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CONSTANT CHARACTERISTIC FUNCTIONS AND HOMOGENEOUS OPERATORS

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ABSTRACT. We explore the relationship between homogeneity and characteristic function for the class of completely non unitary contractive operators on a Hilbert space.

KEYWORDS: *Homogeneous operators, characteristic functions, completely non unitary contractions, projective representations.*

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0. INTRODUCTION

To a completely non unitary (cnu) contraction T on a separable Hilbert space $\mathcal{H}$, Sz.-Nagy and Foiaş (cf. [8]) associate a contraction valued holomorphic function Θ_T on the open unit disk $\mathbb{D}$ such that $\Theta_T(0)$ is a pure contraction. This is called the characteristic function of the operator T. Conversely, given any such holomorphic function Θ on $\mathbb{D}$, there is a completely nonunitary contraction T_Θ whose characterstic function coincides with Θ. The Sz.-Nagy–Foiaş theory provides an explicit construction of the model operator T_Θ for any given characteristic function Θ. Moreover, as is well known ([8], Theorem 3.4, p. 257), two of these operators S and T are unitarily equivalent if and only if the two functions Θ_S and Θ_T coincide. However, it is not easy to determine when two functions Θ and Ψ coincide. This limits the use of Θ as a complete unitary invariant. Besides, the model T_Θ is not necessarily the best possible description of a cnu contraction with characteristic function Θ. In fact, in a recent paper ([1]), the models associated with the constant characteristic functions Θ were described following the Sz.-Nagy–Foiaş construction. Even in this simple situation the model obscures the nature of the associated operators.

In the first part of this note we shall describe up to unitary equivalence all the cnu contractions which possess a constant characteristic function. In case one of the two defect indices is finite, we show that the characteristic function is constant if and only if the operator admits a direct sum decomposition such that each summand is one of the bilateral weighted shifts with weight sequence $\{\ldots, 1, \lambda, 1, \ldots\}$, $0 < \lambda < 1$, or the unilateral shift or the adjoint of the unilateral shift. In the general case, we appeal to direct integral theory and obtain a similar result. One consequence of this general result is that the characteristic function of an irreducible contraction is constant if and only if it is one of the shift operators described above. It was shown in [1] that these operators are examples of homogeneous contractions. In the second part we extend this class of examples to produce homogeneous operators which are not necessarily contractions. We also show that a cnu contraction with either one of the defect indices finite is homogeneous if and only if the characteristic function is constant. More generally, it turns out that the restriction of a homogeneous contraction to its defect space is in the Hilbert-Schmidt class if and only if its characteristic function is a constant. One striking consequence is that, except for the weighted shifts mentioned above, any other irreducible homogeneous contraction has the bilateral shift of infinite multiplicity as its minimal unitary dilation.

Along the way, we put the notion of a homogeneous operator on a rigorous footing and obtain a usable criterion for testing the homogeneity of an operator which does not require prior knowledge about the spectrum of the operator. As an application, we give an abstract nonsense construction of homogeneous operators which covers all the hitherto known examples of irreducible homogeneous operators. We also show that, if the conditions are right, one can get an entire continuum of homogeneous operators starting with a single operator in this class. Last, but not the least, we explicitly record a characterisation of homogeneity of cnu contractions in terms of their characteristic functions. This result was implicit in the proof of Theorem 2.1 in [1].

1. CONSTANT CHARACTERISTIC FUNCTIONS

All Hilbert spaces in this paper are separable and all operators are bounded linear operators between Hilbert spaces. For a Hilbert space $\mathcal{H}$, $\mathcal{U}(\mathcal{H})$ will denote the group of unitary operators on $\mathcal{H}$. Recall that $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ are the defect operators associated with a cnu contraction T. The range closures $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ of D_T and D_{T^*}, respectively, are called the defect spaces. The dimension of these subspaces are said to be the defect indices. A contraction C is said to be pure if $\|Cx\| < \|x\|$ for all non-zero vectors x. We will say that two operators $C_i : \mathcal{L}_i \to \mathcal{K}_i$, $i = 1, 2$, coincide if there exist unitary operators $\tau : \mathcal{L}_2 \to \mathcal{L}_1$ and $\tau_* : \mathcal{K}_1 \to \mathcal{K}_2$ such that $\tau_* C_1 \tau = C_2$. The operator valued functions $\Theta_i(z) : \mathcal{L}_i \to \mathcal{K}_i, i = 1, 2$, are said to coincide if there exist unitary operators $\tau : \mathcal{L}_2 \to \mathcal{L}_1$ and $\tau_* : \mathcal{K}_1 \to \mathcal{K}_2$ such that $\tau_* \Theta_1(z) \tau = \Theta_2(z)$ for all z. Note that this is stronger than merely requiring that $\Theta_1(z)$ and $\Theta_2(z)$ coincide for each z. An operator is quasi invertible if it has trivial kernel and dense range. $\mathbb{D}$ and $\mathbb{T}$ will denote the open unit disc and the unit circle, respectively.

