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DUAL OPERATOR ALGEBRAS AND CONTRACTIONS WITH FINITE DEFECT INDICES

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ABSTRACT. We charaterize the membership in the various dual operator algebra classes $\mathbb{A}_{m,n}$ of absolutely continuous contraction operators T on Hilbert space with finite defect indices; the characterizations involue multiplicity measures like the difference of the defects or the number of copies of the bilateral shift in the minimal coisometric extension of T. We then give examples of operators in no class $\mathbb{A}_{m,n}$ with m or n infinite.

KEYWORDS: *Dual operator algebra, defect index, minimal coisometric extension.*

AMS SUBJECT CLASSIFICATION: Primary 47D27; Secondary 47A45.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. The study of contractions on Hilbert space via dual algebras was initiated in [3]; crucial have been certain classes $\mathbb{A}_{m,n}$, $1 \leqslant m, n \leqslant \aleph_0$ (definitions reviewed below). Yet the classes with both indices finite are not well understood. Further, there are few classes of operators whose membership is known exactly. In this note we give a characterization of membership in the classes $\mathbb{A}_{m,n}$ for operators with finite defect, and provide a family of operators not in any class with an infinite index.

We begin with some standard notation and preliminaries. Let $\mathbb{T}$ denote the unit circle in the complex plane, $\mathbb{D}$ the open unit disk, and m Lebesgue measure on

$\mathbf{T}$. For $1 \leqslant p < \infty$, let $L^p = L^p(\mathbf{T})$ be the usual Lebesgue space, and $H^p = H^p(\mathbf{T})$ the usual Hardy subspace of L^p. It is well known that H^∞ is the dual space of a certain quotient space L^1/H_0^1 of L^1. For Γ some measurable subset of $\mathbf{T}$, let $L^2(\Gamma)$ denote the subspace of $L^2(\mathbf{T})$ consisting of those functions with support on Γ, and denote by M_Γ the operator of multiplication by z on $L^2(\Gamma)$. We will use $S_\mathbf{T}$ to denote the unilateral shift (that is, multiplication by z on $H^2(\mathbf{T})$). Finally, for some subset Λ of $\mathbf{D}$, denote by $\mathrm{NTL}(\Lambda)$ the subset of $\mathbf{T}$ consisting of non-tangential limits of sequences of points from Λ.

We shall have concern solely with *singly generated* dual algebras (see [1] for a full discussion), so suppose T is an operator in $\mathcal{L}(\mathcal{H})$, and denote by $\mathcal{A}_T$ the dual algebra it generates. One knows that $\mathcal{A}_T$ is the dual space of a certain quotient Q_T of the trace class operators; denote elements of Q_T by $[L]_{Q_T}$ or simply $[L]$ if no confusion will arise. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator defined by $(x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}$. The elements $[x \otimes y]_{Q_T}$ constitute particularly important elements of Q_T.

Recall that a contraction T is completely non-unitary (abbreviated c.n.u.) if it has no unitary direct summand, and is *absolutely continuous* if its unitary part has spectral measure absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$ or acts on the space (0). For an absolutely continuous contraction T we use without further comment the Sz.-Nagy–Foiaş Functional Calculus (cf. [1], Theorem 4.1 and [17], Theorem 4.1) $\Phi_T : H^\infty \to \mathcal{A}_T$. Recall that there exists a bounded, linear, one-to-one map ϕ_T of Q_T into L^1/H_0^1 such that $\Phi_T = \phi_T^*$.

The class $\mathbb{A} = \mathbb{A}(\mathcal{H})$ consists of those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ is an isometry. It is by now standard that for T in $\mathbb{A}$ and each λ in $\mathbf{D}$ there exists an element $[C_\lambda]_{Q_T}$ such that $\langle f(T), [C_\lambda]_{Q_T}\rangle = f(\lambda)$, $f \in H^\infty$ (where we identify f with its extension to $\mathbf{D}$).

Suppose that m and n are any cardinal numbers satisfying $1 \leqslant m, n \leqslant \aleph_0$. A dual algebra $\mathcal{A}_T$ is said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}]$, $0 \leqslant i < m$, $0 \leqslant j < n$, where $\{[L_{ij}]\}_{\substack{0\leqslant i<m \\ 0\leqslant j<n}}$ is an arbitrary $m \times n$ array from Q_T, has a solution $\{x_i\}_{0\leqslant i<m}$, $\{y_j\}_{0\leqslant j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. As usual we denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. Furthermore, we denote by $\mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$, and usually simplify $\mathbb{A}_{m,n}(\mathcal{H})$ to $\mathbb{A}_{m,n}$ and $\mathbb{A}_{n,n}$ to $\mathbb{A}_n$.

Recall that $\mathcal{M}$ is a semi-invariant subspace for $T \in \mathcal{L}(\mathcal{H})$ if there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2 = \mathcal{N}_1 \cap \mathcal{N}_2^\perp$. We write $\mathrm{Lat}(T)$ and $\mathrm{SLat}(T)$ respectively for the lattices of invariant and semi-invariant subspaces of T. For $\mathcal{L}$ an element of Lat(T) we let $T|\mathcal{L}$ denote the restriction of T to $\mathcal{L}$; if S is (unitarily equivalent to) a restriction of T to an invariant subspace we may say instead that T is an extension of S. For $\mathcal{M}$ in SLat(T), let $T_{\mathcal{M}}$ denote the *compression* of T to $\mathcal{M}$, that is, $T_{\mathcal{M}} = P_{\mathcal{M}}T|\mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$. We say that T *dilates* an operator S, or S is a *compression* of T, if S is unitarily equivalent to a compression of T.

