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DENSE BARRELED SPACES IN HARDY SPACES

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ABSTRACT. For $1 \leqslant q \leqslant \infty$ let K^q be the set of all the noncyclic vectors for the backward shift operator acting on the Hardy space H^q. We show that a closed operator densely defined on H^q with values in a Banach space is bounded if and only if its natural domain contains K^q.

KEYWORDS: *Backward shift, noncyclic vectors, Banach-Steinhaus manifold.*

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0. INTRODUCTION

As usual, H^p $(1 \leqslant p \leqslant \infty)$ will denote the Hardy space on the unit disc $\mathbf{D}$. If $1 < p < \infty$ and $\varphi \in L^\infty$ $(= L^\infty(\partial\mathbf{D}))$, the Toeplitz operator T_φ on H^p is defined as $T_\varphi(f) = P_+(\varphi f)$ $(f \in H^p)$, where $P_+ : L^p \to H^p$ is the *analytic projection*.

The backward shift operator S^* is then defined as $S^* = T_{\overline{z}}$, where z denotes the identity function in (H^∞), and the bar means complex conjugation. By Beurling's theorem, the nontrivial invariant subspaces of S^* in H^p are exactly $H^p \cap u\overline{z}\overline{H}^p = \operatorname{Ker} T_{\overline{u}}$, where u is an inner function. We denote these spaces by K^p_u.

By a subspace of a Banach space we mean a closed subspace; a linear subspace will be called a *linear manifold*. We write $\mathcal{U}$ for the set of inner functions. For $\mathcal{V} \subset \mathcal{U}$ we put

$$K^p(\mathcal{V}) = \text{linear span of } \bigcup\{K^p_u : u \in \mathcal{V}\},$$

or simply K^p when $\mathcal{V} = \mathcal{U}$. From the equality $K^p_u = \operatorname{Ker} T_{\overline{u}}$ it is easy to see that $K^p_u + K^p_v \subset K^p_{uv}$ and that $K^p_u \cap K^p_v = K^p_{u\wedge v}$, where $u \wedge v$ denotes the maximum common divisor of u and v (except for unimportant multiplicative constants of modulus one). Therefore, when $\mathcal{V}$ is a semigroup (i.e. $uv \in \mathcal{V}$ for every u, $v \in \mathcal{V}$), we have that $K^p(\mathcal{V}) = \bigcup\{K^p_u : u \in \mathcal{V}\}$.

If $T : E \to F$ is a bounded operator between two Banach spaces, a vector $x \in E$ is called a *cyclic vector* for T if the T-invariant subspace of F generated by x is F. So, Beurling's theorem implies that the set of *noncyclic* vectors for S^* acting on H^p is precisely K^p.

The cyclic and noncyclic vectors for S^* in H^2 have been extensively studied by Douglas, Shapiro and Shields in [3]. In particular, they proved that K^2 is a set of first category. A similar proof works for $1 < p < \infty$.

If F is a Banach space and $M \subset F$ is a linear manifold, M is called a *Banach operator range* if there is a bounded operator T from some Banach space into F such that $M = R(T)$ (the range of T). The survey paper of Fillmore and Williams ([5]) is an excellent source to get acquainted with this concept. It is known that any proper Banach operator range is a set of first category ([5], p. 257). In accordance to this fact, in [3] the authors raise the question whether K^2 is or is not a Banach operator range.

The problem was solved negatively by Lotto and Sarason in [8]. Actually, they proved something stronger. We reproduce here their theorem. Let E, F be Banach spaces and $T : E \to F$ be a bounded operator. The range norm in $R(T)$ is defined as $\|Tx\|_* = \inf\{\|y\|_E : y \in E,\ y - x \in \operatorname{Ker} T\}$ $(x \in E)$.

