J. OPERATOR THEORY
36(1996), 317–334

MAXIMAL ABELIAN SUBALGEBRAS OF THE GROUP FACTOR OF AN $\tilde{A}_2$ GROUP

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Communicated by Şerban Strătilă

ABSTRACT. An $\tilde{A}_2$ group Γ acts simply transitively on the vertices of an affine building $\triangle$. We study certain subgroups $\Gamma_{\text{D}} \cong \mathbf{Z}^2$ which act on certain apartments of $\triangle$. If one of these subgroups acts simply transitively on an apartment, then the corresponding subalgebra of the group von Neumann algebra is maximal abelian and singular. Moreover the Pukánszky invariant contains a type I_∞ summand.

KEYWORDS: *Group factor, abelian subalgebra, affine building.*
AMS SUBJECT CLASSIFICATION: Primary 46L10; Secondary 22D25, 51E24, 20E99.

0. INTRODUCTION

Let Γ be a group acting simply transitively on the vertices of a homogeneous tree $\mathbf{T}$ of degree $n+1<\infty$. Then, by [10], Chapter I, Theorem 6.3,

$$\Gamma \cong \mathbf{Z} * \cdots * \mathbf{Z} * \mathbf{Z}_2 * \cdots * \mathbf{Z}_2$$

where there are s factors of $\mathbf{Z}$, t factors of $\mathbf{Z}_2$, and $2s+t=n+1$.

We can identify $\mathbf{T}$ with the Cayley graph of Γ constructed from right multiplication by the natural generators for Γ. The action of Γ on $\mathbf{T}$ is equivalent to the natural action of Γ on its Cayley graph via left multiplication. With this geometric interpretation, certain geodesics in $\mathbf{T}$ arise as Cayley graphs of subgroups of Γ. That is, there are subgroups of Γ which act simply transitively on geodesics in $\mathbf{T}$.

EXAMPLE 0.1. $\Gamma \cong \mathbf{Z} * \mathbf{Z}_2 * \mathbf{Z}_2$ has generators a,b,c, and relations $b^2 = c^2 = e$. Here $\mathbf{T}$ is a homogeneous tree of degree four. The Cayley graph of the subgroup $\Gamma_0 = \langle a \rangle \cong \mathbf{Z}$ relative to the generators a, a^{-1} is a geodesic in $\mathbf{T}$. The Cayley graph of the nonabelian subgroup $\Gamma_1 = \langle b, c \rangle \cong \mathbf{Z}_2 * \mathbf{Z}_2$ relative to the generators b, c is also a geodesic.

We denote by $W^*(\Gamma)$ the group von Neumann algebra of a group Γ. It is generated by the left regular representation of Γ on $l^2(\Gamma)$. We regard $W^*(\Gamma)$ as a convolution algebra embedded in $l^2(\Gamma)$ ([11], vol. II, 6.7). Then $W^*(\Gamma) = \{f \in l^2(\Gamma) : f * l^2(\Gamma) \subset l^2(\Gamma)\}$. If Γ_0 is a subgroup of Γ then $W^*(\Gamma_0)$ may be identified with the set of functions in $W^*(\Gamma)$ whose support is contained in Γ_0.

In the example above, $W^*(\Gamma_0)$ is an abelian von Neumann subalgebra of $W^*(\Gamma)$. It is actually a maximal abelian subalgebra or *masa*. In fact it follows from [14], Proposition 4.1 that it is a *singular* masa. (The definition will be given later.)

The homogeneous tree $\mathbf{T}$ may be regarded as a one dimensional affine building of type $\widetilde{A}_1$. Our purpose is to generalize the above observations to the two dimensional case where Γ is a group acting simply transitively on the vertices of an affine building Δ of type $\widetilde{A}_2$. The building Δ is a chamber system consisting of vertices, edges and triangles. Each edge lies on $q+1$ triangles, where $q \geqslant 2$ is the *order* of Δ. An *apartment* is a subcomplex of Δ isomorphic to the Euclidean plane tesselated by equilateral triangles (i.e. a Coxeter complex of type $\widetilde{A}_2$).

An $\widetilde{A}_2$ group can be constructed as follows ([6], I, Section 3). Let (P, L) be a projective plane of order q. There are q^2+q+1 points (elements of P) and q^2+q+1 lines (elements of L). Each point lies on $q+1$ lines and each line contains $q+1$ points. Let $\lambda : P \to L$ be a bijection (a *point-line correspondence*). Let $\mathcal{T}$ be a set of triples (x, y, z) where $x, y, z \in P$, with the following properties:

(i) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for some $z \in P$ if and only if y and $\lambda(x)$ are incident (i.e. $y \in \lambda(x)$).

(ii) $(x, y, z) \in \mathcal{T} \Rightarrow (y, z, x) \in \mathcal{T}$.

(iii) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for at most one $z \in P$.

$\mathcal{T}$ is called a *triangle presentation* compatible with λ. A complete list is given in [6] of all triangle presentations for $q = 2$ and $q = 3$.

Let $\{a_x : x \in P\}$ be q^2+q+1 distinct letters and form the group

$$\Gamma = \langle a_x, x \in P \mid a_x a_y a_z = e \text{ for } (x, y, z) \in \mathcal{T} \rangle.$$

The Cayley graph of Γ with respect to the generators $a_x, x \in P$, and their inverses is the 1-skeleton of an affine building of type $\widetilde{A}_2$.

It is convenient to identify the point $x \in P$ with the generator $a_x \in \Gamma$. The lines in L correspond to the inverses of the generators of Γ according to the point-line correspondence: $a_{\lambda(x)} = a_x^{-1}$ for $x \in P$ ([6]). We may therefore write x^{-1} for a_x^{-1} and identify x^{-1} with $\lambda(x)$. From now on we use the concise notation x and $\lambda(x)$ instead of a_x and $a_{\lambda(x)}$ respectively. It is important to note that $\mathcal{T}$ can be recovered from Γ :

$$\mathcal{T} = \{(x, y, z) : x, y, z \in P \text{ and } xyz = e\}.$$

This implies that if $x, y \in P$ then $y \in \lambda(x)$ if and only if $xyz = e$ for some $z \in P$.