Recall from [17] that a contraction T has an isometric dilation which is, in a natural sense, minimal, and we denote this by U_T^+; its unitary extension yields U_T, the minimal unitary dilation of T. There is as well a minimal co-isometric extension of T which we denote by B_T. Since U_T^+ is an isometry it has a (maximal) unitary summand R_T, and we denote by $\Sigma(T)$ the Borel subset of $\mathbf{T}$ (unique up to sets of measure zero) such that $m|\Sigma(T)$ is a scalar spectral measure for R_T. Similarly, for the co-isometry B_T, denote by $\Sigma_*(T)$ the analogous set for its maximal unitary summand.

Recall as well from [17] that the class $C_{0\cdot}$ consists of those operators T such that $\|T^n x\| \to 0$ for all $x \in \mathcal{H}$, and $C_{\cdot 0} = (C_{0\cdot})^*$. Also, $C_{1\cdot}$ is the class of those T such that $\|T^n x\| \to 0$ only for $x = 0$, $C_{\cdot 1} = (C_{1\cdot})^*$, and the class $C_{\alpha\beta}$ is the intersection of $C_{\alpha\cdot}$ and $C_{\cdot\beta}$ for each pair (α, β), $\alpha, \beta \in \{0, 1\}$. For a contraction T the defect index d_T is defined to be $d_T = \dim\{(I - T^*T)^{\frac{1}{2}}\mathcal{H}\}^{-}$. If both d_T and d_{T^*} are finite, we say T has finite defects.

Finally, we shall use two notions associated with any absolutely continuous contraction T which arise from efforts to study its "boundary behavior" on the circle $\mathbf{T}$. First, for an absolutely continuous contraction T define as in [4] a subset Σ of $\mathbf{T}$ to be essential for T if $\|f(T)\| \geqslant \|f|\Sigma\|_\infty \triangleq \operatorname{ess\,sup}|f|\Sigma|$ for every function f in $H^\infty(\mathbf{T})$. Define ess(T) to be the maximal essential set for T (up to sets of Lebesgue measure zero). (Observe then that $T \in \mathbb{A}$ if and only if $\mathbf{T}$ is essential for T.) Second, in [7] a subset X_T of $\mathbf{T}$ is defined for any absolutely continuous contraction T. (Roughly, X_T is the subset of $\mathbf{T}$ on which T "wants" to be in the most restrictive class $\mathbb{A}_{\aleph_0,\aleph_0}$.) For our purposes we need merely note from that work that X_T is empty for an operator with finite defects, and that $\mathbf{T} = \text{ess}(T) = X_T \cup \Sigma(T) \cup \Sigma_*(T)$, $T \in \mathbb{A}$.

2. OPERATORS WITH FINITE DEFECTS

We embark upon our description of contraction operators T, with both defect indices d_T and d_{T^*} finite, in the various classes $\mathbb{A}_{m,n}$. Recall from [17] that for an absolutely continuous contraction T the operators $\Delta_T(t)$, $t \in (0, 2\pi)$, are defined in terms of the characteristic function Θ_T by $\Delta_T(t) = (I - \Theta_T(e^{it})^* \Theta_T(e^{it}))^{1/2}$, $t \in (0, 2\pi)$. Denote by $r(t)$ the rank of $\Delta_T(t)$.

LEMMA 2.1. *Let T be a* c.n.u. *contraction with d_T or d_{T^*} finite. Then*

$$U_T^+ \cong \underbrace{S_{\mathbf{T}} \oplus \cdots \oplus S_{\mathbf{T}}}_{(d_{T^*})} \oplus \underbrace{M_{\Gamma_1} \oplus \cdots \oplus M_{\Gamma_n}}_{(d_T)}$$

acting on $\underbrace{H^2(\mathbf{T}) \oplus \cdots \oplus H^2(\mathbf{T})}_{(d_{T^*})} \oplus \underbrace{L^2(\Gamma_1) \oplus \cdots \oplus L^2(\Gamma_n)}_{(d_T)}$, *where* $\Gamma_i = \{e^{it} : t \in (0, 2\pi), r(t) \geqslant i\}$, $i = 1, 2, \ldots, n$.

Proof. For the case of d_T and d_{T^*} finite this holds from [17], Theorem VI.6.1, and if one defect index is infinite combine Theorem VI.3.1, Proposition VI.2.1, II.(2.6), II.(2.7), and II.(1.3) of that work. ∎

A proposition for the finite defect case follows immediately.