LEMMA 1.1. *Let C be a contraction between two Hilbert spaces. Then the following are equivalent.*

(i) *C is a pure contraction;*

(ii) *C^* is a pure contraction;*

(iii) *$(I - C^*C)^{1/2}$ is quasi invertible;*

(iv) *$(I - CC^*)^{1/2}$ is quasi invertible.*

Proof. To show that (i) and (ii) are equivalent, first note that the contraction C is pure if and only if the kernel of the operator $(I - C^*C)^{1/2}$ is trivial. However, if C is quasi invertible then polar decomposition shows that $(I - CC^*)^{1/2}$ and $(I - C^*C)^{1/2}$ are unitarily equivalent, which implies the stated equivalence in this case. For the general case, write $C = \tilde{C} \oplus 0$ with $\tilde{C}$ quasi invertible, and note that C is pure if and only if $\tilde{C}$ is pure.

Clearly, if C is a pure contraction then the kernel of the self adjoint operator $(I - C^*C)^{1/2}$ is trivial and hence this operator has dense range. Thus $(I - C^*C)^{1/2}$ is quasi invertible.

The equivalence of (i) and (ii) shows that the the operator $(I - CC^*)^{1/2}$ is quasi invertible as well.

Finally, if $(I - C^*C)^{1/2}$ is quasi invertible then it is obvious that C is pure. ∎

NOTATION 1.2. Let $C : \mathcal{L} \to \mathcal{K}$ be a bounded operator from the Hilbert space $\mathcal{L}$ into the Hilbert space $\mathcal{K}$. Put

$$\mathcal{H}_n = \begin{cases} \mathcal{L} & \text{if } n \leqslant 0 \\ \mathcal{K} & \text{if } n > 0. \end{cases}$$

Let $\mathcal{H} = \bigoplus\limits_{n \in \mathbb{Z}} \mathcal{H}_n$. Define an operator $T_C : \mathcal{H} \to \mathcal{H}$ as follows :

$$T_C|\mathcal{H}_n = \begin{cases} I : \mathcal{H}_n \to \mathcal{H}_{n+1} & n \neq 0; \\ C & n = 0. \end{cases}$$

(Clearly, $\|T_C\| = \max\{\|C\|, 1\}$.) In particular, for scalars $\lambda \geqslant 0$, T_λ will denote the bilateral shift with weight sequence $\{\ldots, 1, \lambda, 1, \ldots\}$.

THEOREM 1.3. *For each pure contraction C, the operator T_C is a cnu contraction with constant characteristic function. Conversely, each cnu contraction with constant characteristic function is unitarily equivalent to T_C for some pure contraction C.*

Proof. Let $\widetilde{\mathcal{H}}$ be a reducing subspace for $T = T_C$. If T were unitary on $\widetilde{\mathcal{H}}$ then for any $x \in \widetilde{\mathcal{H}}$, we would have $\|T^n x\| = \|x\| = \|T^{*n} x\|$, $n = 1, 2, \ldots$. From the definition of the operator T_C and the fact that C is pure it follows that $x = 0$. This shows that $\widetilde{\mathcal{H}}$ is the trivial subspace. Therefore, T_C has no unitary part and hence is a cnu contraction.

It is easy to verify that the adjoint $T^* = T_C^* : \mathcal{H} \to \mathcal{H}$ is given by

$$T^*|\mathcal{H}_n = \begin{cases} I : \mathcal{H}_n \to \mathcal{H}_{n-1} & n \neq 1; \\ C^* & n = 1. \end{cases}$$

Consequently,

$$(I - T^*T)^{\frac{1}{2}}|\mathcal{H}_n = \begin{cases} 0 : \mathcal{H}_n \to \mathcal{H}_n & n \neq 0; \\ (I - C^*C)^{\frac{1}{2}} & n = 0. \end{cases}$$

Similarly,

$$(I - TT^*)^{\frac{1}{2}}|\mathcal{H}_n = \begin{cases} 0 : \mathcal{H}_n \to \mathcal{H}_n & n \neq 1; \\ (I - CC^*)^{\frac{1}{2}} & n = 1. \end{cases}$$

Since C is a pure contraction, by Lemma 1.1 both the operators $(I - C^*C)^{1/2}$ and $(I - CC^*)^{1/2}$ have dense range. It follows that

$$\mathcal{D}_T = \{x \in \mathcal{H} : x_n = 0 \quad \text{for } n \neq 0\}$$

and

$$\mathcal{D}_{T^*} = \{x \in \mathcal{H} : x_n = 0 \quad \text{for } n \neq 1\}.$$

It is now evident that $D_{T^*}(T^*)^{n-1}D_T|\mathcal{D}_T = 0$ for $n \geqslant 1$. Thus the characteristic function of T_C is given by

$$\Theta_{T_C}(z) \stackrel{\text{def}}{=} \Big[-T + \sum_{n=1}^{\infty} z^n D_{T^*}(T^*)^{n-1}D_T\Big]\Big|\mathcal{D}_T = -T_C\big|\mathcal{D}_T,$$

for all $z \in \mathbf{D}$. The fact that $-T|\mathcal{D}_T$ coincides with the operator $-C : \mathcal{L} \to \mathcal{K}$ completes the first half of the proof.

Conversely, let T be a cnu contraction with constant characteristic function $\Theta_T(z) = -C : \mathcal{L} \to \mathcal{K}$ for all $z \in \mathbf{D}$. Then $C = -\Theta_T(0)$ is a pure contraction. Furthermore, by the direct part of this theorem, the characteristic function of the operator T_C coincides with Θ_T. Therefore, T_C is unitarily equivalent to T. This completes the proof of the theorem. ∎

We shall show that the cnu contraction T_C is a direct integral of ordinary weighted shift operators.

