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GLOBAL RESOLVENT ESTIMATES FOR MULTIPLICATION OPERATORS

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ABSTRACT. We give a method to obtain global smoothness estimates and boundary values of the resolvent for a large class of multiplication operators. It relies on the positivity of a commutator and the method of differential inequalities.

KEYWORDS: *Boundary values of the resolvent, conjugate operator, smooth operator, multiplication operator.*

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1. INTRODUCTION

Let H be the operator of multiplication with a real function $H(x)$ in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^n)$. We shall study the behaviour of $(H-z)^{-1}$ when $|\operatorname{Im} z| \to 0$. If z approaches a real number λ which is not a critical value for H (i.e. $(\nabla H)(x) \neq 0$ for $x \in H^{-1}(\{\lambda\})$), then there are very precise results in this direction (see [9] for example). One of the ways of proving such type of results is the method of Mourre ([12], [1]). If H satisfies some weak conditions, we are able to prove a Mourre inequality in the neighbourhood of the regular values (i.e. those which are not critical). But what can be said if λ is a critical value of H? Some partial answers to this problem are given in some recent papers (let us quote [3], [2], [7], [11] and [15] for example) where global estimates for the resolvent of H are given in certain particular cases. In fact, these estimates were obtained for partial differential operators with constant coefficients which are, via the Fourier transform, unitarily equivalent to multiplication operators. Most of these papers consider the case of the Laplacian $H(x) = |x|^2$ and the proofs make use of its special properties.

The method we propose in this paper is based on the positivity of the commutator of H with an appropriate operator (so it is a variant of the method of Mourre). Since in the neighbourhood of the critical values it is not possible to obtain a sharp positivity (i.e. a standard Mourre inequality), we shall content ourselves to exploit a weaker positivity and this will force us to introduce some auxiliary spaces. In the general case, these spaces are rather complicated. But if H satisfies some supplementary conditions, we are able to describe them more precisely and to obtain explicit criteria of smoothness.

The paper contains two sections. In the first of them, we shall prove a general abstract result concerning the behaviour of $(H-z)^{-1}$ when z is in the neighbourhood of the real axis. The approach is close to that from [5], where the weak positivity allowed proving a criterium of pure absolute continuity for Schrödinger operators with anisotropic potentials. In the second section examples are given of smooth operators with respect to H (for various functions H). As a consequence of the abstract result we can also solve in spaces of distributions the equation $Hf = 1$. If f is in S', its Fourier transform is an elementary solution of $H(P)$, with $P = -i\nabla$. At the end we indicate possible extensions and comment on the results.

In what follows, Ω_0 is a fixed, open set in $\mathbf{R}^n$. We shall denote by $\|f\|$ the norm of the function $f \in L^2(\Omega_0) \equiv \mathcal{H}$. If f and g are two measurable functions on Ω_0 such that $f \cdot g$ is absolutely integrable, then $\langle f, g\rangle$ will be by definition $\int_{\Omega_0} \bar{f} \cdot g \, dx$. The space $B(\mathcal{X}_1, \mathcal{X}_2)$ is the space of bounded, linear operators from $\mathcal{X}_1$ to $\mathcal{X}_2$. If x is in $\mathbf{R}^n$, $\langle x\rangle = (1 + |x|^2)^{1/2}$.

2. A GENERAL RESULT

The function $H : \Omega_0 \to \mathbf{R}$ is called *admissible* if it satisfies the conditions:

(i) There is on open subset Ω of Ω_0 with negligible complement (with respect to the Lebesgue measure) such that H is C^∞ in Ω.

(ii) There is a completely integrable C^∞ vector field $F : \Omega \to \mathbf{R}^n$ and a constant $C > 0$ such that for all $x \in \Omega$:

$$B(x) \equiv (F(x), (\nabla H)(x)) > 0 \tag{2.1}$$

and

$$|(F(x), (\nabla B)(x))| \leqslant CB(x). \tag{2.2}$$

Here $(\cdot,\cdot)$ is the scalar product in $\mathbf{R}^n$. We denote by $\mathcal{O}$ the space of C^∞ functions with a compact support included in Ω and by $\mathcal{B}$ its completion in the Hilbert norm $\|f\|_{\mathcal{B}} = \|B^{1/2}f\|$. It will be a Hilbert space, generally not comparable with $\mathcal{H}$. But if we identify $\mathcal{H}$ with its topological antidual $\mathcal{H}^*$ by means of the Riesz isomorphism, $\mathcal{B}^*$ will be identical to the completion of $\mathcal{O}$ in $\|g\|_{\mathcal{B}^*} = \|B^{-1/2}g\|$. $B : \mathcal{O} \to \mathcal{O}$ is a bijection which extends to an unitary operator: $\mathcal{B} \to \mathcal{B}^*$. It is easy to give an explicit description of $\mathcal{B}^*$ (for instance): $\mathcal{B}^* = \{g : \Omega_0 \to \mathbf{C} | g$ is measurable and $B^{-1/2}g \in \mathcal{H}\}$. It will be convenient to note that $B = (F, \nabla H)$ is in fact the commutator $[H, \mathrm{i}A]$, where on $\mathcal{O}$ the operator A acts as