THEOREM 0.1. (L-S) *Let M be a Banach operator range in H^2 and let N be the S^*-invariant subspace of H^2 generated by M. Then the cyclic vectors for $S^*|N$ that lie in M form a dense G_δ subset of M relative to the range norm.*

The theorem implies that every Banach operator range M contained in K^2 is contained in K^2_u for some $u \in \mathcal{U}$. So, if $M \supset K^2$, then the inclusion must be proper. Alternatively, Lotto and Sarason ask if the above inclusion forces M to be the whole space H^2. The same problem can be posed for $1 < p < \infty$. It is easy to prove that Banach operator ranges coincide with domains of closed operators whose codomains are Banach spaces ([5]). Therefore, by the closed graph theorem, Lotto and Sarason's question can be restated in terms of closed operators as follows. Suppose that $T : D \subset H^p \to F$ is a closed operator, where $1 < p < \infty$ and F is a Banach space. Does the inclusion $K^p \subset D$ imply that T is bounded? The purpose of this paper is to give an affirmative answer to this question, and to study this kind of phenomena from the more general point of view provided by the theory of topological vector spaces.

1. BANACH-STEINHAUS MANIFOLDS

Let E be a Banach space. A linear manifold $M \subset E$ will be called a *Banach-Steinhaus manifold* (for E) if for every set $\mathcal{L} \subset E^*$,

$$(1.1) \qquad \sup_{\lambda \in \mathcal{L}} |\lambda(x)| < \infty \ (\forall\, x \in M) \ \Rightarrow \sup_{\lambda \in \mathcal{L}} \|\lambda\| < \infty.$$

Although with a different name, this concept has been extensively studied in the theory of topological vector spaces. Specifically, let E be a topological vector space over the field $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$. A set $G \subset E$ is called *circled* if $\lambda G \subset G$ for all $\lambda \in \mathbf{K}$, $|\lambda| \leqslant 1$, and it is called *absorbing* if for all $x \in E$ there is $\lambda \in \mathbf{K}$ such that $x \in \lambda G$. A locally convex vector space E is called *barreled* if every set $G \subset E$ which is closed, convex, circled and absorbing (a barrel) is also a neighborhood of 0 ([9]). It is known that a normed space E is barreled if and only if it is a Banach-Steinhaus manifold for its completion. The next theorem summarizes some other well known equivalences. For two topological vector spaces E and F, we denote by $\mathcal{L}(E,F)$ the space of continuous linear functions from E into F.

THEOREM 1.1. *Let M be a linear submanifold of the Banach space E. Then the following conditions are equivalent:*

(i) *M is a Banach-Steinhaus manifold for E.*

(ii) *M is dense in E and the normed space $(M, \|\cdot\|_E)$ is barreled.*

(iii) *For every topological vector space F, every set $\mathcal{L} \subset \mathcal{L}(E,F)$ which is pointwise bounded on M is also equicontinuous.*

(iv) *For every Banach space F, every set $\mathcal{L} \subset \mathcal{L}(E,F)$ which is pointwise bounded on M is bounded.*

(v) *The only Banach operator range in E that contains M is E.*

(vi) *Let F be a Frechet space. Then every closed operator $T : (M, \|\cdot\|_E) \to F$ is continuous.*

All these equivalences and many others can be found in the book of Pérez Carreras and Bonet ([9]). Notice that every linear manifold that contains a Banach-Steinhaus manifold is also a Banach-Steinhaus manifold.

2. BEURLING'S SPACES OF CARLESON PRODUCTS

For $r < 1$, let $Q = \{z \in \overline{\mathbf{D}} : 1 - r \leqslant |z| \leqslant 1,\ e^{i\theta} \leqslant \arg z \leqslant e^{i(\theta+2\pi r)}\}$ be an angular square. We put $l(Q) = 1 - r$ for the side length of Q. A positive measure μ on $\overline{\mathbf{D}}$ is called a *Carleson measure* if there is a positive constant $C > 0$ such that

$$\mu(Q) \leqslant Cl(Q) \tag{2.1}$$

for every angular square Q. It is well known ([6], p. 63) that the above condition is equivalent to

$$\int |f|^p \, d\mu \leqslant C_p \|f\|_p^p \quad \forall f \in H^p \ (1 \leqslant p < \infty), \tag{2.2}$$

where $C_p > 0$ is a constant that depends only on p. Also, if condition (2.2) holds for some p, then it holds for every p.

