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SMOOTHING TECHNIQUES IN C^*-ALGEBRA THEORY

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ABSTRACT. We show that any algebraic element in a C^*-algebra A can be approximated by a smooth algebraic element, with the same minimal polynomial. By a smooth element we mean an element of a given dense subalgebra A_∞ that is closed under C^∞ functional calculus. Smoothing results for a variety of other C^*-relations are obtained. These serve to prove the density of the smooth homomorphisms in $\hom(CM_n, A)$. From this, smoothing results in mod-p K-theory may be derived. We also prove two closure properties for stable, smoothable relations.

KEYWORDS: *C^*-algebra, algebraic element, perturbation, stable relations, smoothing.*

AMS SUBJECT CLASSIFICATION: Primary 46L05; Secondary 41A65, 46L89, 47C15.

1. INTRODUCTION

Our inspiration for smoothing results comes from analogous lifting theorems. A simple lifting theorem is that every nilpotent of order two lifts to a nilpotent of order two. This has two substantial generalizations: the lifting theorem for nilpotents of any finite order, proven by C. Olsen and the second author, and secondly the projectivity of the cones CM_n, proven by the first author in [6], and given a vastly simpler proof in [9]. That the second result is, indeed, a generalization requires comment.

The universal C^*-algebra generated by a single contraction of order two is isomorphic to the cone $CM_2 = C_0(0,1] \otimes M_2$. Knowing well the C^*-algebra CM_n is critical to the study of $M_n(A)$ when A is not unital. This is because a copy of this algebra operates as a would-be set of matrix units in $M_n(A)$ whenever A is a σ-unital C^*-algebra.

Let us now focus on the easy result that the relation $x^2 = 0$ is smoothable, while at the same time defining precisely this notion. Any reasonable notion of smooth elements in a C^*-algebra A should allow that every element x in A can be approximated closely in norm by a smooth element y. Should $x^2 = 0$ then $y^2 \approx 0$. How might we modify y to make $y^2 = 0$?

One way to force square zero is to sandwich the element between orthogonal positive elements, such as defining

$$\tilde{x} = f(y^*y)yg(y^*y)$$

where f and g are positive functions on the real line with $fg = 0$. We want to ensure $f(y^*y)y \approx y$ and $yg(y^*y) \approx y$ which we may do by assuming $f(0) = 1$ and $g \equiv 1$ outside a small neighborhood of zero. There are many such functions, so we have produced $\tilde{x} \approx x$ with again $\tilde{x}^2 = 0$. But what of smoothness?

If we assume, as is reasonable, that f and g are smooth, then $\tilde{x}$ will remain in the set of smooth elements if these form a $*$-algebra that is at least closed under C^∞-functional calculus for self-adjoint elements. Although in examples one expects better closure properties than this, since this is all we need, it is all we assume in this paper.

DEFINITION 1.1. A subset A_∞ of a C^*-algebra A is called a *real C^∞-subalgebra* if it is a dense $*$-subalgebra and $h = h^* \in A_\infty$ implies that $f(h) \in A_\infty$ whenever $f : \mathbf{R} \to \mathbf{R}$ is smooth and $f(0) = 0$.

The existence of a plethora of interesting real C^∞-subalgebras, and other more specialized notions of smooth structures on a C^*-algebra are amply discussed in work of Sakai ([12]), Connes ([3]), and Blackadar and Cuntz ([2]), to name but a few.

If we think of these subalgebras as defining an analog of algebras of smooth functions, it is natural to ask whether, in the non-unital case, these structures can, in a generalized sense, be restricted to create algebras of smooth functions of compact support. This we can answer. In particular, the minimal dense ideal of A is always an example of a real C^∞-subalgebra.

PROPOSITION 1.2. *If A_∞ is a real C^∞-subalgebra of a C^*-algebra A and A_c. denotes the minimal dense ideal of A, then $A_c \cap A_\infty$ is also a real C^∞-subalgebra of A.*

Proof. In [11], Section 5.6 the ideal A_c is defined, and it is proved that if a is any positive element of A, then $f(a) \in A_c$ whenever $f : \mathbf{R} \to \mathbf{R}$ is continuous and vanishes on a neighborhood of zero. Should it happen that $a \in A_\infty$ and f is smooth, then also $f(a) \in A_\infty$. It follows that $A_c \cap A_\infty$ is dense. ∎

We shall be content to work with $*$-polynomial relations on finitely many *noncommuting* variables, together perhaps with a finite number of positivity and norm conditions. It should be true that there are relations of a more general type that are smoothable, but we avoid for now making a broad definition. Indeed, we state our main definition only in terms of $*$-polynomials and norm bounds and leave it to the good judgement of our readers to adjust this for other types of relations.

Henceforth we shall let A denote an arbitrary C^*-algebra, and A_∞ an arbitrary real C^∞-subalgebra of A.

DEFINITION 1.3. A set $\mathcal{P}$ of $*$-polynomials, in n noncommuting variables, is *smoothable* if, given contractions $x_1, \ldots, x_n$ in A which satisfy $p(x_1, \ldots, x_n) = 0$ for every p in $\mathcal{P}$, there exist contractions $y_1, \ldots, y_n$ in A_∞, arbitrarily close to these elements, such that again $p(y_1, \ldots, y_n) = 0$.

A now standard warning is in order. The universal C^*-algebra for $\mathcal{P}$ is not likely to exist; it will exist only if norm conditions are imposed. Unless stated otherwise, it should be assumed that generators are restricted to having norm at most one.

Our proof that $x^n = 0$ is smoothable borrows ideas from [1] and [10]. From this special case we prove a smoothing result for any polynomial in x. See also the paper [5] by Hadwin for a closely related discussion regarding the lifting of algebraic elements. Our method of attack on the relation $x^n = 0$ has the advantage that we derive a smoothing result for certain sets of nilpotents of order two along the way.

The proof that CM_n is smoothable is best tackled in the context of mapping telescopes for finite-dimensional C^*-algebras. We will use the presentation by generators and relations of such telescopes given in [9].