$$Af = -\frac{1}{2}\{(F,P)f + (P,F)f\}, \quad P = -\mathrm{i}\nabla. \tag{2.3}$$

It is easy to see that, as a consequence of Nelson's lemma, A is essentially self-adjoint on $\mathcal{O}$ and that its closure is the infinitesimal generator of the unitary group in $\mathcal{H}$:

$$[W(t)f](x) = \sqrt{J_t(x)}f[w_t(x)] \quad \text{for all } t \in \mathbf{R}, x \in \Omega, f \in \mathcal{H}. \tag{2.4}$$

Here $\{w_t \mid t \in \mathbf{R}\}$ is the C^∞ flow in Ω generated by $F(F(x) = \mathrm{d}/\mathrm{d}t|_{t=0}\, w_t(x))$ and $J_t(x) = \det[(\nabla w_t)(x)]$. It is also useful to observe that $(F, \nabla B)$ is the second commutator $[B, \mathrm{i}A] = [[H, \mathrm{i}A], \mathrm{i}A]$. From the assumption $|(F, \nabla B)| \leqslant CB$ and Gronwall's lemma we easily infer that the operators $W(t)|\mathcal{O}$ extend to bounded operators in $\mathcal{B}$ and $\mathcal{B}^*$ and, by some standard arguments, that we get c_0-groups in $\mathcal{B}$ and $\mathcal{B}^*$, denoted by the same letter W. Their infinitesimal generators will also be denoted by the single symbol A, but it will be convenient to distinguish between their domains by using the natural notations $D(A;\mathcal{B})$ and $D(A;\mathcal{B}^*)$. Since this latter space will appear frequently, it will be denoted briefly by $\mathcal{A}$. It is a Hilbert space with the graph-norm $\|f\|_{\mathcal{A}} = [\|f\|^2_{\mathcal{B}^*} + \|Af\|^2_{\mathcal{B}^*}]^{1/2}$. $\mathcal{O}$ is dense in it by a general form of Nelson's lemma, to be found in [6] for example.

We can now state our main result.

THEOREM 2.1. *Let H be an admissible function.*

(i) *There exists a finite constant C such that for all λ in $\mathbf{R}$, $\mu > 0$ and f, g in $\mathcal{H} \cap \mathcal{A}$:*

$$|\langle f, (H - \lambda \mp \mathrm{i}\mu)^{-1} g\rangle| \leqslant C\|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}.$$

(ii) *Let T be a closed operator in $\mathcal{H}$ such that T^* admits a continuous extension in $B(\mathcal{H}, \mathcal{A})$. Then T is H-smooth.*

(iii) *The limits $\langle f, (H - \lambda \mp \mathrm{i}\,0)^{-1} g\rangle \equiv \lim_{\mu \searrow 0} \langle f, (H - \lambda \mp \mathrm{i}\mu)^{-1} g\rangle$ exist for any $f, g \in \mathcal{A}$, uniformly in $\lambda \in \mathbf{R}$.*

REMARK 2.2. As a consequence of (i), we obtain that the sesquilinear form $h_{\lambda,\mu}$ defined on $\mathcal{H} \cap \mathcal{A}$ by $h_{\lambda,\mu}(f,g) = \langle f, (H - \lambda \mp i\mu)^{-1} g\rangle$ admits a continuous extension to a bounded form on $\mathcal{A}$. This is the sense of the notation $\langle \cdot, \cdot \rangle$ in (iii). In particular, the operator $(H - \lambda \mp i\mu)^{-1}$ is in $B(\mathcal{A}, \mathcal{A}^*)$, which is not a priori evident.

