J. OPERATOR THEORY
37(1997), 155–181

TRIPLETS OF HILBERT SPACES AND FRIEDRICHS EXTENSIONS ASSOCIATED WITH THE SUBCLASS $\mathbf{N}_1$ OF NEVANLINNA FUNCTIONS

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Communicated by Florian-Horia Vasilescu

ABSTRACT. The selfadjoint extensions of a closed symmetric operator S with defect numbers $(1,1)$ are described when S has a Q-function belonging to the subclass $\mathbf{N}_1$ of all Nevanlinna functions. With the associated triplet of Hilbert spaces $\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}$ all but one of the selfadjoint extensions of S are interpreted as rank one perturbations of a fixed operator extension; the exceptional extension corresponds to a proper relation extension. Each nonexceptional selfadjoint extension gives rise to the same triplet of Hilbert spaces. The exceptional extension is characterized in a similar way as the Friedrichs extension of a semibounded operator.

KEYWORDS: *Symmetric operator, selfadjoint extension, rank one perturbation, Friedrichs extension, Q-function, Nevanlinna function, triplet of Hilbert spaces.*

AMS SUBJECT CLASSIFICATION: Primary 47A70, 47B15, 47B25; Secondary 47A55, 47A57.

0. INTRODUCTION

In a Hilbert space $\mathfrak{H}$ with the inner product $[\cdot,\cdot]$ we consider a closed, symmetric operator S with defect numbers $(1,1)$. The assumption that the defect numbers are equal implies that S has canonical selfadjoint extensions, i.e., selfadjoint extensions which take place inside the Hilbert space $\mathfrak{H}$. If S is densely defined all its canonical selfadjoint extensions are operators. If S is nondensely defined the adjoint S^* is a multivalued operator with $\dim \operatorname{mul} S^* = 1$ (here $\operatorname{mul} S^*$ denotes the multivalued part of S^*) and all but one of the canonical selfadjoint extensions of S are operators.

Let A be a canonical selfadjoint extension of S in $\mathfrak{H}$. For some $\mu \in \mathbb{C} \setminus \mathbb{R}$ we choose a nontrivial element $\chi(\mu) \in \ker(S^* - \mu)$ and define

$$\chi(\ell) = (I + (\ell - \mu)(A - \ell)^{-1})\chi(\mu), \quad \ell \in \rho(A), \tag{0.1}$$

where $\rho(A)$ denotes the resolvent set of A. Then $\ker(S^* - \ell)$ is spanned by $\chi(\ell)$. In fact, in terms of $\chi(\ell)$ and A, the operator S can be recovered as follows:

$$S = \{\{h, k\} \in A : [k - \bar{\ell}h, \chi(\ell)] = 0\}, \quad \ell \in \rho(A). \tag{0.2}$$

A Q-function of S and A is a solution of

$$\frac{Q(\ell) - \overline{Q(\lambda)}}{\ell - \bar{\lambda}} = [\chi(\ell), \chi(\lambda)], \quad \ell, \lambda \in \rho(A). \tag{0.3}$$

This defines $Q(\ell)$ uniquely, up to a real constant. From (0.3) follows the operator representation

$$Q(\ell) = \overline{Q(\mu)} + (\ell - \bar{\mu})[(I + (\ell - \mu)(A - \ell)^{-1})\chi(\mu), \chi(\mu)], \quad \ell \in \rho(A). \tag{0.4}$$

The theory of Q-functions goes back to M.G. Kreĭn. It was presented in a general form in [15]. The identity (0.3) implies that $Q(\ell)$ is a Nevanlinna function. The interest in Q-functions lies in the fact that we can translate facts about the operator S and its selfadjoint extensions into purely function-theoretic terms.

The class $\mathbf{N}$ of Nevanlinna functions is the set of all functions $Q(\ell)$ which are holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfy $\overline{Q(\ell)} = Q(\bar{\ell})$, $\operatorname{Im} Q(\ell)/\operatorname{Im} \ell \geqslant 0$, $\ell \in \mathbb{C} \setminus \mathbb{R}$. The class $\mathbf{N}$ contains a subclass $\mathbf{N}_1$ of functions $Q(\ell)$ which belong to $\mathbf{N}$ and for which $\int_1^\infty \operatorname{Im} Q(iy)/y \, dy < \infty$. The class $\mathbf{N}_1$, which was introduced by I.S. Kac, contains the subclass $\mathbf{N}_0$ consisting of all functions $Q(\ell)$ which belong to $\mathbf{N}$ and for which $\sup_{y>0} \operatorname{Im} Q(iy) < \infty$. We refer for these classes and their integral representations to [9], [10], see also [5], [13].

The canonical selfadjoint extensions of S are in one-to-one correspondence with $\tau \in \mathbb{R} \cup \{\infty\}$. If the Q-function $Q(\ell)$ of S and A is normalized by $\operatorname{Re} Q(\mu) = 0$, then

$$Q_\tau(\ell) = \frac{Q(\ell) - \tau(\operatorname{Im} Q(\mu))^2}{\tau Q(\ell) + 1}, \quad \tau \in \mathbb{R} \cup \{\infty\}, \tag{0.5}$$

is the Q-function of S and the canonical selfadjoint extension $A(\tau)$ which corresponds to τ. It is normalized by $\operatorname{Re} Q_\tau(\mu) = 0$. In [5] it is shown that there is an alternative: either S has a canonical selfadjoint operator extension with a

Q-function belonging to $\mathbf{N}_1 \setminus \mathbf{N}_0$ or $\mathbf{N}_0$, in which case all but one of its Q-functions belong to $\mathbf{N}_1 \setminus \mathbf{N}_0$ or $\mathbf{N}_0$, respectively, or there is no canonical selfadjoint extension of S with a Q-function belonging to $\mathbf{N}_1$.

If there is a canonical selfadjoint extension A with a Q-function in $\mathbf{N}_0$, then the restriction in (0.2) is a domain restriction, so that S is not densely defined, and all other canonical selfadjoint extensions of S with a Q-function belonging to $\mathbf{N}_0$ can be described as rank one perturbations of A, so that the domains are all equal to $\operatorname{dom} A$, while the exceptional extension is the only selfadjoint extension of S which is not an operator, see [5] and [6]. If S is semibounded this exceptional extension is the Friedrichs extension, cf. [3], [4], [11], and [12]. In the case where a canonical selfadjoint extension A has a Q-function in $\mathbf{N} \setminus \mathbf{N}_0$ the restriction in (0.2) is a graph restriction and S is densely defined. Now all other canonical selfadjoint extensions of S with a Q-function belonging to $\mathbf{N} \setminus \mathbf{N}_0$ are described as graph perturbations of A, cf. [6].

In the present paper we show that the case where a canonical selfadjoint extension A has a Q-function in $\mathbf{N}_1 \setminus \mathbf{N}_0$ is similar to the case where a Q-function belongs to $\mathbf{N}_0$, when the canonical selfadjoint extensions are extended by means of a triplet of Hilbert spaces

$$\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}, \tag{0.6}$$

generated by the operator $|A|^{\frac{1}{2}}$. In this sense the restriction (0.2) is again a domain restriction and all but one of the selfadjoint extensions — extended to operators from $\mathfrak{H}_{+1}$ to $\mathfrak{H}_{-1}$ — are rank one perturbations of (the extended) A. The (extended) exceptional extension is the only extension which is not an operator. We show an invariance property of the domains related to the nonexceptional extensions and we characterize the exceptional extension in the original space. If S is semibounded, our description reduces to a known description of the Friedrichs extension, see [14]. Another approach to generalize the Friedrichs extension by means of sesquilinear forms is due to A.G.R. McIntosh, cf. [17], [18], [19]. For our use of triplets of Hilbert spaces we refer to [2], [7], [8], [16], and [20].

1. Q-FUNCTIONS OF CLASS $\mathbf{N}_0$

Let S be a closed symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint operator extension of S. Let $Q(\ell)$ be a Q-function of S and A. Then $Q(\ell)$ belongs to $\mathbf{N}_0$ if and only if $Q(\ell)$ has the integral representation

$$Q(\ell) = \gamma + \int_{\mathbf{R}} \frac{1}{t-\ell}\, \mathrm{d}\sigma(t), \tag{1.1}$$

where the function $\sigma(t)$ is nondecreasing on $\mathbf{R}$ and satisfies $\int_{\mathbf{R}} \mathrm{d}\sigma(t) < \infty$. An equivalent condition is that $\ker(S^* - \ell) \subset \operatorname{dom} A$, for some (and, hence, for all) $\ell \in \mathbf{C} \setminus \mathbf{R}$. In this section we present in the language of $\mathbf{N}_0$ functions some known facts (see [5], [6]), which will later be interpreted in terms of triplets of Hilbert spaces when $Q(\ell)$ belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$.

THEOREM 1.1. *Let S be a closed, symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint extension of S. The following conditions are equivalent:*

(i) *S and A have a Q-function $Q(\ell)$ belonging to $\mathbf{N}_0$.*

(ii) *A is an operator and there exists an element $\omega \in \mathfrak{H}$ such that the operator S is defined by*

$$\operatorname{dom} S = \{h \in \operatorname{dom} A : [h, \omega] = 0\}. \tag{1.2}$$

(iii) *S and A have a Q-function $Q(\ell)$ with the operator representation*

$$Q(\ell) = \gamma + [(A-\ell)^{-1}\omega, \omega], \quad \ell \in \mathbf{C} \setminus \mathbf{R}, \tag{1.3}$$

where $\omega \in \mathfrak{H}$ and $\gamma \in \mathbf{R}$.

(iv) $\operatorname{dom} S^* \subset \operatorname{dom} A$.

In each of these cases, $\chi(\ell) = (A-\ell)^{-1}\omega$, $\gamma = \lim_{y\to\infty} Q(\mathrm{i}y)$ and S is not densely defined.

COROLLARY 1.2. *The spectral measure $\mathrm{d}\sigma(t)$ and $\omega \in \mathfrak{H}$ are related by*

$$\|\omega\|^2 = \int_{\mathbf{R}} \mathrm{d}\sigma(t).$$

Proof. It follows from (1.3) and the resolvent identity, that

$$y \operatorname{Im} Q(\mathrm{i}y) = y^2[(A-\mathrm{i}y)^{-1}\omega, (A-\mathrm{i}y)^{-1}\omega].$$

The result follows by taking $y \to \infty$ in this identity. ∎

Note that the element ω in (1.2) is a module element for S, i.e., for all $\ell \in \mathbb{C} \setminus \mathbb{R}$,

$$\operatorname{ran}(S-\ell) + \operatorname{span}\{\omega\} = \mathfrak{H}.$$

The codimension of $\operatorname{ran}(S-\ell)$ is one, since we assume that S has defect numbers $(1,1)$. So it is sufficient to show that $\omega \notin \operatorname{ran}(S-\ell)$: if not, then $\omega = (S-\ell)f$ for some nontrivial $f \in \operatorname{dom} S$ and it follows from (1.2) that $0 = [Sf,f] - \ell[f,f]$, a contradiction. The adjoint S^* of S is given by

$$S^* = A \dotplus (\{0\} \oplus \operatorname{span}\{\omega\}). \tag{1.4}$$

Here the symbol $\dotplus$ denotes the componentwise sum in $\mathfrak{H} \oplus \mathfrak{H}$. We will describe the canonical selfadjoint extensions of S as rank one perturbations of A, in terms of the decomposition (1.4).