Any element $g \in \Gamma \setminus \{e\}$ can be written uniquely in the *left normal form*

$$g = x_1^{-1}x_2^{-1} \cdots x_m^{-1} y_1 y_2 \cdots y_n$$

where there are no obvious cancellations and $x_i, y_j \in P$, $1 \leqslant i \leq m, 1 \leqslant j \leqslant n$ ([4], Lemma 6.2). The absence of "obvious" cancellations means that $x_i \notin \lambda(x_{i+1})$ $(1 \leqslant i < m)$, $y_{j+1} \notin \lambda(y_j)$ $(1 \leqslant j < n)$, and $x_m \neq y_1$. Also any such word for g is a minimal word for g in the generators $x \in P$ and their inverses ([4], Lemma 6.2). We write $|g| = m + n$. An exactly analogous statement is true for the *right normal form* for g in which the inverse generators are on the right of the word ([6], I, Proposition 3.2). We shall use these facts repeatedly. The reader is referred to [11] for background information on von Neumann algebras and to [2], [22] for buildings. Operator algebras associated with $\widetilde{A}_2$ buildings are studied in [21] and [20]. S. Mozes ([13]) has also been concerned with automorphism groups of affine buildings and corresponding actions of $\mathbb{Z}^2$ on apartments.

From now on, unless otherwise stated, Γ will denote an $\widetilde{A}_2$ group with associated projective plane of order $q \geqslant 2$, and Δ will denote the corresponding affine building whose vertices are identified with the elements of Γ. The following result shows that $W^*(\Gamma)$ is a factor ([11], Theorem 6.7.5).

LEMMA 0.2. *Γ is an i.c.c. group. That is, each conjugacy class in Γ, except for $\{e\}$, is infinite.*

Proof. Let $g \in \Gamma \setminus \{e\}$. Assume that g has left normal form

$$g = x_1^{-1}x_2^{-1} \cdots x_m^{-1} y_1 y_2 \cdots y_n$$

where $m, n \geqslant 1$. (If $m = 0$ or $n = 0$ the argument is simpler.) Thus $|g| = m + n$. We may choose $z \in P$ such that $z \notin \lambda(x_1)$ and $z \notin \lambda(y_n)$. This is possible because there are $q+1$ points on each of the lines $\lambda(x_1)$ and $\lambda(y_n)$. Since there are q^2+q+1 points altogether, and $q^2 + q + 1 > 2(q + 1)$, we may choose a point z not lying on either of these lines. Then $z^{-1}x_1^{-1}x_2^{-1} \cdots x_n^{-1} y_1 y_2 \cdots y_m z$ is in left normal form. It follows that $|z^{-1}gz| = |g| + 2$. Repeating the process, we see that the conjugacy class of g contains a sequence of elements of length $|g| + 2n, n = 1, 2, \ldots$ and hence is infinite. ∎

The $\widetilde{A}_2$ groups have Kazhdan's property (T) ([5], Theorem 4.6). It is therefore particularly interesting to investigate how properties of such groups Γ are reflected in the structure of $W^*(\Gamma)$, in view of the following rigidity conjecture of A. Connes ([7], V.B.ε).

CONJECTURE. If Γ_1 and Γ_2 are i.c.c. groups with property (T) and Γ_1 is not isomorphic to Γ_2, then $W^*(\Gamma_1)$ is not isomorphic to $W^*(\Gamma_2)$.

1. SOME ABELIAN SUBGROUPS OF $\widetilde{A}_2$ GROUPS

Recall that an apartment is a subcomplex of Δ isomorphic to the Euclidean plane tesselated by equilateral triangles. An abelian subgroup of Γ which acts simply transitively on an apartment will be the analogue of the subgroup $\mathbb{Z}$ in the tree case (Example 0.1). In the $\widetilde{A}_2$ case, such an abelian subgroup Γ_0 necessarily contains three distinct elements a, b, c of P. We begin by elucidating the structure of Γ_0.

LEMMA 1.1. *If $x, y \in P$, $x \neq y$ and $xy = yx$ then $xyz = e$, where $\{z\} = \lambda(x) \cap \lambda(y)$. Moreover $x \in \lambda(y)$ and $y \in \lambda(x)$.*

Proof. Suppose that $xy = yx$. Then the left side of the equation $y^{-1}xy = x$ is not in left normal form, by the uniqueness assertion of Lemma 6.2, [4]. Since $y \neq x$ it follows that $y \in \lambda(x)$. Thus $xyz = e$ for some $z \in P$. The fact that $yxz = e$ then shows that $x \in \lambda(y)$. It is also clear from these equations that $z \in \lambda(x) \cap \lambda(y)$ ∎

LEMMA 1.2. *Let a, b, c be distinct mutually commuting generators in P. Then $abc = e$ and $\lambda(a) \cap \lambda(b) = \{c\}$.*

Proof. By Lemma 1.1 we have $abz = e$, where $\{z\} = \lambda(a) \cap \lambda(b)$. Since $ac = ca$, the same lemma shows that $c \in \lambda(a)$. Since $bc = cb$, we also have $c \in \lambda(b)$. Therefore $z = c$, which proves the result. ∎

REMARK 1.3. It follows from this result that a set of pairwise commuting elements in P can have no more than three elements.

LEMMA 1.4. *Let a, b, c be distinct mutually commuting generators in P. Then the subgroup $\Gamma_0 = \langle a, b, c \rangle$ of Γ is isomorphic to $\mathbb{Z}^2$. The Cayley graph of Γ_0 relative to the generators a, b, c and their inverses is the 1-skeleton of an apartment in Δ.*

Proof. By Lemma 1.2, $abc = e$. Therefore an isomorphism $\theta : \langle a, b \rangle \to \mathbb{Z}^2$ is defined by $\theta(a) = (1, 0)$, $\theta(b) = (0, 1)$, $\theta(c) = (-1, -1)$. The vertices adjacent to e in the corresponding Cayley graph are labelled as in Figure 1. ∎

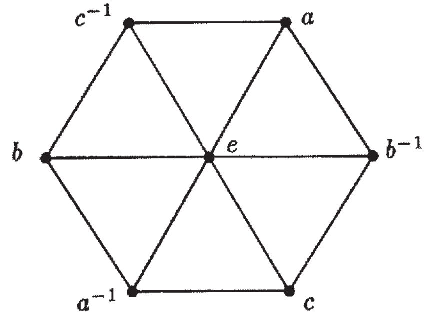

Figure 1. The star of e in the Cayley graph of $\mathbf{Z}^2$.

REMARK 1.5. There are many examples of such subgroups. For example, the groups (4.1), (5.1), (6.1), (9.2), (13.1), (28.1) in the tables at the end of [6] contain such subgroups. (However there are no examples when $q = 2$.)

Note that each element g of the group $\langle a, b, c\rangle$ has left (right) normal form $g = x^{-k}y^l$ $(= y^l x^{-k})$ where $x, y \in \{a, b, c\}$, $k, l \geqslant 0$ and $x \neq y$.

