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STATIONARY DENSE OPERATORS AND GENERATION OF NON-DENSE DISTRIBUTION SEMIGROUPS

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ABSTRACT. We introduce the notion of stationary dense operators and show how they can be used to extend a result of V. Wrobel on the spectrum and generation theorems for distribution semigroups due to T. Ushijima and D. Fujiwara from densely defined operators to not necessarily densely defined operators. Other applications are sketched.

KEYWORDS: *Non-dense operators, spectrum, distribution semigroups.*

AMS SUBJECT CLASSIFICATION: 47A10, 47D03.

INTRODUCTION

Let A be a closed linear operator in a Banach space E. Then for each $n \in \mathbb{N}_0$ the space $D(A^n)$ supplied with the norm $\|x\|_n := \sum_{k=0}^{n} \|A^k x\|$ is complete, as can shown by induction using the closedness of A. Thus the projective limit $D_\infty(A)$ of $(D(A^n), \|\cdot\|_n)$, i.e. $\bigcap_n D(A^n)$ supplied with the family of norms $(\|\cdot\|_n)_n$, is a Fréchet space. Moreover A_∞, i.e. A restricted to $D_\infty(A)$, is a continuous linear operator in $D_\infty(A)$.

Instead of studying the behaviour of A in E we can study the behaviour of A_∞ in $D_\infty(A)$ and hope that both are somehow related. For some properties and under additional assumptions this is indeed the case. A quite natural assumption seems to be that A is densely defined and has non-empty resolvent set $\rho(A)$ since in this case $D_\infty(A)$ is dense in E (see [15] and [16]).

But in the last years when dealing with closed operators in a Banach space there has been a growing interest in dropping the usually imposed densely-defined assumption. This has been the case for generators of integrated semigroups, then for generators of regularized semigroups, and quite recently for generators of distribution semigroups (see [8]). But when A is not densely defined then a fortiori $D_\infty(A)$ is not dense in E, we "lose information" and in general it is impossible to retrieve it.

We give an example which we are especially interested in. For the generators of distribution semigroups as introduced in [10], i.e. for the "dense case" (see Definition 3.1), a result of T. Ushijima ([15]) relates the operators A in E and A_∞ in $D_\infty(A)$ in the following way.

THEOREM 0.1. (Ushijima) *Let A be a closed operator in E. Then A generates a dense distribution semigroup if and only if A is densely defined with non-empty resolvent set and A_∞ generates a C_0-semigroup in $D_\infty(A)$.*

In this theorem the conditions "dense" and "densely defined" can not be simultaneously dropped (see Section 3); we lose "too much" information.

We therefore introduce in Section 1 the new notion *stationary dense* for closed linear operators in a Banach space. If A is stationary dense with $\rho(A)$ non-empty then the information lost in passing from A and E to A_∞ and $D_\infty(A)$ can be retrieved. We give some examples in the following sections. In Section 2 we extend a result on the equality of the spectrum of A and A_∞ due to V. Wrobel from dense to stationary dense operators. In Section 3 we drop "dense" from Theorem 0.1 and have to replace "densely defined" by "stationary dense". In the same way we extend a similar result of D. Fujiwara ([6]) that characterizes the generators of exponentially bounded distribution semigroups which are exactly the generators of exponentially bounded integrated semigroups. Further applications are sketched. Throughout this paper E will denote a Banach space.

1. STATIONARY DENSE OPERATORS

DEFINITION 1.1. Let A be a closed linear operator in E. We define

$$n(A) := \inf\{k \in \mathbb{N}_0; \forall m \geqslant k : D(A^m) \subset \overline{D(A^{m+1})}\}$$

and call A *stationary dense* if $n(A)$ is finite.

REMARK 1.2. (i) If $D_\infty(A)$ is dense in $D(A^n)$ for the norm of E then A is stationary dense with $n(A) \leqslant n$. For $n = 0$ those operators have appeared in [15].

(ii) If A is densely defined with non-empty resolvent set then $D_\infty(A)$ is dense in E (see [15] or [16]), and according to (i) A is stationary dense with $n(A) = 0$.

(iii) If A has non-empty resolvent set then

$$n(A) = \inf\{k \in \mathbb{N}_0;\ D(A^k) \subset \overline{D(A^{k+1})}\}.$$

We want to remark that (ii) and (iii) are false in general if the assumption $\rho(A) \neq \emptyset$ is dropped, as the following example shows.