THEOREM 1.3. *Let S be a closed, symmetric operator with defect numbers $(1,1)$ as in (1.2). The selfadjoint extensions of S in $\mathfrak{H}$ are in one-to-one correspondence with $\tau \in \mathbb{R} \cup \{\infty\}$ via*

$$A(\tau) = A + \frac{1}{\frac{1}{\tau}+\gamma}[\,\cdot\,,\omega]\omega, \quad \frac{1}{\tau}+\gamma \neq 0, \tag{1.5}$$

and

$$A(\tau) = S \dotplus (\{0\} \oplus \operatorname{mul} S^*), \quad \frac{1}{\tau}+\gamma = 0. \tag{1.6}$$

Moreover, the selfadjoint extensions $A(\tau)$, $1/\tau+\gamma \neq 0$, have a Q-function in $\mathbf{N}_0$. When $1/\tau+\gamma = 0$, the Q-function of $A(\tau)$ belongs to $\mathbf{N} \setminus \mathbf{N}_1$. The resolvent operator of $A(\tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$, is given by

$$(A(\tau)-\ell)^{-1} = (A-\ell)^{-1} - \chi(\ell)\frac{1}{Q(\ell)+\frac{1}{\tau}}[\,\cdot\,,\chi(\bar{\ell})], \quad \ell \in \mathbb{C}\setminus\mathbb{R}. \tag{1.7}$$

It follows from (1.5) that $\operatorname{dom} A(\tau) = \operatorname{dom} A$ for $1/\tau+\gamma \neq 0$, while it follows from (1.6) that for $1/\tau+\gamma = 0$, $\operatorname{dom} A(\tau) = \operatorname{dom} S$. The exceptional extension coincides with the Friedrichs extension S_F of S, when S is semibounded. This follows from the representation (1.6) and Theorem 2, [4], and the preceding Lemma. In this case another representation of S_F can be obtained in the following way. Assume without loss of generality that $S \geqslant 0$ and define on $\operatorname{dom} S$ a new inner product by

$$[f,g]_S = [(S+I)f,g], \quad f,g \in \operatorname{dom} S.$$

The completion of $\operatorname{dom} S$ in this new inner product is a Hilbert space, which we denote by $\mathfrak{H}_S$. The following result was proved in [3], [4].

PROPOSITION 1.4. *Assume that S is a nondensely defined, closed nonnegative operator. Then the Friedrichs extension S_F of S has the representation*

$$S_F = \{\{f,g\} \in S^* : f \in \mathfrak{H}_S\}. \tag{1.8}$$

This extension S_F is the only selfadjoint extension H of S with the property $\operatorname{dom} H \subset \mathfrak{H}_S$.

A discussion of the canonical selfadjoint extensions of a densely defined nonnegative operator can be found in [7] and [14] via triplets of Hilbert spaces. In the next sections we will prove similar results for densely defined symmetric, not necessarily semibounded, operators which have a selfadjoint extension with a Q-function in $\mathbf{N}_1 \setminus \mathbf{N}_0$.

2. SELFADJOINT EXTENSIONS IN TRIPLET SPACES

We recall a few basic facts about triplet spaces associated with selfadjoint operators. Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ with inner product $[\cdot,\cdot]$. On the Hilbert space $\mathfrak{H}$ we define a new inner product $[\cdot,\cdot]_{-1}$ by $[f,g]_{-1} = [(I+|A|)^{-1}f,g]$, $f,g \in \mathfrak{H}$. The completion of $\mathfrak{H}$ with respect to the inner product $[\cdot,\cdot]_{-1}$ is denoted by $\mathfrak{H}_{-1}$. On $\operatorname{dom}|A|^{\frac{1}{2}}$ we define the inner product $[\cdot,\cdot]_{+1}$ by $[f,g]_{+1} = [f,g] + [|A|^{\frac{1}{2}}f, |A|^{\frac{1}{2}}g]$, $f,g \in \operatorname{dom}|A|^{\frac{1}{2}}$. The subspace $\operatorname{dom}|A|^{\frac{1}{2}}$ provided with the norm $\|\cdot\|_{+1}$ is a Hilbert space, denoted by $\mathfrak{H}_{+1}$, which is isometrically isomorphic to the graph of $|A|^{\frac{1}{2}}$. The domain $\operatorname{dom} A$ is dense in $\mathfrak{H}_{+1}$. The mapping $I+|A|$ is isometric from $\operatorname{dom} A$ onto $\mathfrak{H}$, with respect to the new topologies. The Riesz operator V_{+1} is the unique isometric extension of $I+|A|$ from $\mathfrak{H}_{+1}$ onto $\mathfrak{H}_{-1}$. The duality $(\cdot,\cdot)$ between the Hilbert spaces $\mathfrak{H}_{+1}$ and $\mathfrak{H}_{-1}$ is expressed by

$$(f,g) = [V_{+1}f,g]_{-1} = [f,V_{+1}^{-1}g]_{+1}, \quad f \in \mathfrak{H}_{+1},\ g \in \mathfrak{H}_{-1}, \tag{2.1}$$

and, in particular,

$$(f,g) = [f,g], \quad f \in \mathfrak{H}_{+1},\ g \in \mathfrak{H}. \tag{2.2}$$

We define $(g,f) = \overline{(f,g)}$ when $f \in \mathfrak{H}_{+1}$ and $g \in \mathfrak{H}_{-1}$. Clearly, the Cauchy-Schwarz inequality is valid:

$$|(f,g)| \leqslant \|f\|_{+1}\|g\|_{-1}, \quad f \in \mathfrak{H}_{+1},\ g \in \mathfrak{H}_{-1}. \tag{2.3}$$

The operator A is contractive, and has a unique contractive extension $\widetilde{A}$ from $\mathfrak{H}_{+1}$ into $\mathfrak{H}_{-1}$. Also the identity operator on $\operatorname{dom} A$ can be uniquely extended to all of $\mathfrak{H}_{+1}$. Hence, for any $\ell \in \mathbb{C}$, the operator $A - \ell$ can be continuously extended to a continuous operator $(A-\ell)^{\sim} = \widetilde{A} - \ell$, from $\mathfrak{H}_{+1}$ into $\mathfrak{H}_{-1}$. Moreover,

$$((\widetilde{A}-\ell)f, g) = (f, (\widetilde{A}-\bar{\ell})g), \quad f, g \in \mathfrak{H}_{+1}, \quad \ell \in \mathbb{C}, \tag{2.4}$$

which follows from (2.2), (2.3), and the symmetry of A.

The symmetry of $\widetilde{A}$ can also be expressed in terms of the polar decomposition $A = U|A|$ and the spectral decomposition $A = \int_{\mathbb{R}} t\, dE(t)$ of the selfadjoint operator A:

$$(\widetilde{A}f, g) = (f, \widetilde{A}g) = [U|A|^{\frac{1}{2}}f, |A|^{\frac{1}{2}}g] = \int_{\mathbb{R}} t\, d([E(t)f, g]), \quad f, g \in \mathfrak{H}_{+1}. \tag{2.5}$$

This sesquilinear form is continuous on $\mathfrak{H}_{+1}$. For $\ell \in \rho(A)$, the operator $R(\ell) = (A-\ell)^{-1}$ is bounded, and has a unique bounded extension $\widetilde{R}(\ell)$ from $\mathfrak{H}_{-1}$ into $\mathfrak{H}_{+1}$. Moreover,

$$(\widetilde{R}(\ell)f, g) = (f, \widetilde{R}(\bar{\ell})g), \quad f, g \in \mathfrak{H}_{-1}, \quad \ell \in \rho(A), \tag{2.6}$$

which follows from (2.3) and $[R(\ell)f, g] = [f, R(\bar{\ell})g]$, $f, g \in \mathfrak{H}$. In addition, a continuity argument shows that

$$((\widetilde{A}-\ell)^{-1}V_{+1}f, V_{+1}g) = \int_{\mathbb{R}} \frac{(|t|+1)^2}{t-\ell}\, d([E(t)f, g]), \quad \ell \in \rho(A), \ f, g \in \mathfrak{H}_{+1}. \tag{2.7}$$

The extensions $\widetilde{A}$ and $\widetilde{R}(\ell)$ are connected via $\widetilde{R}(\ell)(\widetilde{A}-\ell) = I_{\mathfrak{H}_{+1}}$, $(\widetilde{A}-\ell)\widetilde{R}(\ell) = I_{\mathfrak{H}_{-1}}$. They satisfy the symmetry relation

$$(f, g) = ((\widetilde{A}-\ell)f, \widetilde{R}(\bar{\ell})g), \quad f \in \mathfrak{H}_{+1}, \ g \in \mathfrak{H}_{-1}. \tag{2.8}$$

Moreover, for $\ell, \lambda \in \rho(A)$

$$\widetilde{R}(\ell) - \widetilde{R}(\lambda) = (\ell - \lambda)R(\ell)\widetilde{R}(\lambda). \tag{2.9}$$

It is useful to observe that $A = \widetilde{A} \cap \mathfrak{H}^2$, so that

$$A = \{\{f, \widetilde{A}f\} : f \in \mathfrak{H}_{+1}, \ \widetilde{A}f \in \mathfrak{H}\}. \tag{2.10}$$

Now we will consider rank one perturbations of a (not necessarily semibounded) selfadjoint operator, in the sense of triplets. Let A be a selfadjoint operator and let $\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}$ be the associated triplet of Hilbert spaces. Assume that $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$ and define the restriction $\widetilde{S}$ of $\widetilde{A}$ as follows

$$\operatorname{dom} \widetilde{S} = \{f \in \mathfrak{H}_{+1} : (f, \omega) = 0\}. \tag{2.11}$$

In order to consider selfadjoint extensions of $\widetilde{S}$ in the triplet, we introduce some terminology. The Cartesian product of $\mathfrak{H}_{+1}$ and $\mathfrak{H}_{-1}$, denoted by $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$, is provided with the usual topology. The duality $(\cdot, \cdot)$ between $\mathfrak{H}_{+1}$ and $\mathfrak{H}_{-1}$ induces the notion of adjoint in the following way. Let T be any subset of the Cartesian product $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$. Then the adjoint T^* of T is defined by

$$T^* = \{\{h, k\} \in \mathfrak{H}_{+1} \times \mathfrak{H}_{-1} : (g, h) = (f, k) \text{ for all } \{f, g\} \in T\}.$$

This is a closed linear subset of $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$. The relation T is called symmetric if $T \subset T^*$ and selfadjoint if $T = T^*$. In the following lemma $\{0\} \times \operatorname{span}\{\omega\}$ denotes the Cartesian product of $\{0\}$ in $\mathfrak{H}_{+1}$ and of $\operatorname{span}\{\omega\}$ in $\mathfrak{H}_{-1}$. The symbol $\dot{+}$ denotes a componentwise sum in $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$. The next result and its proof are parallel to the case of a nondensely defined symmetric operator acting in a Hilbert space, cf. (2.4) and (2.8).