Having done so, we show that smoothable relations are closed under constructions corresponding, at the level of universal C^*-algebras, to direct sum and taking matrices. (Actually, we require also the assumption that the relations are weakly stable.) These results become much simpler when the relations involved determine a unital C^*-algebra. In this case, the essential facts are that the relations

$$p_i^2 = p_i^* = p_i \quad (i = 1, \ldots, n),$$
$$p_i p_j = 0 \quad (i \neq j)$$

are smoothable, as are the relations describing n by n matrix units. That these are smoothable, we leave to the reader to verify by the usual tricks.

2. ALGEBRAIC ELEMENTS

Recall from [6] that a set of relations $\mathcal{P}$ is *weakly stable* if any finite set $(x_1, \ldots, x_n)$ in a C^*-algebra A, approximately satisfying the relations, can be perturbed to a set $(y_1, \ldots, y_n)$ in A satisfying $\mathcal{P}$ exactly. More precisely: for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x_1, \ldots, x_n$ are elements in A with $\|x_k\| \leqslant 1$ and

$$\|p(x_1, \ldots, x_n)\| \leqslant \delta, \quad \forall p \in \mathcal{P},$$

then we can find $y_1, \ldots, y_n$ in A with $\|x_k - y_k\| < \varepsilon$ for all k, such that

$$p(y_1, \ldots, y_n) = 0, \quad \forall p \in \mathcal{P}.$$

This is weaker than stability, which corresponds to semiprojectivity for the universal C^*-algebra $C^*\langle g_1, \ldots, g_n \mid \mathcal{P}\rangle$.

(This is assuming $\mathcal{P}$ consists only of $*$-polynomials in noncommuting variables $g_1, \ldots, g_n$ with additional implicit relations on $\|g_j\|$. Also, the definition above differs from that given in [6], but is equivalent. The distinction is weakening $\|x_k\| \leqslant 1$ to $\|x_k\| \leqslant 1 + \varepsilon$ in the definition of approximate representation. The distinction only becomes important when considering stronger forms of stability. See [8] for an account of the various forms of stability.)

We now define a set $\mathcal{P}$ to be *smoothly stable* if in addition we can choose the elements $y_1, \ldots, y_n$ in A_∞. Evidently, this is stronger than smoothability, but at the moment — and this may not be a coincidence — all proofs of smoothable relations pass through smooth stability.

The next result is an approximate version of (a corollary to) [10], Lemma 6.3. Whereas the exact result — writing a nilpotent in a pseudo-triangular form — is only valid in special C^*-algebras like corona algebras, the approximate version holds in any C^*-algebra.

LEMMA 2.1. *For each $n \geqslant 2$ and $\varepsilon > 0$ there is a $\delta > 0$, such that if $x \in A$, $\|x\| \leqslant 1$, and $\|x^n\| \leqslant \delta$, then*

$$x = x_1 + \cdots + x_{n-1}$$

for some contractions x_k, $1 \leqslant k \leqslant n-1$, in A satisfying

$$\|x_j x_k\| \leqslant \varepsilon \quad \text{for all} \quad j \leqslant k.$$

Proof. If $\|x^n\| \leqslant \delta$, choose $0 < \alpha < \varepsilon_1 < \varepsilon$ and let f be a continuous, increasing function on $\mathbb{R}$, such that $f(t) = 0$ for $t \leqslant \alpha$ and $f(t) = 1$ for $f \geqslant \varepsilon_1$. Let $e = f(|x|)$ and define

$$y = ex, \quad x_{n-1} = (1-e)x;$$

so that $x = y + x_{n-1}$. Since $\|x(1-e)\| \leqslant \varepsilon_1$, by spectral theory, it follows that

$$\|x_{n-1}^2\| \leqslant \varepsilon_1 \quad \text{and} \quad \|yx_{n-1}\| \leqslant \varepsilon_1.$$

Moreover,

$$\|y^{n-1}\| \leqslant (n-2)\varepsilon_1 + \|ex^{n-1}\| \leqslant (n-2)\varepsilon_1 + \alpha^{-1}\delta,$$

because $\alpha e \leqslant |x|$.

Taking $\delta' = (n-2)\varepsilon_1 + \alpha^{-1}\delta$ we may repeat the procedure to write $y = y' + x_{n-2}$ with $y' = e'y$ and $x_{n-2} = (1-e')y$. But the new elements will still satisfy

$$\|y'x_{n-2}\| \leqslant \varepsilon_1 \quad \text{and} \quad \|x_{n-2}x_{n-1}\| \leqslant \varepsilon_1,$$

so the argument may proceed by induction. ∎

PROPOSITION 2.2. *For each n the relations $x_j x_k = 0$, for $1 \leqslant j \leqslant k \leqslant n$, are smoothly stable.*

Proof. We are given contractions $x_1, \ldots, x_n$ in A, such that $\|x_j x_k\| \leqslant \delta$ for all $j \leqslant k$. Since A_∞ is dense in A we may assume that the x_j's already belong to A_∞.

Now take $0 < \alpha < \beta < \varepsilon$ and choose C^∞ functions f and g on $\mathbb{R}$ such that $0 \leqslant f, g \leqslant 1$, $fg = f$, both f and g vanish on $[0, \alpha]$ and $f(t) = 1$ for $t \geqslant \beta$. Put $h = \sum_{j=1}^{n} x_j^* x_j$ and define

$$y_n = (1 - g(h))x_n f(h)$$

and, for $1 \leqslant j < n$,

$$y_j = x_j f(h).$$

The new elements are still contractions belonging to A_∞ and now

$$y_n^2 = 0 \quad \text{and} \quad y_j y_n = 0 \quad \text{for } j < n.$$

Moreover, for $j < n$,

$$\|x_j - y_j\| = \|x_j(1 - f(h))\| \leqslant \|h^{\frac{1}{2}}(1 - f(h))\| \leqslant \beta^{\frac{1}{2}}.$$

In particular, $\|y_i y_k\| \leqslant 2\beta^{1/2} + \delta$ for $1 \leqslant j \leqslant k < n$. Similarly, using the triangle inequality,

$$\|x_n - y_n\| \leqslant \beta^{\frac{1}{2}} + \|g(h)x_n\| \leqslant \beta^{\frac{1}{2}} + \alpha^{-1}n\delta,$$

because $\alpha g(h) \leqslant h$ and $\|hx_n\| \leqslant n\delta$.