LEMMA 1.6. *Let $a, b \in P$ with $ab^2 = e$. If $a \neq b$ then $\langle a, b\rangle \cong \mathbf{Z}$ and Cayley graph of the group $\langle a, b\rangle$ relative to the generators a, b and their inverses is an infinite strip in an apartment of $\triangle$. If $a = b$ then $\langle a, b\rangle = \langle a\rangle$ is cyclic of order three and the Cayley graph is a triangle.*

Proof. If $a \neq b$, an isomorphism $\theta : \langle a, b\rangle \to \mathbf{Z}$ is defined by $\theta(a) = 2$, $\theta(b) = -1$. It is easy to verify the remaining assertions. ∎

REMARK 1.7. Lemma 1.6 is a degenerate analogue of the Lemma 1.4. The case when $\Gamma_0 \cong \mathbf{Z}$ and the Cayley graph is an infinite strip occurs for subgroups of many of the groups given explicitly in [6]. For example, the groups (B3) (where $q = 2$) and (9.3), (38.1), (63.1), (64.1) (where $q = 3$). A counting argument ([6], II, Section 3) shows that if q is divisible by 3 then Γ has a generator of order three.

REMARK 1.8. In the case of a group Γ acting simply transitively on the vertices of a tree there may exist a nonabelian subgroup of Γ (necessarily isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_2$) which acts simply transitively on a geodesic in the tree. In Example 0.1 the Cayley graph of the subgroup $\langle b, c\rangle \cong \mathbf{Z}_2 * \mathbf{Z}_2$ relative to the generators b, c and their inverses is a geodesic in the tree.

Analogously, a nonabelian subgroup of an $\widetilde{A}_2$ group can act simply transitively on an apartment. There are two examples. One is the Dyck group (or triangle group)

$$T(3,3,3) = \langle x, y, z \mid x^3 = e,\ y^3 = e,\ z^3 = e,\ xyz = e \rangle.$$

$T(3,3,3)$ is a subgroup of index 2 of the Coxeter group W of type $\widetilde{A}_2$. In fact $T(3,3,3)$ is the *rotation subgroup* of W consisting or words of even length in the canonical generators of W ([12], II.3 and II.4).

The other possibility is the amalgam

$$\mathbf{Z} *_{\mathbf{Z}} \mathbf{Z} = \langle y, z \mid y^2 = z^2 \rangle = \langle x, y, z \mid xy^2 = xz^2 = e \rangle.$$

This is the nonabelian Bieberbach group in two dimensions, namely the *fundamental group of the Klein bottle* ([3], Chapter 1). The Cayley graph of each of the groups $T(3,3,3)$ and $\mathbf{Z} *_{\mathbf{Z}} \mathbf{Z}$ relative to the generators x,y,z and their inverses is the 1-skeleton of the Coxeter complex of type $\widetilde{A}_2$.

Many of the $\widetilde{A}_2$ groups enumerated in [6] contain triples of generators which generate a subgroup isomorphic to one of these groups. Detailed enumeration of the possibilities shows that the groups $\mathbf{Z}^2$, presented as in Lemma 1.4, and the groups $T(3,3,3)$, $\mathbf{Z} *_{\mathbf{Z}} \mathbf{Z}$, presented as above, are the only possible subgroups of an $\widetilde{A}_2$ group which can act simply transitively on an apartment.

More precisely, let Γ be a group of automorphisms of a Coxeter complex of type $\widetilde{A}_2$ which acts simply transitively and in a type rotating way on the vertices. Then the generators satisfy relations of the form $xyz = e$ and the 1-skeleton of the Coxeter complex is the Cayley graph of Γ. There are three generators a,b,c and the neighbours of e lie as shown in Figure 2, where $\{g_1, g_2, g_3\} = \{a^{-1}, b^{-1}, c^{-1}\}$.

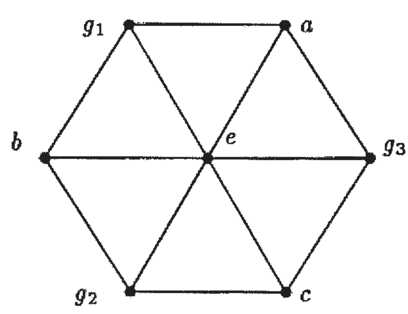

Figure 2. The star of e in a Coxeter complex.

There is a relation of the form $xyz = e$ if and only if $yz = x^{-1}$, that is y is adjacent to x^{-1} in the graph. We now have a degenerate version of [6], I, Section 3. Namely, the link of e is the incidence graph of a *geometry* with set of points $P = \{a, b, c\}$ and set of lines $L = \{a^{-1}, b^{-1}, c^{-1}\}$. Moreover, $y \in x^{-1} \iff xyz = e$, for some generator z. There are essentially three cases : (g_1, g_2, g_3) is one of the triples (c^{-1}, a^{-1}, b^{-1}), (a^{-1}, b^{-1}, c^{-1}), (b^{-1}, a^{-1}, c^{-1}). Detailed enumeration of the possibilities shows that each geometry gives rise to exactly one group whose Cayley graph tesselates the plane by equilateral triangles. The groups are respectively $\mathbf{Z}^2$, $T(3,3,3)$, and $\mathbf{Z} *_{\mathbf{Z}} \mathbf{Z}$. It is instructive to sketch the labelled Coxeter complex in each case. These groups may be regarded as degenerate $\widetilde{A}_2$ groups corresponding to a degenerate projective plane (P, L) of order $q = 1$.

We now consider a much more general situation. The apartment of Lemma 1.4 which is spanned by two commuting generators of Γ is an example of a periodic apartment. An apartment $\mathcal{A}$ in Δ is *doubly periodic* if, when regarded as a copy of the Euclidean plane tesselated by equilateral triangles, with directed edges labelled by generators of Γ, it has two independent periodic directions. This means that the edge-labelled apartment is invariant under an action of $\mathbf{Z}^2$ on $\mathcal{A}$ by translation. This definition depends on the choice of the group Γ acting on Δ.

This concept coincides with that of a *rigidly periodic* apartment in the sense of [20]. (See [20], Lemma 2.10.) A *rigidly periodic limit point* $\omega \in \Omega$ is a boundary point of a rigidly periodic apartment. It is important to note the elementary fact that each doubly periodic apartment is uniquely determined by any one of its sectors ([20], Lemma 2.4).