LEMMA 2.1. *The operator $\widetilde{S}$ is closed, symmetric and its adjoint is given by*

$$\widetilde{S}^* = \widetilde{A} \dot{+} (\{0\} \times \operatorname{span}\{\omega\}),$$

where $\widetilde{A}$ is a selfadjoint operator. Moreover, $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$ is a module element for $\widetilde{S}$:

$$\operatorname{ran}(\widetilde{S} - \ell) + \operatorname{span}\{\omega\} = \mathfrak{H}_{-1}, \quad \ell \in \mathbb{C} \setminus \mathbb{R},$$

where the sum is direct.

The following lemma is a straightforward generalization of known results for selfadjoint relations in a Hilbert space to the case of a triplet associated with $|A|^{\frac{1}{2}}$.

LEMMA 2.2. *Let $\widetilde{H}$ be a selfadjoint extension of $\widetilde{S}$ in $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$. Then*

$$\ker(\widetilde{H} - \ell) = \{0\}, \quad \operatorname{ran}(\widetilde{H} - \ell) = \mathfrak{H}_{-1}, \quad \ell \in \mathbb{C} \setminus \mathbb{R}.$$

In particular, $(\widetilde{H} - \ell)^{-1}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, is a bounded linear mapping from $\mathfrak{H}_{-1}$ into $\mathfrak{H}_{+1}$. If $\widetilde{H}$ is an operator then $\operatorname{dom} \widetilde{H} = \operatorname{dom} \widetilde{A}$ and if $\widetilde{H}$ is not an operator then $\operatorname{dom} \widetilde{H} = \operatorname{dom} \widetilde{S}$.

Now we give an explicit description of all selfadjoint relations $\widetilde{H}$ in $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$ which extend the symmetric operator $\widetilde{S}$, i.e., $\widetilde{S} \subset \widetilde{H}$. For $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$ we define the function $\chi(\ell)$ by $\chi(\ell) = \widetilde{R}(\ell)\omega$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, and the function $Q(\ell)$ by $Q(\ell) = (\widetilde{R}(\ell)\omega, \omega)$, $\ell \in \mathbb{C} \setminus \mathbb{R}$. It follows from (2.6) and (2.9) that $\overline{Q(\ell)} = Q(\bar{\ell})$ and

$$\frac{Q(\ell) - \overline{Q(\lambda)}}{\ell - \bar{\lambda}} = [\chi(\ell), \chi(\lambda)],$$

so that $Q(\ell)$ is a Nevanlinna function.

THEOREM 2.3. *The selfadjoint extensions of* $\widetilde{S}$ *in* $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$ *are in one-to-one correspondence with* $\tau \in \mathbb{R} \cup \{\infty\}$ *via*

$$\widetilde{A}(\tau) = \widetilde{A} + \tau(\cdot\,,\omega)\omega, \quad \tau \in \mathbb{R}, \tag{2.12}$$

and

$$\widetilde{A}(\infty) = \widetilde{S} \dotplus (\{0\} \times \operatorname{span}\{\omega\}). \tag{2.13}$$

The extension $\widetilde{A}(\infty)$ *is the only selfadjoint extension which is not an operator. For each* $\tau \in \mathbb{R} \cup \{\infty\}$*, the operator* $(\widetilde{A}(\tau) - \ell)^{-1}$*,* $\ell \in \mathbb{C} \setminus \mathbb{R}$*, is given by*

$$(\widetilde{A}(\tau) - \ell)^{-1} = (\widetilde{A} - \ell)^{-1} - \chi(\ell)\,\frac{1}{Q(\ell) + \frac{1}{\tau}}\,(\cdot\,,\chi(\bar{\ell})), \quad \ell \in \mathbb{C} \setminus \mathbb{R}. \tag{2.14}$$

Proof. By Lemma 2.1 every linear relation $\widetilde{H}$ in the Cartesian product $\mathfrak{H}_{+1} \times \mathfrak{H}_{-1}$ with the property $\widetilde{S} \subset \widetilde{H} \subset \widetilde{S}^*$ is automatically closed. If, in addition, it is a proper symmetric extension of $\widetilde{S}$, a dimension argument shows that $\widetilde{H}$ is selfadjoint.

Now, it is easily verified that $\widetilde{A}(\tau)$ given by (2.12) and (2.13), respectively, is symmetric and a proper extension of $\widetilde{S}$. Hence, $\widetilde{A}(\tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$, is a selfadjoint extension of $\widetilde{S}$.

Next we prove the converse. Let $\widetilde{H}$ be a selfadjoint extension of $\widetilde{S}$. Choose a fixed element $\{h, k\} \in \widetilde{H} \setminus \widetilde{S}$. It follows from Lemma 2.1 that $k = \widetilde{A}h + c\omega$ for some $c \in \mathbb{C}$. The symmetry of $\widetilde{H}$ gives

$$0 = (\widetilde{A}h + c\omega, h) - (h, \widetilde{A}h + c\omega) = c(\omega, h) - (h, \omega)\bar{c}. \tag{2.15}$$

If $c = 0$, then clearly $\{h, k\} \in \widetilde{A}$. This implies that $\widetilde{H} = \widetilde{A}$, which corresponds to $\tau = 0$. If $c \neq 0$, then (2.15) shows that $(h, \omega)/c \in \mathbb{R}$. We denote this number by $1/\tau$, $\tau \in \mathbb{R} \cup \{\infty\}$, $\tau \neq 0$. In the case $\tau \in \mathbb{R} \setminus \{0\}$ we see that $\{h, k\} \in \widetilde{A}(\tau)$ and we conclude $\widetilde{H} = \widetilde{A}(\tau)$. In the case $\tau = \infty$ we have $(h, \omega) = 0$ so that $\{h, k\} \in \widetilde{A}(\infty)$, which gives $\widetilde{H} = \widetilde{A}(\infty)$. Hence, we have shown that each selfadjoint extension $\widetilde{H}$ of $\widetilde{S}$ is of the form (2.12) or (2.13), respectively. Clearly, $\widetilde{A}(\infty)$ in (2.13) is the only selfadjoint extension of $\widetilde{S}$ which is not an operator, see Lemma 2.2.

In order to prove (2.14) we first assume that $\tau \in \mathbb{R}$. For $f \in \mathfrak{H}_{+1}$ we define $g = (\widetilde{A}(\tau) - \ell)f$. Then $g \in \mathfrak{H}_{-1}$ and it follows from (2.12) that $g = (\widetilde{A} - \ell)f + \tau(f, \omega)\omega$. Hence, $f = \widetilde{R}(\ell)g - \tau(f, \omega)\widetilde{R}(\ell)\omega$ and $(f, \omega) = (\widetilde{R}(\ell)g, \omega) - \tau(f, \omega)(\widetilde{R}(\ell)\omega, \omega)$. This gives

$$(f, \omega) = \frac{(\widetilde{R}(\ell)g, \omega)}{1 + \tau(\widetilde{R}(\ell)\omega, \omega)}.$$

On the other hand $f = (\widetilde{A}(\tau) - \ell)^{-1}g$. Now, using the symmetry condition (2.6) we obtain (2.14).

Next we consider the case $\tau = \infty$. For $g \in \mathfrak{H}_{-1}$ we define $f = (\widetilde{A}(\infty) - \ell)^{-1}g$, cf. Lemma 2.2. Then $\{f, g + \ell f\} \in \widetilde{A}(\infty)$, so that $f \in \operatorname{dom} \widetilde{S}$ and $g + \ell f = \widetilde{A}f + c\omega$ for some $c \in \mathbb{C}$. Thus $g = (\widetilde{A} - \ell)f + c\omega$ with $(f, \omega) = 0$, which gives with (2.8)

$$(g, \widetilde{R}(\bar{\ell})\omega) = ((\widetilde{A} - \ell)f, \widetilde{R}(\bar{\ell})\omega) + c(\omega, \widetilde{R}(\bar{\ell})\omega) = c(\widetilde{R}(\ell)\omega, \omega). \tag{2.16}$$

It follows that

$$(\widetilde{A} - \ell)^{-1}g = f + c(\widetilde{A} - \ell)^{-1}\omega = (\widetilde{A}(\infty) - \ell)^{-1}g + c(\widetilde{A} - \ell)^{-1}\omega. \tag{2.17}$$

Solving c from (2.16) and substituting the result in (2.17) we obtain (2.14) for $\tau = \infty$. ∎

3. Q-FUNCTIONS OF CLASS $\mathbf{N}_1 \setminus \mathbf{N}_0$

Let S be a closed symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint operator extension of S. Let $Q(\ell)$ be a Q-function of S and A. Then $Q(\ell)$ belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$ if and only if $Q(\ell)$ has the integral representation (1.1) where the function $\sigma(t)$ is nondecreasing on $\mathbb{R}$ and satisfies $\int_{\mathbb{R}} (1 + |t|)^{-1}\, d\sigma(t) < \infty$ and $\int_{\mathbb{R}} d\sigma(t) = \infty$. Moreover, γ is a real number satisfying $\gamma = \lim_{y \to \infty} Q(iy)$, see [10]. An equivalent condition is that $\ker(S^* - \ell) \subset \operatorname{dom} |A|^{\frac{1}{2}} \setminus \operatorname{dom} A$, for some (and, hence, for all) $\ell \in \mathbb{C} \setminus \mathbb{R}$, see [5] and Proposition 1.2, [6]. We translate these conditions in terms of the triplet of Hilbert spaces $\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}$, associated with the extension A.

