(aI) - aI$. The mapping φ is surjective and preserves square-zero operators in both directions. So, $XA + AX = 0$ holds true for any square-zero operator X. Since every operator can be written as a sum of five square-zero operators we have $XA + AX = 0$ for every operator X. It follows that $A = 0$, or equivalently, $\varphi(I) = I$. This completes the proof. ∎

Let us next consider linear mappings preserving $\mathcal{V}_f$ with $f(x) = (x-a)(x-b)$, $a \neq b$, $a \neq -b$.

PROPOSITION 3.2. *Let H be an infinite-dimensional Hilbert space, a and b complex numbers, $a \neq b$, $a \neq -b$, and let $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear mapping. Assume that for every $A \in \mathcal{B}(H)$ we have $(A - aI)(A - bI) = 0$ if and only if $(\varphi(A) - aI)(\varphi(A) - bI) = 0$. Then φ is either an automorphism or an antiautomorphism.*

Proof. Once again we only need to prove that $\varphi(I) = I$. At least one of a and b is a nonzero number. With no loss of generality we will assume that $a \neq 0$. Obviously, we have $\varphi(aI) = aP + b(I - P)$ for some idempotent $P \in \mathcal{B}(H)$. We will show that P is either I or 0. If this is not true, then we can find a nonzero square-zero operator N satisfying $PN = N$ and $NP = 0$. The surjectivity assumption yields the existence of a square-zero operator M such that $\varphi(M) = N$. As $f(\varphi(aI) + N) = 0$ we have $f(aI + M) = 0$ which further yields $M = 0$. This contradiction shows that either $\varphi(aI) = aI$ or $\varphi(aI) = bI$.

In the first case we have the desired relation $\varphi(I) = I$. In the second case we get $\varphi(bI) = \varphi((b/a)aI) = b^2a^{-1}I$. But $f(\varphi(bI)) = 0$, and therefore, b^2a^{-1} is equal either to a or to b. The mapping φ preserves square-zero operators in both directions and must be therefore bijective. It follows that $\varphi(aI) \neq \varphi(bI)$, and consequently, $b^2a^{-1} = a$. Under our assumptions this is not possible. So, the second possibility can not occur. This completes the proof. ∎

Under the assumption that $a = -b$ we have two possibilities: $\varphi(I)$ is equal either to I or to $-I$. So, we have the following result.

PROPOSITION 3.3. *Let H be an infinite-dimensional Hilbert space, a a nonzero complex number, and let $\varphi : B(H) \to B(H)$ be a surjective linear mapping. Assume that for every $A \in B(H)$ we have $A^2 = aI$ if and only if $(\varphi(A))^2 = aI$. Then $\varphi = c\theta$ where $c \in \{-1, 1\}$ and θ is either an automorphism or an antiautomorphism.*

Untill now we have proved that Main Theorem holds true whenever $\deg(f) = 2$ or $f(x) = x^k$. So, let us assume from now on that $\deg(f) = m > 2$ and $f(x) \neq x^m$. Then we have already proved that φ preserves $\mathcal{N}_m(H)$ in both directions. So, by Theorem 2.4, φ is either an automorphism or an antiautomorphism multiplied by a nonzero constant c. Assume that $c \neq 1$ and write $f(x) = (x - x_1) \cdots (x - x_m)$ where $x_1, \ldots, x_m$ are possibly repeated complex numbers. If x_j is any of the roots of f then $\varphi(x_jI) = cx_jI$; but $f(\varphi(x_jI)) = 0$ and therefore $cx_j = x_p$ for some p, $1 \leqslant p \leqslant m$. This shows that the finite set $\{x_1, \ldots, x_m\}$ is closed under multiplication by c, and consequently, c must be a kth root of unity for some positive integer k. Moreover, the polynomial f must be of the form $f(x) = x^l g(x^k)$, $l \geqslant 0$. This gives us together with the previous propositions the following result which together with Theorems 2.1 and 2.4 yields the Main Theorem.

THEOREM 3.4. *Let H be an infinite-dimensional Hilbert space, and let $\varphi : B(H) \to B(H)$ be a surjective linear mapping. Let f be a complex polynomial satisfying $\deg(f) = m \geqslant 2$ and $f(x) \neq x^m$. Assume that for every $A \in B(H)$ we have $f(A) = 0$ if and only if $f(\varphi(A)) = 0$. Then either φ is an automorphism, or an antiautomorphism, or for some $l \geqslant 0$, $k \geqslant 2$, $g(x) \in \mathbb{C}[x]$, we have $f(x) = x^l g(x^k)$, and for some kth root of unity c, $\varphi = c\theta$ where θ is either an automorphism or an antiautomorphism.*

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