Taking $\delta' = 2\beta^{1/2} + \delta$ we can now repeat the argument with the elements $y_1, \ldots, y_{n-1}$. The new perturbations involve multiplications on the right with factors $f'(h')$ where $h' = \sum_{i=1}^{n-1} y_i^* y_i$. Since, however, $y_i y_n = 0$ for all i, we have $f'(h')y_n = 0$ so that the previously established relations will prevail. By induction we can therefore perturb all the elements $x_1, \ldots, x_n$ inside A_∞ to satisfy the desired relations exactly. ∎

REMARK 2.3. The proposition above, coupled with Lemma 2.1, may be regarded as a clever tool that reduces higher nilpotents to sums of quadratic nilpotent elements. There may, however, be compelling reasons for this: squares may be all we can handle. Although not explicitly stated, it is easy to see — given the techniques in [9] — that the universal C^*-algebra of contractions satisfying the relations in Proposition 2.2 is projective. By contrast, it remains an open question whether the universal C^*-algebra generated by a nilpotent contraction (of order $n > 2$) is projective.

LEMMA 2.4. (cf. [5], Theorem 2) *If $e_1, \ldots, e_n$ are pairwise orthogonal idempotents in a unital C^*-algebra (i.e. $e_j e_k = \delta_{jk} e_j$ for all j and k) with $\sum_{j=1}^{n} e_j = 1$, there is an invertible element s in A such that the elements $p_j = s e_j s^{-1}$ form a set of pairwise orthogonal projections with sum 1. Moreover,*

$$\|s\|\,\|s^{-1}\| \leqslant \left(n \prod_{j=1}^{n-1} (1 + \|e_j\|) \right)^{\frac{1}{2}}.$$

Proof. Put $s^2 = \sum e_j^* e_j$, so that $s \in A_+$. Then

$$1 = \left(\sum e_j\right)^* \left(\sum e_j\right) \leqslant n \sum e_j^* e_j = ns^2,$$

so that s is invertible, with

$$\|s\|\,\|s^{-1}\| \leqslant \left(n \sum \|e_j\|^2\right)^{\frac{1}{2}}.$$

Since $s^2 e_j = e_j^* e_j = e_j^* s^2$, it follows that $p_j = s e_j s^{-1}$ is self-adjoint, and hence a projection, and evidently $\sum p_j = 1$. ∎

PROPOSITION 2.5. (cf. [5], Theorem 2) *If x is an algebraic element in a unital C^*-algebra A, there is a similar element $y = sxs^{-1}$ and pairwise orthogonal projections $p_1, \dots, p_n$ in A with sum 1, commuting with y, such that*

$$(y - \lambda_k)^{m_k} p_k = 0$$

for all k, where $\lambda_1, \dots, \lambda_n$ are the distinct roots in the minimal polynomial for x with multiplicities $m_1, \dots, m_n$.

Proof. We have $f(x) = 0$, where $f(\lambda) = \prod(\lambda - \lambda_j)^{m_j}$. Set

$$r_j(\lambda) = \left(\prod_{k\neq j}(\lambda_j - \lambda_k)^{m_k}\right)^{-1}\left(\prod_{k\neq j}(\lambda - \lambda_k)^{m_k}\right)$$

and put $a_j = r_j(x)$, $1 \leqslant j \leqslant n$. Then, by construction, $a_1, \dots, a_n$ are pairwise orthogonal elements in A, commuting with x, such that

$$(x - \lambda_k)^{m_k} a_k = 0$$

for every k. The polynomial $g(\lambda) = 1 - \sum r_j(\lambda)$ has (at least) the roots $\lambda_1, \dots, \lambda_n$. Therefore the element $g(x)$ is nilpotent and, in particular, the element $b = \sum a_k$ is invertible. Thus, if we set $e_j = a_j b^{-1}$ for $1 \leqslant j \leqslant n$, we have pairwise orthogonal elements with sum 1. But then the e_j's must be idempotent.

By Lemma 2.4 there is an invertible element s in A, such that, with $p_j = se_js^{-1}$, we obtain an orthogonal family of projections with sum 1. Take $y = sxs^{-1}$ and note that y commutes with the p_j's. Moreover,

$$(y - \lambda_k)^{m_k} p_k = s(x - \lambda_k)^{m_k} a_k b^{-1} s^{-1} = 0$$

for every k. ∎

THEOREM 2.6. *If p is a complex polynomial in one variable and x is an element of A satisfying $p(x) = 0$, then x can be approximated by elements x_∞ in A_∞ for which $p(x_\infty) = 0$.*

Proof. If A is not unital, then necessarily one of the roots of p is zero. If $x_n \to x$ with $x_n \in \tilde{A}$, and $p(x_n) = 0$ then $p(\varepsilon(x_n)) = 0$ if $\varepsilon : \tilde{A} \to \mathbb{C}$ is the usual map. Since the roots of p are isolated, after a finite number of terms $\varepsilon(x_n) = 0$ so $x_n \in A$. Therefore, the non-unital case reduces to the unital case.

Assume then that A is unital and $p(x) = 0$ with $p(\lambda) = \prod_{k=1}^{n}(\lambda - \lambda_k)^{m_k}$. By applying Proposition 2.5 we obtain $y = sxs^{-1}$ such that

$$(y - \lambda_k)^{m_k} p_k = 0$$

for all k, where $p_1, \ldots, p_n$ is an orthogonal family of projections in A, commuting with y and having sum 1.