We say that a sequence $\xi = \{\omega_n\} \subset \mathbf{D}$ is a Carleson sequence if $\mu_\xi = \sum(1 - |\omega_n|)\delta_{\omega_n}$ is a Carleson measure, where δ_ω is the point mass measure at ω. For μ_ξ conditions (2.1) and (2.2) read as

$$\sum_{\omega_n \in Q} (1 - |\omega_n|) \leqslant Cl(Q) \tag{2.3}$$

and

$$\sum (1 - |\omega_n|)|f(\omega_n)|^p \leqslant C_p \|f\|_p^p \quad \forall f \in H^p \ (1 \leqslant p < \infty), \tag{2.4}$$

respectively. Since μ_ξ is a finite measure then $\sum(1 - |\omega_n|) < \infty$, and consequently ξ is the zero sequence of some Blaschke product. We call such a product a *Carleson product.*

In what follows, $1 < p, q < \infty$ are conjugate numbers of each other (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). For $f \in H^p$, $g \in H^q$, $\{a_n\} \in l^p$ and $\{b_n\} \in l^q$, we write

$$\langle f, g \rangle = \int f(e^{i\theta})\overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \quad \text{and} \quad \langle \{a_n\}, \{b_n\} \rangle = \sum a_n \bar{b}_n.$$

Also, by $\|\cdot\|_p$ we mean the p-norm in either of the spaces H^p or l^p. Consider the functions $k_\omega(z) = 1/(1 - \overline{\omega} z) \in (H^\infty)$ $(\omega \in \mathbf{D})$. These functions are called *reproducing kernels,* because if $f \in H^p$ then $\langle f, k_\omega \rangle = f(\omega)$. It is well known that the linear span of $\{k_\omega : \omega \in \mathbf{D}\}$ is dense in H^p. Also, if b is a Blaschke product with zeros $\{\omega_n\}$, a straightforward calculation shows that $T_{\bar{b}} k_{\omega_n} = 0$ for all n. So, $k_{\omega_n} \in K_b^p$ for all n.

LEMMA 2.1. *Let b be a Carleson product with zeros $\{\omega_n\}$. Then for every $\{a_n\} \in l^q$,*

$$g = \sum a_n (1 - |\omega_n|)^{\frac{1}{p}} k_{\omega_n} \in K_b^q \tag{2.5}$$

and for every $f \in H^p$,

$$\langle f, g \rangle = \langle \{(1 - |\omega_n|)^{\frac{1}{p}} f(\omega_n)\}, \{a_n\} \rangle. \tag{2.6}$$

Proof. Since $k_{\omega_n} \in K_b^q$ for every n and K_b^q is closed in H^q, then the proof of (2.5) amounts to showing that $g \in H^q$. For N a positive integer, put $g_N = \sum_{1 \leqslant n \leqslant N} a_n (1 - |\omega_n|)^{1/p} k_{\omega_n}$. Then for $f \in H^p$,

$$\begin{aligned}
|\langle f, g_N \rangle| &= \left| \langle f, \sum_{1 \leqslant n \leqslant N} a_n (1 - |\omega_n|)^{\frac{1}{p}} k_{\omega_n} \rangle \right| \\
&= \left| \sum_{1 \leqslant n \leqslant N} \overline{a}_n (1 - |\omega_n|)^{\frac{1}{p}} f(\omega_n) \right| \\
&\leqslant \left[\sum_{1 \leqslant n \leqslant N} |\overline{a}_n|^q \right]^{\frac{1}{q}} \left[\sum_{1 \leqslant n \leqslant N} (1 - |\omega_n|) |f(\omega_n)|^p \right]^{\frac{1}{p}} \\
&\leqslant \|\{a_n\}_{n=1}^N\|_q \, \|\{(1 - |\omega_n|)^{\frac{1}{p}} f(\omega_n)\}\|_p \\
&\leqslant C \|\{a_n\}_{n=1}^N\|_q \, \|f\|_p,
\end{aligned} \tag{2.7}$$