Proof. **Step 1.** For $\lambda \in \mathbf{R}$, $\mu > 0$, $\varepsilon > 0$ we set $G_\varepsilon^\pm \equiv (H - \lambda \mp i\mu \mp i\varepsilon B)^{-1}$. They are smooth functions on Ω, which are extended arbitrary on Ω_0. As operators, they satisfy $G_\varepsilon^\pm \mathcal{O} \subset \mathcal{O}$. By a simple calculation, they may be extended as bounded operators: $\mathcal{B}^* \to \mathcal{B}$, verifying

$$\|G_\varepsilon^\pm\|_{\mathcal{B}^* \to \mathcal{B}} \leqslant \frac{1}{\varepsilon}, \tag{2.5}$$

uniformly in λ and μ and $\langle f, G_\varepsilon^\mp g\rangle = \langle G_\varepsilon^\pm f, g\rangle$ for any $f, g \in \mathcal{B}^*$.

Step 2. For f belonging to $\mathcal{O}$, we write

$$\mp \operatorname{Im} \langle G_\varepsilon^\pm f, f\rangle = \mp \operatorname{Im} \langle G_\varepsilon^\pm f, (H - \lambda \pm i\mu \pm i\varepsilon B) G_\varepsilon^\pm f\rangle = \mu \|G_\varepsilon^\pm f\|^2 + \varepsilon \|G_\varepsilon^\pm f\|_{\mathcal{B}}^2$$

which gives immediately

$$\|G_\varepsilon^\pm f\|_{\mathcal{B}} \leqslant \frac{1}{\sqrt{\varepsilon}} |\langle f, G_\varepsilon^\pm f\rangle|^{\frac{1}{2}}. \tag{2.6}$$

This extends by continuity to any $f \in \mathcal{B}^*$. Since we do not rely on $B \geqslant a > 0$, there is no analogous inequality for $\|G_\varepsilon^\pm f\|$ and this justifies the introduction of the space $\mathcal{B}$.

Step 3. Now, let us fix $f \in \mathcal{O}$ and set $F_\varepsilon \equiv F_\varepsilon(\lambda, \mu; f) = \langle f, G_\varepsilon f\rangle$, where we shall prefer the notations $G_\varepsilon \equiv G_\varepsilon^+$ and $G_\varepsilon^* \equiv G_\varepsilon^-$. By differentiating with respect to ε we get $F_\varepsilon' = \langle f, iG_\varepsilon^2 B f\rangle$. Since $B = i[H, A]$, this can be easily put in the form

$$F_\varepsilon' = \langle G_\varepsilon^* f, Af\rangle - \langle Af, G_\varepsilon f\rangle + i\varepsilon \langle G_\varepsilon^* f, [B, A] G_\varepsilon f\rangle. \tag{2.7}$$

Step 4. The assumption $|(F, \nabla B)| \leqslant CB$ means exactly that $i[B, A] \in B(\mathcal{B}, \mathcal{B}^*)$. By this and by (2.6), we infer from (2.7) the differential inequality

$$|F_\varepsilon'| \leqslant \frac{2}{\sqrt{\varepsilon}} \|f\|_{\mathcal{A}} |F_\varepsilon|^{\frac{1}{2}} + \|[B, A]\|_{\mathcal{B} \to \mathcal{B}^*} |F_\varepsilon|. \tag{2.8}$$

By a version of the Gronwall lemma given in [4], Appendix B and by (2.5), we integrate (2.8) to get

$$|F_\varepsilon(\lambda, \mu; f)| \leqslant C \|f\|_{\mathcal{A}}^2, \tag{2.9}$$

with C independent of λ, μ, f. It is easy to see that we have, in fact, for f, g in $\mathcal{A}$

$$|\langle f, G_\varepsilon g\rangle| \leqslant C\|f\|_{\mathcal{A}} \cdot \|g\|_{\mathcal{A}}. \tag{2.9'}$$

But if f and g are also in $\mathcal{H}$, then $\lim\limits_{\varepsilon \searrow 0}\langle f, G_\varepsilon g\rangle = \langle f, (H-\lambda-\mathrm{i}\mu)^{-1}g\rangle$.

This finishes the proof of the point (i) of the Theorem.

Step 5. The point (ii) is a direct consequence of (i). For the relevant definitions about H-smooth operators, we send (for instance) to [13].