Let $\mathcal{L}_0$ (resp. $\mathcal{K}_0$) be the kernel (resp. range closure) of C and let $\mathcal{L}_1$ (resp. $\mathcal{K}_1$) be its orthocomplement in $\mathcal{L}$ (resp. in $\mathcal{K}$). The operator C admits a 2×2 matrix representation with respect to this decomposition. Indeed, $C = \widetilde{C} \oplus \mathbf{0}$. It is clear that the operator T_C is unitarily equivalent to the direct sum $T_{\widetilde{C}} \oplus T_{\mathbf{0}}$. Let V_i be the unilateral shift or its adjoint according as $i > 0$ or $i < 0$. It is not hard to verify that the operator $T_{\mathbf{0}}$ is unitarily equivalent to $\bigoplus_{i=-m}^{n} V_i, i \neq 0$, where $m = \dim \mathcal{L}_0$ and $n = \dim \mathcal{K}_0$.

Thus it is enough to obtain a simple representation for the cnu contraction T_C with the assumption that $C : \mathcal{L} \to \mathcal{K}$ is quasi invertible. In this case, let $U : \mathcal{K} \to \mathcal{L}$ be any unitary operator. Since $CU : \mathcal{K} \to \mathcal{K}$ has dense range, it follows that the operator W in the polar decomposition $CU = WP$ is unitary. Hence the operator C coincides with the positive operator $P : \mathcal{K} \to \mathcal{K}$. The characteristic functions of the operators T_C and T_P are the constant functions $-C$ and $-P$ respectively, which coincide. Hence the cnu contraction T_C is unitarily equivalent to T_P by [8], Theorem 3.4, p. 257.

If either the dimension of $\mathcal{L}$ or that of $\mathcal{K}$ is finite then both $\mathcal{L}$ and $\mathcal{K}$ are of the same finite dimension k. In this case, the positive operator P is unitarily equivalent to a diagonal operator Λ. Another appeal to [8], Theorem 3.4, p. 257 shows that the operators T_P and T_Λ are unitarily equivalent. However, it is easy to construct a unitary operator intertwining T_P and T_Λ explicitly using the unitary implementing the equivalence of P and Λ. Let $\{\lambda_1, \ldots, \lambda_k\}$ be the eigenvalues of Λ arranged in decreasing order. Again, it is straightforward to verify that T_Λ and

$\bigoplus_{\ell=1}^{k} T_{\lambda_\ell}$ are unitarily equivalent. We point out that the operator T_{λ_ℓ} is the weighted bilateral shift with weight sequence $\{\ldots, 1, \lambda_\ell, 1 \ldots\}$. Further, the characteristic function of the operator $\bigoplus_{\ell=1}^{k} T_{\lambda_\ell}$ is constant.

Before we discuss the case where both the defect indices are possibly infinite, it is good to record our observations so far as

COROLLARY 1.4. *Let T be any cnu contraction with at least one finite defect index. The operator T has constant characteristic function if and only if it is unitarily equivalent to*

$$\Big(\bigoplus_{\substack{i=-m\\ i\neq 0}}^{n} V_i\Big) \oplus \Big(\bigoplus_{\ell=1}^{k} T_{\lambda_\ell}\Big)$$

for uniquely determined integers $m, n, k \geqslant 0$, and uniquely determined positive scalars $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k < 1$.

Now we allow the possibility that both the defect indices for the operator T_C may be infinite. If C is quasi invertible then the discussion preceeding Corollary 1.4 shows that the operator T_C is unitarily equivalent to T_P for some positive operator P. In the present situation P need not be a finite dimensional operator. However, the spectral theorem guarantees a direct integral decomposition for P. This will allow us to obtain an analogue of Corollary 1.4 in case both defect indices are infinite. First, we recall some relevant facts from the theory of direct integrals.

Let (Λ, m) be a measure space and for $\lambda \in \Lambda$, let $\mathcal{H}_\lambda$ be a non-zero separable Hilbert space. A section is a map $s : \Lambda \to \bigcup_{\lambda\in\Lambda} \mathcal{H}_\lambda$ such that $s(\lambda) \in \mathcal{H}_\lambda$. We will denote the linear space of all sections by $\mathcal{S}$. We adopt the following definition from [2].

DEFINITION 1.5. The pair $(\mathcal{H}_\lambda : \lambda \in \Lambda, \Gamma)$ is said to be a *measurable field of Hilbert spaces* if Γ is a linear subspace of $\mathcal{S}$ such that

(i) for each $s \in \Gamma$, the function $\lambda \to \|s(\lambda)\|$ is measurable;

(ii) if s_0 is in $\mathcal{S}$ and for every $s \in \Gamma$, the function $\lambda \to \langle s_0(\lambda), s(\lambda)\rangle$ is measurable then s_0 is in Γ;

(iii) there exists a sequence $s_n \in \Gamma$ such that $\{s_n(\lambda)\}$ spans $\mathcal{H}_\lambda$ for each $\lambda \in \Lambda$.

