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ALMOST MULTIPLICATIVE MORPHISMS AND SOME APPLICATIONS

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ABSTRACT. We show that, for any $\varepsilon > 0$ and an integer $n > 0$, there exists $\delta > 0$ such that if $x_1, x_2, \ldots, x_n$ are normal elements in the unit ball of a purely infinite simple C^*-algebra A with

$$\|x_i x_j - x_j x_i\| < \delta \quad i = 1, 2, \ldots, n$$

then there exist mutually commuting normal elements $y_1, y_2, \ldots, y_n \in A$ such that

$$\|x_i - y_i\| < \varepsilon \quad i = 1, 2, \ldots, n.$$

KEYWORDS: *C^*-algebra, C^*-algebra homomorphism, almost multiplicative morphisms.*

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0. INTRODUCTION

We study the problem when an almost multiplicative contractive unital positive linear map $\varphi : C(X) \to A$, where X is a compact metric space and A is a unital C^*-algebra, is close to a homomorphism. One of the original questions which leads to the problem is the question whether a pair of almost commuting selfadjoint matrices is close to a pair of commuting selfadjoint matrices, an once long standing problem which is now solved affirmatively (see [42]). This question can be viewed as special case of the problem we mentioned above (when X is a compact subset of the plane). Clearly the problem is closely related to the study of homomorphisms from $C(X)$ into a (non-abelian) C^*-algebra. The first known significant result of the study of homomorphisms from $C(X)$ into a (non-abelian) C^*-algebra is perhaps the BDF-theory ([5], [6]) which studies the homomorphisms from $C(X)$ into the Calkin algebra $B(l^2)/\mathcal{K}$ as well as into $B(l^2)$, where $\mathcal{K}$ is the compact operators on l^2. Recent development (see [22]) in the classification theory of C^*-algebras also requires deep understanding of homomorphisms from $C(X)$ to a C^*-algebra. Let $\psi : C(X) \to A$ be a homomorphism and p be a projection in A. If p approximately commutes with ψ, then the map $p\psi p : C(X) \to A$ defined by $p\psi(f)p$ is an almost multiplicative positive linear map. An often occurred question is whether there is a homomorphism $h : C(X) \to pAp$ which approximates $p\psi p$. The problem also occurs when one attempts to lift a homomorphism $\psi : C(X) \to A/I$ to a linear map $\varphi : C(X) \to A$, where I is an ideal of A. Here is another version of the problem. Let $x_1, x_2, \ldots, x_n$ be normal elements in a C^*-algebra A. If $\|x_ix_j - x_jx_i\|$ is small, will it follow that $x_1, x_2, \ldots, x_n$ are approximated by n commuting normal elements in A ?

A number of results related to this problem have been obtained. Important and interesting applications have been given. We will not try to attempt to give a history tour. However, we will give some description of certain results which we believe that are most relevant to this paper. But, before that, let us state the problem more precisely.

DEFINITION 0.1. Let A and B be two (unital) C^*-algebras and let $\psi : A \to B$ be a contractive positive linear map, δ be a positive number, and let $\mathcal{F}$ be a finite subset of A. The map ψ is said to be *δ-$\mathcal{F}$-multiplicative*, if

$$\|\psi(fg) - \varphi(f)\psi(g)\| < \delta$$

for all f and g in $\mathcal{F}$. Now here is the precise statement of the problem:

QUESTION 0.2. Let X be a compact metric space, $\varepsilon > 0$ and let $\mathcal{G}$ be a finite subset of $C(X)$. When do there exist $\delta > 0$ and a finite subset $\mathcal{F}$ of $C(X)$ satisfying the following: for any unital C^*-algebra A and any unital contractive positive linear map $\varphi : C(X) \to A$ which is δ-$\mathcal{F}$-multiplicative, there is a unital homomorphism $\psi : C(X) \to A$ such that

$$\|\varphi(f) - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{G}$?

We note that we could consider bounded, almost linear, almost $*$-preserving maps. However, it is more convenient to consider contractive maps. We point out in Proposition 1.7 that a contractive, almost linear, almost $*$-preserving, and almost multiplicative map is close to a contractive linear positive and almost multiplicative map. Therefore, it suffices to consider only almost multiplicative contractive positive linear maps. It is shown ([40]) that when X is a compact subset of $\mathbb{S}^1 \times \mathbb{S}^1$ and A is assumed to be purely infinite and simple, δ and $\mathcal{F}$ in Question 0.2 always exist. This is equivalent to say that a pair of almost commuting unitaries in a unital purely infinite simple C^*-algebra is is close to a pair of commuting unitaries in the C^*-algebra. It is also shown ([44]) that when A is assumed to be a purely infinite simple C^*-algebra with trivial K-theory, δ and $\mathcal{F}$ always exist. Applications of these results to classification of C^*-algebras are given in [40] and [44].

On the other hand, it is shown ([57], [13], [14], [46] and [47]) that in general there are K-theoretical obstacles for the existence of δ and $\mathcal{F}$. Theorem 1.6 gives an answer to a stable version of Question 0.2 (see also Notation 1.8). Let α be an element in $KK(C(X), A)$, where X is a finite CW complex and A is a unital purely infinite simple C^*-algebra. It has been shown that for some special case, α can be realized by a homomorphism ([48], [25] and [40]). Using results in C^*-algebra classification theory, we also improve these results by showing that there is always a homomorphism $\varphi : C(X) \to A$ such that $KK(\varphi) = \alpha$. Combining with Theorem 1.6, we show (Theorem 1.19) that when A is purely infinite simple C^*-algebra, δ and $\mathcal{F}$ in Question 0.2 always exist, i.e., an almost multiplicative contractive positive linear map is close to a homomorphism . There is no doubt that there are many applications of Theorem 1.19. However, we only give applications related to extensions. We give a stable type of Weyl-Von Neumann-Berg-Voiculescu's approximate diagonalization theorem. Other applications will appear elsewhere. We also point out that this result gives a classification of unital essential extensions of $C(X)$ by purely infinite simple C^*-algebras.

1. ALMOST MULTIPLICATIVE MORPHISMS

We begin with some notation and known facts related to K-theory.

DEFINITION 1.1. The standard definition of mod-p K-theory for C^*-algebras as given by Schochet in [56], is

$$K_i(A; \mathbb{Z}/n) = K_i(A \otimes C_0(C_n)),$$

where C_n is the 2-dimensional CW complex obtained by attaching a 2-cell to $\mathbb{S}^1$ via the degree n map from $\mathbb{S}^1$ to $\mathbb{S}^1$ (note that $K_0(C_0(C_n)) = \mathbb{Z}/n\mathbb{Z}$ and

$K_1(C(C_n)) = \{0\})$. Let A be a C^*-algebra, following Dadarlat and Loring ([15]), we denote

$$\underline{K}(A) = K_0(A) \oplus K_1(A) \bigoplus_{i=0,1,n\geqslant 2} K_i(A;\mathbf{Z}/n).$$

Let B be a C^*-algebra, following Rørdam ([53]), we denote by $KL(C(X),B)$ the quotient of $KK(C(X),B)$ by the subgroup of pure extensions in $\mathrm{Ext}(K_*(C(X)), K_{*-1}(B))$.

Note that

$$K_0(A\otimes C(C_m\times \mathbb{S}^1)) \cong K_0(A)\oplus K_1(A)\oplus K_0(A;\mathbf{Z}/m)\oplus K_1(A;\mathbf{Z}/m).$$

We define $\underline{K}(A)_+$ to be the semigroup of $\underline{K}(A)$ generated by $K_0(A\otimes C(C_m\times\mathbb{S}^1))_+$, $m\geqslant 2$.

The following is a result of Dadarlat and Loring.

THEOREM 1.2. ([15]) *Let X be a compact metric space. Suppose that $\varphi,\psi: C(X)\to B$, where B is a unital C^*-algebra, are homomorphisms such that φ and ψ induce the same homomorphisms from $K_i(C(X)\otimes C_k)\to K_i(B\otimes C(C_k))$, $k=1,2,\dots$ and $i=0,1$. Then $[\varphi]=[\psi]$ in $KL(C(X),B)$.*

In fact, Dadarlat and Loring ([15]) show that there is an isomorphism from $KL(C(X),B)$ onto $\mathrm{Hom}(\underline{K}(C(X)),\underline{K}(B))$ for any unital C^*-algebra B. We will use this result very often.

The following is standard. We state here for the convenience.

THEOREM 1.3. *Let A_n be a sequence of C^*-algebras. Then, for any projection p and unitaries u in $\prod_{n=1}^{\infty} A_n/\bigoplus_{n=1}^{\infty} A_n$, there are projection P and U in $\prod_{n=1}^{\infty} A_n$ such that*

$$\pi(P)=p \quad \textit{and} \quad \pi(U)=u.$$

If $p_1,p_2,\dots,p_k$ are mutually orthogonal projections in $\prod_{n=1}^{\infty} A_n/\bigoplus_{n=1}^{\infty} A_n$, then there are mutually orthogonal projections $P_1,P_2,\dots,P_k\in\prod_{n=1}^{\infty} A_n$ such that $\pi(P_i)=p_i$, $i=1,2,\dots,k$. Furthermore, if v and $\{u_t\}\in\prod_{n=1}^{\infty} A_n/\bigoplus_{n=1}^{\infty} A_n$, where $0\leqslant t\leqslant 1$ such that $v^*v=p_1$ and $vv^*=q$ are two projections, and $\{u_t\}$ is a continuous path of unitaries, there is a partial isometry V and a continuous path of unitaries $\{U_t\}$ in $\prod_{n=1}^{\infty} A_n$ such that

$$\pi(V)=v, \pi(U_0)=u_0 \quad \text{and} \quad \pi(U_1)=u_1.$$

Proof. Let $1 \geqslant b \geqslant 0$ and c be elements in $\prod_{n=1}^{\infty} A_n$ such that $\pi(b) = p$ and $\pi(c) = u$. Let $b = \{b_n\}$ and $c = \{c_n\}$, where $b_n, c_n \in A_n$. We have $b^2 - b \in \bigoplus_{n=1}^{\infty} A_n$. Therefore

$$\|b_n^2 - b_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

It is now standard that there are projections $p_n \in A_n$ such that

$$\|b_n - p_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Set $P = \{p_n\}$ then $P - b \in \bigoplus_{n=1}^{\infty} A_n$. Therefore $\pi(P) = p$. Suppose that $p_1, p_2, \dots, p_k \in \prod_{n=1}^{\infty} A_n / \bigoplus_{n=1}^{\infty} A_n$ are mutually orthogonal, and $P_1 \in \prod_{n=1}^{\infty} A_n$ is a projection such that $\pi(P_1) = p_1$. Then $p_2 \in (1 - P_1) \prod_{n=1}^{\infty} A_n (1 - P_1)/(1 - P_1) \bigoplus_{n=1}^{\infty} A_n (1 - P_1)$. We can then apply what has been proved and use induction to produce required projections $p_2, \dots, p_k$. The rest of the proof uses a similar argument. We will be brief. The unitary U can be constructed the same way using polar decomposition. For the last part, the partial isometry V can be constructed the same way as U. Also, there are unitaries $\{W_t\}$ in $\prod_{n=1}^{\infty} A_n$ such that $\pi(W_t) = u_t$. There is a partition:

$$0 < t_1 < t_2 < \cdots < t_m = 1$$

such that $\|u_{t_{i+1}} - u_{t_i}\| < 1/2$. Let $W_t = \{w_t(n)\} \in \prod_{n=1}^{\infty} A_n$. Then each $w_t(n)$ is a unitary. There is an integer $n_0 > 0$ such that

$$\|w_{t_{i+1}}(n) - w_{t_i}(n)\| < 1$$

for all $n \geqslant n_0$. This implies that there is a continuous path of unitaries $\{U_t\}$ in $\prod_{n=1}^{\infty} A_n$ such that $U_0 = W_0$ and $U_1 = W_1$. ∎

REMARK 1.4. Let A be a unital C^*-algebra. Let $\mathcal{P}$ be a finite subset of projections in $\bigcup_{m \geqslant 0} M_\infty(A \otimes C(C_m \times \mathbb{S}^1))$. There are a finite subset $\mathcal{G}(\mathcal{P}) \subset A$ and $\delta(\mathcal{P}) > 0$ such that if B is any unital C^*-algebra and $\varphi : A \to B$ is a contractive positive linear map which is $\delta(\mathcal{P})$-$\mathcal{G}(\mathcal{P})$ multiplicative, then

$$\|(\varphi \otimes \mathrm{id})(p^2) - (\varphi \otimes \mathrm{id})(p)\| < \frac{1}{4}$$

for all $p \in \mathcal{P}$. Hence, for each $p \in \mathcal{P}$, there is a projection $q \in \bigcup_{m \geqslant 0} M_\infty(B \otimes C(C_m \times \mathbb{S}^1))$ such that

$$\|(\varphi \otimes \mathrm{id})(p) - q\| < \frac{1}{2}.$$

Furthermore, if q' is another projection satisfying the same condition, then $\|q - q'\| < 1$, hence q is unitarily equivalent to q'. Let $\overline{\mathcal{P}}$ be the image of $\mathcal{P}$ in $\underline{K}(A)$. For each $p \in \mathcal{P}$, we set $\varphi_*([p]) = [q]$. This defines a map $\varphi_* : \overline{\mathcal{P}} \to \underline{K}(B)$.