EXAMPLE 1.3. Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and let A be an unbounded closed linear operator in H which is densely defined. Choose $x_0 \in H \setminus D(A)$ and set $B := (\cdot, x_0)x_0 \in L(H)$. Then $D(AB) = \{x_0\}^\perp$ and $AB = 0$. Define for each $n \geqslant 2$ the operator A_n in the Hilbert space $E_n := H^n$ by

$$A_n(x_1, \ldots, x_n) = (x_2, x_3, \ldots, x_{n-1}, Bx_n, Ax_1)$$

with $D(A_n) := D(A) \times H^{n-1}$. Then A_n is a closed and densely defined operator in E_n, and for all $k \in \{1, \ldots, n-1\}$ we have

$$A_n^k(x_1, \ldots, x_n) = (x_{k+1}, \ldots, x_{n-1}, Bx_n, BAx_1, BAx_2, \ldots, BAx_{k-1}, Ax_k)$$

and $D(A_n^k) = D(A)^k \times H^{n-k}$. We further have

$$A_n^n(x_1, \ldots, x_n) = (BAx_1, \ldots, BAx_{n-1}, ABx_n)$$

with $D(A_n^n) = D(A)^{n-1} \times \{x_0\}^\perp$. Hence $D(A_n^n)$ is not dense in E_n. Suppose in addition that $\operatorname{im} A \perp x_0$, which happens e.g. for $H := l^2$, and $A(\xi_k) := (\xi_2, 0, 2\xi_4, 0, 3\xi_6, 0, 4\xi_8, 0, \ldots)$ with maximal domain and $x_0 = (0, 1, 0, 1/2, 0, 1/3, 0, 1/4, 0, \ldots)$. Since $BA = 0$ with $D(BA) = D(A)$ we then have $A_n^n = 0$ from which $D(A_n^k) = D(A_n^n)$ for all $k \geqslant n$. Thus A_n is stationary dense with $n(A_n) = n$ and

$$\forall k \in \mathbb{N}_0 : \overline{D(A_n^k)} \neq \overline{D(A_n^{k-1})} \iff k = n.$$

Let now I be a non-empty subset of $\mathbb{N}$, denote by E_I the l^2-direct sum of the spaces E_j, $j \in I$, and define

$$A_I := \{((x_j), (y_j)) \in E_I \times E_I; \forall j \in I : (x_j, y_j) \in A_j\}.$$

Then A_I is a densely defined closed operator in E_I satisfying

$$\forall k \in \mathbb{N}_0 : \overline{D(A_I^k)} \neq \overline{D(A_I^{k-1})} \iff k \in I.$$

Letting $I = \mathbb{N}$ we see that a densely defined operator need not be stationary dense.

EXAMPLE 1.4. Let $m \in \mathbb{N}_0$ and $E := C^m[0,1]$. Let $A := -\mathrm{d}/\mathrm{d}s$ with $D(A) = \{f \in C^{m+1}[0,1]; f(0) = 0\}$. Then $n(A) = m + 1$.

We now give some lemmas to provide further examples of stationary dense operators.

LEMMA 1.5. *Let A be a closed linear operator in E. Suppose that there is a sequence (λ_n) in $\rho(A)$ with $\lim_{n\to\infty} |\lambda_n| = \infty$ and there is a constant $C > 0$ and an integer $k \geqslant -1$ such that $\|R(\lambda_n, A)\| \leqslant C|\lambda_n|^k$ holds for all $n \in \mathbb{N}$. Then A is stationary dense with $n(A) \leqslant k+2$.*

Proof. We may assume $|\lambda_n| \geqslant 1$ for all $n \in \mathbb{N}$. By the resolvent equation there is a $C' > 0$ such that $\|\lambda_n R(\lambda_n, A)x\| \leqslant C'\|x\|_{k+1}$ for all $n \in \mathbb{N}$ and $x \in D(A^{k+1})$. If $x \in D(A^{k+2})$ then $\lambda_n R(\lambda_n, A)x \in D(A^{k+3})$ for all n and

$$\|\lambda_n R(\lambda_n, A)x - x\| = \|R(\lambda_n, A)Ax\| \leqslant C'|\lambda_n|^{-1}\|Ax\|_{k+1} \leqslant C''|\lambda_n|^{-1}\|x\|_{k+2}.$$