Step 6. We prove now (iii); by the polarization formula is enough to take $f = g \in \mathcal{O}$. Let us start with

$$\langle f, (H-\lambda-\mathrm{i}\mu)^{-1}f\rangle = \lim_{\varepsilon \searrow 0} F_\varepsilon(\lambda, \mu; f) = F_1(\lambda, \mu; f) - \int_0^1 F_\tau'(\lambda, \mu; f)\,\mathrm{d}\tau. \tag{2.10}$$

It is clear that $\lim\limits_{\mu \searrow 0} F_1(\lambda, \mu; f) = F_1(\lambda, 0; f)$, uniformly with respect to λ. On the other hand

$$\begin{aligned}
&|F_\tau'(\lambda, \mu; f) - F_\tau'(\lambda, 0; f)| \\
&\quad = \left|\int |f(x)|^2 B(x)\left[\frac{1}{(H(x)-\lambda-\mathrm{i}\mu-\mathrm{i}\tau B(x))^2} - \frac{1}{(H(x)-\lambda-\mathrm{i}\varepsilon B(x))^2}\right]\mathrm{d}x\right| \\
&\quad \leqslant C(f)\cdot \mu \cdot \tau^{-3},
\end{aligned}$$

for all $\lambda \in \mathbf{R}$ and from (2.8) and (2.9)

$$|F_\tau'(\lambda, \mu; f)| + |F_\tau'(\lambda, 0; f)| \leqslant C(f)\tau^{-\frac{1}{2}}$$

for all $\lambda \in \mathbf{R}$. Then, by the dominated convergence theorem, the integral in (2.10) also has a limit when $\mu \to 0$ and this limit is uniform whith respect to λ. ∎

REMARK 2.3. Suppose that $C_0^m(\mathbf{R}^n)$ is a subspace of $\mathcal{A}$. Then Theorem 2.1 (iii) implies that

$$(H-\lambda \pm \mathrm{i}\,0)^{-1} = \lim_{\mu \searrow 0}(H-\lambda \pm \mathrm{i}\mu)^{-1}$$

exist in the weak topology of $\mathcal{D}'_m$, the space of distributions of order m on $\mathbf{R}^n$. Moreover, for a smooth H, $(H-\lambda)(H-\lambda \pm \mathrm{i}\,0)^{-1}$ is well-defined in $\mathcal{D}'_m$ and is equal to 1. If the supplementary conditions in Section 3 are fulfilled, we can replace $\mathcal{D}'$ by $\mathcal{S}'$, the space of tempered distributions. Notice that H is not necessarily a polynomial and may have critical points. Suppose now that ν is a rapidly decreasing distribution and that its Fourier transform H is an admissible function which satisfies the additional conditions from Section 3. Then, as a consequence of the previous considerations, we deduce the existence of an elementary solution for the operator of convolution with ν.

REMARK 2.4. There are some improvements which can be made. First, instead of the assumption (2.2) we may ask only

$$\int_0^1 \frac{d\tau}{\tau} \int_\Omega \left| \frac{B[w_\tau(x)]}{B(x)} - 1 \right| dx < \infty. \tag{2.11}$$

In order to have some insight, we remark that (2.2) is equivalent to the fact that the application $\mathbf{R} \ni t \to (B \circ w_t) \cdot f \in \mathcal{B}^*$ is C^1 in norm for any $f \in \mathcal{B}$, but that (2.11) is weaker than its Hölder continuity of order θ for any $\theta \in (0,1)$. Second, the space $\mathcal{A}$ may be replaced by the real interpolation space $(\mathcal{B}^*, \mathcal{A})_{\frac{1}{2},1}$, whose definition can be found for example in [16]. This would have given a significantly better control upon the behaviour of the resolvent. We give up this possible improvement not only because the proof of Theorem 2.1. would have become more involved (and we want to stress the simplicity of our method) but also because the nature of the space $(\mathcal{B}^*, \mathcal{A})_{\frac{1}{2},1}$ remains rather obscure even in the fortunate cases when $\mathcal{A}$ can be replaced by some simple and explicit space (see the next section).

REMARK 2.5. In the articles dealing with commutator methods in spectral analysis, there is a regularity assumption saying roughly that "B is not too big with respect to H". The same type of condition appears in [5] — which contains an abstract counterpart of the present work, directed mainly towards the spectral analysis of quantum Hamiltonians. The precise form of this condition in our context is $B \leqslant C(1 + |H|)$. The fact that (due to some concrete features of our case) we are dispensed to use it, is rather amazing and of great help in enlarging the class of functions H to which Theorem 2.1 applies.