Now let $\mathcal{A}$ be a doubly periodic apartment, and *assume that $\mathcal{A}$ contains the vertex e*. By double periodicity of the apartment we can choose independent periodic directions in $\mathcal{A}$. This means that there are vertices u, v such that the edge-labelling of the apartment is identical when viewed from each vertex u^m, v^n, $m, n \in \mathbf{Z}$. Then $u^m\mathcal{A} = \mathcal{A}$, $v^n\mathcal{A} = \mathcal{A}$ and left multiplication by $u^m v^n$ defines a periodic action of $\mathbf{Z}^2$ on $\mathcal{A}$ by translation. The *period group* Γ_0 of a periodic apartment $\mathcal{A}$ containing e is the abelian subgroup of Γ consisting of all vertices of the apartment which are periodic directions in the above sense. Clearly Γ_0 is isomorphic to $\mathbf{Z}^2$. By analogy with the tree case, one would expect $W^*(\Gamma_0)$ to be a masa of $W^*(\Gamma)$.

2. MASA'S OF $W^*(\Gamma)$

Suppose that Γ_0 is a maximal abelian subgroup of a countable discrete group Γ. Suppose furthermore that the following two conditions are satisfied:

(I) If $\Gamma_0 x_0 \subset y_1\Gamma_0 \amalg \cdots \amalg y_n\Gamma_0$, where $y_1, \ldots, y_n \in \Gamma$, then $\Gamma_0 x_0 = y_j\Gamma_0$ for some j.

(II) If φ is an automorphism of Γ_0 which fixes pointwise some finite index subgroup $K < \Gamma_0$, then φ is the identity automorphism of Γ_0.

Conditions (I) and (II) are quite different in nature: condition (I) depends on how Γ_0 lies in Γ, while condition (II) is a purely algebraic condition on Γ_0.

PROPOSITION 2.1. *Under the above hypotheses, $W^*(\Gamma_0)$ is a masa of $W^*(\Gamma)$.*

Proof. Let $u \in W^*(\Gamma_0)'$ be unitary. Suppose that $x_0 \in \operatorname{supp} u$. We show that $x_0 \in \Gamma_0$. This will prove that $u \in W^*(\Gamma_0)$.

For $g_0 \in \Gamma_0$ we have $\delta_{g_0} \star u \star \delta_{g_0^{-1}} = u$.

Therefore $u(g_0x_0g_0^{-1}) = u(x_0)$. Since $u(x_0) \neq 0$ and $u \in l^2(\Gamma)$, the set $\{g_0x_0g_0^{-1} : g_0 \in \Gamma_0\}$ is finite. Write its elements as $y_1, \ldots, y_n$ and if necessary delete repetitions which define the same cosets in Γ/Γ_0. Then $\Gamma_0x_0 \subset y_1\Gamma_0 \amalg \cdots \amalg y_n\Gamma_0$.

By condition (I), $\Gamma_0x_0 = y_j\Gamma_0$ for some j. Now $y_j = g_0x_0g_0^{-1}$ for some $g_0 \in \Gamma_0$. Thus $\Gamma_0x_0 = g_0x_0\Gamma_0$. In other words, $\Gamma_0x_0 = x_0\Gamma_0$. That is $x_0\Gamma_0x_0^{-1} = \Gamma_0$. We may therefore define an automorphism φ of Γ_0 by $\varphi(g_0) = x_0g_0x_0^{-1}$. Then for each $g_0 \in \Gamma_0$ $\varphi(g_0)x_0 = x_0g_0$ and so $u(x_0) = u(g_0^{-1}x_0g_0) = u(g_0^{-1}\varphi(g_0)x_0)$.

Again, since $u(x_0) \neq 0$ and $u \in l^2(\Gamma)$, it follows that the set $\{g_0^{-1}\varphi(g_0) : g_0 \in \Gamma_0\}$ is finite. Let $K = \{g_0 \in \Gamma_0 : \varphi(g_0) = g_0\}$. Then K is the kernel of the homomorphism $g_0 \mapsto g_0^{-1}\varphi(g_0)$ on Γ_0, which has finite range. Therefore Γ_0/K is finite. Thus φ fixes the finite index subgroup $K < \Gamma_0$.

It follows from condition (II) that φ is the identity automorphism of Γ_0. This means that $x_0g_0 = g_0x_0$ for all $g_0 \in \Gamma_0$. Since we assumed that Γ_0 is a maximal abelian subgroup of Γ, it follows that $x_0 \in \Gamma_0$. This proves that $u \in W^*(\Gamma_0)$ and hence that $W^*(\Gamma_0)$ is a masa of $W^*(\Gamma)$. ∎

LEMMA 2.2. *Fix $C > 0$ and let S and S' be sectors (Weyl chambers) in an affine building. Either S and S' share a common subsector, or S has a subsector all of whose points are at distance $> C$ from S'.*

Proof. Choose subsectors S_1 and S_1' of S and S' respectively which lie in a common apartment ([22], Chapter 9, Proposition 9.5). If S_1 and S_1' point in the same direction, then they have a common subsector, which is also a common subsector of S and S'.

Otherwise, fix a finite $C_1 > 0$ so that $d(v, S_1') \leqslant C_1$ for any $v \in S'$ ([22], Chapter 9, Lemma 9.2). Choose a subsector S_2 of S_1 all of whose points are at distance $> C + C_1$ from S_1'. Then those points are all at distance $> C$ from S'. ∎

LEMMA 2.3. *Let Γ be an $\widetilde{A}_2$ group and $\mathcal{A}$ a doubly periodic apartment in the corresponding building. Suppose that there exist $y_1, \dots, y_n \in \Gamma$ such that the Hausdorff distance from $\mathcal{A}$ to $y_1\mathcal{A} \cup \cdots \cup y_n\mathcal{A}$ is finite. Then $\mathcal{A}$ coincides with some $y_j\mathcal{A}$.*

Proof. Take a sector S in $\mathcal{A}$. Write each of the apartments $y_j\mathcal{A}$ as a finite union of sectors S'. Suppose that S does not have a subsector in common with any of those sectors S'. Fix $C > 0$. By Lemma 2.2, some finite intersection of subsectors of S has all its points at distance $> C$ from $y_1\mathcal{A} \cup \cdots \cup y_n\mathcal{A}$. A finite intersection of subsectors of S is nonempty (in fact, another subsector) so this contradicts the main hypothesis.