where the last inequality holds by (2.4), with a constant $C > 0$ only depending on p and the sequence $\{\omega_n\}$. Therefore, $\|g_N\|_q \leqslant C \|\{a_n\}_{n=1}^N\|_q$ for all $N \geqslant 1$, from which it follows that $\{g_N\}$ is a Cauchy sequence in H^q. Hence, $g_N \to g \in H^q$ and $\|g\|_q \leqslant C \|\{a_n\}\|_q$. Besides, the first two equalities in (2.7), without the modulus bars, say that (2.6) holds if we take g_N instead of g. The assertion now follows because $g_N \to g$. ∎

Since the sum of two Carleson measures is also a Carleson measure, the set $C = \{b : b \text{ is a Carleson product}\}$ is a semigroup. Hence, $K^q(C) = \bigcup_{u \in C} K_u^q$, and the comments preceding Lemma 2.1 assure that it is dense in H^q. Even more, our next theorem says that it is a Banach-Steinhaus manifold for H^q. First we need a lemma. For an interval $I \subset \partial \mathbf{D}$, $|I|$ will denote the normalized Lebesgue measure of I.

LEMMA 2.2. *Suppose that $\mathcal{L}$ is an unbounded subset of L^p ($= L^p(\partial\mathbb{D})$). Then for every ε, $j > 0$ there is a function $f \in \mathcal{L}$, an interval $I \subset \partial\mathbb{D}$ and a positive integer n such that:*

(i) *if $r = 1 - (1/n)|I|$ then $\int_I |f(re^{i\theta})|^p \frac{d\theta}{2\pi} > j$, and*

(ii) $|I| < \varepsilon$.

Proof. For k an integer such that $1/k < \varepsilon$ consider the intervals in $\partial\mathbb{D}$,

$$I_s = [e^{\frac{i2\pi s}{k}}, e^{\frac{i2\pi(s+1)}{k}}], \quad 0 \leqslant s \leqslant k-1.$$

Since the set $\mathcal{L}$ contains functions of norm arbitrarily big, there is an $f \in \mathcal{L}$ such that

$$2kj < \|f\|_p^p = \sum_{s=0}^{k-1} \int_{I_s} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}.$$

Then there is some interval $I = I_s$ such that $2j < \int_I |f(e^{i\theta})|^p \frac{d\theta}{2\pi}$. Furthermore, for $0 < r < 1$ put $f_r(e^{i\theta}) = f(re^{i\theta})$. It is well known that $f_r \to f$ in p-norm when $r \to 1$ ([4], Theorem 2.6). So, if χ_I denotes the characteristic function of I, then $\|\chi_I f_r - \chi_I f\|_p \leqslant \|f_r - f\|_p \to 0$ when $r \to 1$. Thus $\|\chi_I f_r\|_p^p \to \|\chi_I f\|_p^p > 2j$. So, by taking $r = 1 - (1/n)|I|$ with n big enough, we can assure that $\|\chi_I f_r\|_p^p > j$, and since $|I| = 1/k < \varepsilon$, the lemma follows. ∎

THEOREM 2.3. *$K^q(C)$ is a Banach-Steinhaus manifold for H^q. That is, if $\mathcal{L} \subset H^p$ is a family such that for every $g \in K^q(C)$ there is a constant $C_g > 0$ satisfying*

$$\sup_{f \in \mathcal{L}} |\langle f, g\rangle| \leqslant C_g,$$

then

$$\sup_{f \in \mathcal{L}} \|f\|_p < \infty.$$

Proof. Let b be a Carleson product with zeros $\{\omega_n\}$. By Lemma 2.1 if $\{a_n\} \in l^q$ then $g = \sum a_n(1 - |\omega_n|)^{1/p} k_{\omega_n} \in K_b^q$. Besides, our hypothesis together with (2.6) imply that there is a constant $C_g > 0$ such that for all $f \in \mathcal{L}$,

$$\sup_{f \in \mathcal{L}} |\langle f, g\rangle| = \sup_{f \in \mathcal{L}} |\langle \{(1 - |\omega_n|)^{\frac{1}{p}} f(\omega_n)\}, \{a_n\}\rangle| \leqslant C_g.$$