PROPOSITION 2.2. *Suppose $T \in \mathbb{A}$, d_T, $d_{T^*} < \infty$. Then*
(i) *If $d_T < d_{T^*}$ then $T \in \mathbb{A}_{d_{T^*} - d_T, \aleph_0}$;*
(ii) *If $d_{T^*} < d_T$ then $T \in \mathbb{A}_{\aleph_0, d_T - d_{T^*}}$.*

Proof. From duality (ii) follows easily from (i), and for (i) we consider only $d_T = 1$ and $d_{T^*} = 2$ for ease of exposition. From [17], Theorem VI.6.1 we have that the minimal unitary dilation U_T of T is

$$U_T \cong M_{\mathbf{T}} \oplus M_{\mathbf{T}} \oplus M_{\Gamma_1} \tag{2.1}$$

acting on $L^2(\mathbf{T}) \oplus L^2(\mathbf{T}) \oplus L^2(\Gamma_1)$. It follows also from [17], Theorem VI.6.1 that $U_{T^*}^+$, the minimal isometric dilation of T^*, is

$$U_{T^*}^+ \cong S_{\mathbf{T}} \oplus M_{\Gamma_1'} \oplus M_{\Gamma_2'}$$

acting on $H^2(\mathbf{T}) \oplus L^2(\Gamma_1') \oplus L^2(\Gamma_1')$, where $\Gamma_1' \supseteq \Gamma_2'$. But it is well known that the minimal coisometric extension B_T of T is the adjoint of the minimal isometric dilation of T^*, so

$$B_T \cong (U_{T^*}^+)^* \cong S_{\mathbf{T}}^* \oplus M_{\Gamma_1'}^* \oplus M_{\Gamma_2'}^* \tag{2.2}$$

acting on $H^2(\mathbf{T}) \oplus L^2(\Gamma_1') \oplus L^2(\Gamma_1')$, where $\Gamma_1' \supseteq \Gamma_2'$. And since the minimal isometric dilation of B_T is the minimal unitary dilation of T, we have

$$U_T \cong M_{\mathbf{T}} \oplus M^*_{\Gamma_1'} \oplus M^*_{\Gamma_2'} \tag{2.3}$$

acting on $L^2(\mathbf{T}) \oplus L^2(\Gamma_1') \oplus L^2(\Gamma_1')$. A comparison of (2.1) and (2.3) shows that $\Gamma_1' = \mathbf{T}$ (except possibly on a set of measure zero). But then from (2.2) B_T contains a copy of the bilateral shift, and it follows easily from [14] that $T \in \mathbb{A}_{1,\aleph_0}$, as desired. ∎

We may also make some progress for c.n.u. contractions even with one defect infinite by use of Lemma 2.1. For any operator T and integer n, denote by $T^{(n)}$ the n-fold ampliation of T. We first dispose of a special case: note that if $d_T = 0$, then T is a c.n.u. isometry, so $T = S_{\mathbf{T}}^{(d_{T^*})}$. It is well known that for $n < \infty$, $S_{\mathbf{T}}^{(n)} \in \mathbb{A}_{n,\aleph_0}$ and $S^{(n)} \notin \mathbb{A}_{n+1,1}$ (via [9]), and so the case $d_T = 0$ (and, via duality, $d_{T^*} = 0$) need not be further considered.

The following is the extension one would expect of Proposition 2.2.

PROPOSITION 2.3. *If T is a* c.n.u. *contraction having one finite and non-zero defect index and one infinite defect, then $T \in \mathbb{A}_{\aleph_0}$.*

Proof. Suppose first that $T \in \mathbb{A}$, and consider the case $1 < d_T < \infty$ and $d_{T^*} = \infty$. Using the fact from [14] that if B_T contains a bilateral shift of infinite multiplicity, then $T \in \mathbb{A}_{\aleph_0}$, the proof is just as that for Proposition 2.2, *mutatis mutandis*. If we assume merely that T is an absolutely continuous contraction, Lemma 2.1 and [14] show that in fact $T \in \mathbb{A}$. ∎

This yields the following corollary, which we then strengthen.

COROLLARY 2.4. *Let $T \in \mathbb{A} \setminus (\mathbb{A}_{1,\aleph_0} \cup \mathbb{A}_{\aleph_0,1})$. Then $d_T = d_{T^*}$.*

PROPOSITION 2.5. *If T is a* c.n.u. *contraction in $\mathbb{A} \setminus (\mathbb{A}_{1,\aleph_0} \cup \mathbb{A}_{\aleph_0,1})$ then both defects are infinite.*

Proof. Suppose that T is a c.n.u. contraction in $\mathbb{A} \setminus (\mathbb{A}_{1,\aleph_0} \cup \mathbb{A}_{\aleph_0,1})$ and with finite defects, so by Corollary 2.4 we know $d_T = d_{T^*}$. Consider T in its upper triangular $C_{\cdot 1}$–$C_{\cdot 0}$ decomposition, say