THEOREM 3.1. *Let S be a closed, symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint extension of S. Then the following conditions are equivalent:*

(i) *S and A have a Q-function $Q(\ell)$ belonging to $\mathbf{N}_1 \setminus \mathbf{N}_0$.*

(ii) *There exists an element $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$ such that the operator S is given by*

$$\operatorname{dom} S = \{h \in \operatorname{dom} A : (h, \omega) = 0\}. \tag{3.1}$$

(iii) *S and A have a Q-function $Q(\ell)$ with the operator representation*

$$Q(\ell) = \gamma + ((\widetilde{A} - \ell)^{-1}\omega, \omega), \quad \ell \in \mathbb{C} \setminus \mathbb{R}, \tag{3.2}$$

where $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$ and $\gamma \in \mathbb{R}$.

(iv) $\operatorname{dom} S^* \subset \operatorname{dom} |A|^{\frac{1}{2}}$, $\operatorname{dom} S^* \not\subset \operatorname{dom} A$.

In each of these cases $\chi(\ell) = (\widetilde{A} - \ell)^{-1}\omega$, $\gamma = \lim_{y\to\infty} Q(\mathrm{i}y)$ *and* S *is densely defined.*

Proof. (i) $\Rightarrow$ (ii) Assume that $Q(\ell)$ belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$. Then A is an operator and $\chi(\ell) \in \operatorname{dom} |A|^{\frac{1}{2}} \setminus \operatorname{dom} A$ for all $\ell \in \mathbb{C} \setminus \mathbb{R}$. Fix $\mu \in \mathbb{C} \setminus \mathbb{R}$ and define $\omega = (\widetilde{A} - \mu)\chi(\mu)$, so that $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$. For $h \in \operatorname{dom} A$ it follows from (2.2) and the symmetry condition (2.4) that $[(A-\overline{\mu})h, \chi(\mu)] = ((A-\overline{\mu})h, \chi(\mu)) = (h, (\widetilde{A}-\mu)\chi(\mu)) = (h, \omega)$. Therefore (3.1) follows from (0.2).

(ii) $\Rightarrow$ (iii) Assume that S is given by (3.1) with $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$. Let $\chi(\mu) = (\widetilde{A} - \mu)^{-1}\omega$. It follows from the definition (0.1) of $\chi(\ell)$ and the resolvent identity (2.9) that

$$\frac{\chi(\ell) - \chi(\mu)}{\ell - \mu} = (A - \ell)^{-1}\chi(\mu) = (A - \ell)^{-1}(\widetilde{A} - \mu)^{-1}\omega = \frac{\widetilde{R}(\ell)\omega - \widetilde{R}(\mu)\omega}{\ell - \mu}.$$

This gives $\chi(\ell) = \widetilde{R}(\ell)\omega$ and since $[(A - \overline{\ell})h, \chi(\ell)] = (h, \omega)$, it follows that $\chi(\ell) = \widetilde{R}(\ell)\omega$ maps onto the defect subspace $\ker(S^* - \ell)$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, of S. The resolvent identity (2.9) and the symmetry condition (2.6) give

$$\frac{(\widetilde{R}(\ell)\omega, \omega) - (\widetilde{R}(\overline{\lambda})\omega, \omega)}{\ell - \overline{\lambda}} = (R(\overline{\lambda})\widetilde{R}(\ell)\omega, \omega) = [\widetilde{R}(\ell)\omega, \widetilde{R}(\lambda)\omega], \quad \ell, \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Hence the function $Q(\ell) = \gamma + (\widetilde{R}(\ell)\omega, \omega)$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, with $\gamma \in \mathbb{R}$ satisfies (0.3) and therefore it is a Q-function of S and A.

(iii) $\Rightarrow$ (i) Assume that $Q(\ell)$ has the representation (3.2). Let $\omega = V_{+1}\chi$, where $\chi \in \mathfrak{H}_{+1} \setminus \operatorname{dom} A$. Denoting $\mathrm{d}\sigma(t) = (|t| + 1)^2\, \mathrm{d}([E(t)\chi, \chi])$ we have

$$\int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{|t| + 1} = \int_{\mathbb{R}} (|t| + 1)\, \mathrm{d}([E(t)\chi, \chi]). \tag{3.3}$$

The last integral is finite as $\chi \in \operatorname{dom} |A|^{\frac{1}{2}}$ and, furthermore

$$\int_{\mathbb{R}} \mathrm{d}\sigma(t) = \int_{\mathbb{R}} (|t| + 1)^2 \mathrm{d}([E(t)\chi, \chi]) = \infty,$$

since $\chi \notin \operatorname{dom} A$. Therefore, it follows from (2.7) that

$$Q(\ell) = \gamma + ((\widetilde{A} - \ell)^{-1}V_{+1}\chi, V_{+1}\chi)$$

has the integral representation

$$Q(\ell) = \gamma + \int_{\mathbb{R}} \frac{1}{t-\ell}\,\mathrm{d}\sigma(t), \tag{3.4}$$

with $\int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{|t|+1} < \infty$ and $\int_{\mathbb{R}} \mathrm{d}\sigma(t) = \infty$. Thus, $Q(\ell) \in \mathbf{N}_1 \setminus \mathbf{N}_0$.

Since (i) is equivalent to $\ker(S^* - \ell) \subset \operatorname{dom}|A|^{\frac{1}{2}} \setminus \operatorname{dom} A$ for some (and, hence, for all) $\ell \in \mathbb{C} \setminus \mathbb{R}$, it follows from von Neumann's formula that (iv) is equivalent to (i).

The equivalence of (i), (ii), (iii) and (iv) has been established. The last statement follows by applying the dominated convergence theorem to the second term on the righthand side of (3.4). ∎

We amplify that the Q-function $Q(\ell)$ in Theorem 3.1 has the operator representation (3.2) and the integral representation (3.4), with $\int_{\mathbb{R}} \frac{1}{|t|+1}\,\mathrm{d}\sigma(t) < \infty$ and $\int_{\mathbb{R}} \mathrm{d}\sigma(t) = \infty$. Here $\mathrm{d}\sigma(t) = (|t|+1)^2 \mathrm{d}([E(t)\chi, \chi])$ and $\omega = V_{+1}\chi$ with $\chi \in \mathfrak{H}_{+1}$. By definition of V_{+1} and $[\,\cdot\,,\cdot\,]_{+1}$, we obtain $\|\omega\|_{-1}^2 = \|\chi\|_{+1}^2 = \int_{\mathbb{R}} (|t|+1)\,\mathrm{d}([E(t)\chi, \chi])$. By applying (3.3) we arrive at the following result.

COROLLARY 3.2. *The spectral measure* $\mathrm{d}\sigma(t)$ *and* ω *are related by*

$$\|\omega\|_{-1}^2 = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{|t|+1}.$$

THE HILBERT SPACE $\mathfrak{H}_S$. Assume that S has a selfadjoint extension A with a Q-function in $\mathbf{N}_1 \setminus \mathbf{N}_0$. Since the operator S is a restriction of the contraction A, we have $\|Sf\|_{-1} \leqslant \|f\|_{+1}$, $f \in \operatorname{dom} S$. Let $\mathfrak{H}_S$ be the closure of $\operatorname{dom} S$ in $\mathfrak{H}_{+1}$. Then S has a unique contractive extension $\widetilde{S}$ from $\mathfrak{H}_S$ into $\mathfrak{H}_{-1}$.

LEMMA 3.3. *The Hilbert spaces* $\mathfrak{H}_S$ *and* $\mathfrak{H}_{+1}$ *are related by*

$$\mathfrak{H}_S = \{h \in \mathfrak{H}_{+1} : (h, \omega) = 0\}. \tag{3.5}$$

In particular, $\dim \mathfrak{H}_{+1}/\mathfrak{H}_S = 1$.

Proof. Since $(h, \omega) = [h, V_1^{-1}\omega]_{+1}$, it follows from (3.1) that

$$\operatorname{dom} S = \{h \in \operatorname{dom} A : [h, V_{+1}^{-1}\omega]_{+1} = 0\}.$$

Now take closures in the Hilbert space $\mathfrak{H}_{+1}$. ∎

Let S be a densely defined, closed symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint extension of S. Assume that the Q-function $Q(\ell)$ of S and A belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$ as in Theorem 3.1 with $Q(\ell) = ((\widetilde{A}-\ell)^{-1}\omega, \omega)$. It follows from Lemma 3.3 and identity (2.10) that $S = \widetilde{S} \cap \mathfrak{H}^2$, so that

$$S = \{\{h, \widetilde{S}h\} : h \in \mathfrak{H}_{+1},\ \widetilde{S}h \in \mathfrak{H}\}. \tag{3.6}$$

We have $\operatorname{dom} S^* \subset \mathfrak{H}_{+1}$ and, hence, it follows from the definitions that $S^* = \widetilde{S}^* \cap \mathfrak{H}^2$. Therefore the adjoint S^* in $\mathfrak{H}$ can also be written as:

$$S^* = \{\{h, \widetilde{A}h + c\omega\} : h \in \mathfrak{H}_{+1},\ \widetilde{A}h + c\omega \in \mathfrak{H},\ c \in \mathbf{C}\}. \tag{3.7}$$

This makes it possible to translate Theorem 2.3 into a description of all selfadjoint extensions of S in the original Hilbert space $\mathfrak{H}$.

THEOREM 3.4. *Let S be a densely defined, closed symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint extension of S. Assume that the Q-function $Q(\ell)$ of S and A belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$ as in Theorem 3.1. Then the selfadjoint extensions of S in $\mathfrak{H}$ are in one-to-one correspondence with $\tau \in \mathbf{R} \cup \{\infty\}$ via*

$$A(\tau) = \left\{ \left\{ f, \widetilde{A}f + \frac{1}{\frac{1}{\tau}+\gamma}(f,\omega)\omega \right\} : f \in \mathfrak{H}_{+1},\ \widetilde{A}f + \frac{1}{\frac{1}{\tau}+\gamma}(f,\omega)\omega \in \mathfrak{H} \right\}, \tag{3.8}$$

when $1/\tau + \gamma \neq 0$, and

$$A(\tau) = \{\{f, \widetilde{S}f + c\omega\} : f \in \mathfrak{H}_S,\ \widetilde{S}f + c\omega \in \mathfrak{H},\ c \in \mathbf{C}\}, \tag{3.9}$$

when $1/\tau + \gamma = 0$. The selfadjoint extensions $A(\tau)$, $1/\tau+\gamma \neq 0$, have a Q-function in $\mathbf{N}_1 \setminus \mathbf{N}_0$. When $1/\tau + \gamma = 0$, the Q-function of $A(\tau)$ belongs to $\mathbf{N} \setminus \mathbf{N}_1$. The resolvent operator of $A(\tau)$, $\tau \in \mathbf{R} \cup \{\infty\}$, is given by

$$(A(\tau)-\ell)^{-1} = (A-\ell)^{-1} - \chi(\ell)\,\frac{1}{Q(\ell)+\frac{1}{\tau}}\,[\,\cdot\,, \chi(\bar{\ell})], \quad \ell \in \mathbf{C} \setminus \mathbf{R}. \tag{3.10}$$

Proof. Without loss of generality we take $\gamma = 0$, so that $1/\tau + \gamma = 0$ corresponds to $\tau = \infty$.