DEFINITION 1.5. Let $r : \mathbb{N} \to \mathbb{N}$ be a map. Denote by $\mathbf{A}_r$ the collection of those C^*-algebras A satisfying the following:

For any integer $K > 0$, $m > 0$, and any pair of projections p and q in $M_K(A \otimes C(C_m \times \mathbb{S}^1))$ such that $[p] = [q]$, there is a projection $f \in M_{r(K)}(A \otimes C(C_m \times \mathbb{S}^1))$ such that

$$q \oplus f \sim p \oplus f.$$

Clearly, $\mathbf{A}_r \subset \mathbf{A}_{r'}$ if $r(K) \leqslant r'(K)$ for all $K \in \mathbb{N}$. It follows from [10] that every unital purely infinite simple C^*-algebra belongs to every $\mathbf{A}_r$. In fact, if $[p] = [q]$, then

$$q \oplus \mathbf{1}_L \sim p \oplus \mathbf{1}_L,$$

where $\mathbf{1}_L$ is the identity of $M_L(A \otimes C(C_m \times \mathbb{S}^1))$ for some integer L (which may depend on p, q and m). The projection $\mathbf{1}_L$ is a constant function over $C(C_m \times \mathbb{S}^1)$ (with value in $M_L(A)$). By [10], there is a constant function $v(t)$, where for each t, $v(t)$ is a (same) partial isometry in $M_L(A)$ such that

$$v^* v = \mathbf{1}_L \quad \text{and} \quad vv^* = d$$

for some constant projection d with value in A. This implies that $q \oplus d \sim p \oplus d$. Thus $A \in \mathbf{A}_r$, for all r. Moreover, by results of Rieffel ([52]), if A is of stable rank s, there exist r (depends on s) such that $A \in \mathbf{A}_r$.

The following is a generalization of 1.6 in [44] and 2.3 in [40], and its proof is almost the same as that of 1.6 in [44].

THEOREM 1.6. *Let X be a compact metric space, $r : \mathbb{N} \to \mathbb{N}$, and let $\mathcal{F}$ be a finite subset of the unit ball of $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\bigcup_{n=1}^{\infty} M_\infty(C(X) \oplus C(C_n \times \mathbb{S}^1))$, a finite subset $\mathcal{G}$ of the unit ball of $C(X)$ and $\delta > 0$ such that whenever A is a unital C^*-algebra in $\mathbf{A}_r$ and whenever $\psi, \varphi : C(X) \to A$ with $\psi_* = \varphi_* : \overline{\mathcal{P}} \to \underline{K}(A)$ are two unital contractive positive linear maps which are δ-$\mathcal{G}$ -multiplicative then there are an integer L and unital* *homomorphisms* $\phi_1 : C(X) \to M_L(A)$ *with finite dimensional range and a unitary* $u \in M_{2L}(A)$ *such that*

$$\|u^*(\psi(f) \oplus \phi_1(f))u - \varphi(f) \oplus \phi_1(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. There is an increasing sequence of finite subsets $\mathcal{P}(n)$ of projections in $\bigcup_{m=1}^{\infty} M_\infty(C(X) \otimes C(C_m \times \mathbb{S}^1))$ such that $\bigcup_{m=1}^{\infty} \overline{\mathcal{P}}(n)$ forms a generating set of the semigroup $\underline{K}(C(X))_+$. Suppose that the theorem is false. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots$ be a sequence of finite subsets of unit ball of $C(X)$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and the union $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in the unit ball of $C(X)$, and $\mathcal{G}(\mathcal{P}(n)) \subset \mathcal{F}_n$, where $\mathcal{G}(\mathcal{P})$ is defined in Remark 1.4. Then there are a positive number $\varepsilon > 0$, a finite subset $\mathcal{F}$, a sequence of positive numbers $\delta_n \to 0$ with $\delta_n \leqslant \delta(\mathcal{P}(n))$, unital C^*-algebras $B_n \in \mathbf{A}_r$ and unital contractive positive linear maps $\psi_n, \varphi_n : C(X) \to B_n$ which are δ_n-$\mathcal{F}_n$-multiplicative and $(\psi_n)_* = (\varphi_n)_*$ on $\overline{\mathcal{P}}(n)$, and for all n,

$$\inf_{k,\varphi,u} \{\sup_{f \in \mathcal{F}} \{\|u^*(\psi_n(f) \oplus \phi(f))u - \varphi_n(f) \oplus \phi(f)\|\}\} \geqslant \varepsilon$$

for some $\varepsilon > 0$. Here the infimum is taken for all $k \in \mathbb{N}$, all $\phi : C(X) \to M_k(B_n)$ homomorphisms with finite dimensional range and all unitaries $u \in M_{k+1}(B_n)$.

Now let

$$B = \bigoplus_{n=1}^{\infty} B_n,$$

the set of all sequences b with $b_n \in B_n$ and $\|b_n\| \to 0$. Then B is a σ-unital C^*-algebra. The multiplier algebra $M(B)$ of B is

$$M(B) = \prod_{n=1}^{\infty} B_n,$$

the set of all sequences b with $b_n \in B_n$ and $\sup_n \|b_n\| < \infty$. Let $\pi : M(B) \to M(B)/B$ be the quotient map. Let $\Psi = \{\psi_n\}, \Phi = \{\varphi_n\} : C(X) \to M(B)$ be the contractive positive linear maps defined by the sequences $\{\psi_n\}$ and $\{\varphi_n\}$, respectively. Since $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in the unital ball of $C(X)$ and $\delta_n \to 0$, it follows that $\pi \circ \Psi$ and $\pi \circ \Phi$ are homomorphisms. Set $\overline{\Psi} = \pi \circ \Psi$ and $\overline{\Phi} = \pi \circ \Phi$. We claim that $\overline{\Psi}_* = \overline{\Phi}_* : \underline{K}(C(X)) \to \underline{K}(M(B)/B)$. Since $\bigcup_{n=1}^{\infty} \overline{\mathcal{P}}(n)$ generates

$\underline{K}(C(X))_+$, it suffices to show that $\overline{\Psi}_*([p]) = \overline{\Phi}_*([p])$ for all projections $p \in \mathcal{P}(s)$ and all $s \in \mathbb{N}$. We have

$$\|(\varphi_n \otimes \mathrm{id})(p^2) - (\varphi_n \otimes \mathrm{id})(p)\| \to 0.$$

There is a sequence of projections $p_n \in M_L(B_n \otimes C(C_m \times \mathbb{S}^1))$ such that

$$\|(\psi_n \otimes \mathrm{id})(p) - p_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Similarly, there is a sequence of projections $q_n \in M_L(B_n \otimes C(C_m \times \mathbb{S}^1))$ such that

$$\|(\varphi_n \otimes \mathrm{id})(p) - q_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Since $(\psi_n)_* = (\varphi_n)_*$ on $\overline{\mathcal{P}}(n)$ and $[p] \in \overline{\mathcal{P}}(n)$ for all $n \geqslant s$, we conclude that

$$[p_n] = [q_n] \quad \text{in} \quad K_0(B_n \otimes C(C_m \times \mathbb{S}^1))$$

for all $n \geqslant s$. Since $B_n \in \mathbf{A}_r$, there is $K > 0$ projections $e_n \in M_K(B_n \otimes C(C_m \times \mathbb{S}^1))$ and unitaries $v \in M_{K+1}(B_n \otimes C(C_m \times \mathbb{S}^1))$ such that

$$v_n^*(p_n \oplus e_n)v_n = q_n \oplus e_n$$

for all $n \geqslant s$. It follows that

$$\overline{\Psi}_*([p]) = \overline{\Phi}_*([p]).$$

Therefore $\overline{\Psi}_* = \overline{\Phi}_* : \underline{K}(C(X)) \to \underline{K}(M(B)/B)$. By applying Theorem 1.2 and Theorem A in [12], we obtain an integer $L \in \mathbb{N}$, a homomorphism $F : C(X) \to M_L(M(B)/B)$ with finite dimensional range and a unitary $u \in M_{L+1}(M(B)/B)$ such that

$$\|u^*(\overline{\Psi}(f) \oplus F(f))u - \overline{\Phi}(f) \oplus F(f)\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$. There are points $\xi_1, \xi_2, \ldots, \xi_k \in X$ and mutually orthogonal projections $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_k \in M_L(M(B)/B)$ such that

$$F(f) = \sum_{j=1}^{k} f(\xi_j)\overline{d}_j$$

for $f \in \mathcal{F}$. By Theorem 1.3, we obtain mutually orthogonal projections $d_i \in M_L(M(B))$ such that $\pi(d_i) = \overline{d}_i$ for $i = 1, 2, \ldots, k$ and unitary $U \in M(B)$ such that $\pi(U) = u$. Hence there are $b_f \in M_{L+1}(B)$ for $f \in \mathcal{F}$ such that

$$\left\| U^*\Big(\{\psi_n(f)\} \oplus \sum_{i=1}^{k} f(\xi_i)d_i\Big)U - \{\varphi_n(f)\} \oplus \sum_{i=1}^{k} f(\xi_i)d_j - b_f \right\| < \frac{\varepsilon}{2}.$$

Write $U = \{u_n\}$, where each u_n is a unitary in B_n, $b_f = \{b_n^{(f)}\}$, then $\lim_{n\to\infty} \|b_n^{(f)}\| = 0$. This implies that there exists a sequence of homomorphisms $\phi_n^{(1)} : C(X) \to M_L(B_n)$ with finite dimensional range such that for sufficiently large n we have

$$\|u_n^*(\psi_n(f) \oplus \phi_n^{(1)}(f))u_n - \varphi_n(f) \oplus \phi_n^{(1)}(f)\| < \frac{\varepsilon}{2}$$

for $f \in \mathcal{F}$. ∎

REMARK 1.7. The requirement that $A \in \mathbf{A}_r$ is used to show that $\overline{\Psi}_* = \overline{\Phi}_*$. If X is a compact contractive space, then one always has $\overline{\Psi}_* = \overline{\Phi}_* = 0$. Therefore, if X is a compact contractive space, the condition that $A \in \mathbf{A}_r$ can be removed in Theorem 1.6.

The following lemma justifies our assumption that maps are linear and positive. It is certainly known. We state here without proof.