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}. \tag{2.4}$$

Observe that both T_1 and T_2 have finite defect indices by [17], Proposition VII.3.6. Suppose now that $\operatorname{ess}(T_2) \neq \emptyset$. Since $T_2 \in C_{\cdot 0}$, from [7],

$$\operatorname{ess}(T_2) = X_{T_2} \cup \Sigma_*(T_2), \tag{2.5}$$

and since T_2 has finite defects $X_{T_2} = \emptyset$, and thus $\text{ess}(T_2) = \Sigma_*(T_2) \neq \emptyset$. Then T_2 has a non-zero unitary part of its minimal coisometric extension, so since T_2 has finite defects we know by [9], Lemma 1.3 that the minimal coisometric extension contains a bilateral shift. Using [14] as usual, we have T_2, and hence T, in $\mathbb{A}_{1,\aleph_0}$, a contradiction. Thus we may assume that $\text{ess}(T_2) = \emptyset$. Since for any absolutely continuous contraction V we have from [7] that $\text{ess}(V) = X_V \cup \Sigma_*(V) \cup \Sigma(V)$, we deduce finally that

$$X_{T_2} = \Sigma_*(T_2) = \Sigma(T_2) = \emptyset. \tag{2.6}$$

Since $T \in \mathbb{A}$ we know that $\text{ess}(T) = \mathbb{T}$, and since T has finite defects $X_T = \emptyset$. Then

$$\begin{aligned} \mathbb{T} &= X_T \cup \Sigma_*(T) \cup \Sigma(T) \\ &= \Sigma_*(T) \cup \Sigma(T) \\ &= \Sigma_*(T_1) \cup \Sigma(T_1) \cup \Sigma_*(T_2) \cup \Sigma(T_2) \\ &= \Sigma_*(T_1) \cup \Sigma(T_1), \end{aligned}$$

where the third inequality follows from [2], Lemma 1.4 and the fourth from (2.6). Thus $\text{ess}(T_1) = \mathbb{T}$, so $T_1 \in \mathbb{A}$, and of course T_1 has finite defect indices.

Consider now T_1 in its upper triangular $C_{\cdot 1}$–$C_{\cdot 0}$ decomposition, say

$$T_1 = \begin{pmatrix} T_{11} & * \\ 0 & T_{12} \end{pmatrix}. \tag{2.7}$$

A repetition of the argument above, and some adjoints, shows that necessarily either T_{11}, and hence T, is in $\mathbb{A}_{\aleph_0,1}$ (a contradiction) or $T_{12} \in \mathbb{A}$ and T_{12} has finite defect indices. But note that $T_{12} \in C_{11}$; from [10] we have T_{12}, hence T, in $\mathbb{A}_{1,\aleph_0}$, and this final contradiction finishes the proof. ∎

The following lemma is from [5], and will enable us to complete our characterization of the operators with finite defect indices in the classes $\mathbb{A}_{n,\aleph_0}$.

LEMMA 2.6. *For any absolutely continuous contraction T and each integer $n \geqslant 1$, the following are equivalent:*

(i) $T \in \mathbb{A}_{n,\aleph_0}$;

(ii) *there exists $\mathcal{M} \in \text{Lat}(T)$ such that $T|\mathcal{M} \in C_{\cdot 0} \cap \mathbb{A}_{n,\aleph_0}$.*

Proof. That (ii) implies (i) is obvious, so suppose that $T \in \mathbb{A}_{n,\aleph_0}$ for some n. From [5], there exist pairwise orthogonal subspaces $\mathcal{M}_i$, $i = 1, 2, \ldots, n$ such that

$\mathcal{M}_1 \in \mathrm{Lat}(T)$ and $\mathcal{M}_i \in \mathrm{SLat}(T)$ $(i = 2, \dots, n)$ and so that $T_{\mathcal{M}_i} \in \mathbf{A}_{1,\aleph_0}$ for $i = 1, \dots, n$. These subspaces may in fact be chosen so that T has the decomposition

$$T = \begin{pmatrix} T_1 & & & & \\ & T_2 & & * & \\ & & \ddots & & \\ & 0 & & T_n & \\ & & & & T_{n+1} \end{pmatrix} \tag{2.8}$$

with respect to $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n \oplus \left(\bigvee_{i=1}^{n} \mathcal{M}_i\right)^{\perp}$, where $T_i \triangleq T_{\mathcal{M}_i}$ $(i = 1, \dots, n)$ and T_{n+1} is the compression of T to $\left(\bigvee_{i=1}^{n} \mathcal{M}_i\right)^{\perp}$ (possibly T_{n+1} is absent).

Since T_1 is in $\mathbf{A}_{1,\aleph_0}$, there exists (from [6]) a subspace $\mathcal{N}_1$ invariant for T_1 (and hence for T) such that $T_1|\mathcal{N}_1 \in C_{\cdot 0} \cap \mathbf{A}_{1,\aleph_0}$. Further, from the techniques of [5] the compression of T to $(\mathcal{N}_1)^{\perp}$ is in $\mathbf{A}_{n-1,\aleph_0}$ (essentially, because of the $n-1$ blocks T_i, $i = 2, \dots, n$, on the diagonal in (2.8)).