It can be verified directly that $A(\tau)$ in (3.8) is a closed symmetric extension of S. In order to show that $A(\tau)$ is selfadjoint, we verify that $A(\tau)$ is a proper extension of S, i.e., that $A(\tau) \neq S$. By Lemma 2.2 and Theorem 2.3 the resolvent operator $(\widetilde{A}(\tau)-\ell)^{-1}$ maps $\mathfrak{H}_{-1}$ continuously onto $\mathfrak{H}_{+1}$. Since $\mathfrak{H}$ is dense in $\mathfrak{H}_{-1}$, we conclude that $\operatorname{dom} A(\tau)$ is dense in $\mathfrak{H}_{+1}$. By Lemma 3.3 this proves the claim. Therefore, each $A(\tau)$ in (3.8) defines a selfadjoint extension of S.

Next we show that $A(\infty)$ given by (3.9) is a selfadjoint extension of S. Clearly, $S \subset A(\infty)$ and the symmetry of $A(\infty)$ follows by using Lemma 3.3. Again, to show that $A(\infty)$ is selfadjoint we verify that $A(\infty)$ is a proper extension of S, i.e., that $A(\infty) \neq S$. For this it suffices to show that there exists $f \in \mathfrak{H}_S$ such that $\widetilde{S}f + c\omega \in \mathfrak{H}$ for some $c \neq 0$, cf. (3.6). Observe that $\operatorname{ran}(\widetilde{S} - \ell)$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, is not dense in $\mathfrak{H}_{-1}$ while $\mathfrak{H}$ is dense in $\mathfrak{H}_{-1}$. This implies that there exists a nontrivial element $v \in \mathfrak{H} \setminus \operatorname{ran}(\widetilde{S} - \ell)$, $\ell \in \mathbb{C} \setminus \mathbb{R}$. According to Lemma 2.1 we can write

$$v = (\widetilde{S} - \ell)f + c\omega, \quad f \in \mathfrak{H}_S, \ c \in \mathbb{C}, \ c \neq 0.$$

Hence, the vector $h = v + \ell f$ belongs to $\mathfrak{H}$ and satisfies $h = \widetilde{S}f + c\omega$, $c \neq 0$. This proves the claim $A(\infty) \neq S$. Therefore, $A(\infty)$ in (3.9) also defines a selfadjoint extension of S.

Next we prove the converse. Let H be a selfadjoint extension of S and choose a fixed element $\{h, k\} \in H \setminus S$. By (3.7) $k = \widetilde{A}h + c\omega$ for some $c \in \mathbb{C}$. The symmetry of H gives $0 = c(\omega, h) - (h, \omega)\bar{c}$. If $c = 0$ then $k = \widetilde{A}h \in \mathfrak{H}$ and hence $\{h, k\} \in A(0)$. Since H and $A(0)$ both are selfadjoint this forces $H = A(0) = A$. If $c \neq 0$, then $(h, \omega)/c \in \mathbb{R}$. We denote this number by $1/\tau$, $\tau \in \mathbb{R} \cup \{\infty\}$, $\tau \neq 0$. In the case $\tau \in \mathbb{R} \setminus \{0\}$ we see that $\{h, k\} \in A(\tau)$ and conclude that $H = A(\tau)$. In the case $\tau = \infty$ we have $(h, \omega) = 0$ and hence $h \in \mathfrak{H}_S$, i.e., $\{h, k\} \in A(\infty)$, which implies that $H = A(\infty)$. Hence, we have shown that every selfadjoint extension H of S in $\mathfrak{H}$ is of the form (3.8) or (3.9), respectively.

If $\tau \in \mathbb{R}$, then $f \in \operatorname{dom} A(\tau)$ is equivalent to $(\widetilde{A}(\tau) - \ell)f \in \mathfrak{H}$ so that $(A(\tau) - \ell)^{-1}$ is the restriction of $(\widetilde{A}(\tau) - \ell)^{-1}$ to $\mathfrak{H}$. Similarly, $(A(\infty) - \ell)^{-1}$ is the restriction of $(\widetilde{A}(\infty) - \ell)^{-1}$ to $\mathfrak{H}$. Hence, the identity (3.10) for the resolvents of $A(\tau)$ follows by restricting the formula (2.14) to $\mathfrak{H}$.

The statements about the Q-functions of S and $A(\tau)$ follow from the bilinear transform, see [5]. ∎

The exceptional extension can be characterized in terms of the Hilbert space $\mathfrak{H}_S$. In the next section it will be shown that $\mathfrak{H}_S$ (as a topological space) is independent of the choice of the nonexceptional extension A of S.

PROPOSITION 3.5. *Let S be a closed symmetric operator with defect numbers $(1,1)$. Suppose that S has a selfadjoint extension A whose Q-function $Q(\ell)$ belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$. Then the exceptional selfadjoint extension is given by*

$$S_F = \{\{h, k\} \in S^* : h \in \mathfrak{H}_S\}. \tag{3.11}$$

It is the only selfadjoint extension H of S with the property that $\operatorname{dom} H \subset \mathfrak{H}_S$.

Proof. We will show that $A(\infty) = T$ where T is given by

$$T = \{\{h, k\} \in S^* : (h, \omega) = 0\}. \tag{3.12}$$

Since $\operatorname{dom} S^* \subset \mathfrak{H}_{+1}$ the representation (3.11) is a direct consequence of (3.12), cf. Lemma 3.3. Now let $1/\tau + \gamma = 0$. Then the selfadjoint extension $A(\tau)$ in (3.9) has the property $\operatorname{dom} A(\tau) \subset \mathfrak{H}_S$, and hence, $A(\tau) \subset T$. The identity (3.7) implies that T is symmetric. We conclude that $T = A(\tau)$. Clearly, if H is a selfadjoint extension of S in $\mathfrak{H}$ such that $\operatorname{dom} H \subset \mathfrak{H}_S$, then $H \subset T$. Since both operators are selfadjoint, it follows that $H = T$. ∎

Let S be a densely defined symmetric operator with defect numbers $(1, 1)$ and assume that S is semibounded. We show that there is a selfadjoint extension whose Q-function belongs to $\mathbf{N}_1$, cf. [5]. Let A be a selfadjoint extension of S, different from the Friedrichs extension S_F. Without loss of generality we assume that both S and A have a nonnegative lower bound. Then $|A| = A$ and the topologies generated by $S + I$ and $|A| + I$ coincide on $\operatorname{dom} S$. Therefore, the Hilbert space $\mathfrak{H}_S$ in Section 4 is the completion of $\operatorname{dom} S$ with respect to the inner product $[(S+I)\cdot, \cdot]$ and coincides with the space $\mathfrak{H}_S$ as introduced in Section 1. If $\mathfrak{H}_{+1} = \mathfrak{H}_S$, then $\operatorname{dom} A \subset \mathfrak{H}_S \cap \operatorname{dom} S^* = \operatorname{dom} S_F$ implies that $A = S_F$, a contradiction. Since S has defect numbers $(1, 1)$, we conclude that $\dim \mathfrak{H}_{+1}/\mathfrak{H}_S = 1$. Therefore S is a restriction of A of the form (3.1) and Theorem 3.1 implies that the Q-function of A and S belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$. Note that the exceptional extension in (3.11) coincides with the Friedrichs extension of S.

Let S be nondensely defined and let A be a selfadjoint operator extension of S. Then the results of this section parallel the results of Section 1, when suitably interpreted. Let the space $\mathfrak{H}_S$ be the closure of $\operatorname{dom} S$ in $\mathfrak{H}_{+1}$. When S is also nonnegative, $\mathfrak{H}_S$ coincides with the space defined at the end of Section 1. The analog of Proposition 3.5 now takes the following form.

PROPOSITION 3.6. *Let S be a closed symmetric operator with defect numbers $(1, 1)$. Suppose that S has a selfadjoint extension A whose Q-function $Q(\ell)$ belongs to $\mathbf{N}_0$. Then the exceptional selfadjoint extension $S \dotplus (\{0\} \oplus \operatorname{mul} S^*)$ is given by (3.11). It is the only selfadjoint extension H of S with the property that $\operatorname{dom} H \subset \mathfrak{H}_S$.*

4. INVARIANCE PROPERTIES OF EXTENSIONS ASSOCIATED WITH $\mathbf{N}_1$

Let S be a symmetric operator with defect numbers $(1,1)$. By Theorem 1.1 the operator S is not densely defined if and only if there is a Q-function $Q(\ell)$ of S belonging to $\mathbf{N}_0$. If S is nondensely defined, then all but one of the selfadjoint extensions of S are operators whose domains coincide. In fact these domains are all equal to $\operatorname{dom} S^*$, cf. Theorem 1.3. This property is characteristic for this situation. Let A_1 and A_2 be two different selfadjoint extensions of S. Applying von Neumann's formula and a dimension argument, we see that $S^* \subset A_1 \dot{+} A_2$; in particular $\operatorname{dom} S^* \subset \operatorname{dom} A_1 + \operatorname{dom} A_2$. Hence, if we assume that $\operatorname{dom} A_1 = \operatorname{dom} A_2$, then $\operatorname{dom} S^* = \operatorname{dom} A_i$, $i = 1,2$, and Theorem 1.1 shows that S is not densely defined.

The Q-function $Q(\ell)$ of S belongs to $\mathbf{N}_1$ if and only if $\operatorname{dom} S$ is not dense in $\mathfrak{H}_{+1}$, cf. Theorem 3.1. We give invariance properties for the domain $\operatorname{dom} |A(\tau)|^{\frac{1}{2}}$, when $A(\tau)$ is not an exceptional selfadjoint extension.