PROPOSITION 1.8. (see p. 371 in [14] and also 2.1 in [44]) *Let A be a nuclear C^*-algebra. For any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there is $\delta > 0$, a finite $G \subset \mathbf{C}$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any contractive map $\psi : A \to B$, where B is a C^*-algebra, if*

$$\|\psi(\lambda f + g) - \lambda\psi(f) - \psi(g)\| < \delta,$$

$$\|\psi(f^*) - \psi(f)^*\| < \delta \quad \text{and} \quad \|\psi(fg) - \psi(f)\psi(g)\| < \delta$$

for all $f, g \in \mathcal{G}$ and $\lambda \in G$, then there is a contractive completely positive linear map $\varphi : A \to B$ which is ε-$\mathcal{F}$-multiplicative such that

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

NOTATION 1.8. Let X be a compact metric space and A be unital C^*-algebra. Fix a finite subset of projections $\mathcal{P}$ in $\bigcup_{n=1} M_\infty(C(X) \otimes C(C_n \times \mathbb{S}^1))$. Let $\mathbf{h}(X_{\mathcal{P}}, A)$ be the set of those maps $\alpha : \overline{\mathcal{P}} \to \underline{K}(A)$ such that there is a homomorphism $\varphi : C(X) \to M_m(A)$ for some integer m which has the property that $\varphi_* = \alpha : \overline{\mathcal{P}} \to \underline{K}(A)$. Let $\mathbf{h}_0(X_{\mathcal{P}}, A)$ be the set of those maps $\alpha \in \mathbf{h}(X_{\mathcal{P}}, A)$ such that $\alpha = \varphi_*$ for some homomorphism φ which has finite dimensional range. Let $\mathbf{ah}_{\delta\text{-}\mathcal{F}}(X_{\mathcal{P}}, A)$ be the set those maps $\alpha : \overline{\mathcal{P}} \to \underline{K}(A)$ such that there is a contractive positive linear map $\psi : C(X) \to M_m(A)$ for some integer m which is δ-$\mathcal{F}$-multiplicative, where $\delta = \delta(\mathcal{P})$ as defined in Remark 1.4, $\mathcal{G}(\mathcal{P}) \subset \mathcal{F}$ and $\mathcal{G}(\mathcal{P})$ as defined in Remark 1.4. We denote by $\mathbf{aah}(X_{\mathcal{P}}, A)$ the set those maps α such that there is an asymptotic (positive linear) morphism $\{\psi_t\} : C(X) \dashrightarrow M_m(A)$ with the property that $\{\varphi_t\}_* = \alpha : \overline{\mathcal{P}} \to \underline{K}(A)$.

COROLLARY 1.10. *Let X be a compact metric space, $r : \mathbb{N} \to \mathbb{N}$, and let $\mathcal{F}$ be a finite subset of the unit ball of $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\bigcup_{n=1}^{\infty} M_\infty(C(X) \otimes C(C_n \times \mathbb{S}^1))$, a finite subset $\mathcal{G}$ of the unit ball of $C(X)$ and $\delta > 0$ such that whenever A is a unital C^*-algebra in $\mathbf{A}_r$ and whenever $\psi : C(X) \to A$ is a unital contractive positive linear map which is δ- $\mathcal{G}$-multiplicative and $\psi_* \in \mathbf{h}(X_{\mathcal{P}}, A)$, then there are an integer L and a unital homomorphism $\phi_1 : C(X) \to M_L(A)$ with finite dimensional range and a unital homomorphism $\phi_2 : C(X) \to M_{L+1}(A)$ such that*

$$\|\psi(f) \oplus \phi_1(f) - \phi_2(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

The following is an easy corollary of Corollary 1.10. It also follows from a result in [24].

COROLLARY 1.11. *For any $\varepsilon > 0$ and integer $n > 0$, there exists $\delta > 0$ satisfying the following: If A is a unital C^*-algebra and $x_1, x_2, \ldots, x_n$ are normal elements in the unit ball of A such that*

$$\|x_i x_j - x_j x_i\| < \delta$$

for all i and j, then there exist mutually commuting normal elements $y_1, y_2, \ldots, y_n \in M_L(A)$ with $\|y_i\| \leqslant 1$ and with finite spectrum, and mutually commuting normal elements $z_1, z_2, \ldots, z_n \in M_{L+1}(A)$ with $\|y_i\| \leqslant 1$ and with finite spectrum for some integer $L > 0$ such that

$$\|x_i \oplus y_i - z_i\| < \varepsilon \quad i = 1, 2, \ldots, n.$$

Proof. Let X be the product of n unit disk. Define a bounded linear map $\varphi : C(X) \to A$ such that $\varphi(g_i) = x_i$, where g_i is the standard generator of the unit disk, $i = 1, 2, \ldots, n$. Without loss of generality, we may assume that φ is contractive. For any finite subset $\mathcal{G}$ of $C(X)$ and $\sigma > 0$, if δ is small enough, by Proposition 1.8, without loss of generality, we may assume that φ is positive and γ-$\mathcal{G}$-multiplicative. Note that X is contractible. We then apply Remark 1.7. ∎

DEFINITION 1.12. (cf. 1.2 of [45]) Let ψ be a contractive linear map from $C(X)$ to C^*-algebra A, where X is a compact metric space. Fix a finite subset $\mathcal{F}$ contained in the unit ball of $C(X)$. For $\varepsilon > 0$, we denote by $\Sigma_\varepsilon(\psi, \mathcal{F})$ (or simply $\Sigma_\varepsilon(\psi)$) the closure of the set of those points $\lambda \in X$ for which there is a nonzero hereditary C^*-subalgebra B of A which satisfies

$$\|(f(\lambda) - \psi(f))b\| < \varepsilon \quad \text{and} \quad \|b(f(\lambda) - \psi(f))\| < \varepsilon$$

for $f \in \mathcal{F}$ and $b \in B$ with $\|b\| \leqslant 1$. Note that if $\varepsilon < \sigma$, then $\Sigma_\varepsilon(\psi) \subset \Sigma_\sigma(\psi)$.

PROPOSITION 1.13. *Let X be a compact metric space. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that whenever A is a unital C^*-algebra and whenever $\psi : C(X) \to A$ is a unital contractive positive linear map which δ-$\mathcal{G}$-multiplicative then $\Sigma_\varepsilon(\psi, \mathcal{F})$ is nonempty.*

Proof. Suppose that the proposition is false. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n, \dots$ be a sequence of finite subsets of the unit ball of $C(X)$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and the union $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in the unit ball of $C(X)$. Then there are a positive number $\varepsilon > 0$, a finite subset $\mathcal{F}$ of $C(X)$, a sequence of positive numbers $\delta_n \to 0$, unital C^*-algebras B_n, and unital contractive positive linear maps $\psi_n : C(X) \to B_n$ which are δ_n-$\mathcal{F}_n$-multiplicative, such that for any $n \in \mathbf{N}$, any hereditary C^*-subalgebra $C_n \subset B_n$ and any $\lambda \in X$ there is $f \in \mathcal{F}$ and there is $c_n \in C_n$ with $\|c_n\| \leqslant 1$ such that

$$\|[f(\lambda) - \psi_n(f)]c_n\| \geqslant 2\varepsilon.$$

Now let

$$B = \bigoplus_{n=1}^{\infty} B_n.$$

Then

$$M(B) = \prod_{n=1}^{\infty} B_n.$$

Let $\pi : M(B) \to M(B)/B$ be the quotient map. Let $\Psi = \{\psi_n\} : C(X) \to M(B)$ be the contractive positive linear map defined by the sequence $\{\psi_n\}$. Since $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in the unital ball of $C(X)$ and $\delta_n \to 0$, it follows that $\pi \circ \Psi$ is a homomorphism. We have $\ker(\pi \circ \Psi) = C_0(U)$ for some open subset $U \subset X$. Set $Y = X \setminus U$, and let $\overline{\Psi} : C(Y) \to M(B)/B$ be the map induced by $\pi \circ \Psi$. Since $\pi \circ \Psi \neq 0$, $Y \neq \emptyset$. Suppose that $\lambda \in Y$. Let h be a nonnegative function in $C(Y)$ such that $h(\lambda) = 0$ and $h(\xi) > 0$ elsewhere. Set $g = \overline{\Psi}(h)$. Denote by $\mathrm{Her}(g)$ the hereditary C^*-subalgebra of $M(B)/B$ generated by g. Note that $\mathrm{Her}(g)$ is σ-unital. It follows from Theorem 15 in [49] that the hereditary C^*-subalgebra

$$\mathrm{Her}(g)^\perp = \{x \in M(B)/B : xg = gx = 0\} \neq \{0\}.$$

It is easy to see that

$$\overline{\Psi}(f|Y)c = f(\lambda)c$$

for all $f \in C(Y)$ and $c \in \mathrm{Her}(g)^{\perp}$. Take a nonzero positive element $a \in \mathrm{Her}(g)^{\perp}$ with $\|a\| = 1$. There is a positive element $b = \{b_n\} \in M(B)$ such that $\|b\| = 1$, $\pi(b) = a$. Without loss of generality, we may assume that $\|b_n\| \geqslant 3/4$ for all n. Let $f_{1/2}$ and $f_{3/4}$ be positive function in $C_0((0,1])$ such that $0 \leqslant f_{1/2}, f_{3/4} \leqslant 1$, $f_{1/2}(t) = 1$ for all $t \geqslant 1/2$, $f_{3/4}(t) = 1$ for all $t \geqslant 3/4$ and $f_{3/4}(t) = 0$ for all $t \leqslant 1/2$. Set $H_n = \mathrm{Her}(f_{3/4}(b_n))$. Then, for any $c_n \in H_n$,

$$f_{1/2}(b_n)c_n = c_n f_{1/2}(b_n) = c_n$$

$n = 1, 2, \ldots$. Since

$$[f(\lambda) - \overline{\Psi}(f|Y)]f_{1/2}(a) = 0,$$

we conclude that, if $\|c_n\| \leqslant 1$,

$$\|[f(\lambda) - \psi_n(f)]c_n\| = \|[f(\lambda) - \psi_n(f)]f_{1/2}(b_n)c_n\| \to 0$$

as $n \to \infty$. This leads to a contradiction. ∎

DEFINITION 1.14. Fix a finite subset $\mathcal{F}$ of the unit ball of $C(X)$ and $\varepsilon > 0$. A contractive linear map $\psi : C(X) \to A$ is said to be *η-injective* (with respect to $\mathcal{F}$ and $\varepsilon > 0$) if

$$\mathrm{dist}(\lambda, \Sigma_\varepsilon(\psi, \mathcal{F})) < \eta$$

for all $\lambda \in X$.

LEMMA 1.15. *Let X be a compact metric space and let $\mathcal{F}$ be a finite subset of the unit ball of $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\bigcup_{n=1}^{\infty} M_\infty(C(X) \otimes C(C_n \times \mathbb{S}^1))$, $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G}$ of the unit ball of $C(X)$ such that whenever A is a purely infinite simple unital C^*-algebra and whenever $\psi : C(X) \to A$ is a contractive unital positive linear map which is δ-$\mathcal{G}$-multiplicative and is σ-injective with respect to δ and $\mathcal{F}$ and $\psi_* \in \mathbf{h}(X_{\mathcal{P}}, A)$, then there exists a unital homomorphisms $\phi : C(X) \to A$ such that*

$$\|\psi(f) - \phi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. The proof is the same as that of 1.6 in [44]. We use Proposition 1.13 and apply Theorem 1.6 in this paper instead of 1.4 in [44]. Note that, in fact, we only need to require that ψ is σ-injective with respect to some η (and $\mathcal{F}$), but we can make δ is smaller than η (so we do not need to introduce a new number $\eta > 0$). ∎

REMARK 1.16. From the proof of Lemma 1.15 (see the proof of 1.4 in [44]), it suffices to require the number σ satisfying the following:

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$, provided that $\operatorname{dist}(x, y) < \sigma$. Let us name this σ by $\sigma_X(\varepsilon, \mathcal{F})$. Let F be a compact subset of X. We note that one can take $\sigma_F(\varepsilon, h(\mathcal{F})) = \sigma_X(\varepsilon, \mathcal{F})$, where $h : C(X) \to C(F)$ is the quotient map, i.e., h is induced by the embedding $F \to X$.