Now we know that for some j the doubly periodic apartments $\mathcal{A}$ and $y_j\mathcal{A}$ have a common subsector, and therefore that they coincide. ∎

LEMMA 2.4. *Assume that $\Gamma_0 \cong \mathbb{Z}^2$ is a subgroup of the period group of a doubly periodic apartment $\mathcal{A}$ containing the vertex e. Then conditions* (I) *and* (II) *hold. If, furthermore, Γ_0 is the full period group of $\mathcal{A}$, then Γ_0 is a maximal abelian subgroup of Γ.*

Proof. (I) Suppose that $\Gamma_0 x_0 \subset y_1\Gamma_0 \amalg \cdots \amalg y_n\Gamma_0$, where $y_1, \dots, y_n \in \Gamma$. Then the Hausdorff distance $\mathrm{d}(\Gamma_0, y_1\Gamma_0 \amalg \cdots \amalg y_n\Gamma_0) \leqslant |x_0|$ and so there exists $C > 0$ such that

$$\mathrm{d}(\mathcal{A}, y_1\mathcal{A} \amalg \cdots \amalg y_n\mathcal{A}) \leqslant C.$$

It follows from Lemma 2.3 that $\Gamma_0 x_0 = y_j\Gamma_0$.

(II) Let K be a subgroup of $\Gamma_0 \cong \mathbb{Z}^2$. There is a basis $\{e_1, e_2\}$ of Γ_0 such that $K = \langle e_1^m | \, e_2^n \rangle$. If K has finite index, then $m \neq 0, n \neq 0$. If φ is an automorphism of Γ_0 which fixes K pointwise, then since Γ_0 is torsion free, $\varphi(e_1) = e_1, \varphi(e_2) = e_2$, and so φ is the identity automorphism of K.

Now assume that Γ_0 is the full period group of $\mathcal{A}$ and suppose that an element $g \in \Gamma$ commutes with Γ_0. We must show that $g \in \Gamma_0$. If $g_0 \in \Gamma_0$ then $gg_0 = g_0g$ and so $\mathrm{d}(gg_0, g_0) \leqslant |g|$. Thus

$$\mathrm{d}(g\Gamma_0, \Gamma_0) \leqslant |g|.$$

Therefore there is a constant $C > 0$ such that

$$d(g\mathcal{A}, \mathcal{A}) \leqslant C.$$

It follows from Lemma 2.3 that $g\mathcal{A} = \mathcal{A}$. The elements of Γ_0 are vertices of the apartment $\mathcal{A}$ and since g commutes with Γ_0, g acts on Γ_0 by translation. Hence g must act on $\mathcal{A}$ by translation, rather than rotation or glide-reflection. Thus g is a period of the apartment $\mathcal{A}$ and so g belongs to Γ_0. ∎

COROLLARY 2.5. *Assume that Γ_0 is the full period group of a doubly periodic apartment $\mathcal{A}$ containing the vertex e. Then $W^*(\Gamma_0)$ is a masa of $W^*(\Gamma)$.*

REMARK 2.6. It was shown in [4], Section 6 that the weak closure of the algebra of biradial functions on Γ is a masa of $W^*(\Gamma)$.

DEFINITION 2.6.1. If A is an abelian von Neumann subalgebra of a von Neumann algebra M then the *normalizer* $N(A)$ is the set of unitaries u in M such that $u^*Au = A$. The subalgebra A is called *regular* (or a *Cartan subalgebra*) if $N(A)'' = M$. It is called *singular* if $N(A)'' = A$.

Regular masa's are easy to construct from the classical group measure space construction in which M is the crossed product of an abelian von Neumann algebra A by a discrete group G which acts ergodically on A. Then A is a regular masa in M. It is more difficult to construct singular masa's. See [18] and [1] for recent examples. The masa of biradial functions on an $\widetilde{A}_2$ group has been studied in [5]. The authors have convinced themselves that this masa is singular. Singularity has been proved in [18] and [1] for certain masa's of radial functions, in the context of trees and similar rank 1 structures.

LEMMA 2.7. *Suppose that Γ is a maximal abelian subgroup of a countable discrete group Γ. Assume in addition to conditions* (I) *and* (II) *that no element of $\Gamma \setminus \Gamma_0$ normalizes Γ_0. Then $W^*(\Gamma_0)$ is a singular masa of $W^*(\Gamma)$.*

Proof. Let $u \in W^*(\Gamma)$ be a unitary satisfying $u^*W^*(\Gamma_0)u \subset W^*(\Gamma_0)$. Suppose that $x_0 \in \operatorname{supp} u$. We must prove that $x_0 \in \Gamma_0$.

There are only a finite number of cosets $y\Gamma_0$ such that $\|u|y\Gamma_0\|_2 \geqslant |u(x_0)|$. Call them $y_1\Gamma_0, \ldots, y_n\Gamma_0$. We claim that $\Gamma_0 x_0 \subset y_1\Gamma_0 \amalg \cdots \amalg y_n\Gamma_0$. To prove this, note that if $z \in \Gamma_0$ then $u^{-1} \star \delta_z \star u = f \in W^*(\Gamma_0)$ is unitary. Therefore

$$\begin{aligned} |u(x_0)| &= |(\delta_z \star u)(zx_0)| = |(u \star f)(zx_0)| \\ &\leqslant \|(u \star f)|zx_0\Gamma_0\|_2 = \|u|zx_0\Gamma_0 \star f\|_2 = \|u|zx_0\Gamma_0\|_2. \end{aligned}$$

(The last two equalities are valid because $\operatorname{supp} f \subset \Gamma_0$ and f is unitary.) This shows that $zx_0 \in y_1\Gamma_0 \amalg \cdots \amalg y_n\Gamma_0$, as claimed above.

Suppose that $x_0 \notin \Gamma_0$. It follows from condition (I) that $\Gamma_0 x_0 = y_j \Gamma_0$ for some j. In particular $x_0 \in y_j\Gamma_0$, and so $x_0\Gamma_0 = y_j\Gamma_0 = \Gamma_0 x_0$. Thus $x_0^{-1}\Gamma_0 x_0 = \Gamma_0$ with $x_0 \notin \Gamma_0$, contrary to the assumption of the lemma. This contradiction shows that $W^*(\Gamma_0)$ is singular. ∎

THEOREM 2.8. *Let a, b, c be distinct mutually commuting generators of an $\widetilde{A}_2$ group Γ and let $\Gamma_0 = \langle a, b, c\rangle \cong \mathbb{Z}^2$. (See Lemma 1.4.) Then $W^*(\Gamma_0)$ is a singular masa of $W^*(\Gamma)$.*

Proof. Note that the elements of Γ_0 are the vertices of a doubly periodic apartment $\mathcal{A}$. According to the preceding results, we need only show that no element of $\Gamma \setminus \Gamma_0$ normalizes Γ_0. Let $g \in \Gamma$ satisfy $g\Gamma_0 g^{-1} = \Gamma_0$. If $g_0 \in \Gamma_0$ then $gg_0 = g_0' g$ for some $g_0' \in \Gamma_0$ and so $\mathrm{d}(gg_0, g_0') \leqslant |g|$. Thus

$$\mathrm{d}(g\Gamma_0, \Gamma_0) \leqslant |g|.$$

In other words,

$$\mathrm{d}(g\mathcal{A}, \mathcal{A}) \leqslant |g|.$$

It follows from Lemma 2.3 that $g\mathcal{A} = \mathcal{A}$. Since the elements of Γ_0 are the vertices of the apartment $\mathcal{A}$, this means $g\Gamma_0 = \Gamma_0$. Therefore $g \in \Gamma_0$, as required. ∎

REMARK 2.9. Suppose that Γ_0 is the full period group of a doubly periodic apartment $\mathcal{A}$ containing the vertex e, as in Corollary 2.5. Then the masa $W^*(\Gamma_0)$ of $W^*(\Gamma)$ may be nonsingular.