Hence $x = \lim_{n\to\infty} \lambda_n R(\lambda_n, A)x$ belongs to the closure of $D(A^{k+3})$. We are finished by Remark 1.2 (iii). ∎

In Example 1.4 with $m \in \mathbb{N}_0$ fixed we have $n(A) = m+1$, $]0,\infty[\subset \rho(A)$ and $\|R(\lambda, A)\| \leqslant C\lambda^{m-1}$ for $\lambda > 1$ as can be seen using (1.1) below, which is also valid here. Thus the estimate $n(A) \leqslant k+2$ can not be improved.

Moreover, the condition of polynomial growth in Lemma 1.5 is sharp as the following example, inspired by Theorem 3 in [11], shows.

EXAMPLE 1.6. Let $\omega : [0,\infty[\to]1,\infty[$ be a continuous function with $\lim_{r\to\infty} r^{-k}\omega(r) = \infty$ for each $k \in \mathbb{N}$. Then there is a Banach space E and a closed linear operator A in E which is not stationary dense such that all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$ belong to $\rho(A)$ and satisfy

$$\|R(\lambda, A)\| \leqslant \omega(|\lambda|).$$

Indeed, we may construct inductively a sequence $(M_n)_{n\in\mathbb{N}_0}$ of positive numbers satisfying $M_0 = 1$, $M_1 \geqslant 1$ and

$$\forall\, r \geqslant 0 : \sum_{k=0}^{\infty} \frac{r^k}{M_k} \leqslant \omega(r); \quad \forall\, p, q \in \mathbb{N}_0 : M_{p+q} \geqslant M_p M_q.$$

Let now

$$E_{(M_n)} := \{f \in C^\infty[0,1]; \|f\|_{(M_n)} := \sup_n M_n^{-1}\|f^{(n)}\|_\infty < \infty\}$$

and $A_{(M_n)} := -\mathrm{d}/\mathrm{d}s$ with $D(A_{(M_n)}) = \{f \in E_{(M_n)}; f' \in E_{(M_n)}, f(0) = 0\}$. Define for $\lambda \in \mathbb{C}$ the operator R_λ by

$$(R_\lambda f)(t) = \mathrm{e}^{-\lambda t} \int_0^t f(s)\mathrm{e}^{\lambda s}\, \mathrm{d}s.$$

If $\operatorname{Re}\lambda \geqslant 0$ and $g := R_\lambda f$ then $g(0) = 0$ and $\lambda g + g' = f$. Moreover g is a C^∞-function with

$$g^{(n)} = f^{(n-1)} + \sum_{k=1}^{n-1}(-\lambda)^k f^{(n-1-k)} + (-\lambda)^n g \tag{1.1}$$

for all $n \in \mathbb{N}$, as can be shown by induction. Using $\|g\|_\infty \leqslant \|f\|_\infty$ and $M_k M_{n-1-k} \leqslant M_{n-1}$ for $k \leqslant n-1$ we thus get

$$M_n^{-1}\|g^{(n)}\|_\infty \leqslant M_{n-1}^{-1}\|f^{(n-1)}\|_\infty + \sum_{k=1}^{n-1} \frac{|\lambda|^k}{M_k M_{n-1-k}}\|f^{(n-1-k)}\|_\infty + \frac{|\lambda|^n}{M_n}\|f\|_\infty.$$

Hence

$$\|g\|_{(M_n)} \leqslant \sum_{k=0}^{\infty} \frac{|\lambda|^k}{M_k}\|f\|_{(M_n)} \leqslant \omega(|\lambda|)\|f\|_{(M_n)}.$$

Thus $g \in E_{(M_n)}$, moreover $g \in D(A_{(M_n)})$, and finally $R_\lambda = R(\lambda, A_{(M_n)})$ with $\|R(\lambda, A_{(M_n)})\| \leqslant \omega(|\lambda|)$ for $\operatorname{Re}\lambda \geqslant 0$. From

$$D(A^k_{(M_n)}) = \{f \in E_{(M_n)}; \forall j \in \{0, \dots, k-1\} : f^{(j+1)} \in E_{(M_n)}, f^{(j)}(0) = 0\}$$

we get

$$\overline{D(A^k_{(M_n)})} \subset \{f \in E_{(M_n)}; \forall j \in \{0, \dots, k-1\} : f^{(j)}(0) = 0\}$$

for each $k \in \mathbb{N}_0$. The function $f : t \mapsto t^k/k!$ belongs to $D(A^k_{(M_n)})$, but not to $\overline{D(A^{k+1}_{(M_n)})}$ since $f^{(k)}(0) = 1$. Hence $A_{(M_n)}$ is not stationary dense.