Since the dual space of l^q is l^p, a direct application of the Banach-Steinhaus theorem gives that $\{\{(1 - |\omega_n|)^{1/p} f(\omega_n)\} : f \in \mathcal{L}\}$ must be a bounded set in l^p. That is,

$$\sum (1 - |\omega_n|)|f(\omega_n)|^p \leqslant C \quad \forall f \in \mathcal{L}, \tag{2.8}$$

where the constant $C > 0$ depends only on the sequence $\{\omega_n\}$. Suppose that the conclusion of the theorem does not hold, that is, suppose that $\mathcal{L}$ is unbounded. The theorem will follow if we prove that this implies the existence of a Carleson sequence $\{\omega_n\}$ and a sequence $\{f_j\} \subset \mathcal{L}$ such that

$$\sum (1 - |\omega_n|)|f_j(\omega_n)|^p \to \infty \quad \text{when } j \to \infty.$$

Fix an arbitrary α, $0 < \alpha < 1$. By a recursive argument we can construct three sequences, a sequence $\{f_j\}$ of functions in $\mathcal{L}$, of positive integers $\{n_j\}$ and of closed intervals $\{I_j\}$ in $\partial\mathbb{D}$, such that at the j-step we have

(i) if $r_j = 1 - (1/n_j)|I_j|$ then $\int_{I_j} |f_j(r_j e^{i\theta})|^p \frac{d\theta}{2\pi} > j$, and

(ii) $|I_j| < \alpha(1/n_{j-1})|I_{j-1}|$.

Indeed, for $j = 1$ condition (ii) is empty, and Lemma 2.2 assures that there are f_1, I_1 and n_1 satisfying (i). Suppose now that we have constructed the three sequences up to the $(j-1)$-step. Then we apply Lemma 2.2 with $\varepsilon = \alpha(1/n_{j-1})|I_{j-1}|$.

Next, for each j consider the partition of I_j in n_j closed intervals $J_{j,1}, \ldots, J_{j,n_j}$ of measure $|J_{j,s}| = (1/n_j)|I_j|$ for $s = 1, \ldots, n_j$. By (i),

$$\sum_{s=1}^{n_j} \int_{J_{j,s}} |f_j(r_j e^{i\theta})|^p \frac{d\theta}{2\pi} = \int_{I_j} |f_j(r_j e^{i\theta})|^p \frac{d\theta}{2\pi} > j, \tag{2.9}$$

where $r_j = 1 - |J_{j,s}|$ for all $1 \leqslant s \leqslant n_j$. Since $f_j(r_j e^{i\theta})$ is a continuous function of $e^{i\theta}$, the mean value theorem for integrals implies that for every $1 \leqslant s \leqslant n_j$ there is a point $\omega_{j,s} = r_j \exp(i\theta_{j,s})$, with $\exp(i\theta_{j,s}) \in J_{j,s}$, such that

$$\int_{J_{j,s}} |f_j(r_j e^{i\theta})|^p \frac{d\theta}{2\pi} = |f_j(\omega_{j,s})|^p |J_{j,s}| = |f_j(\omega_{j,s})|^p (1 - |\omega_{j,s}|). \tag{2.10}$$

So, by (2.9) and (2.10)

$$\sum_{s=1}^{n_j} (1 - |\omega_{j,s}|)|f_j(\omega_{j,s})|^p > j \tag{2.11}$$

for every $j \geqslant 1$. Thus, we obtain a double index sequence

$$\xi = \{\omega_{j,s} : 1 \leqslant j,\ s = 1, \ldots, n_j\}$$

such that

$$\sum_{\omega \in \xi} (1 - |\omega|)|f_j(\omega)|^p \geqslant \sum_{s=1}^{n_j} (1 - |\omega_{j,s}|)|f_j(\omega_{j,s})|^p \geqslant j.$$

We still must show that the sequence ξ is a Carleson sequence.