THEOREM 4.1. *Assume that the Q-function $Q(\ell)$ of S and A belongs to $\mathbf{N}_1$ and let $\gamma = \lim_{y\to\infty} Q(iy)$. Then*

$$\operatorname{dom} |A(\tau)|^{\frac{1}{2}} = \operatorname{dom} |A|^{\frac{1}{2}}, \quad \tau \in \mathbb{R} \cup \{\infty\}, \quad \frac{1}{\tau} + \gamma \neq 0. \tag{4.1}$$

Proof. Assume that $1/\tau + \gamma \neq 0$, $\tau \neq 0$, and let $\psi \in \operatorname{dom} |A|^{\frac{1}{2}}$. Then $\varphi = (\widetilde{A} - \mu)\psi \in \mathfrak{H}_{-1}$ and according to (2.14)

$$(\widetilde{A}(\tau) - \ell)^{-1}\varphi = (\widetilde{A} - \ell)^{-1}\varphi - \chi(\ell)\frac{(\varphi, \chi(\bar{\ell}))}{Q(\ell) + \frac{1}{\tau}}. \tag{4.2}$$

We claim

$$(\widetilde{A}(\tau) - \ell)^{-1}\varphi \in \operatorname{dom} |A(\tau)|^{\frac{1}{2}}. \tag{4.3}$$

Since the Q-function of $A(\tau)$ and S belongs to $\mathbf{N}_1$ we have also $\chi(\ell) \in \operatorname{dom} |A(\tau)|^{\frac{1}{2}}$. It then follows from (4.2) (with $\ell = \mu$) that $\psi = (\widetilde{A} - \mu)^{-1}\varphi \in \operatorname{dom} |A(\tau)|^{\frac{1}{2}}$. This shows $\operatorname{dom} |A|^{\frac{1}{2}} \subset \operatorname{dom} |A(\tau)|^{\frac{1}{2}}$.

In order to prove the claim (4.3) we note that $\widetilde{A}(\tau)$ satisfies the (resolvent) identity (2.9) by continuity (cf. Theorem 2.3). Hence the mapping $(\widetilde{A}(\tau) - \ell)^{-1}\varphi$ satisfies (0.1) and defines a symmetric restriction $S_{\tau,\varphi}$ of $A(\tau)$ whose defect subspaces are thus spanned by $(\widetilde{A}(\tau) - \ell)^{-1}\varphi$, cf. (0.2). Our claim is equivalent to saying that the corresponding Q-function of $A(\tau)$ and $S_{\tau,\varphi}$ belongs to $\mathbf{N}_1$, cf. (0.3). This means

$$\|(\widetilde{A}(\tau) - iy)^{-1}\varphi\|^2 \in L^1(1, \infty). \tag{4.4}$$

To show (4.4) we note that it follows from (4.2) that

$$(4.5)\qquad \|(\widetilde{A}(\tau)-\ell)^{-1}\varphi\|^2 \leqslant 2\left(\|(\widetilde{A}-\ell)^{-1}\varphi\|^2+\|\chi(\ell)\|^2\left|\frac{(\varphi,\chi(\bar{\ell}))}{Q(\ell)+\frac{1}{\tau}}\right|^2\right).$$

By (2.3), $|(\varphi,\chi(-\mathrm{i}y))| \leqslant \|\varphi\|_{-1}\|\chi(-\mathrm{i}y)\|_{+1}$. Observe that with $\chi = V_{+1}^{-1}\omega \in \operatorname{dom}|A|^{\frac{1}{2}}$

$$\|\chi(-\mathrm{i}y)\|_{+1}^2 = \int_{\mathbb{R}} \frac{(|t|+1)^3}{t^2+y^2}\,\mathrm{d}([E(t)\chi,\chi]) \leqslant M < \infty, \quad y\in[1,\infty),$$

compare (2.7). Since $\omega,\varphi\in\mathfrak{H}_{-1}$ the functions

$$\|\chi(\mathrm{i}y)\|^2 = \|(\widetilde{A}-\mathrm{i}y)^{-1}\omega\|^2,\ \|(\widetilde{A}-\mathrm{i}y)^{-1}\varphi\|^2,$$

belong to $L^1(1,\infty)$. Furthermore, $\lim_{y\to\infty} Q(\mathrm{i}y)+1/\tau \neq 0$ and hence (4.5) implies (4.4).

The converse inclusion $\operatorname{dom}|A(\tau)|^{\frac{1}{2}} \subset \operatorname{dom}|A|^{\frac{1}{2}}$ follows from the fact that both A and $A(\tau)$ have a Q-function belonging to $\mathbf{N}_1$ so that their roles above can be interchanged. ∎

As a consequence of Theorem 4.1 we observe the following. Let S be a densely defined symmetric operator with defect numbers $(1,1)$ and let A_1 and A_2 be two selfadjoint extensions of S in $\mathfrak{H}$. Assume that the corresponding Q-functions $Q_1(\ell)$ and $Q_2(\ell)$ belong to $\mathbf{N}_1$. Then it follows from (4.1) by applying the closed graph theorem, that the topologies of the spaces $\mathfrak{H}_{+1}(A_1)$ and $\mathfrak{H}_{+1}(A_2)$, and of the spaces $\mathfrak{H}_{-1}(A_1)$ and $\mathfrak{H}_{-1}(A_2)$, are the same. Hence Theorem 4.1 implies that the space triplet associated with A, as well as the space $\mathfrak{H}_S$, do not depend on the choice of the nonexceptional extension.

It is shown in the following theorem that the invariance result in Theorem 4.1 characterizes the class $\mathbf{N}_1$. We prove this converse result in a slightly stronger form.

THEOREM 4.2. *Let S be a closed symmetric operator in $\mathfrak{H}$ with defect numbers $(1,1)$ and let $\alpha\geqslant 0$. If for two different selfadjoint operator extensions A_1 and A_2 of S the inclusion*

$$\operatorname{dom}|A_1|^{\frac{1}{2}} \supset \operatorname{dom}|A_2|^{\frac{1}{2}+\alpha} \quad \textit{or} \quad \operatorname{dom}A_1 \supset \operatorname{dom}|A_2|^{1+\alpha},$$

is satisfied, then all but one of the selfadjoint extensions $A(\tau)$ of S in $\mathfrak{H}$ satisfy the identity

$$\operatorname{dom}|A(\tau)|^{\frac{1}{2}} = \operatorname{dom}|A_1|^{\frac{1}{2}} \quad \textit{or} \quad \operatorname{dom}A(\tau) = \operatorname{dom}A_1,$$

respectively. Moreover, the Q-functions of these extensions of S all belong to $\mathbf{N}_1$ or to $\mathbf{N}_0$, respectively.

Proof. Assume that for the selfadjoint extensions A_1 and A_2 of S the indicated inclusion is satisfied and let $R_1(\ell)$ and $R_2(\ell)$ be their resolvent operators. With A_1 we associate the mapping $\chi(\ell)$ as in (0.1) and let $Q(\ell)$ be the corresponding Q-function of A_1 and S. By Kreĭn's formula

$$\frac{[h,\chi(\bar{\ell})]}{\frac{1}{\tau}+Q(\ell)}\chi(\ell) = R_1(\ell)h - R_2(\ell)h, \quad h \in \mathfrak{H},\ \tau \neq 0. \tag{4.6}$$

Since $|A_2|^\alpha$ is a selfadjoint operator it is densely defined and hence we may select $h \in \operatorname{dom}|A_2|^\alpha$ such that for a fixed $\ell \in \mathbb{C}\setminus\mathbb{R}$, $[h,\chi(\bar{\ell})] \neq 0$. Then we have

$$R_1(\ell)h \in \operatorname{dom} A_1 \subset \operatorname{dom}|A_1|^{\frac{1}{2}}, \quad R_2(\ell)h \in \operatorname{dom}|A_2|^{1+\alpha} \subset \operatorname{dom}|A_2|^{\frac{1}{2}+\alpha}.$$

By assumption, $\operatorname{dom}|A_2|^{\frac{1}{2}+\alpha} \subset \operatorname{dom}|A_1|^{\frac{1}{2}}$ and $\operatorname{dom}|A_2|^{1+\alpha} \subset \operatorname{dom} A_1$. Hence, by the above selection of h, it follows from (4.6), that $\chi(\ell) \in \operatorname{dom}|A_1|^{\frac{1}{2}}$ or $\chi(\ell) \in \operatorname{dom} A_1$, respectively. This means that $Q(\ell) \in \mathbf{N}_1$ or $Q(\ell) \in \mathbf{N}_0$, respectively. Now the claims concerning the domains associated with the selfadjoint extensions of S follow from Theorems 4.1 and 1.3. The last statement follows from Theorems 1.1 and 3.1. ∎

The invariance properties mentioned in Theorem 4.2 can be used to strengthen the characterization in Propositions 3.5 and 3.6 of the (generalized) Friedrichs extension: it is the only selfadjoint extension H of S for which $\operatorname{dom} H^k \subset \mathfrak{H}_S$ for some (and hence for all) $k \geqslant 1$.

5. MULTIPLICATION OPERATORS

In this section we consider a real-valued nondecreasing function ρ on $\mathbb{R}$ and associate with this function the Hilbert space $\mathfrak{H} = L^2(\mathrm{d}\rho)$. The inner product on $L^2(\mathrm{d}\rho)$ is denoted by $[\cdot,\cdot]$, and by abuse of notation we shall denote the integral $\int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}\rho(t)$ also by $[f,g]$ whenever fg is integrable with respect to $\mathrm{d}\rho$. Multiplication by the independent variable:

$$A = \{\{f,g\} \in \mathfrak{H}^2 : g = tf\},$$

is a densely defined selfadjoint operator. Now let ω be a measurable scalar function on $\mathbb{R}$ and consider the restriction S of A defined by

$$\operatorname{dom} S = \{f \in \operatorname{dom} A : [f,\omega] = 0\}. \tag{5.1}$$

It can be shown (see [6]) that S is a closed symmetric operator with defect numbers $(1,1)$ if and only if ω does not vanish almost everywhere with respect to $\mathrm{d}\rho$ and

$$\int_{\mathbb{R}} |\omega(t)|^2 \frac{\mathrm{d}\rho(t)}{t^2+1} < \infty.$$

The Q-function $Q(\ell)$ of S and A is given by

$$Q(\ell) = \alpha + \int_{\mathbb{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) |\omega(t)|^2 \, \mathrm{d}\rho(t), \quad \alpha \in \mathbb{R}.$$

It belongs to $\mathbf{N}_1$ if and only if

$$\int_{\mathbb{R}} |\omega(t)|^2 \frac{\mathrm{d}\rho(t)}{|t|+1} < \infty. \tag{5.2}$$

Note that the Q-function of S and A belongs to $\mathbf{N}_0$ if and only if ω does not vanish almost everywhere with respect to $\mathrm{d}\rho$ and $\omega \in L^2(\mathrm{d}\rho)$, which is the case if and only if S is not densely defined. In this case all but one of the canonical selfadjoint extensions of S are rank one range perturbations of A as in Theorem 1.3.