LEMMA 1.7. *Let $a > 0, n \in \mathbb{N}_0$ and A be a closed linear operator in E such that the Cauchy problem for A has a unique mild solution on $[0, a]$ for all initial values in $D(A^n)$. Then A is stationary dense with $n(A) \leqslant n$.*

Proof. Just as in the proof of Theorem 10.3.4 in [7] we can show that

$$E_0 := \left\{ \int_0^a \varphi(t)u(t;x)\,\mathrm{d}t; x \in D(A^n), \varphi \in C^\infty \text{ with } \operatorname{supp}\varphi \subset [0,a]\right\} \subset D_\infty(A),$$

where $u(\cdot; x)$ denotes the unique mild solution of $u' = Au, u(0) = x$ on $[0, a]$. Hence by Remark 1.2 (i) the proof is done. ∎

Note that Example 1.4 ($n = m + 1$ here and $u(t; f) = f(\cdot - t)$) shows that the estimation $n(A) \leqslant n$ can not be improved.

COROLLARY 1.8. *Let A be the generator of a local n-times integrated semigroup. Then A is stationary dense with $n(A) \leqslant n$.*

For the notion of (local) integrated semigroups we refer to [1] and [14] (see also [12]).

2. SPECTRAL PROPERTIES AND AN EXTENSION OF A RESULT OF V. WROBEL

PROPOSITION 2.1. *Let A be a stationary dense operator in E with non-empty resolvent set. Let $n := n(A)$ and F be the closure of $D(A^n)$ in E. Then*

(i) *A_F is densely defined in F.*

(ii) *$\rho(A; L(E)) = \rho(A_F; L(F))$ and for all $\lambda \in \rho(A)$ holds*

$$\|R(\lambda, A)\|_{L(E)} \leqslant C(1+|\lambda|)^n(\|R(\lambda, A_F)\|_{L(F)} + 1).$$

(iii) *The Fréchet spaces $D_\infty(A)$ and $D_\infty(A_F)$ coincide topologically and $A_\infty = (A_F)_\infty$.*

(iv) *$n(A) = \inf\{k \in \mathbb{N}_0; D_\infty(A) \subset \overline{D(A^k)}\}$.*

Proof. From $D(A^{n+1}) \subset D(A_F)$ we get (i). Since F is invariant under the resolvents of A, $\rho(A; L(E)) \subset \rho(A_F; L(F))$ holds. If $\lambda \in \rho(A_F)$ then $\lambda - A$ is injective, and $(\lambda - A)^{-1}$ is a closed extension of $R(\lambda, A_F)$. If we choose $\mu \in \rho(A)$ then $(\lambda - A)^{-1} = (\mu - A)^n(\lambda - A)^{-1}R(\mu, A)^n$. Thus $\lambda \in \rho(A)$ and

$$\begin{aligned} &\|R(\lambda, A)\|_{L(E)} \\ &\quad\leqslant \|(\mu - A)^n\|_{L(D(A^n),E)}\|R(\lambda, A_F)\|_{L(D(A^n))}\|R(\mu, A)\|_{L(E,D(A^n))}. \end{aligned} \tag{2.1}$$

The resolvent equation yields for $x \in D(A^n)$

$$\begin{aligned} \|R(\lambda, A_F)x\|_n &= \sum_{k=0}^{n} \|A^k R(\lambda, A_F)x\| \\ &\leqslant \sum_{k=0}^{n} \left(|\lambda|^k \|R(\lambda, A_F)\|_{L(F)}\|x\| + \sum_{j=0}^{k-1} |\lambda|^j \|A^{k-1-j}x\| \right) \end{aligned}$$

which together with (2.1) implies (ii). By applying resolvents to $D(A^{n+k}) \hookrightarrow D((A_F)^k) \hookrightarrow D(A^k)$ for $k = 0$ we get this relation for all $k \in \mathbb{N}_0$, hence (iii) holds. (iv) follows from (iii). ∎

For the next section we note a simple consequence.