THEOREM 1.17. *Let A be a unital purely infinite simple C^*-algebra and X be a compact finite CW complex. Then for any $\alpha \in KK(C(X), A)$, there is a homomorphism $\varphi : C(X) \to A$ such that $KK(\varphi) = \alpha$.*

Proof. Clearly, without loss of generality, we may assume that X is connected. Let $\gamma : KK(C(X), A) \to \mathrm{Hom}(K_*(C(X)), K_*(A))$ be in the Universal Coefficient Theorem (see [55]).

We first show that there is a homomorphism $\varphi_1 : C(X) \to A$ such that $(\varphi_1)_*$ as an element in $\mathrm{Hom}(K_*(C(X)), K_*(A))$ coincides with $\gamma \circ \alpha$.

We write $\gamma \circ \alpha = (\alpha_0, \alpha_1)$, where $\alpha_i \in \mathrm{Hom}(K_i(C(X)), K_i(A))$, $i = 0, 1$. We also write $K_0(C(X)) = G \oplus G_0$, where G is a finite generated free group and G_0 is a finitely generated torsion group.

Claim (1). There is a projection $1 \neq p \neq 0$ in A such that $\alpha([1]) = [p]$ and there is a homomorphism $\varphi' : C(X) \to pAp$ such that $\varphi'_{*0}|G = \alpha_0|G$, $\varphi'_{*0}|G_0 = 0$, and $\varphi'_{*1} = 0$.

One first maps $C(X)$ into a unital AF-algebra and then maps the AF-algebra into A. To get the required map, one uses Lemma 7.2 of [53] and Theorem 1.1 (and 1.4) of [13].

Claim (2). There is a homomorphism $\varphi'' : C(X) \to qAq$ such that $\varphi''_{*0}|G = 0$, $\varphi''_{*0}|G_0 = \alpha_0|G_0$, and $\varphi''_{*1} = \alpha_1$, where q is a nonzero projection with $[q] = 0$ in $K_0(A)$.

Suppose that G_0 has generators $g_1, g_2, \ldots, g_l$ and $K_1(C(X))$ has generators $f_1, f_2, \ldots, f_m$.

For each i, let g_i be of order $k_i > 0$. Set $s = k_i + 1$. Let $B_1 = C(X)$, $B_{n+1} = B_n \otimes M_s$ and let $h_n : B_n \to B_{n+1}$ be a monomorphism defined by

$$h_n(f) = \mathrm{diag}(f, f(\lambda_n), f(\lambda_n), \ldots, f(\lambda_n)),$$

where $\{\lambda_n\}$ is dense in X. By [32], the inductive limit $B_\infty = \lim_{n\to\infty}(B_m, h_n)$ is a simple C^*-algebra with real rank zero. Note that $(h_n)_{*0}(g_i) = sg_i = g_i$. Furthermore, since $(h_n)_{*1} = \mathrm{id}$, $K_1(B_\infty) = K_1(C(X))$ and there is a monomorphism

$h : C(X) \to B_\infty$ such that h_{*1} is an isomorphism, and $h_{*0}(g_i)$ is a torsion element with the same order as g_i. Let $C = B_\infty \otimes \mathcal{O}_\infty$. It is easy to see that $\mathcal{O}_\infty$ is approximately divisible (see the proof of 1.7 in [50]), which implies that C is purely infinite and simple, since B_∞ is simple (see [3]). Note also that there is an isomorphism from $K_*(B_\infty)$ onto $K_*(C)$. This gives an monomorphism $F_1 : C(X) \to C$ such that $(F_1)_{*1} = \alpha_1$ and $(F_1)_{*0}(g_i)$ has the same order as g_i. It follows from [51] that C is in Rørdam's classifiable class $\mathcal{C}$ (see [53]). In particular, C is a cross product in 7.1 of [53]. By applying 7.1 of [53] and composing that map with F_1, we obtain a monomorphism $F_2 : C(X) \to eAe$ for any nonzero projection $e \in A$ such that $(F_2)_{*1} = \alpha_1$ and $(F_2)_{*0}(g_i) = \alpha_0(g_i)$. By taking the diagonal form $\mathrm{diag}(F_2, F_3, \dots, F_{l+1})$, from the above construction, we obtain ψ'' as required in the Claim (2).

Now we may assume that p and q are mutually orthogonal and $p + q \leqslant 1$ in A. Set $\varphi_1 = \mathrm{diag}(\psi', \psi'')$. This monomorphism φ_1 meets the requirements.

Now we prove the theorem. By 8.3 of [53], there is a purely infinite simple C^*-algebra C_1 which is a cross product described in [53] with $K_*(C_1) = K_*(C(X))$. From above, there is a monomorphism $h : C(X) \to C_1$ such that h_* gives an isomorphism from $K_*(C(X))$ onto $K_*(C_1)$. Then $KK(h)$ is invertible by Proposition 7.2 in [55]. So this implies that $C(X)$ and C_1 are KK-equivalent. This also implies that $KK(C(X), A) \cong KK(C_1, A)$. Let $\alpha' \in KK(C_1, A)$ be the image of α under the above isomorphism. It follows that there is $\sigma \in KK(C_1, C_1)$ such that

$$KK(h) \times \sigma \times \alpha' = \alpha,$$

where " $\times$ " is Kasparov's product ($KK(\varphi)$ is invertible). Again, by 7.1 in [53], there are monomorphisms $h_1 : C_1 \to C_1$ such that $KK(h_1) = \sigma$ and $h_2 : C_1 \to A$ such that $KK(h_2) = \alpha'$. We then define $\varphi = h_2 \circ h_1 \circ h$. ∎

LEMMA 1.18. *Let X be a compact metric space. For any $\varepsilon > 0$, $\sigma > 0$, $\eta > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that whenever A is a unital C^*-algebraand whenever $\psi : C(X) \to A$ is a unital contractive positive linear map which δ-$\mathcal{G}$-multiplicative then there is a ε-$h(\mathcal{F})$-multiplicative contractive positive linear map $\psi : C(F) \to A$ which is σ-injective with respect to ε and $h(\mathcal{F})$ such that*

$$\|\varphi(f) - \psi \circ h(f)\| < \eta$$

for all $f \in \mathcal{F}$, where F is a compact subset of X and $h : C(X) \to C(F)$ is the quotient map (from $C(X) \to C(X)/I \cong C(F)$, $I = \{f \in C(X) : f(x) = 0$ for $x \in F\}$).

Proof. The proof uses an argument used in Theorem 1.6 and Proposition 1.13. Suppose the proposition is false. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots$ be a sequence of finite subsets of unit ball of $C(X)$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and the union $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in the unit ball of $C(X)$. Then there are positive numbers $\varepsilon > 0$, $\sigma > 0$, $\eta > 0$, a finite subset $\mathcal{F}$ of $C(X)$, a sequence of positive numbers $\delta_n \to 0$, unital C^*-algebras B_n, and for all n,

$$\inf_{F,\psi}\left\{\sup_{f\in\mathcal{F}}\{\|\varphi_n(f) - \psi \circ h(f)\|\}\right\} \geqslant \eta.$$

Here the infimun is taken for all compact subset $F \subset X$ and all ε-$h(\mathcal{F})$-multiplicative contractive linear positive maps $\psi : C(F) \to B_n$ which are σ-injective (with respect to ε and $\mathcal{F}$). Now denote $B = \bigoplus_{n=1}^{\infty} B_n$ and $M(B) = \prod_{n=1}^{\infty} B_n$.

Let $\pi : M(B) \to M(B)/B$ be the quotient map. Let $\Phi = \{\varphi_n\} : C(X) \to M(B)$ be the contractive positive linear map defined by the sequence $\{\varphi_n\}$. Since $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in the unital ball of $C(X)$ and $\delta_n \to 0$, it follows that $\pi \circ \Phi$ is a homomorphism. We have $\ker(\pi \circ \Psi) = \{f \in C(X) : f(x) = 0 \text{ for } x \in F\}$ for some compact subset $F \subset X$. Let $h : C(X) \to C(F)$ be the quotient map. There is a monomorphism $\overline{\Psi} : C(F) \to M(B)/B$ induced by $\pi \circ \Psi$. We have $\pi \circ \Phi = \overline{\Psi} \circ h$. Since $C(F)$ is nuclear, by [9], there is a contractive completely positive linear map $\Psi : C(F) \to M(B)$ such that $\pi \circ \Psi = \overline{\Psi}$. Thus, for each $f \in C(X)$,

$$\Phi(f) - \Psi \circ h(f) \in B.$$

We may write $\Psi = \{\psi_n\}$, where each $\psi_n : C(F) \to B_n$ is a contractive positive linear map. Since $\overline{\Psi}$ is a homomorphism,

$$\|\psi_n(fg) - \psi_n(f)\psi_n(g)\| \to 0$$

as $n \to \infty$ for all $f \in C(F)$. In particular, when n is large enough, ψ_n are ε-$h(\mathcal{F})$-multiplicative. With $Y = F$ in the proof of Proposition 1.13, we see that, when n is large enough, $\Sigma_\varepsilon(\psi_n) = F$, whence ψ_n are σ-injective.

We have

$$\|\varphi_n(f) - \psi_n \circ h(f)\| \to 0$$

as $n \to \infty$ for all $f \in C(X)$. This leads to a contradiction. ∎

THEOREM 1.19. *Let X be a compact metric space and let $\mathcal{F}$ be a finite subset of (the unit ball of) $C(X)$. For any $\varepsilon > 0$, there exist $\delta > 0$, and a finite subset $\mathcal{G}$ of (the unit ball of) $C(X)$ such that whenever A is a purely infinite simple unital C^*-algebra and whenever $\psi : C(X) \to A$ is a contractive unital positive linear map which is δ-$\mathcal{G}$-multiplicative, then there exists a unital homomorphisms $\phi : C(X) \to A$ such that*

$$\|\psi(f) - \phi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. We may assume that $\mathcal{F}$ is in the unit ball of $C(X)$. Since $C(X) = \lim_{n\to\infty} C(X_n)$, where each X_n is a compact finite CW-complex, without loss of generality, we may assume that the finite subset $\mathcal{F}$ lies in (the image of) some $C(X_n)$. So, without loss of generality, we may further assume that X is itself a finite CW-complex.

Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, let $\sigma = \sigma_X(\varepsilon, \mathcal{F})$ be as in Remark 1.16.

Since X is now assumed to be a compact finite CW-complex, there are finitely many compact subsets $X_1, X_2, \dots, X_m \subset X$ which are itself finite CW-complex such that for any compact subset $F \subset X$, there is one X_i satisfying

$$\sup\{\mathrm{dist}(F, x) : \lambda \in F, x \in X_i\} < \frac{\sigma}{2}$$

and $F \subset X_i$.

Let $\delta_1^{(i)}$ be positive and $\mathcal{G}_1^{(i)}$ be a finite subset of $C(X_i)$ such that there is homomorphism $\psi_1^{(i)} : C(X_i) \to A$ satisfying

$$\|\Phi(f) - \psi_1^{(i)}(f)\| < \frac{\varepsilon}{2}$$

for all $f \in h_i(\mathcal{F})$, where $h_i : C(X) \to C(X_i)$ is the quotient map, whenever A is a purely infinite simple C^*-algebra and $\Phi : C(X_i) \to A$ is a $\delta_1^{(i)}$-$\mathcal{G}_1^{(i)}$-multiplicative contractive positive linear map which $\sigma/2$-injective with respect to $\delta_1^{(i)}$ and $h_i(\mathcal{F})$. To get the above statement we apply Lemma 1.15, Remark 1.16 and Remark 1.21.