3. EXAMPLES

The main unpleasant feature of the result above is the fact that the norm

$$\|f\|_{\mathcal{A}} = \left[\left\| \frac{1}{\sqrt{B}} f \right\|^2 + \left\| \frac{1}{\sqrt{B}} \left\{ (F,P) - \frac{i}{2} \mathrm{div} F \right\} f \right\|^2 \right]^{\frac{1}{2}}$$

is too intricate for a convenient interpretation. By assuming that $\mathrm{div} F = \sum_{j=1}^{n} \partial_j F_j$ is a bounded function, $\|f\|_{\mathcal{A}}$ is equivalent to $\left[\|(1/\sqrt{B})f\|^2 + \|(1/\sqrt{B})(F,P)f\|^2 \right]^{1/2}$, but this is not a significant progress. We intend to dominate $\|f\|_{\mathcal{A}}$ by a norm of the form $\|\gamma(P)f\|$ (for a suitable function γ), which is much simpler. With this end in view, we set $\Omega_0 = \mathbf{R}^n$ and introduce *the weak L^p-space*

$$L^p_w(\mathbf{R}^n) \equiv \{\varphi : \mathbf{R}^n \to \mathbf{C} | \mu[\{x \mid |\varphi(x)| > t\}] \leqslant Ct^{-p} \text{ for any } t > 0\}, \quad p \in [1, \infty)$$

where μ is the Lebesque measure on $\mathbf{R}^n$. We send to [14] for details and notice only that $L^p_w(\mathbf{R}^n)$ is slightly larger than $L^p(\mathbf{R}^n)$ (the function $|\cdot|^{-n/p}$ is a typical element of $L^p_w(\mathbf{R}^n)$ which is in no L^p space). For $p=\infty$ we set simply $L^\infty_w(\mathbf{R}^n)\equiv L^\infty(\mathbf{R}^n)$. The relevance of this definition in our context comes from the fact that if $p\in(2,\infty]$, $1/p+1/p'=1, \varphi\in L^p_w(\mathbf{R}^n)$ and $\mathcal{F}\psi\equiv$ the Fourier transform of ψ is in $L^{p'}_w(\mathbf{R}^n)$, then $\varphi\cdot\psi(P)$ extends to a bounded operator on $\mathcal{H}=L^2(\mathbf{R}^n)$ (see [14]). Hence, suppose that $1/\sqrt{B}\in L^{p_1}_w(\mathbf{R}^n)$ for a $p_1\in(2,\infty]$ and that $\alpha:\mathbf{R}^n\to\mathbf{C}$ is a Borel function such that $\mathcal{F}(1/\alpha)\in L^{p_1'}_w(\mathbf{R}^n)(p_1'$ is conjugated to $p_1)$. Then $\|(1/\sqrt{B})f\|\leqslant\|(1/\sqrt{B})(1/\alpha(P))\|\cdot\|\alpha(P)f\|$. Second, assume that $|F|/\sqrt{B}\in L^{p_2}_w(\mathbf{R}^n), p_2>2$ and $\beta:\mathbf{R}^n\to\mathbf{C}$ is such that $\mathcal{F}(1/\beta)\in L^{p_2'}_w(\mathbf{R}^n)$. Then $\|(1/\sqrt{B})(F,P)f\|\leqslant C\|(|F|/\sqrt{B})\cdot(1/\beta(P))\|\cdot\|\,|P|\beta(P)f\|$. By setting $\gamma(x)\equiv\max\{|\alpha(x)|,|x|\,|\beta(x)|\}$ we get $\|f\|_{\mathcal{A}}\leqslant C\|\gamma(P)f\|$, as required. Until now, the argument was formal. Since it is difficult to describe the image of $\mathcal{O}$ by Fourier transform, it is simpler to make the calculations on the Schwartz space $\mathcal{S}$. We give below explicit conditions under which this may be done. They are general enough for the examples we have in view:

(i) F has at most polynomial growth and $\operatorname{div}F$ is bounded;
(ii) $B^{-1/2}(x)\langle x\rangle^{-m}\in L^2(\mathbf{R}^n)$ for some $m\in\mathbf{R}$;
(iii) $(|\alpha(x)|+|\beta(x)|)\langle x\rangle^{-m}\in L^2(\mathbf{R}^n)$ for some $m\in\mathbf{R}$;
(iv) $1/\alpha, 1/\beta\in\mathcal{S}'$.