Mackey ([5], p. 91) calls such a sequence $\{s_n\}$ a pervasive sequence. It can be shown that the existence of a pervasive sequence is equivalent to the measurability of the extended integer valued function d on Λ defined by $d(\lambda) = \dim \mathcal{H}_\lambda$. The direct integral $\int_\Lambda^\oplus \mathcal{H}_\lambda \, dm$ is the obvious Hilbert space on the set of sections s in

Γ such that $\int\limits_{\Lambda} \|s(\lambda)\|^2 \, dm$ is finite (two such sections being identified if they are almost everywhere (m) equal). We refer the reader to [2] for further details.

Suppose for each $\lambda \in \Lambda$, we have an operator $T(\lambda)$ on $\mathcal{H}_\lambda$ such that

(i) The function $\lambda \to \langle T(\lambda)s_1, s_2\rangle$ is measurable for each pair of sections $s_1, s_2 \in \int\limits_{\Lambda}^{\oplus} \mathcal{H}_\lambda \, dm$;

(ii) $\operatorname{ess\,sup} \|T(\lambda)\| < \infty$.

We then define $\int\limits_{\Lambda}^{\oplus} T(\lambda)$, the direct integral of $\{T(\lambda)\}$ by the formula

$$(1.1) \qquad \Big(\int\limits_{\Lambda}^{\oplus} T(\lambda)(s)\Big)(\mu) = T(\mu)s(\mu), \quad s \in \int\limits_{\Lambda}^{\oplus} \mathcal{H}_\lambda \, dm; \quad \lambda, \mu \in \Lambda.$$

Define the multiplication operator $(Ms)(\lambda) = \lambda s(\lambda)$; $s \in \int\limits_{\Lambda}^{\oplus} \mathcal{H}_\lambda \, dm$. It is convenient to use the suggestive notation $\int\limits_{\Lambda}^{\oplus} \lambda \, dm$ for the operator M. The spectral theorem says that every normal operator is unitarily equivalent to a multiplication operator for a suitable choice of a measure m on its spectrum. (m is the so called scalar spectral measure of the given operator.) Thus we may write our positive contraction P as such a multiplication operator, that is, $P = \int\limits_{\Lambda}^{\oplus} \lambda \, dm$, where $\Lambda \subseteq [0,1]$ is the spectrum of P. It is easily seen that P is a pure contraction if and only if $m\{1\} = 0$. One may verify directly that the operators T_P and T_M are unitarily equivalent.

Let $\mathcal{G}_\lambda$ denote the direct sum of infinitely many copies of $\mathcal{H}_\lambda$. Let Γ be the linear space of sections implicit in the direct integral representation of M. Define $\widetilde{\Gamma}$ to be the linear space of all sections $s : \Lambda \to \bigcup\limits_{\lambda \in \Lambda} \mathcal{G}_\lambda$ such that $\lambda \mapsto s_i(\lambda)$ is in Γ for all i. (Here $s_i(\lambda)$ is the projection of $s(\lambda)$ into the i-th component of $\mathcal{G}_\lambda$.) The pair $(\mathcal{G}_\lambda : \lambda \in \Lambda, \widetilde{\Gamma})$ is easily seen to be a measurable field of Hilbert spaces. Let $\int\limits_{\Lambda}^{\oplus} \mathcal{G}_\lambda \, dm$ be the associated direct integral. Define the map $\eta : \bigoplus\limits_{-\infty}^{\infty} \int\limits_{\Lambda}^{\oplus} \mathcal{H}_\lambda \, dm \to \int\limits_{\Lambda}^{\oplus} \mathcal{G}_\lambda \, dm$ by

$$\eta\Big(\bigoplus_{i=-\infty}^{\infty} (\lambda \to s_i(\lambda))\Big) = \Big(\lambda \to \Big(\bigoplus_{i=-\infty}^{\infty} s_i(\lambda)\Big)\Big).$$

It is easily seen that η is unitary. A simple calculation shows that $\eta T_M \eta^* : \bigoplus\limits_{-\infty}^{\infty} \int\limits_{\Lambda}^{\oplus} \mathcal{H}_\lambda \, dm \to \int\limits_{\Lambda}^{\oplus} \mathcal{G}_\lambda \, dm$ is the operator $\int\limits_{\Lambda}^{\oplus} T_{\lambda \cdot I} \, dm$. Let $d(\lambda) = \dim \mathcal{H}_\lambda$ and let

$d(\lambda)\cdot T_\lambda = \bigoplus_1^{d(\lambda)} T_\lambda$. For each fixed λ, the operator $T_{\lambda \cdot I} : \mathcal{G}_\lambda \to \mathcal{G}_\lambda$ is unitarily equivalent to $d(\lambda)\cdot T_\lambda$. This unitary equivalence in turn effects an unitary equivalence of the operators $\int_\Lambda^\oplus T_{\lambda \cdot I}\, dm$ and $\int_\Lambda^\oplus d(\lambda)\cdot T_\lambda\, dm$.