Let $\delta_2 = \min_{1 \leqslant i \leqslant m} \{\delta_2^{(i)}\}$ and $\mathcal{G}_2 = \bigcup_{i=1}^{m} h_i^{-1}(\mathcal{G}_2^{(i)})$. For above $\delta_2 > 0$, $\sigma > 0$ and $\mathcal{G}_2$, let $\delta_3 > 0$, a finite subset $\mathcal{G}_3$ in $C(X)$ be as in Lemma 1.18 such that there is a compact subset $F \subset X$ and a $\delta_2/2$-$h(\mathcal{G}_2)$- multiplicative positive linear map $\psi_2 : C(F) \to A$ which is $\sigma/2$-injective with respect $\delta_2/2$ and $h(\mathcal{G}_2)$ satisfying:

$$\|\varphi(f) - \psi_2 \circ h(f)\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{G}_2$, whenever A is a unital purely infinite simple C^*-algebra and φ is δ_3-$\mathcal{G}_3$-multiplicative positive linear map from $C(X)$ into A, where $h : C(X) \to C(F)$ is the quotient map.

So now if $\varphi : C(X) \to A$ is δ_3-$\mathcal{G}_3$-multiplicative positive linear map from $C(X)$ into A, let ψ_2 as above. Choose X_i above so that $F \subset X_i$ and

$$\sup\{\operatorname{dist}(F, x) : \lambda \in F, x \in X_i\} < \frac{\sigma}{2}.$$

Let $h_0 : C(X_i) \to C(F)$ be the quotient map and $\psi_1 = \psi_2 \circ h_0 : C(F) \to A$. Then ψ_1 is $\delta_2/2$-$h_i(\mathcal{G}_2)$-multiplicative positive linear map which is σ-injective with respect to $\delta_2/2$ and $\mathcal{G}_2$. By our choice of δ_2 and $\mathcal{G}_2$, there is a homomorphism $\varphi_1 : C(X_i) \to A$ such that

$$\|\psi_2 \circ h_0(f) - \varphi_1(f)\| < \frac{\varepsilon}{2}$$

for all $f \in h_i(\mathcal{F})$. Note that $\psi_2 \circ h_0 \circ h_i = \psi_2 \circ h$. We have

$$\|\varphi(f) - \varphi_1 \circ h_i(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. Set $\varphi = \varphi \circ h_i$. ∎

COROLLARY 1.20. *For any $\varepsilon > 0$ and integer $n > 0$, there exists $\delta > 0$ satisfying the following: If A is a unital purely infinite simple C^*-algebra and $x_1, x_2, \ldots, x_n$ are normal elements in the unit ball of A such that*

$$\|x_i x_j - x_j x_i\| < \delta$$

for all i and j, then there exist mutually commuting normal elements $y_1, y_2, \ldots, y_n \in A$ with $\|y_i\| \leqslant 1$ such that

$$\|x_i - y_i\| < \varepsilon$$

for all i.

Proof. The proof is similar to that of Corollary 1.11. But we now apply Theorem 1.19 instead. ∎

REMARK 1.21. Unlike the case that A is a purely infinite simple C^*-algebra there are K-theoretical obstacles for an almost multiplicative contractive positive linear morphism being close to a homomorphism in the case that A is a stable finite C^*-algebra. Let φ be a homomorphism from $C(X)$ into A. Then, φ induces a homomorphism φ_{*0} in $\mathrm{Hom}(K_0(C(X)), K_0(A))$. More important, φ_{*0} preserves the order. It is shown in [48], in general, an asymptotic morphism gives an element in $\mathrm{Hom}(K_0(C(X)), K_0(A))$ which may not preserve the order. Such morphisms can not be approximated by homomorphisms. In Proposition 1.26 below, we see that there are plenty of such examples.

For the rest of this section, we will consider two closely related problems. Given a δ-$\mathcal{G}$-multiplicative morphism ψ which induces ψ_* on $\overline{\mathcal{P}}$, is there a η-$\mathcal{H}$-multiplicative morphism φ such that $\varphi_* = \psi_*$ on $\overline{\mathcal{P}}$ for any $\eta > 0$ and any finite subset $\mathcal{H}$ of $C(X)$? Is there a ε-$\mathcal{F}$-multiplicative morphism $\psi^* : C(X) \to M_k(A)$ such that $\psi \oplus \psi^*$ is close to a point-evaluation $h : C(X) \to M_{k+1}(A)$ for some k?

PROPOSITION 1.22. (cf. 5.3 in [13]) *Let X be a (compact) finite* CW-*complex and A be a unital C^*-algebra. Suppose that $\{\varphi_t\} : C(X) \to A$ is an asymptotic positive (contractive and linear) morphism. Then there exists an asymptotic positive (contractive, linear) morphism $\{\psi_t\} : C(X) \to M_m(A)$ such that $\{\psi_t\} \oplus \{\psi_t\}$ is homotopic to a point evaluation map $\Psi : C(X) \to M_{m+1}(A)$, where m can be chosen to be $2\dim(X) + 1$.*

Proof. Clearly, without loss of generality, we may assume that X is connected. Let Y be a subspace of X by removing one point, say $\xi \in X$. By [16], $\mathrm{id}_{C_0(Y)}$ has an additive inverse which maps $C(X)$ into $M_m(A)$, where m can be chosen to be $2\dim(X) + 1$. Then, by 5.3 in [13], there is an asymptotic (positive, contractive linear) morphism $\{\psi'_t\} : C_0(Y) \to M_m(A)$ such that $\varphi|(C_0(Y)) \oplus \{\psi'_t\}$ is homotopic to zero. Set $\{\psi_t\}$ by defining

$$\psi_t(f) = f(\xi)1_m + \psi'_t(f - f(\xi))$$

for all $f \in C(X)$ and t. Clearly, $\{\varphi_t\} \oplus \{\psi_t\}$ is homotopic to a point evaluation Ψ. ∎

PROPOSITION 1.23. *Let X be a compact metric space. For any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and finite subset $\mathcal{G} \subset C(X)$ such that whenever A is a unital C^*-algebra and whenever $\psi : C(X) \to A$ is a unital contractive positive linear map which is δ-$\mathcal{G}$-multiplicative then there exists a unital contractive positive linear map $\varphi : C(X) \to M_m(A)$ which is ε-$\mathcal{F}$-multiplicative and a unital homomorphism $\varphi_0 : C(X) \to M_{m+1}(A)$ with finite dimensional range such that*

$$\|\psi(f) \oplus \varphi(f) - \varphi_0(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. First note that $C(X) = \lim_{n\to\infty} C(X_n)$, where each X_n is a finite CW-complex. Clearly, without loss of generality, we may assume that $\mathcal{F} \subset C(X_n)$ for some n. Therefore, without loss of generality, we may further assume, as well, that X is a finite CW-complex. Furthermore, without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $C(X)$. The proof uses the similar argument used in both Theorem 1.6 and Proposition 1.13. We will be a little sketchy. Assume that Proposition 1.23 is false. We would have a sequence $\{\psi_n\}$ of δ_n-multiplicative (with respect to $\mathcal{F}_n$) contractive unital positive linear maps, which does not satisfy the conclusion of Proposition 1.23, from $C(X)$ into a sequence of unital C^*-algebras $\{B_n\}$. We will keep the notation in Proposition 1.13. From this sequence of $\{\psi_n\}$, we would have a homomorphism $\pi \circ \Psi : C(X) \to M(B)/B$. By applying Proposition 1.22, we obtain an asymptotic positive (contractive linear) morphism $\{\psi'_t\} : C(X) \to M_m(M(B)/B$ such that $\pi \circ \Psi \oplus \{\psi'_t\}$ is homotopic to a homomorphism $\Phi' : C(X) \to M_{m+1}(M(B)/B)$ with finite dimensional range for some m. We now apply [11]. We obtain a homomorphism $\tilde{\psi} : C(X) \to M_k(M(B)/B)$ with finite dimensional range and a homomorphism $\Phi : C(X) \to M_{k+2}(M(B)/B)$ with finite dimensional range such that

$$\|[\pi \circ \Psi(f) \oplus \psi'_t(f) \oplus \tilde{\psi}(f) - \Phi(f)\| < \frac{\varepsilon}{8}$$

for all $f \in \mathcal{F}$, some $t > 0$ and some $k > 0$. Denote $\psi'_t \oplus \tilde{\psi}$ by $\overline{\psi}$. We note that $\overline{\psi}$ is a unital contractive positive linear map from $C(X)$ into $M_k(M(B)/B)$. By [9], there is a contractive positive linear map $\Psi_0 = \{\psi'_n\}$ from $C(X)$ into $M_k(M(B))$ such that $\pi \circ \Psi_0 = \overline{\psi}$. Since Φ has finite dimensional range, there is a homomorphism $\Phi_{00} = \{\varphi'_n\} : C(X) \to M_{k+1}(M(B))$ with finite dimensional range such that $\pi \circ \Phi_{00} = \Phi$. Therefore, when n is large enough,

$$\|\psi_n(f) \oplus \psi'_n(f) - \varphi'_n(f)\| < \frac{\varepsilon}{4}$$

for all $f \in \mathcal{F}$. Let $p_n = \mathrm{diag}(0, 1_k)$ be the projection in $M_{k+1}(B_n)$. Then,

$$\|\psi'_n(f) - p_n\varphi'_n(f)p_n\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|p_n\psi'_n(f) - \psi_n(f)p_n\| < \frac{\varepsilon}{2}.$$

This implies that ψ'_n is ε-multiplicative with respect to $\mathcal{F}$, since ψ'_n is a homomorphism. By the argument in Proposition 1.13, this ends the proof. ∎

COROLLARY 1.24. *Let X be a finite CW-complex, $m = 2\dim(X) + 1$ and let $\mathcal{P}$ be a finite subset of projections in $\bigcup_{m>0} M_\infty(C(X) \otimes C(C_m \times \mathbb{S}^1))$. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ which contains $\mathcal{G}(\mathcal{P})$ (see Remark 1.4), there exist $\delta > 0$ and finite subset $\mathcal{G} \subset C(X)$ such that whenever A is a unital C^*-algebra and whenever $\psi : C(X) \to A$ is a unital contractive positive linear map which is δ-$\mathcal{G}$-multiplicative then there exists a contractive unital positive linear map $\varphi : C(X) \to M_m(A)$ which is ε-$\mathcal{F}$-multiplicative such that there is an element $\alpha \in KL(C(X), A)$ which is represented by homomorphism which is a point evaluation with the property that*

$$(\psi \oplus \varphi)_* = \alpha \quad \text{on } \overline{\mathcal{P}}.$$

Proof. This is an immediate consequence of the proof of Proposition 1.23. As in the proof of Proposition 1.23, we first assume that Proposition 1.23 is false. We would have a sequence $\{\psi_n\}$ of δ_n-multiplicative (with respect to $\mathcal{F}_n$) contractive unital positive linear maps, which does not satisfy the conclusion of Corollary 1.24, from $C(X)$ into a sequence of unital C^*-algebras $\{B_n\}$. We follow the proof of Proposition 1.23. In the proof of Proposition 1.23, we note that, since $\tilde{\psi}$ is a homomorphism with finite dimensional range, one can write $\psi_n' = \psi_n^{(1)} \oplus \psi_n^{(2)}$, where $\{\psi_n^{(2)}\}$ is a homomorphism lift of $\tilde{\psi}$. Since (see the proof of Proposition 1.23) ψ_n' is ε-$\mathcal{F}$ multiplicative, so is $\psi_n^{(1)}$ which is a positive map from $C(X)$ into $M_m(A_n)$. If n is large enough, we would have

$$\|\psi_n(f) \oplus \psi_n^{(1)}(f) \oplus \psi_n^{(2)}(f) - \varphi_n'(f)\|$$

is small for all $\mathcal{F}$. Since $\mathcal{G}(\mathcal{P}) \subset \mathcal{F}$, the above inequality implies that, when n is large enough, $(\psi_n \oplus \psi_n^{(1)})_*$ on $\overline{\mathcal{P}}$ is the same as some required α in $KL(C(X), A_n)$. This leads to a contradiction. ∎