Let us denote by $\dot{D}[\gamma(P)]$ the completion of $\mathcal{S}$ in $\|\gamma(P)\cdot\|$. The dot is introduced to make the difference with respect to $D[\gamma(P)]=$ the domain in $\mathcal{H}$ of $\gamma(P)$, which is defined in terms of the full graph norm $(\|\gamma(P)\cdot\|^2+\|\cdot\|^2)^{1/2}$. We have proved

THEOREM 3.1. *Let $H:\mathbf{R}^n\to\mathbf{R}$ be an admissible function (with respect to the vector field F) and $\alpha,\beta:\mathbf{R}^n\to\mathbf{C}$ Borel functions such that* (i), (ii), (iii) *and* (iv) *are fulfilled and for two numbers $p_1,p_2\in(2,\infty]$ one has $1/\sqrt{B}\in L^{p_1}_w(\mathbf{R}^n)$, $|F|/\sqrt{B}\in L^{p_2}_w(\mathbf{R}^n)$, $\mathcal{F}(1/\alpha)\in L^{p_1'}_w(\mathbf{R}^n)$ and $\mathcal{F}(1/\beta)\in L^{p_2'}_w(\mathbf{R}^n)$. Set $\gamma=\max\{|\alpha|,|x|\cdot|\beta|\}$. Then:*

(i) $|\langle f,(H-\lambda\mp i\mu)^{-1}g\rangle|\leqslant C\|f\|_{\dot{D}[\gamma(P)]}\cdot\|g\|_{\dot{D}[\gamma(P)]}$, *with C not depending on $\lambda\in\mathbf{R},\mu>0$ and $f,g\in\dot{D}[\gamma(P)]$.*

(ii) *$\gamma^{-1}(P)$ is H-smooth.*

(iii) For any $f,g\in\dot{D}[\gamma(P)]$, the limits $\langle f,(H-\lambda\mp i\,0)^{-1}g\rangle=\lim_{\mu\searrow0}\langle f,(H-\lambda\mp i\mu)^{-1}g\rangle$ exist uniformly in λ.

In spite of the fact that it follows easily from Theorem 2.1, this result is called "Theorem" because it has the great advantage of having a more transparent conclusion. The function γ which appears in the resolvent estimates can be obtained by the procedure described above from the properties of $B^{-1/2}$ and $|F|\cdot B^{-1/2}$. Since this procedure is not quite straightforeward, let us indicate a simple particular case. Set $\alpha(x) = |x|^a$ and $\beta(x) = |x|^{b-1}$, where $a \in (0, n/2)$ and $b \in (1, 1+(n/2))$. Then $[\mathcal{F}(1/\alpha)](x) = C_{n,\alpha}|x|^{-n+a} \in L_w^{n/(n-a)}(\mathbf{R}^n)$, $[\mathcal{F}(1/\beta)](x) = C_{n,b}|x|^{-n+b-1}$. In consequence, we are allowed to put $\gamma(x) = \max\{|x|^a, |x|^b\}$ if $1/\sqrt{B} \in L_w^{n/a}(\mathbf{R}^n)$ and $|F|/\sqrt{B} \in L_w^{n/(b-1)}(\mathbf{R}^n)$.

We shall now present some examples. In most cases, we shall content ourselves with the calculation of the function γ, leaving the task of formulating results to the reader.

EXAMPLE 3.2. Let us take a look first at the case when the choice $F = \nabla H$ is possible. We shall have $B = |\nabla H|^2$ and $(F, \nabla B) = \sum_{j,k=1}^{n} \partial_j H \cdot \partial_k H \cdot \partial_j \partial_k H$. Hence, we must ask that the closed set of critical points $\mathrm{Cr}(H) = \{x \mid (\nabla H)(x) = 0\}$ is (Lebesgue-) negligible and take $\Omega = \mathbf{R}^n \setminus \mathrm{Cr}(H)$. $B(x)$ is positive for any $x \in \Omega$. Let us also impose that the functions $\partial_j \partial_k H, j, k = 1, \ldots, n$ are bounded. In particular, it follows that $\mathrm{div} F$ is bounded and F is globally Lipschitz, hence completely integrable and growing at most as $|x|$ at infinity. Since $|F|/\sqrt{B} = 1 \in L_w^\infty(\mathbf{R}^n) = L^\infty(\mathbf{R}^n)$, we may take $\mathcal{F}(1/\beta) \in L_w^1(\mathbf{R}^n)$. But a simpler choice, which we adopt, is $\beta = 1$. Assume now that $1/\sqrt{B} \in L_w^{n/a}(\mathbf{R}^n)$ for an $a \in (0, n/2)$, i.e. $\mu[\{x \mid |(\nabla H)(x)| < s\}] \leqslant Cs^{n/a}$ for every $s > 0$. Then, if α grows at most polynomially at infinity and $\mathcal{F}(1/\alpha) \in L_w^{n/(n-a)}(\mathbf{R}^n)$, one can put $\gamma = \max\{\alpha, |x|\}$. A particular case is $\gamma = \max\{|x|^a, |x|\}$. Let us consider, for instance the second order polynomial