If a is a self-adjoint element of A_∞ such that $a \approx p_1$, then $q_1 = f(a)$ is a projection in A_∞ close to a so long as we choose f as a smooth function with $f \equiv 0$ on a neighborhood of 0 and $f \equiv 1$ on a neighborhood of 1. We now proceed to approximate p_2 by elements from $(1-q_1)A_\infty(1-q_1)$ and by induction we see that for every $\varepsilon > 0$ there are orthogonal projections $q_1, \ldots, q_n$ in A_∞ with sum 1 such that $\|p_k - q_k\| < \varepsilon$. A more general version of this problem will reappear in Lemma 3.2.

For each k the element $y_k = q_k y q_k$ in $q_k A q_k$ satisfies $(y_k - \lambda_k)^{m_k} \approx 0$. Moreover, $\sum y_k \approx y$. Combining Lemma 2.1 with Proposition 2.2, and working inside $q_k A q_k$, we can find z_k in $q_k A_\infty q_k$ with $z_k \approx y_k$ such that $(z_k - \lambda_k)^{m_k} = 0$.

The element $z = \sum z_k$ belongs to A_∞ and approximates y. Moreover,

$$p(z)q_k = \prod (z - \lambda_k)^{m_k} q_k = \prod (z_k - \lambda_k)^{m_k} q_k = 0$$

since already the factor $(z_k - \lambda_k)^{m_k}$ equals zero. Since $\sum q_k = 1$ we conclude that $p(z) = 0$. Choose an invertible element $s_0 \approx s$ in A_∞ and put $x_\infty = s_0^{-1} z s_0$. Then $x_\infty \in A_\infty$, $x_\infty \approx s_0^{-1} y s_0 \approx x$ and evidently $p(x_\infty) = 0$. ∎

REMARK 2.7. This paper focuses on smoothing and not stability, but we wish to point out that Lemma 2.4 and Proposition 2.5 have stable versions, which appear in [5]. Hadwin uses these to show various lifting results and also to show that for any polynomial in a single variable, the relation $p(x) = 0$ is weakly stable.

Given the complete success of the theory of algebraic elements, it is natural to ask if any progress can be made for polynomials that include both x and x^*. Jumping to second-order equations we find a few that can be handled, namely $x^*x = 1$ or $xx^* = x^*x = 1$, corresponding to the well known smoothability of isometries and unitaries. However, the relation $x^*x = xx^*$ is non-smoothable (as well as non-stable) as demonstrated in [7]. Smoothability for $*$-relations may be very rare.

3. CONES OF MATRICES

The best way to study the cones CM_n and $C\mathbb{C}^n$ is in the more general setting of finite mapping telescopes for finite-dimensional algebras. Given a finite sequence of (let us restrict our attention a little) unital embeddings of finite-dimensional C^*-algebras

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n,$$

$T(A_1, A_2, \ldots, A_n)$ is defined as

$$\{f \in C_0((0,n], A_n) \mid f(t) \in A_k \text{ if } t \leqslant k\}.$$

For each such telescope, we defined in [9] a canonical set of generators and relations. We will describe specific sets of generators and relations for all the telescopes we use. However, the reader is referred to [9] for proofs and is warned that we don't always use the canonical relations or notation.

An obvious isomorphism is $T(A) \cong CA$. Another is

$$T(A_1, \ldots A_n, A_n) \cong T(A_1, \ldots, A_n).$$

Our canonical generators and relations for these two C^*-algebras differ, however, giving us a key tool for proving results by induction on the number of algebras forming the telescope.

The relations corresponding to $C\mathbb{C}^n$ are easy to imagine; the generators are n orthogonal positive contractions. The next two lemmas will, in particular, show these relations to be smoothable.

LEMMA 3.1. *If $k \in A_\infty$ and $0 \leqslant k$ then $\overline{kAk} \cap A_\infty$ is a real C^∞-subalgebra of $\overline{kAk}$.*

Proof. Clearly $A_\infty \cap \overline{kAk}$ is a $*$-subalgebra closed under the requisite functional calculus. Every element in $\overline{kAk}$ is close to an element of the form kak for some a in A. Since $a \approx \bar{a}$ for some $\bar{a}$ in A_∞, we have established that

$$kak \approx k\bar{a}k \in \overline{kAk} \cap A_\infty. \quad \blacksquare$$

In the next lemma, given $h_1, \ldots, h_n$, the needed element k in A_∞ can always be manufactured, so in particular the smoothability of n orthogonal positive contractions is established.

LEMMA 3.2. *For every $\varepsilon > 0$, there exists, independent of A, a positive number δ such that: if $h_1, \ldots, h_n$ are elements of A and k is a positive element of A_∞ such that*

$$0 \leqslant h_i \leqslant 1 \qquad (i = 1, \ldots, n),$$

$$h_i h_j = 0 \quad (i \neq j), \qquad \left\| \sum h_j - k \right\| \leqslant \delta$$

then there exist $\bar{h}_1, \ldots, \bar{h}_n$ in $A_\infty \cap \overline{kAk}$ such that $\|\bar{h}_i - h_i\| \leqslant \varepsilon$ and

$$0 \leqslant \bar{h}_i \leqslant 1 \quad (i = 1, \ldots, n), \quad \bar{h}_i \bar{h}_j = 0 \quad (i \neq j).$$

Proof. The simplest interesting case of the lemma is $n = 2$. In this case, first choose an element z of A_∞ with $-1 \leqslant z \leqslant 1$ and $z \approx h_2^{1/3} - h_1^{1/3}$. Let f denote a smooth, positive function on $\mathbb{R}$, with $0 \leqslant f \leqslant 1$, $f(0) = 0$, and, for $0 \leqslant t \leqslant 1$, $f(t) \approx t^{1/3}$. Define

$$b = f(k) z f(k) \in A_\infty \cap \overline{kAk},$$

so that $-1 \leqslant b \leqslant 1$ and $b \approx h_2 - h_1$. Moreover $b_+ \approx b_2$ and $b_- \approx h_1$. Therefore, we may take

$$\bar{h}_i = f_i(b) \in A_\infty \cap \overline{kAk}$$

for f_1 and f_2 appropriate smooth functions chosen with

$$f_1 f_2 = 0, \quad f_1(t) \approx -t \vee 0 \quad \text{and} \quad f_2(t) \approx t \vee 0.$$