The above argument, applied now to the compression of T to $(\mathcal{N}_1)^{\perp}$ which is in $\mathbf{A}_{n-1,\aleph_0}$, produces a subspace $\mathcal{N}_2$ invariant for this compression such that $(T_{(\mathcal{N}_1)^{\perp}})_{\mathcal{N}_2} = T_{\mathcal{N}_2} \in C_{\cdot 0} \cap \mathbf{A}_{1,\aleph_0}$, while the compression of T to $(\mathcal{N}_1 \cup \mathcal{N}_2)^{\perp}$ is in $\mathbf{A}_{n-2,\aleph_0}$. Iteration of this argument shows that T has the form

$$T = \begin{pmatrix} T_1' & & & & \\ & T_2' & & * & \\ & & \ddots & & \\ & 0 & & T_n' & \\ & & & & T_{n+1}' \end{pmatrix} \tag{2.9}$$

with respect to the decomposition $\mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_n \oplus \left(\bigvee_{i=1}^{n} \mathcal{N}_i\right)^{\perp}$, and so that T_i' is in $C_{\cdot 0} \cap \mathbf{A}_{1,\aleph_0}$, $i = 1, \dots, n$. From [5] $\mathcal{M} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_n$ is as required. ∎

We may finally characterize membership of a contraction with finite defect indices in some class $\mathbf{A}_{n,\aleph_0}$.

THEOREM 2.7. *Suppose T is an absolutely continuous contraction with finite defect indices d_T and d_{T^*}. Then the following are equivalent:*

(i) $T \in \mathbf{A}_{n,\aleph_0}$;

(ii) *there exists $\mathcal{M} \in \mathrm{Lat}(T)$ such that $\widetilde{T} \triangleq T|\mathcal{M}$ is* c.n.u. *and $d_{\widetilde{T}^*} - d_{\widetilde{T}} \geqslant n$;*

(iii) *$R_*(T)$ contains a copy of the bilateral shift of multiplicity n.*

Proof. For (i) implies (ii), note that the restriction of T to the subspace produced from Lemma 2.6 has finite defects by [17], Proposition VII.3.6 and is in

$C_{\cdot 0} \cap \mathbb{A}_{n,\aleph_0}$, and so by [9] satisfies the defect inequality in (ii). That (ii) implies (iii) follows from [9], which shows that $R_*(T|\mathcal{M})$ contains a copy of $M_{\mathbf{T}}^{(n)}$, and the obvious fact that the unitary part of $R_*(T)$ has $R_*(T|\mathcal{M})$ as a summand. Finally, (iii) implies (i) follows from [14]. ∎

Observe that there is as usual a dual version of the result for the class $\mathbb{A}_{\aleph_0,n}$, which we leave to the interested reader. We remark also that the characterization of membership in the $\mathbb{A}_{m,n}$ with both m and n finite even in the case of finite defects awaits the determination, for example, of whether $S_{\mathbf{T}} \oplus S_{\mathbf{T}}^*$ is in $\mathbb{A}_{2,2}$, a long standing but still open problem. Note also that (i) does not imply that $d_{T^*} \dot{-} d_T \geqslant n$, as is obvious from the example $T = S_{\mathbf{T}}^{(n)} \oplus S_{\mathbf{T}}^*$, and that since a normal operator with eigenvalues dense in the disk is in $\mathbb{A}_{\aleph_0}$ but has no unitary part of its minimal coisometric extension, some condition like finite defects is required for the results above.

Indeed, one may produce a T merely in $\mathbb{A}_{1,\aleph_0}$ (not even in $\mathbb{A}_{2,2}$) for which there is no $\mathcal{M} \in \mathrm{Lat}(T)$ so that $T_{\mathcal{M}} \in \mathbb{A}_{1,\aleph_0}$ and $d_{T_{\mathcal{M}}}, d_{T_{\mathcal{M}}^*}$ are both finite. Let B' be constructed by decreasing each weight of $M_{\mathbf{T}}$ in such fashion as to have all weights of B' strictly less than one but B' similar to $M_{\mathbf{T}}$ (see [15]). Observe that B' is a contraction, $\mathcal{A}_{B'}$ has property $(\mathbb{A}_{1,\aleph_0})$ (since $\mathcal{A}_{M_{\mathbf{T}}}$ does), and $B' \in \mathbb{A}$. Hence $B' \in \mathbb{A}_{1,\aleph_0}$. But any compression of B' shrinks the norm of every vector, so any compression to an infinite dimensional subspace has infinite defect indices.

The following corollary improves a theorem in [10]. Denote by J_T the Jordan model of T (see [16] for the relevant definitions).

COROLLARY 2.8. *Suppose $T \in \mathbb{A}_{n,\aleph_0}$ and T has finite defect indices. Then there exists $\mathcal{M} \in \mathrm{Lat}(T)$ such that $J_{T|\mathcal{M}}$ has the unilateral shift of multiplicity n as a summand.*

3. SOME EXAMPLES

We consider in this section a family of examples illustrative for $\mathbb{A}$ and its subclasses and providing examples of operators not in $\mathbb{A}_{\aleph_0,1} \cup \mathbb{A}_{1,\aleph_0}$. Let

$$L = \{z \in \mathbf{C} : (|z| = 1 \text{ and } \mathrm{Re}(z) \leqslant 0) \text{ or } (\mathrm{Re}(z) = 0 \text{ and } |\mathrm{Im}(z)| \leqslant 1)\}.$$

Thus L is the boundary of the open left half disk; let $\mathbf{D}_L$ denote this left half disk (so $\mathbf{D}_L$ is the simply connected component of $\mathbf{C} \setminus L$). Put arclength measure ℓ on L, and define $L^2(L, \mathrm{d}\ell)$ to be the space of (equivalence classes of) square integrable complex functions on L. Define $H^2(L, \mathrm{d}\ell)$ to be the closure of the polynomials in $L^2(L, \mathrm{d}\ell)$. Let N_L be the (normal) operator of multiplication by z on $L^2(L, \mathrm{d}\ell)$, and T_L its (subnormal) restriction to $H^2(L, \mathrm{d}\ell)$. Similarly, define R to be the reflection of L across the imaginary axis, and $\mathbb{D}_R$, $L^2(R, \mathrm{d}\ell)$, $H^2(R, \mathrm{d}\ell)$, N_R, and T_R in the obvious way.