For example in the group (B3) of [6] there exist generators a, b, c satisfying $ab^2 = ac^2 = e$. Let $\Gamma_0 = \langle a, bc\rangle \cong \mathbb{Z}^2$. It is a subgroup of the group $\langle a, b, c\rangle \cong \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ whose 1-skeleton is the Cayley graph of an apartment $\mathcal{A}$. (See Remark 1.8.)

Then Γ_0 is the period lattice of $\mathcal{A}$ and so $W^*(\Gamma_0)$ is a masa. However $b \notin \Gamma_0$ and b normalizes Γ_0. For $bab^{-1} = a \in \Gamma_0$ and $b(bc)b^{-1} = c^3b^{-1} = cb^2b^{-1} = cb = a^{-1}(bc)^{-1}a^{-1} \in \Gamma_0$. Therefore $W^*(\Gamma_0)$ is not singular.

REMARK 2.10. An $\widetilde{A}_2$ group Γ has Kazhdan's property T ([5]). It therefore follows from [15], Corollary 4.5 that $W^*(\Gamma)$ contains an *ultrasingular* masa A. This means that the only automorphisms of $W^*(\Gamma)$ which normalize A are the inner automorphisms implemented by unitaries in A. We do not know whether the masa's we have been considering are ultrasingular.

REMARK 2.11. H. Yoshizawa has given an explicit decomposition of the regular representation of the free group on two generators into two inequivalent families of irreducible representations. These representations are induced from characters on the abelian subgroup generated by one of the generators. See, for example [19], Chapter 19. Our results may be re-interpreted to give a two dimensional analogue of Yoshizawa's construction. For example, if Γ, Γ_0 are as in Theorem 2.8, then $\widehat{\Gamma}_0 \cong \mathbf{T}^2$ and we have a direct integral decomposition

$$l^2(\Gamma) = \int_{\mathbf{T}^2}^{\oplus} \mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi \, d\chi.$$

Since $W^*(\Gamma_0)$ is a masa of $W^*(\Gamma)$, it follows that the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}\chi$ is irreducible for almost all characters $\chi \in \widehat{\Gamma}_0$ ([9], Section 8.5).

3. THE PUKÁNSZKY INVARIANT

Let A be a masa of a type II_1 factor M. Suppose that M acts in its standard representation on the Hilbert space $L^2(M)$, which is the completion of M relative to the inner product defined by the trace on M. Denote by L the left regular representation of M on $L^2(M)$ defined by $L_x y = xy$, and by R the anti-representation defined by $R_x y = yx$. Let $\mathcal{A}$ be the (abelian) von Neumann subalgebra of $\mathcal{B}(L^2(M))$ generated by L_A and R_A. A result of Popa ([16], Corollary 3.2) asserts that if A is a regular masa then $\mathcal{A}$ is a masa of $\mathcal{B}(L^2(M))$. Let p_1 denote the orthogonal projection of $L^2(M)$ onto the closed subspace generated by A. Then by [16], Lemma 3.1, p_1 is in the centre of $\mathcal{A}'$ and $\mathcal{A}'p_1$ is abelian. The Pukánszky invariant is the type of the (type I) von Neumann algebra $\mathcal{A}'(1-p_1)$. It is an isomorphism invariant of the pair (A, M), since any automorphism of M is implemented by a unitary in $\mathcal{B}(L^2(M))$. The Pukánszky invariant has been computed for some particular examples in [17], [18], [1]. Popa showed that if $\mathcal{A}'(1-p_1)$ is homogeneous of type I_n where $2 \leqslant n \leqslant \infty$ then A is a singular masa ([16], Remark 3.4).

Now let Γ be a group and $M = W^*(\Gamma)$. Then $L^2(M)$ is naturally identified with $l^2(\Gamma)$ and L and R respectively become the left regular representation $L_g\delta_h = \delta_{gh}$ and the right regular anti-representation $R_g\delta_h = \delta_{hg}$.

Let Γ_0 be an abelian subgroup of Γ such that $A = W^*(\Gamma_0)$ is a masa of $M = W^*(\Gamma)$. Then $\mathcal{A} = (L_{\Gamma_0} \cup R_{\Gamma_0})''$. Denote by $\mathcal{D}$ the set of double cosets $D = \Gamma_0 g \Gamma_0 \in \Gamma_0 \setminus \Gamma / \Gamma_0$ which have the property that

$$g^{-1}\Gamma_0 g \cap \Gamma_0 = \{e\}. \tag{3.1}$$

Note that this condition depends only on the double coset $\Gamma_0 g \Gamma_0$ and not on the representative element g.

For each $D \in \mathcal{D}$ let p_D denote the orthogonal projection from $l^2(\Gamma)$ onto $l^2(D)$. Since $l^2(D)$ is invariant under L_{Γ_0} and R_{Γ_0} it follows that p_D lies in the commutant $\mathcal{A}'$ of $\mathcal{A}$.

LEMMA 3.1. *If $C, D \in \mathcal{D}$ and $C \neq D$ then the projections p_C, p_D are mutually orthogonal and equivalent in $\mathcal{A}'$.*

Proof. Let $C = \Gamma_0 c \Gamma_0$ and $D = \Gamma_0 d \Gamma_0$, where $c, d \in \Gamma$, and assume that $C \neq D$. Define a map $\varphi : C \to D$ by $\varphi(ucv) = udv$ for $u, v \in \Gamma_0$. The condition (3.1) ensures that φ is a well defined bijection. Moreover

$$\varphi(uk) = u\varphi(k), \quad \varphi(kv) = \varphi(k)v \quad \text{for } u, v \in \Gamma_0,\ k \in C.$$

Define a partial isometry s on $l^2(\Gamma)$ by

$$s\delta_k = \begin{cases} \delta_{\varphi(k)} & \text{if } k \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Then $s^*s = p_C$ and $ss^* = p_D$. We must show that $s \in \mathcal{A}'$.