Now assume, for $n \leqslant 3$, we are given such $k, h_1, \ldots, h_n$. Apply the above case to find orthogonal, positive contractions $\bar{k}$ and $\bar{h}_2$ in $A_\infty \cap \overline{kAk}$ such that

$$\bar{k} \approx \sum_{j=1}^{n-1} h_j \quad \text{and} \quad \bar{h}_n \approx h_n.$$

By induction, and Lemma 3.1, we can find $\bar{h}_1, \ldots, \bar{h}_{n-1}$ in $A_\infty \cap \overline{\bar{k}A\bar{k}}$ such that $0 \leqslant \bar{h}_j \leqslant 1$ and $\bar{h}_j \bar{h}_k = 0$ for $i \neq j$ less than n. Since $\bar{k} \in \overline{kAk}$, we have $\bar{h}_j \in \overline{kAk}$ for all j. Since $\bar{k}\bar{h}_n = 0$, we also have $\bar{h}_j \bar{h}_n = 0$ for $j < n$. ∎

The next lemma is an improvement on the fact that $x^2 = 0$ is a smoothable relation. Roughly speaking, it says that the smoothing can proceed in stages.

LEMMA 3.3. *Suppose h_1, h_2 are elements of A_∞ and x is an element of A such that*

$$0 \leqslant h_1, h_2 \leqslant 1, \quad \|x\| \leqslant 1, \quad h_1 h_2 = 0,$$

$$|x| \approx h_1 \quad \text{and} \quad |x^*| \approx h_2.$$

Then there exists $\bar{x} \approx x$ in A_∞ with

$$\bar{x}^2 = 0, \quad \|x\| \leqslant 1, \quad |\bar{x}| \in \overline{h_1 A h_1} \quad \text{and} \quad |\bar{x}^*| \in \overline{h_2 A h_2}.$$

Proof. First factor x as

$$x = |x^*|^{\frac{1}{3}} y |x|^{\frac{1}{3}}$$

for some y in A of norm at most one. (Since by polar decomposition $x = |x^*|^{1/3} v |x|^{2/3}$, we can take $y = v|x|^{1/3}$.) Pick a smooth function f with $f(0) = 0$,

$$0 \leqslant f(t) \leqslant 1 \quad \text{and} \quad f(t) \approx t^{\frac{1}{3}}.$$

Let $\bar{y}$ be any contraction in A_∞ that is close to y. The required approximation in A_∞ to x is

$$\bar{x} = f(h_2) \bar{y} f(h_1).$$

Indeed,

$$\bar{x} \approx h_2^{\frac{1}{3}} \bar{y} h_1^{\frac{1}{3}} \approx |x^*|^{\frac{1}{3}} y |x|^{\frac{1}{3}} = x;$$

clearly $\|\bar{x}\| \leqslant 1$ and

$$\bar{x}^* \bar{x} \in \overline{f(h_1) A f(h_1)} \subseteq \overline{h_1 A h_1},$$
$$\bar{x} \bar{x}^* \in \overline{f(h_2) A f(h_2)} \subseteq \overline{h_2 A h_2}. \quad \blacksquare$$

THEOREM 3.4. *For each natural number n, any of the following three sets of relations is smoothable:*

(3.4.1) *Generators: $x_1, \ldots, x_n, h_1, \ldots, h_n$. Relations: $\|x_j\| \leqslant 1$ and $0 \leqslant h_j \leqslant 1$ for $j = 1, \ldots, n$ and*

$$x_1^2 = x_1 h_1 = h_1 x_1 = 0,$$
$$|x_j|\,|x_{j+1}| = |x_{j+1}|,$$
$$h_j |x_{j+1}^*| = |x_{j+1}^*|,$$
$$h_j h_{j+1} = h_{j+1} \qquad (j = 1, \ldots, n-1),$$
$$h_j x_j = 0 \qquad (j = 1, \ldots, n).$$

This is a presentation for $T(M_2 \oplus \mathbb{C}, \ldots, M_{n+1} \oplus \mathbb{C})$ *with embeddings*

$$a \oplus \beta \mapsto \begin{bmatrix} a & 0 \\ 0 & \beta \end{bmatrix} \oplus \beta.$$

(3.4.2) *Generators:* $x_1, \ldots, x_n, h_1, \ldots, h_n, k, l$. *Relations:* $\|x_j\| \leqslant 1$ *and* $0 \leqslant h_j \leqslant 1$, *for* $j = 1, \ldots, n$, *plus* $0 \leqslant k \leqslant 1$, $0 \leqslant l \leqslant 1$ *and*

$$\begin{aligned} &x_1^2 = 0, \\ &x_1 h_1 = h_1 x_1 = 0, \\ &|x_j|\,|x_{j+1}| = |x_{j+1}|, \\ &h_j |x_{j+1}^*| = |x_{j+1}^*|, \\ &h_j h_{j+1} = h_{j+1} \qquad (j = 1, \ldots, n-1), \\ &h_j x_j = 0 \qquad (j = 1, \ldots, n), \\ &|x_n| k = k, \\ &h_n l = l. \end{aligned}$$

This is a presentation for $T(M_2 \oplus \mathbb{C}, \ldots, M_{n+1} \oplus \mathbb{C}, M_{n+1} \oplus \mathbb{C})$.