We will consider $T_L \oplus T_R^*$ and variants in what follows; the basic example below stems from [6] but was never published.

PROPOSITION 3.1. *Let $T = T_L \oplus T_R^*$. Then $T \in \mathbb{A}$, but T is not in $\mathbb{A}_{2,2}$, $\mathbb{A}_{1,\aleph_0}$, or $\mathbb{A}_{\aleph_0,1}$.*

Proof. To show that $T_L \oplus T_R^*$ is in $\mathbb{A}$, observe that $\sigma(T_L)$ is the closure of the region $\mathbb{D}_L$ bounded by L, and $\sigma(T_R^*)$ is the closure of the region $\mathbb{D}_R$ bounded by R. Thus $\sigma(T) = \overline{\mathbb{D}}$, which is sufficient for membership in $\mathbb{A}$.

To show that $T_L \oplus T_R^*$ is not in $\mathbb{A}_{2,2}$, observe that $N_L \oplus N_R^*$ dilates $T_L \oplus T_R^*$. But $N_L \oplus N_R^*$ is a normal operator with $\mathrm{NTL}(\sigma(N_L \oplus N_R^*) \cap \mathbb{D}) = \emptyset$ (almost everywhere), and whose unitary part has multiplicity one (a.e.) on $\mathbb{T}$. By [11], $N_L \oplus N_R^* \notin \mathbb{A}_{2,2}$, so neither is its compression $T_L \oplus T_R^*$.

The proof that $T = T_L \oplus T_R^*$ is not in $\mathbb{A}_{\aleph_0,1}$ is similar to that for $\mathbb{A}_{1,\aleph_0}$, and so we prove only the latter. Suppose that $T = T_L \oplus T_R^*$ is in $\mathbb{A}_{1,\aleph_0}$. Then T has property $E_{0,1}^r$ (see [8] for a definition), and it follows (via the proof of [6], Theorem 6.2 and taking adjoints) that for each $\lambda \in \mathbb{D}$, there exist sequences $\{x_n^\lambda\}_{n=1}^\infty$ and $\{y_n^\lambda\}_{n=1}^\infty$ in the unit ball of $\mathcal{H}$ such that

(a) $\lim_{n\to\infty} \|[C_\lambda] - [x_n^\lambda \otimes y_n^\lambda]\|_{Q_T} = 0$,

(b) $\|[x_n^\lambda \otimes w]\|_{Q_T} \to 0$, $w \in \mathcal{H}$, and

(c) $\{y_n^\lambda\}_{n=1}^\infty$ converges weakly to zero.

Let $T = T_L \oplus T_R^*$ act on $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$ and write vectors as $w = w^L \oplus w^R$. For each λ in the disk, fix sequences $\{x_n^\lambda\}_{n=1}^\infty$ and $\{y_n^\lambda\}_{n=1}^\infty$ as above. Define Λ_L by $\Lambda_L = \{\lambda \in \mathbb{D} : \limsup_n \|x_n^{\lambda,L}\| \cdot \|y_n^{\lambda,L}\| > 1/2\}$. We first claim that

$$\mathrm{NTL}(\Lambda_L) \subseteq \mathbb{T} \cap \{\mathrm{Re}(z) \leqslant 0\}. \tag{3.1}$$

To see this, one shows easily that $\mathrm{NTL}(\Lambda_L)$ is essential for T_L. But T_L has no spectrum, thus no essential set, in the right half plane (see [4]).

Set $\Lambda_R = \{\lambda \in \mathbb{D} : \limsup_n \|x_n^{\lambda,R}\| \cdot \|y_n^{\lambda,R}\| \geqslant 1/2\}$. Since T is in $\mathbb{A}$ we have $\mathbb{T}$ is essential for T, and using (3.1) and x_n^λ and y_n^λ in the unit ball, it is easy to deduce $\mathbb{T} \cap \{\mathrm{Re}(z) > 0\} \subseteq \mathrm{NTL}(\Lambda_R)$.

Now consider some conformal map ψ of the right half disk $\mathbb{D}_R$ onto the unit disk $\mathbb{D}$. It is possible to check that for each λ in Λ_R with associated sequences $\{x_n^{\lambda,R}\}_{n=1}^\infty$ and $\{y_n^{\lambda,R}\}_{n=1}^\infty$, one has

(a) $\lim_{n\to\infty} \|[C_{\psi(\lambda)}] - [x_n^{\lambda,R} \otimes y_n^{\lambda,R}]\|_{Q_{\psi(T_R^*)}} \leqslant 1/2$,

(b) $\|[x_n^{\lambda,R}\otimes w]\|_{Q_{\psi(T_R^*)}}\to 0$, $w\in\mathcal{H}_R$, and

(c) $\{y_n^{\lambda,R}\}_{n=1}^{\infty}$ converges weakly to zero.