Before we continue, let us record all the different unitary equivalences we have used so far. Let $\mathbf{V} = \bigoplus_{i=-m}^{n} V_i$, $i \neq 0$, where V_i be the unilateral shift or its adjoint according as $i > 0$ or $i < 0$. We have

$$(1.2)\ T_C \cong T_{\widetilde{C}\oplus 0} \cong T_P \oplus \mathbf{V} \cong T_M \oplus \mathbf{V} \cong \int_\Lambda^\oplus T_{\lambda \cdot I}\, dm \oplus \mathbf{V} \cong \int_\Lambda^\oplus d(\lambda)\cdot T_\lambda\, dm \oplus \mathbf{V}.$$

Note that if $(\Lambda \subseteq [0,1], m)$ is a measure space with $m(\{1\}) =$ and $d : \Lambda \to \mathbf{N}$ is any measurable function then $\int_\Lambda^\oplus d(\lambda)\cdot T_\lambda\, dm \oplus \mathbf{V}$ is a cnu contraction with constant characteristic function. Thus we have proved:

THEOREM 1.6. *The operator T is a cnu contraction with constant characteristic function if and only if*

$$T = \Big(\bigoplus_{i \in I} V_i\Big) \oplus \int_\Lambda^\oplus d(\lambda)\cdot T_\lambda\, dm$$

where m is a Borel measure with support $\Lambda \subset [0,1]$, $m(\{1\}) = 0$, I is a countable indexing set and each V_i $(i \in I)$ is the unilateral shift or its adjoint, d is a (Borel) extended integer-valued dimension function on Λ, and $d(\lambda)\cdot T_\lambda$ denotes the direct sum of $d(\lambda)$ copies of T_λ.

COROLLARY 1.7. *The only irreducible contractions with constant characteristic function are the forward shift, the backward shift and the operators T_λ, $0 < \lambda < 1$.*

Proof. If Λ is not a singleton then the operator $\int_\Lambda^\oplus d(\lambda)\cdot T_\lambda\, dm$ is reducible since any partition of Λ into two Borel sets of positive measure induces a direct sum decomposition of the direct integral into two parts. On the other hand, if $\Lambda = \{\lambda\}$, this operator is a $d(\lambda)$ — fold direct sum of the operator T_λ. So the operator $\int_\Lambda^\oplus d(\lambda)\cdot T_\lambda\, dm$ is irreducible only if $\Lambda = \{\lambda\}$ and $d(\lambda) = 1$. To complete the proof it suffices to note that the operator T_0 is not irreducible (it is the direct sum of the forward and backward shifts). ∎

2. HOMOGENEOUS OPERATORS

Let Möb($\mathbb{D}$) be the group of biholomorphic automorphisms of the unit disk $\mathbb{D}$. Möb($\mathbb{D}$) consists of the functions φ of the form $\varphi = \varphi_{\theta,a}$, where

$$\varphi_{\theta,a}(z) = \mathrm{e}^{\mathrm{i}\theta}\frac{z-a}{1-\bar{a}z}, \quad |a|<1 \quad \text{and} \quad \theta \in [0, 2\pi).$$

DEFINITION 2.1. A bounded operator T is *homogeneous* if its spectrum is contained in the closed unit disc and T is unitarily equivalent to $\varphi(T)$ for each φ in Möb($\mathbb{D}$).

LEMMA 2.2. *Let T be a bounded operator. Suppose that $\varphi(T)$ is unitarily equivalent to T for each φ in some neighbourhood of the identity in* Möb($\mathbb{D}$). *Then T is homogeneous and the spectrum $\sigma(T)$ is either the unit circle or the closed unit disk — according as T is invertible or not.*

(Note that however large the spectrum of a bounded operator T may be, each φ in a sufficiently small neighbourhood of identity has analytic continuation to a neighbourhood of the spectrum — so that $\varphi(T)$ is defined for such φ.)

Proof. We first show that $\sigma(T)$ is contained in the closed unit disc. Suppose not. Let K be the union of the spectrum with the closed unit disk. Then K is a compact set properly containing the closed unit disk. Get hold of a neighbourhood U of the identity in Möb($\mathbb{D}$) such that each element of U extends analytically to some neighbourhood of K. Then $\varphi(T)$ is well-defined for $\varphi \in U$. Replacing U by a smaller neighbourhood if necessary, we may assume that $\varphi(T)$ is unitarily equivalent to T for $\varphi \in U$. By the spectral mapping theorem, each $\varphi \in U$ maps $\sigma(T)$ into $\sigma(\varphi(T))$, but the latter is nothing but $\sigma(T)$. Thus each $\varphi \in U$ maps $\sigma(T)$ into itself and of course, it maps the closed unit disk into itself. Hence each $\varphi \in U$ maps K into itself and is analytic in some neighbourhood of K. It follows that the same is true of the subgroup generated by U. Connectedness of the group Möb($\mathbb{D}$) implies that this subgroup is the whole of Möb($\mathbb{D}$). However, there is no compact set K properly containing the closed unit disk such that each φ in Möb($\mathbb{D}$) maps K into itself. This shows that $\sigma(T)$ is contained in the closed unit disk. Thus $\varphi(T)$ is well-defined for all φ in Möb($\mathbb{D}$) and it is unitarily equivalent to T for all $\varphi \in U$. But for any operator T with the spectrum $\sigma(T)$ contained in the closed unit disc, the set of all φ for which $\varphi(T)$ is unitarily equivalent to T is a subgroup of G. Since this subgroup contains a neighbourhood of identity, it must be the whole of Möb($\mathbb{D}$) by connectedness. So, T is homogeneous. The second half of the lemma is immediate since the above argument shows that the closed set $\sigma(T) \subseteq \overline{\mathbb{D}}$ is invariant under Möb($\mathbb{D}$), while $\overline{\mathbb{D}}$ and $\partial\mathbb{D}$ are clearly the only invariant closed subsets of $\overline{\mathbb{D}}$. ∎

Recall that a *projective representation* of a standard Borel group G on a Hilbert space is a mapping π of G into the group $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, such that

(i) $\pi(e) = 1$, where e is the identity of G;

(ii) $\pi(g)\pi(h) = m(g,h)\pi(gh)$ for all $g, h \in G$, where $m(g,h)$ is in the unit circle $\mathbf{T}$;

(iii) $g \to \langle \pi(g)\zeta, \eta \rangle$ is a Borel function for each $\zeta, \eta \in \mathcal{H}$.