We are interested in the case that the Q-function $Q(\ell)$ of S and A belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$. The triplet structure associated with A can be described explicitly. The Hilbert space $\mathfrak{H}_{-1}$ is a weighted L^2-space, consisting of all scalar functions f on $\mathbb{R}$ for which

$$\int_{\mathbb{R}} |f(t)|^2 \frac{\mathrm{d}\rho(t)}{|t|+1} < \infty,$$

and is provided with the inner product

$$[f,g]_{-1} = \int_{\mathbb{R}} f(t)\overline{g(t)} \frac{\mathrm{d}\rho(t)}{|t|+1}.$$

The Hilbert space $\mathfrak{H}_{+1}$ is a weighted L^2-space, consisting of all scalar functions f on $\mathbb{R}$ for which

$$\int_{\mathbb{R}} |f(t)|^2 (|t|+1) \, \mathrm{d}\rho(t) < \infty,$$

and is provided with the inner product

$$[f,g]_{+1} = \int_{\mathbb{R}} f(t)\overline{g(t)} \, (|t|+1) \, \mathrm{d}\rho(t).$$

The Riesz operator V_{+1} from $\mathfrak{H}_{+1}$ to $\mathfrak{H}_{-1}$ is given by

$$V_{+1}f(t) = (|t|+1)f(t), \quad f \in \mathfrak{H}_{+1},$$

and the duality $(\cdot,\cdot)$ between $\mathfrak{H}_{+1}$ and $\mathfrak{H}_{-1}$ has the form

$$(f,g) = \int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}\rho(t), \quad f \in \mathfrak{H}_{+1},\ g \in \mathfrak{H}_{-1}.$$

Note that $fg \in L^1(\mathrm{d}\rho)$ when $f \in \mathfrak{H}_{+1}$ and $g \in \mathfrak{H}_{-1}$. The extension $\widetilde{A}$ from $\mathfrak{H}_{+1}$ to $\mathfrak{H}_{-1}$ is given by

$$\widetilde{A}f(t) = tf(t), \quad f \in \mathfrak{H}_{+1}.$$

It coincides with the multiplication operator in the Hilbert space $\mathfrak{H}_{-1}$. The extension $\widetilde{S}$ from $\mathfrak{H}_S$ to $\mathfrak{H}_{-1}$ is given by

$$\mathrm{dom}\,\widetilde{S} = \{h \in \mathrm{dom}\,\widetilde{A} : (h,\omega) = 0\}. \tag{5.3}$$

The Q-function of S and A belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$ if and only if $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$. In this case

$$Q(\ell) = \gamma + \int_{\mathbb{R}} \frac{|\omega(t)|^2}{t-\ell}\,\mathrm{d}\rho(t), \quad \gamma \in \mathbb{R}.$$

The following result is a translation of Theorem 3.4, see [5], [6], where these results were obtained without reference to the triplet structure.

PROPOSITION 5.1. *Assume that the function ω satisfies* (5.2) *and that* $\lim_{y\to\infty} Q(\mathrm{i}y) = 0$. *Then all canonical selfadjoint extensions $A(\tau)$ of the symmetric operator S in* (5.3) *are given by $A(0) = A$ and for $\tau \neq 0$, $\tau \neq \infty$, by*

$$A(\tau) = \Big\{\{f, tf + \tau[f,\omega]\omega\} : f \in L^2(\mathrm{d}\rho),\ f\omega \in L^1(\mathrm{d}\rho), \\ tf + \tau[f,\omega]\omega \in L^2(\mathrm{d}\rho)\Big\}, \tag{5.4}$$

while for $\tau = \infty$ the exceptional canonical selfadjoint extension is given by

$$A(\infty) = \Big\{\{f, tf + c\omega\} : f \in L^2(\mathrm{d}\rho),\ f\omega \in L^1(\mathrm{d}\rho), [f,\omega] = 0, \\ tf + c\omega \in L^2(\mathrm{d}\rho)\Big\}. \tag{5.5}$$

Proof. Let $\{f, \widetilde{A}f + \tau(f,\omega)\omega\}$ be an element in $A(\tau)$ as in (3.8), so that $f \in \mathfrak{H}_{+1}$ and $\widetilde{A}f + \tau(f,\omega)\omega \in \mathfrak{H}$. Now $f \in \mathfrak{H}_{+1}$ implies that $\widetilde{A}f = tf$ and also that $f \in L^2(\mathrm{d}\rho)$. Since $\omega \in \mathfrak{H}_{-1}$ it follows that $f\omega \in L^1(\mathrm{d}\rho)$. Hence,

$\widetilde{A}f+\tau(f,\omega)\omega = tf+\tau[f,\omega]\omega$. We conclude that $\{f, \widetilde{A}f+\tau(f,\omega)\omega\}$ belongs to the righthand side of (5.4).

Conversely, let $\{f, tf+\tau[f,\omega]\omega\}$ be an element in the righthand side of (5.4). Then $f\in L^2(\mathrm{d}\rho)$, $f\omega\in L^1(\mathrm{d}\rho)$ and $g=tf+\tau[f,\omega]\omega\in L^2(\mathrm{d}\rho)$. It follows from

$$|t|\,|f(t)|^2 \leqslant |f(t)|\,|g(t)| + |\tau|\,|[f,\omega]|\,|f(t)|\,|\omega(t)|,$$

that the lefthand side is integrable with respect to $\mathrm{d}\rho$. Hence $f\in\mathfrak{H}_{+1}$ and we may write $tf=\widetilde{A}f$ and $[f,\omega]=(f,\omega)$. We conclude that $\{f, tf+\tau[f,\omega]\omega\}$ belongs to $A(\tau)$ as given by (3.8).

Now let $\{f, \widetilde{S}f+c\omega\}$ be an element in $A(\infty)$ as in (3.9), so that $f\in\mathfrak{H}_S$ and $\widetilde{S}f+c\omega\in\mathfrak{H}$. By Lemma 3.3 it follows that $f\in\mathfrak{H}_{+1}$ and that $(f,\omega)=0$. Hence $f\omega\in L^1(\mathrm{d}\rho)$ and $[f,\omega]=0$. Moreover $\widetilde{S}f=\widetilde{A}f=tf$ so that $tf+c\omega\in L^2(\mathrm{d}\rho)$. Hence $\{f,\widetilde{S}f+c\omega\}$ belongs to the righthand side of (5.5).

Conversely, let $\{f, tf+c\omega\}$ belong to the righthand side of (5.5). Then $f\in L^2(\mathrm{d}\rho)$, $f\omega\in L^1(\mathrm{d}\rho)$, $[f,\omega]=0$ and $g=tf+c\omega\in L^2(\mathrm{d}\rho)$. It follows from

$$|t|\,|f(t)|^2 \leqslant |f(t)|\,|g(t)| + |c|\,|f(t)|\,|\omega(t)|$$

that the lefthand side is integrable with respect to $\mathrm{d}\rho$, so that $f\in\mathfrak{H}_{+1}$. Moreover $(f,\omega)=[f,\omega]=0$, so that by Lemma 3.3 $f\in\mathfrak{H}_S$. Therefore $tf=\widetilde{A}f=\widetilde{S}f$ and $\{f,tf+c\omega\}=\{f,\widetilde{S}f+c\omega\}$ belongs to $A(\infty)$ as given by (3.9). This completes the proof of the proposition. ∎

6. CONCLUDING REMARKS

In this final section we consider some special problems concerning graph perturbations and triplets, the subclass $\mathbf{N}_\rho$ of Nevanlinna functions and a description of the exceptional extensions in terms of von Neumann's formula.

GRAPH PERTURBATIONS. We will give an equivalent formulation of Theorem 3.4 in which no reference is made to the triplet structure; in this way we interpret the canonical selfadjoint extensions as graph perturbations of the original extension A, cf. [6].

In Theorems 3.1 and 3.4 appears the element $\omega\in\mathfrak{H}_{-1}\setminus\mathfrak{H}$. It characterizes the symmetric operator S in (3.1). We will now decompose this element, which allows us to interpret the canonical selfadjoint extensions in Theorem 3.4 as graph perturbations of A in the original Hilbert space $\mathfrak{H}$.

LEMMA 6.1. *Let $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$ be as in Theorem 3.4. Then there exist elements $\varphi \in \mathfrak{H}_{+1} \setminus \operatorname{dom} A$ and $\psi \in \mathfrak{H}$ such that*

$$\omega = \widetilde{A}\varphi - \psi. \tag{6.1}$$

The pair $\{\varphi, \psi\}$ is determined uniquely modulo A. Conversely, if $\varphi \in \mathfrak{H}_{+1} \setminus \operatorname{dom} A$ and $\psi \in \mathfrak{H}$, then ω defined by (6.1) belongs to $\mathfrak{H}_{-1} \setminus \mathfrak{H}$.

Proof. Since $\widetilde{A} - \ell$ maps $\mathfrak{H}_{+1}$ in a one-to-one way onto $\mathfrak{H}_{-1}$, there exists a unique element $h \in \mathfrak{H}_{+1}$ such that $\omega = \widetilde{A}h - \ell h$. With $\varphi = h$ and $\psi = \ell h$, the existence has been demonstrated. As to uniqueness, assume that $\omega = \widetilde{A}\varphi_i - \psi_i$, $\varphi_i \in \mathfrak{H}_{+1}$, $\psi_i \in \mathfrak{H}$, $i = 1, 2$. Then $\widetilde{A}(\varphi_1 - \varphi_2) = \psi_1 - \psi_2$. Hence with $h = \varphi_1 - \varphi_2$ and $k = \psi_1 - \psi_2$, it follows that $k \in \mathfrak{H}$ and $k = \widetilde{A}h$, so that $h \in \operatorname{dom} A$ and $k = Ah$, cf. (2.10). ∎

We now present a formulation of Theorem 3.4, which does not refer to the triplet structure. This formulation brings out clearly the fact that in the $\mathbf{N}_1 \setminus \mathbf{N}_0$ situation the symmetric operator S is obtained via a graph restriction of A. The following result, based on the polar decomposition $A = U|A|$ of the selfadjoint operator A, was derived in a different way in Proposition 6.3, [6].