So, let $Q \subset \mathbb{D}$ be an angular square and let j_1 be the first integer such that there is some $\omega_{j_1,s}$ (with $1 \leqslant s \leqslant n_{j_1}$) contained in Q. Then

$$l(Q) \geqslant 1 - |\omega_{j_1,s}| = |J_{j_1,s}| = \left(\frac{1}{n_{j_1}}\right) |I_{j_1}|. \tag{2.12}$$

Since $1 - |\omega_{j_1,s}|$ is the same for $s = 1, \ldots, n_{j_1}$, elementary geometric considerations show that

$$\sum_{\omega_{j_1,s} \in Q} (1 - |\omega_{j_1,s}|) \leqslant 3l(Q). \tag{2.13}$$

The remaining contribution of the sequence ξ to the sum $\sum_{\omega_{j,s} \in Q} (1 - |\omega_{j,s}|)$ necessarily comes from points $\omega_{j,s}$ where $j > j_1$. By (ii) and (2.12),

$$\begin{aligned} \sum_{j_1 < j} \sum_{1 \leqslant s \leqslant n_j} (1 - |\omega_{j,s}|) &= \sum_{j_1 < j} \sum_{1 \leqslant s \leqslant n_j} |J_{j,s}| = \sum_{j=j_1+1}^{\infty} |I_j| \\ &< \sum_{j=j_1+1}^{\infty} \alpha^{j-j_1-1} |I_{j_1+1}| = |I_{j_1+1}| \sum_{n=0}^{\infty} \alpha^n \\ &< \frac{1}{1-\alpha} \left(\frac{\alpha}{n_{j_1}}\right) |I_{j_1}| \leqslant \frac{\alpha}{1-\alpha} l(Q). \end{aligned}$$

This estimation together with (2.13) gives

$$\sum_{\omega_{j,s} \in Q} (1 - |\omega_{j,s}|) \leqslant \left[3 + \frac{\alpha}{1-\alpha}\right] l(Q),$$

which proves the theorem. ∎

By a modification in the proof of Theorem 2.3 we obtain a corollary that will be a good source of examples. If $G \subset \mathbb{D}$, then

$$C_G = \{b : b \text{ is a Carleson product with zeros in } G\}$$

is a semigroup.

COROLLARY 2.4. *Let $G \subset \mathbb{D}$ be a dense set. Then $K^q(C_G)$ is a Banach-Steinhaus manifold for H^q.*

Proof. We need to show that the points $\{\omega_{j,s}\}$ in the proof of Theorem 2.3 can be chosen in the set G. To do so, first assume that the points $\omega_{j,s}$ are already chosen and take $z_{j,s} \in G$ close enough to $\omega_{j,s}$ so that inequality (2.11) holds for the points $z_{j,s} \in G$. This is possible because of the density of G and the continuity of f_j on $\mathbf{D}$.

Fix a number $0 < \lambda < 1$. It is clear that we can choose the points $z_{j,s}$ with the extra restriction

$$\rho(z_{j,s}, \omega_{j,s}) = \left| \frac{z_{j,s} - \omega_{j,s}}{1 - \overline{\omega}_{j,s} z_{j,s}} \right| < \lambda$$

(ρ is called the *pseudohyperbolic metric* in $\mathbf{D}$). Since $\{\omega_{j,s} : 1 \leqslant j,\ 1 \leqslant s \leqslant n_j\}$ is a Carleson sequence, by [6], VII, Exercises 6 and 7, so is $\{z_{j,s} : 1 \leqslant j,\ 1 \leqslant s \leqslant n_j\}$, and we are done. ∎