$$H(x) = (Dx, x) + (b, x) + c, \tag{3.1}$$

where D is a symmetric, non-degenerate matrix, $b \in \mathbf{R}^n$ and $c \in \mathbf{R}$. It obviously satisfies the conditions above with $a = 1$ (here $n \geqslant 3$ is needed). The fact that for $H(x) = |x|^2$ the operator $|P|^{-1}$ is H-smooth is known from [8], [11], [3]. In [15] the best constant in the smoothness estimate is calculated. But for the general form (3.1), as far as we know, there is no such result. Moreover, Theorem 3.1 (iii) implies the existence of $\mathcal{F}^{-1}(H - \lambda \mp i\,0)^{-1}$ in S'; these distributions are elementary solutions of $H(P) - \lambda$. The result is a slight generalization of Theorem 6.2.1. from [9]. Our proof is completely different from that given in [9].

For the degenerate case we send to Example 3.5. Let us notice that the non-elliptical case can be treated just because we succeeded to avoid the condition $B \leqslant C(1 + |H|)$ (cf. Remark 2.5). We are not limited to polynomials; a large class of second-order symbols may be treated. It seems difficult to modify the methods appearing in the literature in order to cover them.

EXAMPLE 3.3. Suppose that H is a homogeneous function of order $2a$ and strictly positive outside the origin, i.e. $H(x) = |x|^{2a}\theta(x/|x|)$, where θ is a smooth, strictly positive function on the unit sphere. If $a \in [1, n/2)$ and $F(x) = x$ then $B = 2aH > 0$ and $(F, \nabla B) = 4a^2 H$. Since obviously $1/\sqrt{B} \in L_w^{n/(a-1)}(\mathbf{R}^n)$, it follows that the choice $\gamma(x) = |x|^a$ is possible (and this is a quite particular option). When $a \in [1/2, 1)$, we must choose a more complicate vector field. Namely, we take $F(x) = x/\langle x\rangle^{2b}$, where $b \in [1-a, a]$. Then it is easy to verify that all our hypothesis on H are satisfied; moreover $1/\sqrt{B} \in L_w^{n/(a-b)}(\mathbf{R}^n) + L_w^{n/a}(\mathbf{R}^n)$ and $|F|/\sqrt{B} \in L_w^{n/(a+b-1)}(\mathbf{R}^n)$. All these lead us to $\gamma(x) = \max\,\{|x|^{a-b}, |x|^{a+b}\}$. It seems that only the case $a = 1$ and $\theta = 1$ was known before.

EXAMPLE 3.4. If H has a direction of monotony, i.e. there is a unit vector ν such that $(\nu, \nabla H) > 0$ a.e., then we may choose simply $F = \nu$ (hence A is the projection on ν of the momentum operator). We leave the details to the reader.

EXAMPLE 3.5. There may be more than one vector field F fitted to a given function H. Assume, for instance, that $\mathbf{R}^n$ is writen as the direct sum $Y \oplus Y^\perp$, where Y is a proper subspace of $\mathbf{R}^n$ of dimension $n_Y \geqslant 3$. For $x \in \mathbf{R}^n$, we set x^Y for the projection in Y and $x^{Y^\perp}$ for its projection in $Y^\perp$. Consider

$$H(x) = \frac{1}{2}|x^Y|^2 + \frac{\delta}{2}|x^{Y^\perp}|^2 + (b^Y, x^Y) + (b^{Y^\perp}, x^{Y^\perp}) + c, \tag{3.2}$$

where $b \in \mathbf{R}^n, c \in \mathbf{R}$ and $\delta \in \mathbf{R}$ (we denoted by $(\cdot,\cdot)$ the scalar product in any of the spaces $X, Y, Y^\perp$). If $\delta \neq 0$, this is a particular case of Example 3.2, hence (for instance) $|P|^{-1}$ is H-smooth. But one may also use the choice $F(x) = x^Y + b^Y$ (it is the gradient of the part depending on x^Y in H). Then $B = |x^Y + b^Y|^2$ and $1/\sqrt{B}$ is in no $L_w^p(\mathbf{R}^n)$ space, but it is the tensor product between a function in $L_w^{n_Y}(Y)$ and the function $\mathbf{1}$ in the $Y^\perp$-variable. This entails immediately that $|P^Y|^{-1} \otimes \mathbf{1}$ is H-smooth (obvious notation) and this is a better result, because $|x^Y|^{-1}$ dominates $|x|^{-1}$ (remark that it doesn't decay at infinity in the $Y^\perp$ direction). In addition, this approach works also in the case $\delta = 0$. But for $\delta = 0$, we also have the option of Example 3.4 by taking $F = b^Y$, hence $B = |b^{Y^\perp}|^2$. This gives the H-smoothness of $\gamma^{-1}(P)$, where $\gamma(x) = \max(\mathbf{1}, \mathbf{1} \otimes |x^{Y^\perp}|)$, irrespectively of the dimension of Y or $Y^\perp$. Note that Y may be varried.