The function m is the multiplier associated with π and is uniquely determined by π. It has the following properties

(a) $m : G \times G \to \mathbf{T}$ is Borel;

(b) $m(g,e) = 1 = m(e,g)$, where e is the identity of the group G, $g \in G$;

(c) $m(k, gh)m(g,h) = m(k,g)m(kg,h)$, g, h, and k in G.

The set of all multipliers M on the group G is itself a group under point-wise multiplication, called the multiplier group. If there is a Borel function $f : G \to \mathbf{T}$ such that

$$m(g,h) = f(g)f(h)f(gh)^{-1},$$

then the multiplier m is said to be trivial. Note that in this case, if we set

$$\sigma(g) = f(g)^{-1}\pi(g),$$

then $g \to \sigma(g)$ is a *linear representation* of the group G, that is a strongly continuous homomorphism ([9], Lemma 5.28, p. 181).

It was shown in [7] that if T is an irreducible homogeneous operator then there exists a projective representation π : Möb($\mathbf{D}$) $\to$ $\mathcal{U}(\mathcal{H})$ such that $\pi(\varphi)^* T \pi(\varphi) = \varphi(T)$. We shall say that $\pi(g)$ is *a representation associated with the homogeneous operator* T whenever this holds — whether or not T is irreducible.

Let Ω be a standard Borel G space (cf. [9], p. 158), G being a fixed locally compact second countable group. Let V be a normed linear space and let $GL(V)$ be the group of invertible bounded linear operators on V. A Borel map $c : G \times \Omega \to GL(V)$ is said to be a $(G, \Omega, GL(V))$ cocycle ([9], p. 174), if the following two properties are satisfied :

(i) $c(e, z) = 1$ for all $z \in \Omega$;

(ii) $c(g_1 g_2, z) = m(g_1, g_2)c(g_1, g_2 z)c(g_2, z)$ for all $(g_1, g_2, z) \in G \times G \times \Omega$ where $m : G \times G \to \mathbf{T}$ is a multiplier on the group G.

Note that the above conditions are slightly different from those in [9].

Let $\mathcal{H}$ be a Hilbert space of functions on Ω with values in V. For each g in G, let $(\pi(g)f)(z) = c(g^{-1}, z) \cdot f(g^{-1}(z))$, $f \in \mathcal{H}$, for a $(G, \Omega, GL(V))$ cocycle c. If $\pi(g)$ is unitary for each $g \in G$ then the fact that $c : G \times \Omega \to GL(V)$ satisfies the cocycle identities implies that π is a projective representation of the group G on $\mathcal{H}$. It is called the multiplier representation with cocycle c.

PROPOSITION 2.3. *Let Ω denote either the unit disk or the unit circle. If π is any multiplier representation of* Möb($\mathbb{D}$) *on a Hilbert space $\mathcal{H}$ of functions defined on Ω with values in V then the multiplication operator M defined by $(Mf)(z) = zf(z)$ on $\mathcal{H}$ is homogeneous with associated representation π — provided M is bounded.*

Proof. The proof merely consists of the verification:

$$(M\pi(\varphi^{-1})f)(z) = (\pi(\varphi^{-1})\varphi(M)f)(z),$$

whenever $\varphi \in$ Möb($\mathbb{D}$) is such that $\varphi(M)$ is defined (so that $\varphi(M)$ is multiplication by φ). (In view of Lemma 2.2 and the parenthetical remark following its statement, this is sufficient for homogeneity.) But the left hand side of this equality evaluates to $z \cdot c(\varphi^{-1}, z) \cdot f(\varphi^{-1}(z))$, whereas the right hand side is $c(\varphi^{-1}, z)(\varphi \cdot f)(\varphi^{-1}(z))$. ∎

In [1], it was shown that any cnu contraction with a constant characteristic function is homogeneous. (This is also immediate from Theorem 2.9 below.) In view of Theorem 1.3 above, this means that the operator T_C is homogeneous for any pure contraction C. We show next that, even if C is just a bounded operator, T_C is homogeneous. We will verify the homogeneity of T_C in two steps. First, we will assume that $\dim \mathcal{K} = 1 = \dim \mathcal{L}$. In this case, T_C is one of the bilateral weighted shift operators T_λ, $\lambda > 0$. Next, we will appeal to direct integral theory to settle the general case.

The fact that T_λ is homogeneous follows from the following general proposition which may be of some independent interest.