PROPOSITION 1.25. *In Proposition 1.23, if $\psi_* \in \mathbf{aah}(X_{\mathcal{P}}, A)$, then we can choose φ to be η-$\mathcal{H}$-multiplicative for any $\eta > 0$ and any finite subset $\mathcal{H}$ of $C(X)$.*

Proof. This is a corollary of Theorem 1.6 and the proof of Proposition 1.22. By the proof of Proposition 1.22, there is an asymptotic (positive contractive linear) morphism $\{\psi_t\} : C(X) \to M_k(A)$ such that

$$(\psi \oplus \{\psi_t\})_* = (\varphi_0)_*$$

for some homomorphism $\varphi_0 : C(X) \to M_{k+1}(A)$ with finite dimensional range. So, for any $\eta > 0$ and finite subset $\mathcal{H}$ of $C(X)$, we choose φ_t for some large $t > 0$ so that ψ_t is η-multiplicative with respect to $\mathcal{H}$. Then apply the proof of

Theorem 1.6 to $\psi \oplus \psi_t$. We will obtain two homomorphisms $\varphi' : C(X) \to M_m(A)$ and $\varphi_{00} : C(X) \to M_{m+1}(A)$ both with finite dimensional range such that

$$\|\psi(f) \oplus [\psi_t(f) \oplus \varphi'(f)] - \varphi_{00}(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$ and some m, if δ is small enough, $\eta \leqslant \delta$, $\mathcal{G}$ is large enough and $\mathcal{G} \subset \mathcal{H}$. Note that $[\psi_t \oplus \varphi']$ is η-multiplicative with respect to $\mathcal{H}$. ∎

PROPOSITION 1.26. *Let X be a finite CW-complex with torsion free K_*-theory. Then, for any unital C^*-algebra A and $\alpha \in KK(C(X), A)$ with the property that $[\alpha([\mathbf{1}])] = [d]$ for some projection in $M_\infty(A)$, and any $\delta > 0$ and finite subset $\mathcal{F}$ of $C(X)$, there is a contractive positive linear map $\psi : C(X) \to M_k(A)$ which is δ-$\mathcal{F}$-multiplicative for some integer k such that the induced map ψ_* coincides with α on the generators of $K_*(C(X))$.*

Proof. Without loss of generality, we may assume that X is connected. Let $\alpha = (\alpha_1, \alpha_2)$ be homomorphisms from $K_i(C(X))$ into $K_i(A)$, $i = 0, 1$. Fix a point $\xi \in X$. We have $K_1(C_0(X \setminus \{\xi\}) = K_1(C(X))$ and there is an embedding from $K_0(C_0(X \setminus \{\xi\})$ into $K_0(C(X))$. Let α' be the induced maps from $K_1(C_0(X \setminus \{\xi\})$ into $K_1(A)$ and from $K_0(C_0(X \setminus \{\xi\})$ into $K_0(C(X))$. So $\alpha' \in \mathrm{Hom}(K_*(C_0(X \setminus \{\xi\})), A \otimes \mathcal{K})$. By 5.4 in [14], there is an asymptotic morphism $\{\psi_t\} : C_0(X \setminus \{\xi\}) \to A \otimes \mathcal{K}$ such that $(\{\psi_t\})_* = \alpha'$. Let $e_n = \mathbf{1} \otimes \sum_{i=1}^{n} e_{ii}$, where $\{e_{ij}\}$ is a matrix unit. Then $\{e_n\}$ forms an approximate identity for $A \otimes \mathcal{K}$. For any fixed finite subset $\mathcal{G}$ of $C(X)$, any $\varepsilon > 0$, and any (large) $t > 0$, there is an integer n such that

$$\|e_n \psi_t(f) - \psi_t(f) e_n\| < \varepsilon$$

for all $f \in \mathcal{G}$. Let p be a projection in $(\mathbf{1} - e_n)(A \otimes \mathcal{K})(\mathbf{1} - e_n)$ such that $[p] = [d]$ in $K_0(A)$.A Define $\psi(f) = f(\xi)p + e_n \psi_t(f - f(\xi))e_n$ for $f \in C(X)$. Then ψ is a contractive positive linear map which δ-$\mathcal{F}$-multiplicative, provided that ε is small enough and $\mathcal{G}$ and t are large enough. Note that $\psi_*([\mathbf{1}]) = \alpha([\mathbf{1}])$. Since $K_i(C(X))$ are torsion free, $i = 0, 1$, by the Universal Coefficient Theorem, $KK(C(X), A) = \mathrm{Hom}(K_*(C(X)), K_*(A))$. Since $K_i(C(X))$ are finitely generated, using Remark 1.4, by taking sufficient large t and $\mathcal{G}$, we have

$$\varphi_* = \alpha$$

on the generators of $K_*(C(X))$. ∎

COROLLARY 1.27. *In Proposition 1.23, if X is as in Proposition 1.26, then one can choose φ to be η-$\mathcal{H}$-multiplicative for any $\eta > 0$ and any finite subset $\mathcal{H}$ of $C(X)$.*

REMARK 1.28. Theorem 1.17 uses, among other things, the results in C^*-algebra classification theory. In fact, without these deeper results, one can still show the following: Let X be a finite CW-complex and A be a unital purely infinite simple C^*-algebra. Then, for any $\alpha \in KK(C(X), A)$, there exists an asymptotic (contractive, positive linear) morphism $\{\psi_t\} : C(X) \to A$ such that $\{\psi_t\}$ induces α. This just requires to observe that, in the case that A is a purely infinite simple C^*-algebra, one can improve slightly a result of Dadarlat and Loring ([14]). Let $b \in A$ be a nonzero positive element such that $b \leqslant q$ for some projection $q \in A$ and b is not invertible in qAq, and let B be a σ-unital hereditary C^*-subalgebra of B generated by b. By [58], B is stable. By [4], $A \otimes \mathcal{K} \cong B \otimes \mathcal{K} \cong B$. Let Y be the subspace of X by removing a point $\xi \in X$. From 5.3 in [14], for any element $\gamma \in E(C_0(Y), B)$ there exists an asymptotic (contractive positive linear) morphism $\{\psi_t'\} : C(X) \to B$ such that $\{\psi_t'\}$ induces γ. Suppose that $\alpha(\mathbf{1}_{C(X)}) = x$ is an element in $K_0(A)$. Since A is purely infinite, there exists a projection $p \geqslant q$ such that $[p] = x$ in $K_0(A)$. Define

$$\psi_t(f) = f(\xi)p + \psi_t'(f - f(\xi))$$

for all $f \in C(X)$ and all t. It is clear that $\{\psi_t\}$ meets the requirement.

With the above, one obtain the following:

COROLLARY 1.29. *Let X be a compact metric space and A be a unital purely infinite simple C^*-algebra. For any finite subset $\mathcal{F} \subset C(X)$ and $\varepsilon > 0$ there exist $\delta > 0$ and a finite subset $\mathcal{G}(\supset \mathcal{F})$ of $C(X)$ such that for any unital contractive positive linear map $\varphi : C(X) \to A$ which is δ-$\mathcal{G}$-multiplicative, there is a unital contractive positive linear map $\varphi_1 : C(X) \to A$ which is η-$\mathcal{H}$-multiplicative and a unital homomorphism $\psi : C(X) \to M_2(A)$ with finite dimensional range satisfying*

$$\|\varphi(f) \oplus \varphi_1(f) - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, and for any $\eta > 0$ and any finite subset $\mathcal{H}$.

Proof. By the fact that $C(X) = \lim_{n\to\infty} C(X_n)$, where each X_n is a finite CW-complex, it suffices to prove the lemma for the case that X is a finite CW-complex.

A fact in [31] shows that, with sufficient small δ and sufficient large $\mathcal{G}$, $\varphi_* = \alpha$ on given $\overline{\mathcal{P}}$ ($\mathcal{P}$ is as in Remark 1.4) for some element $\alpha \in KL(C(X), A)$. We then apply Remark 1.28 and Proposition 1.25. ∎

2. APPLICATIONS

DEFINITION 2.1. Let X be a compact metric space and A be a σ-unital stable C^*-algebra(with an approximate identity consisting of a sequence of increasing projections). A homomorphism $\varphi : C(X) \to M(A)$ is said to be *diagonalizable*, if there exist an approximate identity $\{e_n\}$ consisting of a sequence of projections and a sequence $\{\lambda_n\}$ of X such that

$$\varphi(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1}),$$

where $e_0 = 0$, for all $f \in C(X)$.

An extension $\tau : C(X) \to M(A)/A$ is *diagonalizable* if there is a diagonalizable homomorphism $\varphi : C(X) \to M(A)$ such that $\pi \circ \varphi = \tau$, where $\pi : M(A) \to M(A)/A$ is the quotient map.

THEOREM 2.2. *Let X be a compact metric space, B be a unital C^*-algebra and $A = B \otimes \mathcal{K}$ with the property that for any finite subset $\mathcal{P}$ of projections in $\bigcup_{n=1}^{\infty} M_\infty(C(X)\otimes C(C_n\times\mathbb{S}^1))$, there is $\delta > 0$ such that $\mathbf{ah}_{\delta,\mathcal{P}}(X_\mathcal{P}, B) = \mathbf{aah}(X_\mathcal{P}, B)$. Let $\varphi : C(X) \to M(A)$ be a unital homomorphism. Then there exists a diagonalizable homomorphism $\varphi_1 : C(X) \to M(A)$ and a sequence of isometries $\{W_n\} \subset M_2(M(A))$ with $W_nW_n^* = 1_{M(A)}$ such that*

$$\|\varphi(f) \oplus \varphi_1(f) - W_n^*\varphi_1(f)W_n\| \to 0$$

and

$$\varphi(f) \oplus \varphi_1(f) - W_n^*\varphi_1(f)W_n \in A$$

for all $f \in C(X)$.

Proof. First we show that Theorem 2.2 holds for the case that φ is a diagonalizable homomorphism. Suppose that

$$\varphi(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1})$$

for all $f \in C(X)$, where $\{\lambda_n\}$ is a dense sequence of X. (We will consider later the case where $\{\lambda_n\}$ is not dense.) Without loss of generality, by rewriting φ, we may assume that $e_n - e_{n-1}$ is equivalent to a projection in B.

There is a double sequence $\{p_n^{(k)}\}_{n,k=1}$ of mutually orthogonal projections in A such that each $p_n^{(k)}$ is equivalent to the identity of B, $\sum_{n,k=1} p_n^{(k)}$ converges to the identity of $M(A)$ in the strict topology, and every infinite sums of $p_n^{(k)}$ converges to a projection in $M(A)$ in the strict topology. Define

$$\varphi_1(f) = \sum_{n,k=1} f(\lambda_n)p_n^{(k)}$$

for every $f \in C(X)$. Then $\varphi_1 : C(X) \to M(A)$ is a diagonalizable homomorphism. Let $q_n^{(k)}$ and $d_n^{(k)}$ be projections such that q_n^k is equivalent to $e_n - e_{n-1}$ for each n and k, for fixed n, $d_n^{(k)}$ are mutually equivalent for all k, and $p_n^{(k)} = q_n^{(k)} + d_n^{(k)}$ for each n and k. It follows that there are partial isometries $\{w_n^{(k)}\} \subset A$ such that

$$(w_n^{(1)})^*(w_n^{(1)}) = (e_n - e_{n-1}) \oplus d_n^{(1)}, (w_n^{(k+1)})^*(w_n^{(k+1)}) = q_n^{(k)} \oplus d_n^{(k+1)}$$

and

$$(w_n^{(k)})(w_n^{(k)})^* = p_n^{(k)},$$

$k = 1, 2, \ldots$. It is easy to check that $W = \sum_{n,k=1} w_n^{(k)}$ converges in the strict topology,

$$W^*W = \mathbf{1}_{M_2(A)} \quad \text{and} \quad WW^* = \mathbf{1}_{M(A)}.$$

Furthermore, we see that

$$\varphi(f) \oplus \varphi_1(f) = W^*\varphi_1(f)W$$

for all $f \in C(X)$.