If $k \in C$ and $u \in \Gamma_0$ then

$$L_u s\delta_k = L_u \delta_{\varphi(k)} = \delta_{u\varphi(k)} = \delta_{\varphi(uk)} = s\delta_{uk} = sL_u\delta_k.$$

If $k \notin C$ and $u \in \Gamma_0$ then $uk \notin C$ and

$$L_u s\delta_k = 0 = sL_u\delta_k.$$

It follows that $L_u s = sL_u$ for each $u \in \Gamma_0$. Similarly $R_u s = sR_u$ for each $u \in \Gamma_0$. Therefore $s \in \mathcal{A}'$, as required.

It is clear that p_C, p_D are mutually orthogonal, since $C \cap D = \emptyset$. ∎

LEMMA 3.2. *If $D = \Gamma_0 d \Gamma_0 \in \mathcal{D}$ where $d \in \Gamma$, then p_D is an abelian projection in $\mathcal{A}'$.*

Proof. (cf. [17], Lemma 4) By (3.1) the map $(x, y) \mapsto xdy$ defines a bijection from $\Gamma_0 \times \Gamma_0$ onto D. Define a unitary operator $U : l^2(D) \to l^2(\Gamma_0 \times \Gamma_0)$ by $(Uf)(x, y) = f(xdy)$. Then for $z_1, z_2 \in \Gamma_0$ and $f \in l^2(D)$,

$$(UL_{z_1}R_{z_2}f)(x, y) = f(z_1^{-1}xdyz_2^{-1}) = (Uf)(z_1^{-1}x, yz_2^{-1})).$$

Thus $UL_{z_1}R_{z_2}U^{-1} = \Lambda_{z_1}P_{z_2}$ where the operators on the right hand side are defined on $l^2(\Gamma_0 \times \Gamma_0)$ by $(\Lambda_z f)(x, y) = f(z^{-1}x, y)$ and $(P_z f)(x, y) = f(x, z^{-1}y)$. Therefore $\mathcal{A}p_D$ is spatially isomorphic to the subalgebra of $\mathcal{B}(l^2(\Gamma_0 \times \Gamma_0))$ generated by the operators Λ_z and P_z, for $z \in \Gamma_0$. Taking Fourier transforms, this algebra is itself spatially isomorphic to the algebra $L^\infty(\widehat{\Gamma}_0 \times \widehat{\Gamma}_0)$ acting by multiplication on $L^2(\widehat{\Gamma}_0 \times \widehat{\Gamma}_0)$. The latter algebra is maximal abelian ([11], vol. I, Example 5.1.6) and hence $\mathcal{A}p_D$ is maximal abelian on $l^2(D)$. It follows that $p_D\mathcal{A}'p_D$ is contained in $\mathcal{A}p_D$ and so is abelian. In other words p_D is an abelian projection in $\mathcal{A}'$. ∎

LEMMA 3.3. *Let Γ be an $\widetilde{A}_2$ group and Γ_0 the abelian subgroup generated by three distinct commuting elements $a, b, c \in P$. Then the set $\mathcal{D}$ is infinite. That is, there exists a sequence $\{g_n\}$ of elements of Γ such that:*

(i) $g_n^{-1}\Gamma_0 g_n \cap \Gamma_0 = \{e\}$ *for* $n = 1, 2, \ldots$;

(ii) $\Gamma_0 g_r \Gamma_0 \cap \Gamma_0 g_s \Gamma_0 = \emptyset$ *for* $r \neq s$.

Proof. By Lemma 1.2 we have $\lambda(a) \cap \lambda(b) = \{c\}$, $\lambda(b) \cap \lambda(c) = \{a\}$ and $\lambda(c) \cap \lambda(a) = \{b\}$. Hence

$$\#(\lambda(a) \cup \lambda(b) \cup \lambda(c)) = (q+1) + q + (q-1) = 3q.$$

Since the total number of points in P is q^2+q+1 and $q^2+q+1 > 3q$ for $q \geqslant 2$, we can choose a generator $z_1 \in P$ with $z_1 \notin \lambda(a) \cup \lambda(b) \cup \lambda(c)$. Of course this implies in particular that $z_1 \notin \{a, b, c\}$.

Now choose $z_2 \in P$ such that $z_2 \notin \{a, b, c\} \cup \lambda(z_1)$. This is possible since $\#(\{a, b, c\} \cup \lambda(z_1)) \leqslant q + 4 < q^2 + q + 1 = \#(P)$. By induction we can choose a sequence $z_2, z_3, \ldots$ in P such that $z_{j+1} \notin \{a, b, c\} \cup \lambda(z_j)$. Define $g_n = z_1 z_2 \cdots z_n$.

Note that any element $g \in \Gamma_0$ has left normal form $g = x^{-k}y^l$ where $x, y \in \{a, b, c\}$, $k, l \geqslant 0$ and $x \neq y$.

Suppose that (i) is not true. Then we can find an element

$$g_n^{-1}x_1^{-m_1}y_1^{n_1}g_n = x_2^{-m_2}y_2^{n_2} \in g_n^{-1}\Gamma_0 g_n \cap \Gamma_0$$

where $x_i, y_i \in \{a, b, c\}$, $x_i \neq y_i$, $m_i, n_i \geqslant 0$ and $m_i + n_i > 0$, $i = 1, 2$. This means that

$$z_n^{-1} \cdots z_1^{-1}x_1^{-m_1}y_1^{n_1}z_1 \cdots z_n = x_2^{-m_2}y_2^{n_2}$$

where on each side of the equation at most one of the terms x_i, y_i is absent. With or without such an absence, each side of the equation is in left normal form, which contradicts uniqueness of left normal form. This proves (i).