PROPOSITION 2.4. *Let T be a homogeneous operator on a Hilbert space $\mathcal{H}$ and suppose that π is a representation of the group* Möb($\mathbb{D}$) *on $\mathcal{H}$ which is associated with T. Let $\mathcal{M}$ be a reducing subspace for π and assume that $T(\mathcal{M}) \subseteq \mathcal{M}$. Finally, let $T = \begin{pmatrix} T_1 & 0 \\ S & T_2 \end{pmatrix}$ be the matrix of T and $\pi = \pi_1 \oplus \pi_2$ with respect to the decomposition $\mathcal{H} = \mathcal{M}^\perp \oplus \mathcal{M}$. Then T_1, T_2 are homogeneous with associated representations π_1, π_2 respectively. Also, for any scalar $\alpha \geqslant 0$, $\begin{pmatrix} T_1 & 0 \\ \alpha S & T_2 \end{pmatrix}$ is homogeneous with associated representation π.*

Proof. This is immediate from the following lemma and the observation that if S satisfies the condition of the Lemma 2.5 then so does αS. ∎

LEMMA 2.5. *With notation as above, T is homogeneous with associated representation π if and only if both T_1 and T_2 are homogeneous with associated representation π_1 and π_2, and S satisfies the identity*

$$e^{i\theta}(1-|a|^2)(1-\bar{a}T_2)^{-1}S(1-\bar{a}T_1)^{-1} = \pi_1^*(\varphi)S\pi_2(\varphi),$$

for all $\varphi = \varphi_{\theta,a}$ in some neighbourhood of the identity in Möb($\mathbb{D}$).

Proof. If φ is in a sufficiently small neighbourhood of the identity so that $\varphi(T)$ is defined, then

$$\varphi(T) = \begin{pmatrix} \varphi(T_1) & 0 \\ e^{i\theta}(1-|a|^2)(1-\bar{a}T_2)^{-1}S(1-\bar{a}T_1)^{-1} & \varphi(T_2) \end{pmatrix}$$

(This is verified by multiplying the right hand side by $I-\bar{a}T = \begin{pmatrix} I-\bar{a}T_1 & 0 \\ -\bar{a}S & I-\bar{a}T_2 \end{pmatrix}$.) and

$$\pi(\varphi)^*T\pi(\varphi) = \begin{pmatrix} \pi_1(\varphi)^*T_1\pi_1(\varphi) & 0 \\ \pi_1(\varphi)^*S\pi_2(\varphi) & \pi_2(\varphi)^*T_2\pi_2(\varphi) \end{pmatrix}.$$

Now the lemma follows from Lemma 2.2 by equating the matrix entries on the right hand side of these equations. ∎

COROLLARY 2.6. *All the operators T_λ, $\lambda > 0$ are homogeneous. They are irreducible for $\lambda \neq 1$.*

Proof. For $\lambda \neq 1$, T_λ is a bilateral weighted shift with an aperiodic weight sequence, so that it is irreducible ([4], Problem 129). Note that the unitary representation π of the group Möb($\mathbb{D}$) on $L^2(\mathbb{T})$ defined by $(\pi(\varphi))(f) = ((\varphi^{-1})')^{1/2} f\circ\varphi^{-1}$ has the Hardy space as an invariant subspace. Also, Proposition 2.3 shows that $\pi(\varphi)^*M_z\pi(\varphi) = \varphi(M_z)$ for all φ in Möb($\mathbb{D}$). If we write the operator M_z as $\begin{pmatrix} T_1 & 0 \\ S & T_2 \end{pmatrix}$ then by Proposition 2.4, the operators $\begin{pmatrix} T_1 & 0 \\ \lambda S & T_2 \end{pmatrix}$, $\lambda > 0$ are homogeneous. The multiplication operator M_z is unitarily equivalent to the bilateral shift. Consequently, the operators $\begin{pmatrix} T_1 & 0 \\ \lambda S & T_2 \end{pmatrix}$ are unitarily equivalent to the bilateral weighted shifts T_λ. ∎

To show that T_C is homogeneous, first assume that $C : \mathcal{L} \to \mathcal{K}$ is quasi invertible. We emphasize that C is not necessarily a pure contraction but merely bounded. In this case, let $C = WP$ be the polar decomposition, where $W : \mathcal{L} \to \mathcal{K}$ is unitary and $P : \mathcal{L} \to \mathcal{L}$ is positive ([4], Problem 105). Let $\mathbf{W} = \bigoplus_{i=-\infty}^{\infty} W_i$, where W_i is $W : \mathcal{L} \to \mathcal{K}$ or $I : \mathcal{L} \to \mathcal{L}$ according as $i \geqslant 0$ or $i < 0$. It is easily verified that $\mathbf{W}$ conjugates T_C to T_P.

Recalling the sequence of unitary equivalences in the display (1.2), we note that all of them except for the second one remain valid even if C is merely bounded. The second equivalence was produced via the Sz.-Nagy–Foiaş theory for contractions. However, as the preceding paragraph shows, even in this case we do not require C to be a pure contraction.