We shall indicate now a way of improving the previous results. In the second section, instead of (2.3) we could have used

$$A^{(\xi)} = -\frac{1}{2}\{(F, P-\xi) + (P-\xi, F)\}, \tag{3.3}$$

where ξ is an arbitrary element of $\mathbf{R}^n$. This amounts to replace (2.4) by

$$[W^{(\xi)}(t)f](x) = \exp\Big\{\mathrm{i}\Big(\xi, \int_0^t F[w_\tau(x)]\,\mathrm{d}\tau\Big)\Big\}\sqrt{J_t(x)}f[w_t(x)]. \tag{3.4}$$

The trick consists in the fact that the succesive commutators of H with $A^{(\xi)}$ are the same as those obtained for $\xi = 0$. In fact, one proves easily that Theorem 2.1. stands true if one replaces A by $\mathcal{A}^{(\xi)} = \mathcal{D}(\mathcal{A}^{(\xi)}, \mathcal{B}^*)$ (obvious definition) and that the constants which appear in the estimates do not depend on ξ. This can be used to integrate over ξ with a complex measure. We shall state only a particular case, easy to prove by arguments already given.

COROLLARY 3.6. *Assume that H is admissible for the vector field F, that* (i) *and* (ii) *are satisfied and that there is an $a \in [1, n/2)$ such that $1/\sqrt{B} \in L_w^{n/a}(\mathbf{R}^n)$ and $|F|/\sqrt{B} \in L_w^{n/(a-1)}(\mathbf{R}^n)$. Let ν be a (finite) complex measure on $\mathbf{R}^n$ and set $\varphi_\nu(x) = \int_{\mathbf{R}^n} \nu(\mathrm{d}\xi)/|x-\xi|^a$. Then the operator $\varphi_\nu(P)$ is H-smooth.*

It is easy to see that $\varphi_\nu \in L_w^{n/a}(\mathbf{R}^n)$. But, unfortunately, not any function $\varphi \in L_w^{n/a}(\mathbf{R}^n)$ is of the form φ_ν. It is tempting to conjecture that, in fact, the operator $\varphi(P)$ is H-smooth for all φ in $L_w^{n/a}(\mathbf{R}^n)$. We note that [10] give for (3.1) a Sobolev inequality which implies that $\varphi(P)$ is H-smooth if $\varphi \in L^{n/a}(\mathbf{R}^n)$ (the special case $H(x) = |x|^2$ is also covered from this point of view in [11]). We cannot obtain this, but we do cover many functions φ which are in $L_w^{n/a}(\mathbf{R}^n) \setminus L^{n/a}(\mathbf{R}^n)$ and we are not restricted to second-order polynomials.

Finally, let us make some short considerations on scattering theory. Any time there are smooth operators, there are also statements on the wave operators. By combining a Fourier transformed version of our result with Theorem XIII.26 from [13], we can indicate various situations when the wave operators $H_0 = H(P)$ and $H_\delta = H_0 + \delta V(Q)$ are unitarily equivalent for small δ, the equivalence being implemented by the wave operators. Virtually, any time our smoothness estimates are new, the scattering results are also new. We shall give only an example. It is well-known that Δ (the Laplace operator) and $\Delta + \delta/\langle Q\rangle^2$ are unitarily equivalent if $n \geqslant 3$ and δ is small. Example 1 shows that this is still true if one replaces Δ by $H(P)$, with H given by (3.1). But, by taking into account arguments similar to those given in Example 3.5, a better result is available.

COROLLARY 3.7. *Set $H_0 = H(P)$, with H as in (3.1) and $H_\delta = H_0 + \delta V(Q)$. Assume that there is a subspace Y of $\mathbb{R}^n$ of dimension at least three which is invariant under D, such that the real Borel function V satisfies $|V(x)| \leqslant C/\langle x^Y \rangle^2$ for a finite constant C. Then there exists $\delta_0 > 0$ such that if $|\delta| < \delta_0$, the wave operators $W_\pm = s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_\delta}\mathrm{e}^{-\mathrm{i}tH_0}$ exist and are unitary. In particular, H_δ is purely absolutely continuous.*

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