In order to prove (ii), it is enough to show that if $g_r \in \Gamma_0 g_s \Gamma_0$, then $r = s$. Suppose that $g_r = x_1^{-m_1}y_1^{n_1}g_s x_2^{-m_2}y_2^{n_2}$, where $x_1^{-m_1}y_1^{n_1}$, $x_2^{-m_2}y_2^{n_2}$ are elements of Γ_0 expressed as usual in left normal form. Then

$$z_1 \cdots z_r = x_1^{-m_1}y_1^{n_1}z_1 \cdots z_s x_2^{-m_2}y_2^{n_2}.$$

This implies

$$x_1^{m_1}z_1 \cdots z_r y_2^{-n_2} = y_1^{n_1}z_1 \cdots z_s x_2^{-m_2}.$$

Both sides are in right normal form, which is unique ([6], I, Proposition 3.2). Since $z_j \notin \{a, b, c\}$ for any j, this implies that $r = s$. ∎

COROLLARY 3.4. *Let Γ be an $\widetilde{A}_2$ group and Γ_0 the abelian subgroup generated by three distinct commuting elements $a, b, c \in P$. (So that $W^*(\Gamma_0)$ is a masa of $W^*(\Gamma)$, by Lemma 1.4 and Corollary 2.5.) Then the Pukánszky invariant contains a type I_∞ summand.*

Proof. Lemmas 3.1, 3.2 and 3.3 show that $\mathcal{A}'(1 - p_1)$ contains an infinite family of mutually orthogonal equivalent abelian projections. ∎

REMARK 3.5. Another consequence of Lemma 3.3 (i) is that $W^*(\Gamma_0)$ has no nontrivial central sequences in $W^*(\Gamma)$. This follows from [14], Remark 4.2 (2).

The following result clarifies the situation in a general group Γ.

PROPOSITION 3.6. *Let Γ be an arbitrary group and Γ_0 an abelian subgroup such that $A = W^*(\Gamma_0)$ is a masa of $M = W^*(\Gamma)$. Suppose that $g^{-1}\Gamma_0 g \cap \Gamma_0 = \{e\}$ for all $g \in \Gamma \setminus \Gamma_0$. Let $n = \#(\Gamma_0 \setminus \Gamma/\Gamma_0)$. Then $\mathcal{A}'(1 - p_1)$ is homogeneous of type I_n.*

Proof. By assumption we have $\mathcal{D} = \Gamma_0 \backslash \Gamma/\Gamma_0$. Lemmas 3.1 and 3.2 imply that $\{p_D : D \in \mathcal{D}\}$ is a family of n (possibly $n = \infty$) mutually orthogonal equivalent abelian projections in $\mathcal{A}'$ with sum $1 - p_1$. This proves the result. ∎

Proposition 3.6 does not apply when Γ is an $\widetilde{A}_2$ group. However it does describe the Pukánszky invariant for the abelian subalgebra generated by a generator of a free group as in Example 0.1.

COROLLARY 3.7. *Let $\Gamma = \Gamma_0 \star \Gamma_1$ where $\Gamma_0 = \langle a \rangle \cong \mathbb{Z}$ and Γ_1 is any nontrivial group. Then $\mathcal{A}'(1 - p_1)$ is homogeneous of type I_∞.*

Proof. Choose $b \in \Gamma_1 \setminus \{e\}$. The double cosets $\Gamma_0 b \Gamma_0, \Gamma_0 bab \Gamma_0, \Gamma_0 babab \Gamma_0, \ldots$ are pairwise disjoint. Therefore $\#(\Gamma_0 \setminus \Gamma/\Gamma_0) = \infty$. Clearly $g^{-1}\Gamma_0 g \cap \Gamma_0 = \{e\}$ for all $g \in \Gamma \setminus \Gamma_0$. The result therefore follows from Proposition 3.6. ∎

4. LINE CENTRAL ELEMENTS

In this section we determine how many generators of Γ can commute with a given generator.

PROPOSITION 4.1. *A generator $a \in P$ commutes with at most $q+1$ other generators in $P\setminus\{a\}$. Moreover if a commutes with $q+1$ generators $x_1, x_2, \ldots, x_{q+1}$ in $P \setminus \{a\}$ then*

(i) $a \notin \lambda(a)$;

(ii) $\{x \in P : ax = xa\} = \{a\} \cup \lambda(a)$.

Proof. If a commutes with $x \in P \setminus \{a\}$ then $x \in \lambda(a)$ by Lemma 1.1. The assertion (i) is immediate, since the line $\lambda(a)$ contains $q+1$ points. If a commutes with $q+1$ generators $x_1, x_2, \ldots, x_{q+1} \in P \setminus \{a\}$ then $\lambda(a) = \{x_1, x_2, \ldots, x_{q+1}\}$ and assertions (i) and (ii) follow from this. ∎

Suppose that a generator a commutes with $q+1$ other generators. The following terminology comes from the fact that the points on the line $\lambda(a)$ then correspond to the generators which commute with a.

DEFINITION 4.1.1. If P is the set of generators of a triangle presentation then an element $a \in P$ is called *line central* if it commutes with $q+1$ elements of $P \setminus \{a\}$.

REMARK 4.2. Examination of the tables given at the end of [6] suggest that line central elements may be relatively rare. However they do exist. Here are some examples.

(i) In the presentation (B.3) of [6], the generator a_0 is line central with $\lambda(a_0) = \{a_1, a_2, a_4\}$. In this case $q = 2$ and there are seven generators $a_0, \ldots, a_6$. The relations involving a_0 are given by the following triples : $(a_0a_1a_1), (a_0a_2a_2)$, $(a_0a_4a_4)$. This group acts on the building of $\mathrm{PGL}(3, \mathbb{Q}_2)$.

(ii) The presentation (4.1) of [6] has an unusually large number of line central generators, namely a_2,a_7 and a_9 . In this case $q = 3$. The line corresponding to a_2 is $\lambda(a_2) = \{a_3, a_4, a_7, a_9\}$. There are thirteen generators $a_0, \ldots, a_{12}$ and the relations involving a_2 are: $(a_2a_4a_7), (a_2a_7a_4), (a_2a_3a_9), (a_2a_9a_3)$. This group acts on the building of $\mathrm{PGL}(3, \mathbb{Q}_3)$.

(iii) The presentation (5.1) of [6] has exactly one line central generator a_2.

(iv) In the presentation (64.1) of [6], the generator a_{12} is line central. Again $q = 3$. The building on which this group acts is not that of any linear group ([6]).

It may be significant that the groups which act on the building of $\mathrm{PGL}(3, \mathbb{F}_q((X)))$ do not appear to have a line central generator.

PROPOSITION 4.3. *If an element $a \in P$ is line central and $x \in P$ then*

$$x \in \lambda(a) \Longleftrightarrow a \in \lambda(x).$$

Proof. Suppose $x \in \lambda(a)$. By Corollary 4.1, $ax = xa$. Also $axy = e$ for some $y \in P$ and so $xay = e$ as well. It follows that $a \in \lambda(x)$.

Conversely, suppose that $a \in \lambda(x)$. Then $xay = e$ for some $y \in P$. Therefore $ayx = e$. This implies that $y \in \lambda(a)$. By Corollary 4.1, $ay = ya$. Then $yax = e$, and so $x \in \lambda(a)$ ∎

This research was supported by the Australian Research Council.

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Received December 3, 1995; revised August 7, 1996.