THEOREM 2.7. *For any bounded operator C, the operator T_C is homogeneous.*

Proof. The discussion preceding the theorem shows that T_C is unitarily equivalent to $\mathbf{V} \oplus \int_{\Lambda}^{\oplus} d(\lambda) \cdot T_\lambda \, dm$. The operator $\mathbf{V}$, being the direct sum of copies of the unilateral shift and its adjoint, is homogeneous. This follows from homogeneity of the unilateral shift and the obvious fact that homogeneity is preserved by taking adjoints and direct sums. The fact that the unilateral shift is homogeneous was first noted in [6]. This fact also follows from Lemma 2.3 by restricting the natural representation of the group Möb($\mathbb{D}$) on $L^2(\mathbb{T})$ to the Hardy space $H^2(\mathbb{T})$. We need to verify that $\int_{\Lambda}^{\oplus} d(\lambda) \cdot T_\lambda \, dm$ is homogeneous. There is a representation π_λ such that $\pi_\lambda(\varphi)$ intertwines the two operators T_λ and $\varphi(T_\lambda)$. It is then easy to verify that the representation $\varphi \longmapsto \int_{\Lambda}^{\oplus} d(\lambda) \cdot \pi_\lambda(\varphi) \, dm$ intertwines the operators $\int_{\Lambda}^{\oplus} d(\lambda) \cdot T_\lambda \, dm$ and $\varphi(\int_{\Lambda}^{\oplus} d(\lambda) \cdot T_\lambda \, dm)$. ∎

COROLLARY 2.8. *For any bounded operator C, the spectrum of T_C is the unit circle or the unit disc according as C is invertible or not.*

Proof. It is clear from the definition of T_C that it is invertible if and only if C is invertible. Therefore this is immediate from Lemma 2.2 and Theorem 2.7. ∎

REMARK 2.9. It was pointed out in [8], p. 262, that if the characteristic function of a cnu contraction T is the constant function A where $0 \leqslant A \leqslant I$ and 0 and 1 are not eigenvalues of A then the spectrum of T is either the unit circle or the closed unit disk according as A is invertible or not. Since the operators $\{T_C : C \text{ bounded}\}$ include all cnu contractions with constant characteristic function, Corollary 2.8 is a significant extension of this result of Sz.-Nagy and Foiaş.

The following characterisation of homogeneity is implicit in the proof of Theorem 2.1 in [1]:

THEOREM 2.10. *Let T be a cnu contraction with characteristic function Θ. Then T is homogeneous if and oly if $\Theta \circ \varphi^{-1}$ coincides with Θ for each $\varphi \in$ Möb($\mathbb{D}$).*

Proof. By [8], p. 240, $\Theta_T \circ \varphi^{-1}$ coincides with the characteristic function of $\varphi(T)$ for φ in Möb($\mathbb{D}$), for any cnu contraction T. Further, T is homogeneous iff the characteristic function of $\varphi(T)$ coincides with Θ_T, i.e., iff Θ_T coincides with $\Theta_T \circ \varphi^{-1}$ for any φ in Möb($\mathbb{D}$). ∎

THEOREM 2.11. *Let T be a homogeneous cnu contraction. Then $T|\mathcal{D}_T$ is in the Hilbert–Schmidt class if and only if T is unitarily equivalent to T_C for some pure contraction C in the Hilbert–Schmidt class.*

Proof. By the previous theorem $\Theta_T \circ \varphi^{-1}$ coincides with Θ_T for each $\varphi \in$ Möb($\mathbb{D}$). Since Möb($\mathbb{D}$) is transitive on $\mathbb{D}$ this implies that $\Theta_T(z)$ coincides with $\Theta_T(0)$ for all $z \in \mathbb{D}$. Also our assumption on T means that $\Theta_T(0)$ is in the Hilbert–Schmidt class. This implies that the Hilbert-Schmidt norm of $\Theta(z)$ is constant. That is, viewed as a map into the Hilbert space of Hilbert–Schmidt operators, Θ_T is a Hilbert space valued analytic function of constant norm. Now, an appeal to the strong maximum modulus principle (see [3], Corollary III.1.5, p. 270) yields that Θ_T is a constant, so that Theorem 1.3 completes the direct part of the proof.

The converse is immediate from the fact (proved in the course of the proof of Theorem 1.1) that for $T = T_C$, $\Theta_T(0)$ coincides with $-C$. ∎

COROLLARY 2.12. *The only irreducible homogeneous contractions with at least one defect index finite are the operators T_λ, $0 < \lambda < 1$, and the unweighted forward and backward shifts.*

If one of the defect indices of a cnu contraction is infinite then the minimal unitary dilation is a bilateral shift of infinite multiplicity ([8], Chapter II, Theorem 7.4 (a)). Thus we have:

COROLLARY 2.13. *Except for the operators in the previous corollary, the minimal unitary dilation of any irreducible homogeneous contraction is the direct sum of infinitely many copies of the bilateral shift.*

REMARK 2.14. Let $\{e_n : n \in \mathbb{Z}\}$ be the standard orthonormal basis in the Hilbert space $\ell^2(\mathbb{Z})$. Fix λ, $0 < \lambda < 1$, and put $\rho = \sqrt{1-\lambda^2}$. Let $\mathcal{K}$ be the closed subspace, in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, spanned by the vectors $\lambda e_n \oplus \rho e_n$, $n < 0$ and $e_n \oplus 0$, $n \geqslant 0$. Let U be the bilateral shift acting on $\ell^2(\mathbb{Z})$. An easy verification shows that the compression of the operator $U \oplus U$ to $\mathcal{K}$ is the bilateral shift with weight sequence $\{\ldots, 1, 1, \lambda, 1, 1, \ldots\}$, which is the homogeneous operator T_λ. Thus $U \oplus U$ is a unitary dilation for the operator T_λ. It is not hard to verify that $U \oplus U$ is minimal. We conclude that the minimal unitary dilation of the operator T_λ is a bilateral shift of multiplicity 2. Thus the exceptions made in the statement of Corollary 2.13 are truly exceptional.

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