Corollary 2.4 shows that H^q has several Banach-Steinhaus manifolds of the kind considered here with trivial intersection. In fact, if $G, F \subset \mathbf{D}$ are dense and disjoint, then $K^q(\mathcal{C}_F)$ and $K^q(\mathcal{C}_G)$ are Banach-Steinhaus manifolds for H^q. Besides, if $f \in K^q(\mathcal{C}_F) \cap K^q(\mathcal{C}_G)$ then there are two Carleson products b_G and b_F with zeros in G and F, respectively, such that $f \in K^q_{b_G} \cap K^q_{b_F} = K^q_{b_G \wedge b_F} = \{0\}$. Similarly, we see that the intersection of a decreasing sequence of these Banach-Steinhaus manifolds can be trivial. Take a denumerable set $G = \{z_n : n = 1, \ldots\}$ that is dense in $\mathbf{D}$, and for a positive integer N, put $G_N = \{z_n : n \geqslant N\}$. Then $K^q(\mathcal{C}_{G_{N+1}}) \subset K^q(\mathcal{C}_{G_N})$ and $\bigcap_N K^q(\mathcal{C}_{G_N}) = \{0\}$.

Let m be the Lebesgue measure on $\partial\mathbf{D}$. In [7], Lemma 13.4, Lotto and Sarason proved that if $\sigma \in L^1(\mathrm{d}m)$, $\sigma \geqslant 0$, then $K^2(\mathcal{C})$ is contained in $L^2(\sigma\,\mathrm{d}m)$ if and only if σ is bounded. It is easy to see that a positive finite measure μ on $\partial\mathbf{D}$ is a Carleson measure if and only if $\mathrm{d}\mu = \sigma\,\mathrm{d}m$ with $\sigma \in L^\infty(\mathrm{d}m)$. Therefore, Theorems 1.1 and 2.3 allow us to generalize the above result in two directions.

Corollary 2.5. *Let μ be a positive finite measure on $\overline{\mathbf{D}}$. Then $K^q(\mathcal{C}) \subset L^q(\mathrm{d}\mu)$ for some $1 < q < \infty$ if and only if μ is a Carleson measure.*

Proof. As said before, μ is a Carleson measure if and only if $H^q \subset L^q(\mathrm{d}\mu)$. So, only the necessity needs to be proved. Since the inclusion $\iota : K^q(\mathcal{C}) \to L^q(\mathrm{d}\mu)$ is a closed operator, Theorems 2.3 and 1.1 (vi) assure that ι is bounded. Thus, the density of $K^q(\mathcal{C})$ implies that H^q is contained in $L^q(\mathrm{d}\mu)$. ∎

There are several recent results in the literature showing that some special classes of closed operators whose domains contain K^2 are bounded (see [8], [10] and [11]). All these theorems, together with the analogous versions for $1 < p < \infty$ are particular cases of Theorems 1.1 and 2.3.

Let u be an inner function and $1 < q < \infty$. The operator $P_u = 1 - T_u T_{\overline{u}}$ is a projection from H^q onto K^q_u. It is the orthogonal projection when $q = 2$. It is easy to see that if $g \in H^p$ and $h \in H^q$ then $\langle P_u g, h\rangle = \langle g, P_u h\rangle$. Let $\mathcal{V} \subset \mathcal{U}$ be a set such that $K^q(\mathcal{V})$ is a Banach-Steinhaus manifold for H^q. Suppose that F is a normed space and $Q : F \to H^p$ $(p = q/(q-1))$ is a linear transformation such that $P_u Q$ is bounded for every $u \in \mathcal{V}$. Then Q is bounded. Otherwise there is a sequence $\{f_n\} \in F$, $\|f_n\|_F = 1$ such that $g_n = Qf_n$ is unbounded. Since $K^q(\mathcal{V})$ is a Banach-Steinhaus manifold, there must be some $h \in K^q_u$ (with $u \in \mathcal{V}$) such that

$$\sup_n \|P_u g_n\|_p \, \|h\|_q \geqslant \sup_n |\langle P_u g_n, h\rangle| = \sup_n |\langle g_n, h\rangle| = \infty.$$

Therefore $P_u g_n$ is unbounded, and consequently so is $P_u Q$.