COROLLARY 2.2. *Let the assumptions of Proposition 2.1 hold and $\Lambda \subset \mathbb{C}$. Then $R(\cdot, A)$ exists on Λ and is polynomially bounded there if and only if $R(\cdot, A_F)$ exists on Λ and is polynomially bounded there.*

As an application we prove the following generalization of a result of V. Wrobel (Theorem 3.4 in [16]).

THEOREM 2.3. *Let A be a stationary dense operator in E with non-empty resolvent set. Then $\rho(A; L(E)) = \rho(A_\infty; L(D_\infty(A)))$.*

Proof. We continue the notation of Proposition 2.1. By Proposition 2.1 (i) and (ii) A_F is a densely defined closed operator in F with non-empty resolvent set. Hence $\rho(A_F; L(F)) = \rho((A_F)_\infty; L(D_\infty(A_F)))$ by Theorem 3.4 in [16]. Now the theorem follows from Proposition 2.1 (i) and (iii). ∎

The next example shows that we cannot drop "stationary dense" from Theorem 2.3.

EXAMPLE 2.4. Let $E := l^1$ and $A(x_n) := (x_{n+1})$ with $D(A) := \{x \in l^1; x_1 = 0\}$. Then A is a closed operator in E which is not stationary dense. Moreover $D_\infty(A) = \{0\}$ and $\rho(A_\infty) = \{\lambda \neq 0\}$. For $\lambda \in \mathbb{C}$ we have $(\lambda - A)^{-1}(y_n) = \left(-\sum_{k=1}^{n-1} \lambda^{n-1-k} y_k\right)$ which for $|\lambda| \geqslant 1$ and $(y_n) := (1, 0, 0, \ldots)$ does not belong to E and which for $|\lambda| < 1$ defines a bounded operator in E with norm $(1 - |\lambda|)^{-1}$. Thus $\rho(A) = \{|\lambda| < 1\}$.

3. EXTENDING RESULTS OF D. FUJIWARA AND T. USHIJIMA

DEFINITION 3.1. A *distribution semigroup* is a mapping $G \in \mathcal{D}'(L(E))$ with support in $[0, \infty[$ satisfying

$$\bigcap\{\ker G(\varphi); \varphi \in \mathcal{D}_0\} = \{0\} \quad \text{and} \quad \forall \varphi, \psi \in \mathcal{D} : G(\varphi)G(\psi) = G(\varphi *_0 \psi)$$

where $\varphi *_0 \psi(t) := \int_0^t \varphi(s)\psi(t-s)\,ds$ for all t, and $\mathcal{D}_0$ is the subset of $\mathcal{D}$ of all functions with support in $[0, \infty[$. The distribution semigroup is called *dense* if $\bigcup\{\operatorname{im} G(\varphi); \varphi \in \mathcal{D}_0\}$ is dense in E. The *generator* A of a distribution semigroup G is defined as

$$A := \{(x, y) \in E \times E; \forall \varphi \in \mathcal{D}_0 : G(-\varphi')x = G(\varphi)y\}.$$

REMARK 3.2. (i) If G is a distribution semigroup then its generator A is a closed linear operator in E which is densely defined if and only if G is dense (see [8]).

(ii) If A is a closed linear operator in E, D denotes $D(A)$ supplied with the graph norm, and I denotes the inclusion $D \to E$ then A is the generator of a distribution semigroup if and only if there is a fundamental solution G for the convolution operator $P_A := \delta' \otimes I - \delta \otimes A \in \mathcal{D}'(L(D,E))$, i.e. a distribution $G \in \mathcal{D}'(L(E,D))$ with support in $[0,\infty[$ satisfying

$$P_A * G = \delta \otimes \mathrm{Id}_E \quad \text{and} \quad G * P_A = \delta \otimes \mathrm{Id}_D$$

(see [10] and [8], for the convolution needed here see also [13] and [5]).

From (ii) and Theorem 1.6 in [2] we get the following characterization of the generators of distribution semigroups.