(3.4.3) *Generators:* $x_1, \ldots, x_n, h_1, \ldots, h_n, k, l_1, l_2$. *Relations:* $\|x_j\| \leqslant 1$ *and* $0 \leqslant h_j \leqslant 1$ *for* $j = 1, \ldots, n$ *plus* $0 \leqslant k \leqslant 1$, $0 \leqslant l_j \leqslant 1$ *and*

$$\begin{aligned} &x_1^2 = 0, \\ &x_1 h_1 = h_1 x_1 = 0, \\ &|x_j||x_{j+1}| = |x_{j+1}|, \\ &h_j |x_{j+1}^*| = |x_{j+1}^*|, \\ &h_j h_{j+1} = h_{j+1}, \qquad (j = 1, \ldots, n-1), \\ &h_j x_j = 0, \qquad (j = 1, \ldots, n), \\ &|x_n| k = k, \\ &h_n l_j = l_j, \\ &l_1 l_2 = 0. \end{aligned}$$

This is a presentation for $T(M_2 \oplus \mathbb{C}, \ldots, M_{n+1} \oplus \mathbb{C}, M_{n+1} \oplus \mathbb{C} \oplus \mathbb{C})$.

Proof. The proof is a simultaneous induction. The base case is (3.4.1) with $n = 1$. We then will show, for a fixed n, that (3.4.1) implies (3.4.2), that (3.4.2) implies (3.4.3), and (3.4.3) implies (3.4.1) with n increased by one.

In the base case, the generators are just x and h with relations

$$\|x\| \leqslant 1, \quad 0 \leqslant h \leqslant 1,$$

$$x^2 = 0, \quad xh = hx = 0.$$

Apply Lemma 3.2 to obtain three smooth orthogonal contractions $k_1, k_2, \bar{h}$ with $k_1 \approx |x|$, $k_2 \approx |x^*|$ and $\bar{h} \approx h$. Lemma 3.3 now provides a smooth contraction $\bar{x} \approx x$ with $|\bar{x}| \in \overline{h_1 A h_1}$ and $|\bar{x}^*| \in \overline{h_2 A h_2}$. The orthogonality of h_1 and h_2 with $\bar{h}$ means that these conditions force $\bar{x}\bar{h} = \bar{h}\bar{x} = 0$.

That (3.4.1) implies (3.4.2) is essentially the fact that there is an isomorphism

$$T(M_2 \oplus \mathbb{C}, \ldots, M_{n+1} \oplus \mathbb{C}, M_{n+1} \oplus \mathbb{C}) \to T(M_2 \oplus \mathbb{C}, \ldots, M_{n+1} \oplus \mathbb{C}),$$

namely

$$\varphi(f)(t) = \begin{cases} f(t), & t \in [0, n-1] \\ f(2t - n - 1) & t \in [n-1, n+1]. \end{cases}$$

We must, however, take advantage of the flexibility that exists in associating the abstract generators with concrete algebra elements. On the righthand side, we choose the generators as was done in [9]:

$$x_j = f_j \otimes (e_{j+1,1} \oplus 0), \quad h_j = f_j \otimes (0 \oplus 1),$$

where $f_j : [0, n] \to [0, 1]$ is the obvious piecewise linear function that is zero on $[0, j-1]$ and one on $[j, n]$.

On the lefthand side, we choose $\tilde{x}_j$ and $\tilde{h}_j$ in the same way for $j < n$, (except to extend f_j as 1 on $[n, n+1]$) while we choose (using the tilde ~ to avoid a notational collision).

$$\tilde{x}_n = g_n \otimes (e_{j+1,1} \oplus 0), \quad \tilde{h}_n = g_n \otimes (0 \oplus 1),$$

$$\tilde{k} = g_{n+1} \otimes (e_{1,1} \oplus 0), \quad \tilde{l} = g_{n+1} \otimes (0 \oplus 1),$$

where the g_j are some smooth functions on $[0, n+1]$ that equal zero on $[0, j-1]$, equal one on $[j, n+1]$ and are monotonically increasing on $[j-1, j]$. We can choose these so that there are smooth function η_1 and η_2 on $[0, 1]$ such that

$$f_n(t)\eta_1(f_n(t)^2) = g_n(2t - n - 1)$$

and

$$\eta_2(f_{n+1}(t)^2) = g_{n+1}(t)$$

for $t \in [n-1, n]$. In this case,

$$\varphi(\tilde{x}_n) = x_n \eta_1(x_n^* x_n), \quad \varphi(\tilde{h}_n) = h_n \eta(h_n^* h_n)$$

and

$$\varphi(\tilde{k}) = \eta_2(x_n^* x_n), \quad \varphi(\tilde{l}) = \eta_2(h_n^* h_n)$$

and so φ transfers any smoothing result for $x_1, h_1, \dots$ back to $\tilde{x}_1, \tilde{h}_1, \dots$.

Now suppose we are given x_j, h_j $(j = 1, \dots, n)$, k, l_1, l_2 in A satisfying the relations in (3.4.3). Using (3.4.2) we can find smooth elements

$$\bar{l} \approx l_1 + l_2, \quad \bar{x}_j \approx x_j, \quad \bar{h}_j \approx h_j \quad \text{and} \quad \bar{k} \approx k,$$

satisfying all the relations in (3.4.2). By Lemma 3.2 there exist $\bar{l}_1, \bar{l}_2$ in $\overline{\bar{l}A\bar{l}} \cap A_\infty$ with $0 \leqslant \bar{l}_1, \bar{l}_2 \leqslant 1$ and $\bar{l}_1 \bar{l}_2 = 0$. All the relations in (3.4.3) have been established for $\bar{x}_j, \bar{h}_j, \bar{k}, \bar{l}_1, \bar{l}_2$ except $\bar{h}_n \bar{l}_j = \bar{l}_j$, which follows from the fact that $\bar{h}_n \bar{l} = 0$.