If $\{\lambda_n\}$ is not dense, then we can add φ_1 first, and then apply the above argument. Note that $M(A) \cong M_2(M(A))$.

Now we consider the general case. We fix a sequence $\{\lambda_n\}$ such that $\{\lambda_n\}_{n=k}^{\infty}$ is dense for each k. Let φ_1 be as above but replace the sequence by the new sequence $\{\lambda_n\}$.

It is well known that

$$K_0(M(A)) = K_1(M(A)) = \{0\}.$$

By Theorem A in [11], for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$ of $C(X)$, there are two homomorphisms $\varphi_2 : C(X) \to M_k(M(A))$ and $\varphi_3 : C(X) \to M_{k+1}(A)$ with finite dimensional range such that

$$\|\varphi(f) \oplus \varphi_2(f) - \varphi_3(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. One should note, again, that $M_2(M(A)) \cong M(A)$. From the first part of the proof, we know that $\varphi_2 \oplus \varphi_1$ is unitarily equivalent to φ_1. Therefore $\varphi \oplus \varphi_1$

is approximated by homomorphisms from $C(X) \to M_2(A)$ with finite dimensional range. Now we apply Zhang's quasi-diagonalization (see [59]). From Zhang's argument (see [59]), there exists an approximate identity $\{d_n\}$ for A consisting of projections such that

$$\|d_n(\varphi(f) \oplus \varphi_1(f)) - (\varphi(f) \oplus \varphi_1(f))d_n\| \to 0$$

as $n \to \infty$ for every $f \in C(X)$. We will use Proposition 1.25. Note that $\mathbf{aah}(X_{\mathcal{P}}, A) = \mathbf{ah}_{\delta,\mathcal{P}}(X, A)$. Let $\mathcal{F}$ be a finite subset of $C(X)$. Let $\mathcal{F}_1 = \mathcal{F}$ and let $\mathcal{F}_1, \mathcal{F}_2, \dots$ be an increasing sequence of finite subsets of $C(X)$ such that the union is dense in $C(X)$. Let $\{\varepsilon_n^{(k)}\}$ be a double sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n^{(k)} < \varepsilon/4$. Let $\{\delta_n^{(k)}\}$ be a sequence of positive number which converges to zero and $\{\mathcal{G}_n^{(k)}\}$ be an increasing sequence of finite subsets of $C(X)$ which corresponds to $\{\varepsilon_n^{(k)}\}$ and $\{\mathcal{F}_n\}$ as required in Proposition 1.23. Now we apply Proposition 1.25. By passing to a subsequence, if necessary, we may assume that $\alpha_n = (d_n - d_{n-1})(\varphi \oplus \varphi_1)(d_n - d_{n-1})$ is $\delta_n^{(1)}$-multiplicative with respect to $\mathcal{G}_n$ for each n. We may also assume that

$$\left\|\varphi(f) \oplus \varphi_1(f) - \sum_{n=1}^{\infty} \alpha_n(f)\right\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$. Since for any $a_n \in A$, if $\sum_{n=1}^{\infty} \|a_n\| < \infty$, then $\sum_{n=1}^{\infty} a_n \in A$, we may also assume that

$$\varphi(f) \oplus \varphi_1(f) - \sum_{n=1}^{\infty} \alpha_n(f) \in A$$

for all $f \in C(X)$. Proposition 1.25 provides, for each n, two sequences of $\delta_n^{(k)}$-$\mathcal{G}_n$-multiplicative positive contractive linear maps $\sigma_n^{(k)} : C(X) \to M_{m(n,k)}(A)$ and $\sigma_n^{(k)*} : C(X) \to M_{m(n,k)'}(A)$ such that

$$\|\alpha_n(f) \oplus \sigma_n^{(1)}(f) - \gamma_n^{(1)}(f)\| < \varepsilon_n^{(1)}$$

for some homomorphism $\gamma_n^{(1)} : C(X) \to M_{m(n,1)+1}(A)$ with finite dimensional range and for all $f \in \mathcal{F}_n$,

$$\|\sigma_n^k(f) \oplus \sigma_n^{(k)*}(f) - g_n^{(k)}(f)\| < \varepsilon_n^{(k)}$$

and

$$\|\sigma_n^{(k)*}(f) \oplus \sigma_n^{k+1}(f) - \gamma_n^{(k+1)}(f)\| < \varepsilon_n^{(k)}$$

for some homomorphisms $\gamma_n^{(k)} : C(X) \to M_{m(n,k+1)+m(n,k)'}(A)$ and $g_n^{(k)} : C(X) \to M_{m(n,k)+m(n,k)'}(A)$ both with finite dimensional range and for all $f \in \mathcal{F}_n$, $n, k = 1, 2, \ldots$. We note that A is stable. Set

$$\Phi_1 = \sum_{n,k=1} (\sigma_n^{(k)} + \sigma_n^{(k)*}), \quad \Phi_2 = \sum_{n,k=1} g_n^{(k)} \quad \text{and} \quad \Phi_3 = \sum_{n,k=1} \gamma_n^{(k)}.$$

Here Φ_2 and Φ_3 are homomorphisms from $C(X)$ into $M(A)$. We also have

$$\|\Phi_1(f) - \Phi_2(f)\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$ and

$$\Phi_1(f) - \Phi_2(f) \in A$$

for all $f \in C(X)$.

There is also an isometry $U \in M_2(M(A))$ such that $UU^* = \mathbf{1}_{M(A)}$,

$$\left\| \sum_{n=1}^{\infty} \alpha_n(f) \oplus \Phi_1(f) - U^*\Phi_3(f)U \right\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}$ and

$$\sum_{n=1}^{\infty} \alpha_n(f) \oplus \Phi_1(f) - U^*\Phi_3(f)U \in M_2(A)$$

for all $f \in C(X)$. Now

$$\begin{aligned}\varphi(f) \oplus \varphi_1(f) \oplus \Phi_2(f) - U^*\Phi_3(f)U &= \varphi(f) \oplus \varphi_1(f) \oplus \Phi_2(f) \\ &\quad - \varphi(f) \oplus \varphi_1(f) \oplus \Phi_1(f) \\ &\quad + \varphi(f) \oplus \varphi_1(f) \oplus \Phi_1(f) - U^*\Phi_3(f)U \in A,\end{aligned}$$

for all $f \in C(X)$ and

$$\begin{aligned}&\|\varphi(f) \oplus \varphi_1(f) \oplus \Phi_2(f) - U^*\Phi_3(f)U\| \\ &\qquad \leqslant \|\varphi(f) \oplus \varphi_1(f) \oplus \Phi_2(f) - \varphi(f) \oplus \varphi_1(f) \oplus \Phi_1(f)\| \\ &\qquad\quad + \|\varphi(f) \oplus \varphi_1(f) \oplus \Phi_1(f) - U^*\Phi_3(f)U\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for all $f \in \mathcal{F}$. From the first part of the proof, we know that $\varphi_1 \oplus \Phi_1 \oplus \varphi_1$ and $U^*\Phi_3 U$ are unitarily equivalent to φ_1. This proves the theorem. ∎

REMARK 2.3. Note that $C(X) = \lim_{n\to\infty} C(X_n)$, where X_n is a finite CW-complex. From the proof of Theorem 2.2, we see that Theorem 2.2 holds if the following is true : For any finite CW-complex Y and any finite subset of projections in $\bigcup_{n=1}^{\infty} M_\infty(C(Y)\otimes C(C_n\times\mathbb{S}^1))$, There is $\delta > 0$ such that $\mathbf{ah}_{d,\mathcal{P}}(Y_\mathcal{P}, B) = \mathbf{aah}(Y_\mathcal{P}, B)$. What we really need is that for any $\eta > 0$, any finite subset $\mathcal{H}$ of $C(X)$, any finite subset $\mathcal{P} \in \bigcup_{n=1}^{\infty} M_\infty(C(X)\otimes C(C_n\times\mathbb{S}^1))$, and for each contractive δ-$\mathcal{G}(\mathcal{P})$-multiplicative positive linear map $\psi : C(X) \to A$, there is a contractive η-$\mathcal{H}$-multiplicative positive linear map $\varphi : C(X) \to M_k(A)$ such that $\varphi_* = \psi_*$ on $\overline{\mathcal{P}}$. Therefore, by apply Proposition 1.26 and Theorem 2.2, we obtain the following:

COROLLARY 2.4. *Let X be a compact finite CW-complex with torsion free K-theory, B be a unital C^*-algebra and $A = B\otimes\mathcal{K}$. Let $\varphi : C(X) \to M(A)$ be a unital homomorphism . Then there exists a diagonalizable homomorphism $\varphi_1 : C(X) \to M(A)$ and a sequence of isometries $\{W_n\} \subset M_2(M(A))$ with $W_nW_n^* = \mathbf{1}_{M(A)}$ such that*

$$\|\varphi(f)\oplus\varphi_1(f) - W_n^*\varphi_1(f)W_n\| \to 0$$

and

$$\varphi(f)\oplus\varphi_1(f) - W_n^*\varphi_1(f)W_n \in A$$

for all $f \in C(X)$.

Both Theorem 2.2 and Corollary 2.4 are stable versions of a Weyl-von Neumann-Berg-Voiculescu type theorem. We note that in the case that $A \cong \mathcal{K}$, the map φ_1 can be easily absorbed. In the case that B is a purely infinite simple C^*-algebra, using existing method (Lemma 2.7), we can also absorb the map φ_1.

DEFINITION 2.5. Let $\varphi_1, \varphi_2 : C(X) \to M(A)$ be two homomorphism. We write

$$\varphi_1 \sim \varphi_2$$

if there exists a sequence of unitaries $u_n \in M(A)$ such that

$$\|\varphi_1(f) - u_n^*\varphi_1(f)u_n\| \to 0 \quad \text{and} \quad \varphi_1(f) - u_n^*\varphi_1(f)u_n \in A.$$

for all $f \in C(X)$.

LEMMA 2.6. *Let A be a unital C^*-algebra and I be a (closed) ideal. Assume that A/I satisfies the conditions*

(i) *that two projections with the same class in $K_0(A/I)$ are equivalent; and*

(ii) *that the map from $K_0(A/I)$ to $K_1(I)$ is injective.*

Suppose that p is a projection in A/I with $[p] = 0$ in $K_0(A/I)$, then for any projection $d \in I$, with the property that $(1-d)I(1-d)$ has an approximate identity consisting of projections, there is a projection $P \in A$ such that $P \geqslant d$ and $\pi(P) = p$, where $\pi : A \to A/I$ is the quotient map.