THEOREM 3.3. (Chazarain) *Let A be a closed operator in E. Then A generates a distribution semigroup in E if and only if there are constants $\alpha, \beta, C > 0$ and an integer k such that for all*

$$\lambda \in \Lambda := \{\xi + i\eta; \xi \geqslant \alpha \log(1 + |\eta|) + \beta\}$$

the resolvent operator $R(\lambda, A)$ exists and satisfies $\|R(\lambda, A)\| \leqslant C(1 + |\lambda|)^k$.

Combining Theorem 3.3, Lemma 1.5 and Corollary 2.2 we get the following.

PROPOSITION 3.4. *Let A be a stationary dense operator in E, $n := n(A)$ and F the closure of $D(A^n)$ in E. Then A generates a distribution semigroup in E if and only if A_F generates a distribution semigroup in F.*

This proposition will be used in the proof of our generalization of Theorem 0.1 (Ushijima's result).

THEOREM 3.5. *Let A be a closed operator in E. Then A generates a distribution semigroup in E if and only if A is stationary dense with non-empty resolvent set and A_∞ generates a C_0-semigroup in $D_\infty(A)$.*

Proof. If A generates a distribution semigroup in E then A is stationary dense with non-empty resolvent set by Theorem 3.3 and Lemma 1.5. Letting $n := n(A)$ and $F := \overline{D(A^n)}$, A_F generates a distribution semigroup in F by Proposition 3.4 which is dense since by Proposition 2.1 A_F is densely defined in F. Thus by Theorem 0.1 $(A_F)_\infty$ generates a C_0-semigroup in $D_\infty(A_F)$, and by Proposition 2.1 (iii) one direction is proved.

If A is stationary dense with non-empty resolvent set and A_∞ generates a C_0-semigrooup in $D_\infty(A)$ define n and F as before. By Proposition 2.1 and Theorem 0.1 A_F generates a distribution semigroup in F. Hence the other direction is implied by Proposition 3.4. ∎

Example 2.4 shows in combination with Theorem 3.3 that "stationary dense" can not be dropped from Theorem 3.5. Another example is furnished by taking $(M_n) = (n!)$ in Example 1.6. The operator $A_{(n!)}$ is not stationary dense and hence does not generate a distribution semigroup, but since all functions in $E_{(n!)}$ are real-analytic we have $D_\infty(A_{(n!)}) = \{0\}$.

For a further application we recall the following notion: A C_0-semigroup (T_t) in a Fréchet space Y is called *quasi-equicontinuous* if there is an $a \geqslant 0$ such that the set $\{\exp(-a\cdot)T_t; t \geqslant 0\}$ is equicontinuous. If $a = 0$ is possible then (T_t) is called *equicontinuous*. In the same way as Theorem 3.5 but using a result of D. Fujiwara ([6]) instead of Theorem 0.1 and Theorem 6.1 in [10] instead of Theorem 3.3 we can prove the following.

THEOREM 3.6. *Let A be a closed operator in E. Then A generates an exponentially bounded distribution semigroup in E if and only if A is stationary dense with non-empty resolvent set and A_∞ generates a quasi-equicontinuous C_0-semigroup in $D_\infty(A)$.*

Similar results can be proved for *distribution semigroups of finite growth order*, i.e. those that belong to $\exp(a(\cdot)^p)\mathcal{S}'(L(E))$ for some $a > 0$ and $p > 1$, by using results due to I. Ciorănescu ([3], [4]) and more generally for *distribution semigroups of growth M*, i.e. for those that belong to $\mathcal{K}'_M(L(E))$ where M is a *growth function* (see [9]), i.e. a continuously differentiable function $[0,\infty[\to[0,\infty[$ with strictly increasing derivative satisfying $M(0) = M'(0) = 0$ and $M'(\infty) = \infty$ (growth order p corresponds to $M(t) = t^p/p$).

THEOREM 3.7. *Let A be a closed linear operator in E and M be a growth function. Then A generates a distribution semigroup of growth M in E if and only if A is stationary dense with non-empty resolvent set and A_∞ generates a C_0-semigroup (T_t) in $D_\infty(A)$ such that $\{\exp(-M(at))T_t; t \geqslant 0\}$ is equicontinuous for some $a > 0$.*

We could also state our results in the notion of (local) integrated semigroups instead of distribution semigroups (see [8] and [9]).

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