Now suppose (3.4.3) is established for some fixed n and that we are given $x_1, \dots, x_{n+1}$ and $h_1, \dots, h_{n+1}$ in A satisfying the relations in (3.4.1) (with n replaced by $n+1$). Then, by assumption, there exists $\bar{x}_1, \dots, \bar{x}_n$, $\bar{h}_1, \dots, \bar{h}_n$, $\bar{k}, \bar{l}_1, \bar{l}_2$ in A_∞ with

$$\bar{k} \approx |x_{n+1}|, \quad \bar{l}_1 \approx |x_{n+1}^*|, \quad \bar{l}_2 \approx h_{n+1},$$
$$\bar{x}_j \approx x_j, \quad \bar{h}_j \approx h_j \quad (j = 1, \dots, n),$$

that satisfy the relations in (3.4.3). Lemma 3.3 now applies to give a smooth contraction $\bar{x}_{n+1}$ approximating x_{n+1} such that $|x_{n+1}| \in \overline{\bar{k}A\bar{k}}$ and $|x_{n+1}^*| \in \overline{\bar{l}_1 A \bar{l}_1}$. To specify the last perturbed generator let $\bar{h}_{n+1} = \bar{l}_2$. The relations $|\bar{x}_n|\bar{k} = \bar{k}$ and $\bar{h}_n \bar{l}_j = \bar{l}_j$ imply that

$$|\bar{x}_n||\bar{x}_{n+1}| = |\bar{x}_{n+1}|, \quad \bar{h}_n |\bar{x}_{n+1}^*| = |\bar{x}_{n+1}^*| \quad \text{and} \quad \bar{h}_n \bar{h}_{n+1} = \bar{h}_{n+1},$$

and the relation $\bar{l}_1 \bar{l}_2 = 0$ implies $\bar{h}_{n+1} \bar{x}_{n+1} = 0$. ∎

COROLLARY 3.5. *The relations*

$$\begin{aligned} &0 \leqslant k \leqslant 1, \\ &\|a_j\| \leqslant 1, \quad (j = 2, \dots, n), \\ &a_i a_j = 0, \quad (i = 2, \dots, n), \\ &a_i^* a_j = 0, \quad (i \neq j), \\ &a_i^* a_i = k^2, \quad (i = 2, \dots, n), \end{aligned}$$

are smoothable. Consequently, the $*$-homomorphisms in $\hom(CM_n, A)$ *that map the smooth functions in CM_n into A_∞ are dense.*

Proof. Consider the inclusion

$$\varphi : CM_n \to T(M_2 \oplus \mathbb{C}, \dots, M_{n+1} \oplus \mathbb{C})$$

defined by

$$\varphi(f)(t) = \begin{cases} 0 & t \in [0, n-2] \\ f((t-n+2)^4) \oplus 0 & t \in [n-2, n-1] \\ f(1) \oplus 0 & t \in [n-1, n]. \end{cases}$$

In terms of generators, this is defined by

$$a_j \mapsto x_{j-1}(x_n^* x_n)^2, \quad k \mapsto x_n^* x_n.$$

For any small δ, consider also the map

$$\psi_\delta : T(M_2 \oplus \mathbb{C}, \ldots, M_{n+1} \oplus \mathbb{C}) \to CM_n$$

defined by

$$\psi_\delta(f)(t) = \begin{cases} f\left(\frac{(n-2)t}{\delta^4}\right) & t \in [0, \delta^4] \\ \pi_1\left(f\left(n-2+\frac{t^{1/4}-\delta}{1-\delta}\right)\right) & t \in [\delta^4, 1]. \end{cases}$$

Here $\pi_1(b \oplus \beta) = b$ and, for $k < n$, we identify $b \oplus \beta \in M_k \oplus \mathbb{C}$ with $\begin{bmatrix} b & 0 \\ 0 & \beta I_{n-k} \end{bmatrix}$. One may check that $\psi_\delta \circ \varphi(f)$ converges to f as $\delta \to 0$. It follows that, given a representation $a_2, \ldots, a_n$ in A of the relations for CM_n, we may, after making a small perturbation to another representation, assume that

$$a_j = x_{j-1}(x_n^* x_n)^2 \quad \text{and} \quad k = x_n^* x_n$$

for some representation $x_1, \ldots, x_n, h_1, \ldots, h_n$ of the relations in (3.4.1). Smoothing these, we are done. ∎

REMARK 3.6. The methods above, coupled with the full machinery from [9], can also prove that the canonical relations for a mapping telescope of finitely many finite-dimensional inclusions are smoothable.

We are unable to prove good closure properties for smoothable relations, but can for smoothly stable relations. (Recall that smooth stability is equivalent to smoothability plus weak stability.)

THEOREM 3.7. *Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are sets of relations that are smoothly stable. Let the variables for $\mathcal{R}_i$ be denoted G_i and assume that $G_1 \cap G_2 = \emptyset$. Then the set of relations*

$$\mathcal{R}_1 \cup \mathcal{R}_2 \cup \{xy = yx = 0 \text{ if } x \in G_1 \text{ and } y \in G_2\}$$

is smoothly stable.

Proof. Of course, if $\mathcal{R}$ denotes this larger set of relations then

$$C^*\langle G_1 \cup G_2 \mid \mathcal{R}\rangle \cong C^*\langle G_1 \mid \mathcal{R}_1\rangle \oplus C^*\langle G_2 \mid \mathcal{R}_2\rangle.$$

In [6] it is established that weakly stable relations are closed under such direct sums, so the only question is smoothing.

Suppose G_i is now a subset of A satisfying $\mathcal{R}_i$. Consider the two orthogonal positive elements

$$h_i = \sum_{x \in G_i} x^*x + xx^*.$$

Applying Lemma 3.2 to h_1 and h_2 (technically, applying that lemma to the elements $\|h_i\|^{-1}h_i$) we can approximate h_1 and h_2 by smooth elements $\bar{h}_1$ and $\bar{h}_2$ that are positive and orthogonal.