Clearly NTL$(\{\psi(\lambda):\lambda\in\Lambda_R\})$ has positive measure on $\mathbf{T}$. But also $\psi(T_R^*)$ is (unitarily equivalent to) the backward unilateral shift $S_{\mathbf{T}}^*$ (of multiplicity one). If $S_{\mathbf{T}}^*$ had *any* point $\mu=\psi(\lambda)$ in $\mathbf{D}$ with sequences as above, then using a standard argument and Möbius transforms as in the proof of [1], Proposition 6.1, *each* point of $\mathbf{D}$ has a sequence as above. But then $S_{\mathbf{T}}^*$ would have property $E_{0,\frac{1}{2}}^r$, and so $S_{\mathbf{T}}^*$ would be in $\mathbb{A}_{1,\aleph_0}$, a contradiction. ∎

Some variants of the above operator provide some further examples.

PROPOSITION 3.2. *For any m,n, $1\leqslant m,n\leqslant\aleph_0$, the operator $T_{m,n}=T_L^{(m)}\oplus(T_R^*)^{(n)}\in\mathbb{A}_{m,n}$. If m and n are both finite, let $k=\min(m,n)$; then $T_{m,n}$ is in none of $\mathbb{A}_{k+1,k+1}$, $\mathbb{A}_{1,\aleph_0}$, and $\mathbb{A}_{\aleph_0,1}$.*

Proof. The assertion with both m and n finite is proved as above. For the first assertion, we merely sketch the case $m=1$, $n=2$. Assume then that $T_{1,2}=T_L\oplus(T_R^*)^{(2)}$ acts on the space $\mathcal{H}=\mathcal{H}_L\oplus\mathcal{H}_R$. Let $(T_R^*)^{(2)}=T_R^*\oplus T_R^*$ act on the space $\mathcal{H}_R$ decomposed as $\mathcal{H}_R=\mathcal{H}_1\oplus\mathcal{H}_2$, so $\mathcal{H}=\mathcal{H}_L\oplus\mathcal{H}_1\oplus\mathcal{H}_2$, and write vectors w in $\mathcal{H}$ with respect either to the decomposition $w=w^L\oplus w^R$ or the decomposition $w=w^L\oplus w^1\oplus w^2$, as needed.

Suppose then that $[L_1]$ and $[L_2]$ are arbitrary in $Q_{T_{1,2}}$. Since $T_{1,2}\in\mathbb{A}=\mathbb{A}_{1,1}$ there exist vectors x_1, x_2, y_1, y_2 in $\mathcal{H}$ so that $[L_1]=[x_1\otimes y_1]_{Q_{T_{1,2}}}$ and $[L_2]=[x_2\otimes y_2]_{Q_{T_{1,2}}}$. Note that since T is a direct sum,

$$[L_1]=[(x_1^L\oplus 0)\otimes(y_1^L\oplus 0)]_{Q_{T_{1,2}}}+[(0\oplus x_1^R)\otimes(0\oplus y_1^R)]_{Q_{T_{1,2}}},$$

and

$$[L_2]=[(x_2^L\oplus 0)\otimes(y_2^L\oplus 0)]_{Q_{T_{1,2}}}+[(0\oplus x_2^R)\otimes(0\oplus y_2^R)]_{Q_{T_{1,2}}}.$$

It is clear that $[L_1^L]\triangleq[x_1^L\otimes y_1^L]\in Q_{T_L}$, $[L_2^L]\triangleq[x_2^L\otimes y_2^L]\in Q_{T_L}$, $[L_1^R]\triangleq[x_1^R\otimes y_1^R]\in Q_{(T_R^*)^{(2)}}$, and $[L_2^R]\triangleq[x_2^R\otimes y_2^R]\in Q_{(T_R^*)^{(2)}}$.

Now T_L is subnormal, so $\mathcal{A}_{T_L}$ has property $(\mathbb{A}_{1,\aleph_0})$ from [13] and hence property $(\mathbb{A}_{1,2})$. Thus there exist vectors u, v_1, and v_2 in $\mathcal{H}_L$ so that

$$[u\otimes v_1]=[L_1^L]_{Q_{T_L}}\quad\text{and}\quad[u\otimes v_2]=[L_2^L]_{Q_{T_L}}. \tag{3.2}$$

Consider now $[x_1^R\otimes y_1^R]_{Q_{(T_R^*)^{(2)}}}$. It is easy to verify that, for any polynomial p, one has

$$\begin{aligned}\langle p((T_R^*)^{(2)}),[x_1^R\otimes y_1^R]\rangle&=(p((T_R^*)^{(2)})x_1^R,y_1^R)_{\mathcal{H}_R}\\&=(p(T_R^*)x_1^1,y_1^1)_{\mathcal{H}_1}+(p(T_R^*)x_1^2,y_1^2)_{\mathcal{H}_2}\\&=\langle p(T_R^*),[x_1^1\otimes y_1^1]_{Q_{T_R^*}}\rangle+\langle p(T_R^*),[x_1^2\otimes y_1^2]_{Q_{T_R^*}}\rangle.\end{aligned}\tag{3.3}$$