Proof. From the six-term exact sequence in K-theory, we have $[\pi(1-d)] = [p] = 0$ in $K_0(A)$. Thus, by the assumption, $p \sim \pi(1-d)$. There is a partial isometry $\overline{v} \in (1-d)A(1-d)/I$ such that $\overline{v}^*\overline{v} = \pi(1-d))$ and $\overline{vv^*} = p$. Lifting $\overline{v}$ to an element in $(1-d)A(1-d)$, using the fact $(1-d)I(1-d)$ has approximate identity consisting of projections, an argument of Elliott (see [18]; see also 2.8 in [59] for the general case) implies that there is a partial isometry $v \in (1-d)A(1-d)$ such that $\pi(v) = \overline{v}$ and $\pi(vv^*) = p$. Take $P = vv^* + d$. ∎

LEMMA 2.7. (cf Lemma 1.5 in [38]) *Let A be a non-unital but σ-unital C^*-algebra with real rank zero and with $M(A)/A$ being purely infinite simple and let $\tau : C(X) \to M(A)/A$ be an extension of $C(X)$ by A. Suppose that X is a compact metric space and F is a compact subset of X. Then there exist a projection $p \in M(A)/A$ and injective maps*

$$\tau' : C(X) \to (1-p)(M(A)/A)(1-p) \quad \text{and} \quad \sigma : C(X) \to p(M(A)/A)p$$

such that $\tau = \tau' + i_* \circ \sigma$, where $i : F \to X$ is the embedding and σ is a trivial diagonalizable extension.

Proof. We just need one modification of the proof of Lemma 1.5 in [38].

Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is a dense subset of F such that $\{\lambda_n\}_{n=k}^{\infty}$ is also dense in F for each k. For any $\lambda_n \in F$, set

$$I = \{f \in C(X) : f(\lambda) = 0\},$$

$B = \mathrm{Her}(\pi \circ \varphi(I))$, the hereditary C^*-subalgebra of $M(A)/A$ generated by $\pi \circ \varphi(I)$ and

$$B^{\perp} = \{x \in M(A)/A : xb = bx = 0 \quad \text{for all } b \in B\}.$$

It follows from 15 in [49] that $B^{\perp}$ is a nonzero hereditary C^*-subalgebra of $M(A)/A$. Since $M(A)/A$ is purely infinite and simple, there are two nonzero mutually orthogonal projections $\overline{p}_n$ and $\overline{p}_n{}'$ in $B^{\perp}$ such that $[\overline{p}_n] = 0$ in $K_0(M(A)/A)$. Since A has real rank zero, by Lemma 2.6, for any projection $d \in A$, there is a projection $p_n \in M(A)$ such that $\pi(p_n) = \overline{p}_n$ and $p_n \geqslant d$. We now apply the proof of 1.5 in [38]. The only place that we used the condition that $RR(M(A)) = 0$ is that projections p_n exists. This is done above by applying Lemma 2.6 (note $K_0(M(A)) = 0$). ∎

COROLLARY 2.8. *Let X be a compact metric space and let A be a nonunital but σ-unital purely infinite simple C^*-algebra. Suppose that $\varphi : C(X) \to M(A)$ is a unital homomorphismsuch that $\pi \circ \varphi : C(X) \to M(A)/A$ is a monomorphism, where $\pi : M(A) \to M(A)/A$ is the quotient map. Then there exists a diagonalizable homomorphism $\varphi_1 : C(X) \to M(A)$ such that*

$$\varphi \sim \varphi_1.$$

Proof. First, a couple of known facts. A is stable and has real rank zero ([60]), $K_0(M(A)) = K_1(M(A)) = 0$, $M(A)/A$ is purely infinite simple and has real rank zero ([34] and [60]).

It follows from Lemma 2.7 that there is a nonzero projection p and there are homomorphisms $\psi_1 : C(X) \to pM(A)p$ and $\psi_2 : C(X) \to (1-p)M(A)(1-p)$ such that ψ_1 is diagonalizable,

$$\psi_1(f) \oplus \psi_2(f) - \varphi(f) \in A$$

for all $f \in C(X)$ and both $\pi \circ \psi_1$ and $\pi \circ \psi_2$ are injective.

Now we will apply Remark 2.3. We note that, by Lemma 1.27, the condition in Remark 2.3 is satisfied.

Note that in Remark 2.3, if $\pi \circ \varphi$ is a monomorphism, so is $\pi \circ \varphi_1$. We also note that $(1-p)M(A)(1-p) \cong M(A)$. Applying Remark 2.3 on ψ_2, we obtain two diagonalizable homomorphism $\varphi_1, \varphi_2 : C(X) \to M(A)/A$ such that $\pi \circ \varphi_1$ and $\pi \circ \varphi_2$ are injective and

$$\psi_2(f) \oplus \varphi_1(f) - \varphi_2(f) \in A$$

for all $f \in C(X)$. By the proof of 8.1 in [37], we know that there is a unitary $u \in M(A)$ such that

$$\psi_1(f) - u^*\varphi_1(f)u \in A$$

for all $f \in C(X)$. We then obtain a diagonalizable homomorphism $\varphi_3 : C(X) \to M(A)$ such that

$$\varphi(f) - \varphi_3(f) \in A$$

for all $f \in C(X)$. Write

$$\varphi_3(f) = \sum_{n=1}^{\infty} f(\lambda_n)e_n,$$

where $\{\lambda_n\}_{n=k}^{\infty}$ is dense in X for each k and $\left\{\sum_{i=1}^{n} e_i\right\}$ forms an approximate identity. Let $\mathcal{F}$ be a finite subset of $C(X)$.

We will now repeat some of the arguments used in the proof of Theorem 2.2. Fix a finite subset $\mathcal{G} \subset C(X)$ ($\mathcal{G}$ will be larger than $\mathcal{F}$). Let $\delta_n > 0$ be a sequence such that $\delta_n \to 0$. Since $\varphi(f) - \varphi_1(f) \in A$, without loss of generality, for any $\varepsilon > 0$, we may assume that

$$\|e_1\varphi(f) - \varphi(f)e_1\| < \frac{\delta_1}{2} \quad \text{and} \quad \|\varphi(f) - \varphi_4(f) \oplus e_1\varphi(f)e_1\| < \varepsilon$$

for all $f \in \mathcal{G}$, where $\varphi_4(f) = \sum_{n=2}^{\infty} f(\lambda_n)e_n$ for all $f \in C(X)$. Set $\alpha_1(f) = e_1\varphi(f)e_1$ for all $f \in C(X)$. Then α is a contractive positive linear map from $C(X)$ into e_1Ae_1 which is δ_1-$\mathcal{G}$-multiplicative. We also have

$$\varphi(f) - \varphi_4(f) \oplus \alpha_1(f) \in A$$

for all $f \in C(X)$. Let $\{\mathcal{F}_k\}$ and $\{\mathcal{G}_k\}$ be two increasing sequences of the unit ball of $C(X)$ such that each of their unions is dense in the unit ball of $C(X)$. We also let $\mathcal{F}_1 = \mathcal{F}$. By Corollary 1.28, there is a contractive positive linear map from $\alpha_2 : C(X) \to e_1Ae_1$ which is δ_2-$\mathcal{G}_1$-multiplicative and a homomorphism $\gamma_1 : C(X) \to M_2(e_1Ae_1)$ with finite dimensional range such that

$$\|\alpha_1(f) \oplus \alpha_2(f) - \gamma_1(f)\| < \frac{\varepsilon}{2}$$

for all $f \in \mathcal{F}_1$, provided that $\mathcal{G}_1$ is large enough and δ_1 is small enough. Note that there is no restriction on δ_2 and $\mathcal{G}_1$. By Corollary 1.28, there is a contractive positive linear map $\alpha_3 : C(X) \to e_1Ae_1$ which is δ_3-$\mathcal{G}_2$-multiplicative and a homomorphism $\gamma_2 : C(X) \to M_2(e_1Ae_1)$ with finite dimensional range such that

$$\|\alpha_2(f) \oplus \alpha_3(f) - \gamma_2(f)\| < \frac{\varepsilon}{4}$$

for all $f \in \mathcal{F}_2$, provided that $\mathcal{G}_2$ is large enough and δ_3 is small enough. Again note that there is no restriction on δ_3 and $\mathcal{G}_2$. Continuing this, we obtain $\{\gamma_n\}$ and $\{\alpha_n\}$ such that

$$\|\alpha_n(f) \oplus \alpha_{n+1}(f) - \gamma_n(f)\| < \frac{\varepsilon}{2^n}$$

for all $f \in \mathcal{F}_n$. Let $\Phi_1 = \sum_{n=2}^{\infty} \gamma_n$ and $\Phi_2 = \sum_{n=1}^{\infty} \gamma_n$. Then

$$\alpha_1(f) \oplus \Phi_1(f) - \Phi_2(f) \in A \otimes \mathcal{K} \quad (\cong A)$$

for all $f \in C(X)$, since $\bigcup \mathcal{F}_n$ is dense in $C(X)$ and

$$\|\alpha_1(f) \oplus \Phi_1(f) - \Phi_2(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. Note that Φ_1 and Φ_2 are diagonalizable. It follows from Remark 2.3 (see Theorem 2.2) that

$$\varphi_4 \oplus \Phi_1 \sim \varphi_4.$$

We also have $\psi_3 \oplus \Phi_2 \sim \psi_3$. So the above shows that

$$\varphi \sim \psi_3. \quad \blacksquare$$

REMARK 2.9. Corollary 2.8 is an application of Theorem 2.2 and earlier results in [38]. After this research started but before it was typed, we learned that E. Kirchberg have announced a more general result than Corollary 2.8. His method is completely different from ours.

REMARK 2.10. Let

$$0 \to A \to E \to C(X) \to 0$$

be an (essential unital) extension of $C(X)$ by a non-unital but σ-unital purely infinite simple C^*-algebra A. Such an extension is determined by a monomorphism from $C(X)$ into $M(A)/A$. Let $\tau_1, \tau_2 : C(X) \to M(A)/A$ be two such extensions. We say τ_1 is unitarily equivalent to τ_2 if there is a unitary $u \in M(A)$ such that $\pi(u^*)\tau_1\pi(u)$, where $\pi : M(A) \to M(A)/A$ is the quotient map. Let $\mathrm{Ext}(C(X), A)$ denote the set of unitary equivalence classes of essential extensions of $C(X)$ by A. Let $\mathrm{Ext}(C(X), A)$ (and $\mathrm{Ext}^e(C(X), A)$) be the quotient of unitary equivalence classes of (essential) extensions of $C(X)$ by A by (essential) trivial extensions. It is clear that, when A is a non-unital, σ-unital purely infinite simple C^*-algebra, there are trivial essential extensions (diagonalizable maps from $C(X)$ into $M(A)$). So $\mathrm{Ext}(C(X), A) = \mathrm{Ext}^e(C(X), A)$ (see 15.6.5 in [2]).

COROLLARY 2.11.

$$\mathrm{Ext}(C(X), A) \cong \mathrm{Ext}(C(X), A) \cong KK^1(C(X), A).$$

Furthermore, an extension τ is trivial if and only if $[\tau] = 0$ in $\mathrm{Ext}(C(X), A)$ (or in $KK^1(C(X), A)$) and all trivial extensions are equivalent and equivalent to a diagonalizable one.

Proof. It is enough to consider unital essential extensions. Suppose that $\tau_1, \tau_2 : C(X) \to M(A)/A$ are two essential unital extensions and $[\tau_1] = [\tau_2]$ in $\mathrm{Ext}(C(X), A)$. Then there are trivial extensions τ_0 and τ_{00} such that $\tau_1 + \tau_0$ is unitarily equivalent to $\tau_2 + \tau_{00}$. By Lemma 2.7, τ_1 is unitarily equivalent to $\tau_1' + \tau_1''$, where τ_1 is trivial. By Remark 2.9, all trivial extensions are unitarily equivalent. In particular, τ_1' is unitarily equivalent τ_0. So τ_1 is unitarily equivalent to τ_1. Similarly, $\tau_2 + \tau_{00}$ is unitarily equivalent to τ_2. Thus τ_1 is unitarily equivalent to τ_2. The last part of the statement follows from Remark 2.9. $\blacksquare$

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