Let Q be a closed operator with domain and codomain in H^2. The compression of Q to the space K^2_u is defined by $P_u Q | K^2_u$, wherever this operator is defined. It is clear that this operator could be bounded even when $Q|K^2_u$ is not (it is enough to assume that the range of Q is in $uH^2 = H^2 \ominus K^2_u$.) Notice also that $P_u Q | K^2_u$ is bounded if and only if the operator $P_u Q P_u$ is bounded, with equal norms. ∎

PROPOSITION 2.6. *Let $\mathcal{V}$ be a semigroup of inner functions such that $K^2(\mathcal{V})$ is a Banach-Steinhaus manifold for H^2. Suppose that Q is a closed operator with domain and codomain in H^2 such that $P_u Q | K^2_u$ is bounded for every $u \in \mathcal{V}$. Then Q is bounded.*

Proof. Let $u, v \in \mathcal{V}$. Then $P_u Q P_v = P_u (P_{uv} Q P_{uv}) P_v$ is bounded, because by hypothesis the operator between brackets is bounded. For a fixed v, the comments preceding the proposition say that QP_v is bounded. Therefore Q is a closed operator whose domain contains $K^q(\mathcal{V})$. By Theorem 1.1 (vi) Q is bounded. ∎

If $\mathcal{V}$ denotes the semigroup of $\mathcal{U}$ generated by $u(z) = z$, then $K^2(\mathcal{V})$ is precisely the set of polynomials. It is easy to see that this is not a Banach-Steinhaus manifold for H^q. So, it is natural to ask what happens if we replace $u(z) = z$ by an arbitrary inner function.

In [1] Bernard and Sidney observed that if a normed space M is barreled, then M *is not* the union of an increasing sequence of closed subspaces. Otherwise $M = \bigcup M_n$ with $M_n \subset M_{n+1}$, and there is a unbounded sequence of continuous linear functionals $\{f_n\}$ on M such that $M_n \subset \operatorname{Ker} f_n$. Then $\{f_n\}$ is pointwise bounded on M, showing that M is not a Banach-Steinhaus manifold for its completion. They also showed that the word 'increasing' is required in general. This requirement can be relaxed for the linear manifolds considered here.

PROPOSITION 2.7. *Let $\mathcal{V} \subset \mathcal{U}$ be a semigroup with finite or denumerable generators. Then $K^q(\mathcal{V})$ is not a Banach-Steinhaus manifold for H^q.*

Proof. Suppose that $\mathcal{V}$ is generated by $\{u_k\}$. For m a positive integer put $v_m = \prod_{k=1}^{m} u_k^m$. Then

$$K^q(\mathcal{V}) = \bigcup\{K^q_{u_{j_1}^{s_1}\cdots u_{j_n}^{s_n}} : \text{where } n, s_1, \ldots, s_n, j_1, \ldots, j_n \text{ are positive integers}\}.$$

Let $m = \max\{s_1, \ldots, s_n, j_1, \ldots, j_n\}$. It is clear that $u = u_{j_1}^{s_1} \cdots u_{j_n}^{s_n}$ is a divisor of v_m. Thus $K_u^q \subset K_{v_m}^q$. Furthermore, since v_m is a divisor of v_{m+1} and they are all contained in $\mathcal{V}$, we have

$$K_u^q \subset K_{v_m}^q \subset K_{v_{m+1}}^q \subset K^q(\mathcal{V}).$$

Hence, $K^q(\mathcal{V}) = \bigcup_m K_{v_m}^q$ and consequently it is not a barreled space. We can show this directly by using the sequence of linear functionals $\{mv_m\} \subset H^p$. ∎

The proposition shows that $\mathcal{V}$ must be very big when $K^q(\mathcal{V})$ is a Banach-Steinhaus manifold for H^q. The author does not know how big it should be. That is, what are the semigroups $\mathcal{V} \subset \mathcal{U}$ that make $K^q(\mathcal{V})$ a Banach-Steinhaus manifold for H^q? Is it true that those semigroups do not depend on q? I am almost certain that the answer to this question is affirmative, but so far my attempts to prove it have failed.

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