Suppose $x \in G_i$. By the two-sided factorization in [6], Lemma 5.3, since $x^*x \leqslant h_i$ and $xx^* \leqslant h_i$, we can write $x = h_i^{1/3} y h_i^{1/3}$ for some y in A with $\|y\| \leqslant \|h_i\|^{1/3}$. We can define $\bar{x} = \bar{h}_i^{1/3} y \bar{h}_i^{1/3}$, which is in $\overline{\bar{h}_i A \bar{h}_i}$ and $\|\bar{x} - x\|$ is bounded by a constant that only depends on $\|\bar{h}_i - h_i\|$. After doing this for all $x \in G_i$, we can apply weak stability to the $\bar{x}$'s and find a representation $\tilde{G}_i$ of $\mathcal{R}_i$ in $\overline{\bar{h}_i A \bar{h}_i}$ that is arbitrarily close to G_i. Applying smoothability, we can find nearby $\widehat{G}_i \subseteq A_\infty \cap \overline{\bar{h}_i A \bar{h}_i}$ that form a representation of $\mathcal{R}_i$. Since these smoothings are in orthogonal subalgebras, $\widehat{G}_i \cup \widehat{G}_2$ is a representation of $\mathcal{R}$. ∎

THEOREM 3.8. *Suppose $\mathcal{R}$ is a set of relations in the variables G that is smoothly stable. Let $a_2, \ldots, a_n$ denote additional variables, and let*

$$c = \left\| \sum_{x \in G} x^*x + xx^* \right\|,$$

the norm being taken in $C^*\langle G \mid \mathcal{R}\rangle$. The set $\mathcal{R}_n$ consisting of all the relations of $\mathcal{R}$ plus the relations

$$\begin{aligned} a_i a_j &= 0 \quad (i = 2, \ldots, n), \\ a_i^* a_j &= 0 \quad (i \neq j), \\ a_i^* a_i &= k^2 \quad (i = 1, \ldots, n), \\ c^{-1} \sum_{x \in G} x^*x + xx^* &= k, \end{aligned}$$

is smoothly stable.

Proof. That the universal C^*-algebra for $\mathcal{R}_n$ is $M_n(C^*\langle G \mid \mathcal{R}\rangle)$ and that $\mathcal{R}_n$ is weakly stable was established in [6], so we here prove only smoothability.

Suppose that $G \subseteq A$ and $a_j \in A$, $2 \leqslant j \leqslant n$, and that these elements comprise a representation of $\mathcal{R}_n$. Let $k = a_i^* a_i$ and consider the map $\varphi : CM_n \to A$ defined by $\varphi(t \otimes e_{j1}) = a_j$. Choose a function $f_2 : [0,1] \to [0,1]$ that is close to the identity but zero on a neighborhood of the identity. Now choose a smooth function $f_1 : [0,1] \to [0,1]$ such that $f_1(0) = 0$ and $f_1 f_2 = f_2$. Lemma 3.5 provides a morphism $\bar{\varphi} : CM_n \to A$ such that $\bar{\varphi}(f_1 \otimes e_{ij}) \in A_\infty$ and such that $\bar{k} = \bar{\varphi}(f_2 \otimes e_{11})$ is sufficiently close to k as is desired.

Since any x in G can be factored as

$$x = k^{\frac{1}{3}} y k^{\frac{1}{3}}$$

with $\|y\| \leqslant c^{2/3}\|x\|$, it follows that

$$\bar{x} = \bar{k}^{\frac{1}{3}} y \bar{k}^{\frac{1}{3}}$$

is close to x. Working inside $\overline{\bar{k}A\bar{k}}$ we can apply weak stability and then smoothability to produce $\tilde{x}$ in $A_\infty \cap \overline{\bar{k}A\bar{k}}$ so that $\{\tilde{x} \mid x \in G\}$ is a representation of $\mathcal{R}$.

Let

$$\tilde{k} = c^{-1} \sum_{x \in G} \tilde{x}^* \tilde{x} + \tilde{x}\tilde{x}^*$$

and

$$\tilde{a}_j = \bar{\varphi}(f_1 \otimes e_{j1})\tilde{k},$$

which are all elements of A_∞. Since

$$\bar{k}\bar{\varphi}(f_1 \otimes e_{j1}) = \bar{\varphi}(f_2 \otimes e_{11})\bar{\varphi}(f_1 \otimes e_{j1}) = 0$$

and $\tilde{k} \in \overline{\bar{k}A\bar{k}}$,

$$\tilde{a}_i \tilde{a}_j = \bar{\varphi}(f_1 \otimes e_{i1})\tilde{k}\bar{\varphi}(f_1 \otimes e_{j1})\tilde{k} = 0.$$

For $i \neq j$,

$$\tilde{a}_i^* \tilde{a}_j = \tilde{k}\bar{\varphi}(f_1 \otimes e_{1i})\bar{\varphi}(f_1 \otimes e_{j1})\tilde{k} = 0.$$

For any i,

$$\tilde{a}_i^* \tilde{a}_i = \tilde{k}\bar{\varphi}(f_1^2 \otimes e_{11})\tilde{k} = \tilde{k}^2,$$

the last equality holding because

$$\bar{\varphi}(f_1 \otimes e_{11})\bar{k} = \bar{\varphi}(f_1 f_2 \otimes e_{11}) = \bar{\varphi}(f_2 \otimes e_{11}) = \bar{k}$$

and $\tilde{k}$ is in the hereditary subalgebra generated by $\bar{k}$. ∎

EXAMPLE 3.9. The C^*-algebra $M_n(C_0(0,1))^{\sim}$ is the universal unital C^*-algebra generated by elements $u, a_2, \ldots, a_n$ subject to the relations

$$\begin{aligned} &u^*u = uu^* = 1, \\ &a_j a_k = 0, \quad (j,k = 2,\ldots,n), \\ &a_j^* a_k = 0, \quad (j \neq k), \\ &|1-u|^2 = |a_j, \quad (j = 2,\ldots,n). \end{aligned}$$

(The norm conditions $\|u\| \leqslant 1$ and $\|a_j\| \leqslant 4$ being forced.) These relations are smoothable.

REMARK 3.10. Given this last example, it is straightforward, if lengthy, to go on to prove that the dimension-drop interval $\mathbf{I}_p$ is universally generated by smoothly-stable relations. Thus every element of $K_0(A; Z/p) \cong \hom(\mathbf{I}_p, A \otimes \mathcal{K})$ (cf. [4]) can be represented by a smooth map from $\mathbf{I}_p$ to $A \otimes M_n$, if n is large enough and A has an appropriate smooth structure.

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