Similarly, we have the completely analogous equation involving $[x_2^R \otimes y_2^R]_{Q_{(T_R^*)^{(2)}}}$, $[x_2^1 \otimes y_2^1]_{Q_{T_R^*}}$, and $[x_2^2 \otimes y_2^2]_{Q_{T_R^*}}$. Now $\mathcal{A}_{T_R^*}$ has property $(\mathbb{A}_{\aleph_0,1})$ and hence $(\mathbb{A}_{1,1})$ from [13] and adjoints, so there are vectors w^1, z^1 in $\mathcal{H}_1$ and w^2, z^2 in $\mathcal{H}_2$ so that

$$(3.4) \qquad [w^i \otimes z^i]_{Q_{T_R^*}} = [x_i^1 \otimes y_i^1]_{Q_{T_R^*}} + [x_i^2 \otimes y_i^2]_{Q_{T_R^*}}, \quad i = 1, 2.$$

It is then possible to check, via actions on polynomials and using (3.2), (3.3), its analogue, and (3.4), that $[(u \oplus w^1 \oplus w^2) \otimes (v_1 \oplus z^1 \oplus 0)]_{Q_{T_{1,2}}} = [L_1]_{Q_{T_{1,2}}}$ and $[(u \oplus w^1 \oplus w^2) \otimes (v_2 \oplus 0 \oplus z^2)]_{Q_{T_{1,2}}} = [L_2]_{Q_{T_{1,2}}}$. Since $[L_1]$ and $[L_2]$ were arbitrary in $Q_{T_{1,2}}$, $T_{1,2}$ is in $\mathbb{A}_{1,2}$ as desired. ∎

One can modify the arguments above to gain a little more information about the membership of the various $T_{m,n}$ in the various classes $\mathbb{A}_{i,j}$.

PROPOSITION 3.3. *Let m and n be positive integers. Suppose $T_{m,n} = T_L^{(m)} \oplus (T_R^*)^{(n)}$ is in $\mathbb{A}_{i,j}$ for some i and j (each necessarily finite by Proposition 3.2). Then if $i \leqslant m$, $T_{m,n+1}$ is in $\mathbb{A}_{i,j+1}$, and if $i > m$, $T_{m,n+1}$ is in $\mathbb{A}_{m,j+1}$. Similarly, if $j \leqslant n$, $T_{m+1,n}$ is in $\mathbb{A}_{i+1,j}$, and if $j > n$, $T_{m+1,n}$ is in $\mathbb{A}_{i+1,n}$.*

Proof. For the case $i > m$, the key observation is that for any m, $\mathcal{A}_{T_L^{(m)}}$ has property $(\mathbb{A}_{m,\aleph_0})$, and hence $(\mathbb{A}_{m,j+1})$ (and not merely $(\mathbb{A}_{m,j})$). A modification of the argument above then suffices. ∎

There's one more part of information available almost for free; we need only observe that $(-T_{n,n})^*$ is unitarily equivalent to $T_{n,n}$.

PROPOSITION 3.4. *Let n be any positive integer. If, for some i and j, $T_{n,n} = T_L^{(n)} \oplus (T_R^*)^{(n)}$ is in $\mathbb{A}_{i,j}$, then $T_{n,n}$ is in $\mathbb{A}_{j,i}$ as well.*

It turns out that the information above is not sufficient to place the $T_{m,n}$ uniquely, each in its "maximal" class $\mathbb{A}_{i,j}$, in the $\aleph_0$ by $\aleph_0$ grid of all the $\mathbb{A}_{i,j}$. We record the obvious conjecture, and then the relationship between the conjecture and Proposition 3.3.

CONJECTURE 3.5. *We conjecture that for each m and n positive integers, $T_{m,n} \in \mathbb{A}_{m,n} \setminus (\mathbb{A}_{m+1,1} \cup \mathbb{A}_{1,n+1})$.*

PROPOSITION 3.6. *The assertion in Conjecture 3.5 is equivalent to the condition that, for each m, n, i, and j positive integers, $T_{m,n} = T_L^{(m)} \oplus (T_R^*)^{(n)}$ in $\mathbb{A}_{i,j}$ implies both $T_{m,n+1}$ is in $\mathbb{A}_{i,j+1}$ and $T_{m+1,n}$ is in $\mathbb{A}_{i+1,j}$.*

The force of these examples, which would be increased if the conjecture is correct, is that the family $\{T_{m,n}\}_{m,n=1}^{\infty}$ provides at least some examples of elements populating the various $\mathbb{A}_{i,j}$ but not in either $\mathbb{A}_{1,\aleph_0}$ or $\mathbb{A}_{\aleph_0,1}$ (that is, in no class with at least one index finite). It is known from [12] that the classes $\mathbb{A}_{i,j}$ are distinct, but this distinction is, more properly put, a distinction between "rows" (i.e., the various $\mathbb{A}_{n,\aleph_0}$), and a different distinction between "columns" (or the various $\mathbb{A}_{\aleph_0,n}$), using various ampliations of the unilateral shift $S_{\mathbf{T}}$ and its adjoint.

We remark finally that the family of examples above may be modified to yield operators T with nonempty set X_T (see [7]) still inhabiting only classes $\mathbb{A}_{i,j}$ with strictly finite indices.

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