PROPOSITION 6.2. *Assume that $\varphi \in \operatorname{dom} |A|^{\frac{1}{2}} \setminus \operatorname{dom} A$ and $\psi \in \mathfrak{H}$. Let S be the restriction of A, given by*

$$\operatorname{dom} S = \{h \in \operatorname{dom} A : [Ah, \varphi] - [h, \psi] = 0\}. \tag{6.2}$$

Then S is a densely defined, closed symmetric operator with defect numbers $(1,1)$, and the Q-function of S and A belongs to $\mathbf{N}_1 \setminus \mathbf{N}_0$. Assume that $\lim_{y\to\infty} Q(iy) = 0$. The canonical selfadjoint extensions of S in $\mathfrak{H}$ are in one-to-one correspondence with $\tau \in \mathbf{R} \cup \{\infty\}$ via

$$\begin{aligned} A(\tau) = \Big\{ \{f, A(f + c\varphi) - c\psi\} &: f + c\varphi \in \operatorname{dom} A, \\ c &= \tau \left([U|A|^{\frac{1}{2}} f, |A|^{\frac{1}{2}}\varphi] - [f, \psi] \right) \Big\}, \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} A(\infty) = \Big\{ \{f, A(f + c\varphi) - c\psi\} &: f + c\varphi \in \operatorname{dom} A, \\ &[U|A|^{\frac{1}{2}} f, |A|^{\frac{1}{2}}\varphi] - [f, \psi] = 0 \Big\}. \end{aligned} \tag{6.4}$$

Proof. Define ω by $\omega = \widetilde{A}\varphi - \psi$, so that $\omega \in \mathfrak{H}_{-1} \setminus \mathfrak{H}$, cf. Lemma 6.1. The restriction S defined by (6.2) then coincides with the symmetric operator in

(3.1). Hence, to describe the canonical selfadjoint extensions of S, we may apply Theorem 3.4. Let $\{f, \widetilde{A}f + \tau(f,\omega)\omega\}$ be in $A(\tau)$ as in (3.8). Then $f \in \mathfrak{H}_{+1}$ and $\widetilde{A}f + \tau(f,\omega)\omega \in \mathfrak{H}$. Let $c = \tau(f,\omega)$, so that

$$\widetilde{A}f + \tau(f,\omega)\omega = \widetilde{A}(f + c\,\varphi) - c\,\psi.$$

Since $\psi \in \mathfrak{H}$, it follows that $\widetilde{A}(f + c\,\varphi) \in \mathfrak{H}$, or equivalently, $f + c\,\varphi \in \operatorname{dom} A$, cf. (2.10). Moreover, since $f \in \mathfrak{H}_{+1}$ and $\psi \in \mathfrak{H}$, it follows from (2.5), that c can be written as

$$c = \tau\left([U|A|^{\frac{1}{2}}f, |A|^{\frac{1}{2}}\varphi] - [f,\psi]\right).$$

Hence, $\{f, \widetilde{A}f + \tau(f,\omega)\omega\}$ is contained in the righthand side of (6.3). Conversely, consider an element $\{f, A(f + c\,\varphi) - c\,\psi\}$ in the righthand side of (6.3), so that $f + c\,\varphi \in \operatorname{dom} A$ and $c = \tau\left([U|A|^{\frac{1}{2}}f, |A|^{\frac{1}{2}}\varphi] - [f,\psi]\right)$. Since $\varphi \in \mathfrak{H}_{+1}$ it follows that $f \in \mathfrak{H}_{+1}$. This together with $\psi \in \mathfrak{H}$ implies that $c = \tau(f, \widetilde{A}\varphi - \psi)$. From $f + c\,\varphi \in \operatorname{dom} A$, $\psi \in \mathfrak{H}$ and

$$A(f + c\,\varphi) - c\,\psi = \widetilde{A}(f + c\,\varphi) - c\,\psi = \widetilde{A}f + \tau(f,\omega)\omega,$$

it follows that $\widetilde{A}f + \tau(f,\omega)\omega \in \mathfrak{H}$. Hence, the righthand side of (6.3) is contained in $A(\tau)$ as given by (3.8).

Now let $\{f, \widetilde{S}f + c\omega\}$ be an element in $A(\infty)$ as in (3.9). Then $f \in \mathfrak{H}_S$, $\widetilde{S}f + c\omega \in \mathfrak{H}$ and

$$\widetilde{S}f + c\omega = \widetilde{A}(f + c\,\varphi) - c\,\psi.$$

Since $\psi \in \mathfrak{H}$, it follows that $\widetilde{A}(f + c\,\varphi) \in \mathfrak{H}$, or equivalently, $f + c\,\varphi \in \operatorname{dom} A$, cf. (2.10). Moreover, by Lemma 3.3, $f \in \mathfrak{H}_S$ means that $f \in \mathfrak{H}_{+1}$ and $(f,\omega) = 0$. Since $f, \varphi \in \mathfrak{H}_{+1}$ and $\psi \in \mathfrak{H}$, this last identity can be written as $[U|A|^{\frac{1}{2}}f, |A|^{\frac{1}{2}}\varphi] - [f,\psi] = 0$. Hence, the righthand side of (3.9) is contained in $A(\infty)$ as given by (6.4). Conversely, consider an element $\{f, A(f + c\,\varphi) - c\,\psi\}$ in the righthand side of (6.4), so that $f + c\,\varphi \in \operatorname{dom} A$ and $[U|A|^{\frac{1}{2}}f, |A|^{\frac{1}{2}}\varphi] - [f,\psi] = 0$. Since $\varphi \in \mathfrak{H}_{+1}$ it follows that $f \in \mathfrak{H}_{+1}$. This together with $\psi \in \mathfrak{H}$ implies that $(f,\omega) = 0$, in other words $f \in \mathfrak{H}_S$. Moreover, $A(f + c\,\varphi) - c\,\psi = \widetilde{S}f + c\omega \in \mathfrak{H}$. Hence, the righthand side of (6.4) is contained in $A(\infty)$ as given by (3.9). ∎

THE SUBCLASS $\mathbf{N}_\rho$ OF NEVANLINNA FUNCTIONS. We define the class $\mathbf{N}_\rho$, $0 < \rho < 2$, as the set of functions $Q(\ell)$ which belong to $\mathbf{N}$ and for which

$$\int_1^\infty \frac{\operatorname{Im} Q(iy)}{y^\rho}\,dy < \infty.$$

This class has been introduced in various sources, see for instance [9], [10], and [20]. It coincides with the functions $Q(\ell)$ with the integral representation

$$(6.5)\qquad Q(\ell) = \alpha + \beta\ell + \int_{\mathbb{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1}\right) d\sigma(t),$$

where $\alpha \in \mathbb{R}$, $\beta \geqslant 0$ and $\sigma(t)$ is a nondecreasing function which satisfies

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{1+|t|^\rho} < \infty.$$

If $0 \leqslant \rho \leqslant 1$, then $\lim_{y\to\infty} Q(iy)$ exists as a real number and the representation (6.5) reduces to the representation (1.1).

Let S be a closed symmetric operator with defect numbers $(1,1)$ and let A be a selfadjoint extension of S. The Q-function of S belongs to $\mathbf{N}_\rho$ if and only if $\chi(\ell) \in \operatorname{dom}|A|^{1-\frac{1}{2}\rho}$ for some (and, hence, for all) $\ell \in \mathbb{C}\setminus\mathbb{R}$, cf. [5] and [6]. Now assume that $0 \leqslant \rho \leqslant 1$. Then for each $\tau \in \mathbb{R}\cup\{\infty\}$, $1/\tau + \gamma \neq 0$, the Q-function $Q_\tau(\ell)$ associated with $A(\tau)$ in (6.3) belongs to $\mathbf{N}_\rho$, cf. (0.5). If $1/\tau + \gamma = 0$, the corresponding function $Q_\tau(\ell)$ of the selfadjoint extension $A(\tau)$ in (6.4) belongs to $\mathbf{N}\setminus\mathbf{N}_1$. Now it is easy to see that in this case counterparts of Theorems 4.1 and 4.2 are valid. For example, apart from the exceptional extension, the domains $\operatorname{dom}|A|^{1-\frac{1}{2}\rho}$ of all the selfadjoint extensions of S are the same. The same can be said for the triplets associated with $\operatorname{dom}|A|^{1-\frac{1}{2}\rho}$.

FRIEDRICHS EXTENSIONS AND VON NEUMANN'S FORMULA. Now we come back to Propositions 3.5 and 3.6 and we characterize the exceptional extension via von Neumann's formula. Let S be a closed symmetric operator with defect numbers $(1,1)$. Let $\chi(\mu) \in \ker(S^*-\mu)$ and $\chi(\bar\mu) \in \ker(S^*-\bar\mu)$, $\mu \in \mathbb{C}\setminus\mathbb{R}$, be defect vectors with the same norm. The following characterization follows from von Neumann's formula (see [1] and [3]).

PROPOSITION 6.3. *There is a one-to-one correspondence between the unit circle and all canonical selfadjoint extensions of S via*

$$(6.6)\qquad A(\zeta) = S \dotplus \operatorname{span}\{\chi(\bar\mu) - \zeta\chi(\mu), \bar\mu\chi(\bar\mu) - \zeta\mu\chi(\mu)\}, \quad |\zeta| = 1.$$

Let A be any selfadjoint extension of S and let $\chi(\mu) \in \ker(S^*-\mu)$, $\mu \in \mathbb{C}\setminus\mathbb{R}$, be a nontrivial defect vector. Let the function $\chi(\ell)$ be defined by (0.1), so that $\chi(\ell) \in \ker(S^*-\ell)$. Then $\chi(\bar\mu) - \chi(\mu) \in \operatorname{dom} A$ and $\chi(\bar\mu)$ and $\chi(\mu)$ have the same norm. Now suppose that we are in the $\mathbf{N}_1$ situation. With these defect vectors

$\chi(\bar{\mu})$ and $\chi(\mu)$ we characterize the value of ζ in (6.6) which corresponds to the exceptional selfadjoint extension. This answers a question of the late E.A. Coddington, who asked to characterize the Friedrichs extension in this way when S is semibounded.

If A happens to be the exceptional selfadjoint extension, then, clearly, the corresponding value ζ in (6.6) is given by $\zeta = 1$. If A is a nonexceptional selfadjoint extension, then its Q-function $Q(\ell)$ belongs to $\mathbb{N}_1$. In the next proposition we use Theorem 3.1, [5], where the connection between von Neumann's formula and Kreĭn's formula was made explicit.

PROPOSITION 6.4. *Assume that S has a selfadjoint extension A with a Q-function belonging to $\mathbb{N}_1$ and assume that $\gamma = \lim_{y\to\infty} Q(iy) = 0$. Then the value $Q(\bar{\mu})/Q(\mu)$ on the unit circle corresponds via (6.6) to the exceptional extension of S.*

Proof. The mapping $\zeta(\tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$, defined by

$$\zeta(\tau) = \frac{Q(\bar{\mu}) + \frac{1}{\tau}}{Q(\mu) + \frac{1}{\tau}} \tag{6.7}$$

gives a one-to-one correspondence between $\mathbb{R} \cup \{\infty\}$ and the unit circle. Hence, according to von Neumann's formula (6.6) there is a one-to-one correspondence between $\mathbb{R} \cup \{\infty\}$ and all canonical selfadjoint extensions of S, given by

$$A(\tau) = S \hat{+} \operatorname{span}\{\chi(\bar{\mu}) - \zeta(\tau)\chi(\mu), \bar{\mu}\chi(\bar{\mu}) - \zeta(\tau)\mu\chi(\mu)\}, \quad \tau \in \mathbb{R} \cup \{\infty\}. \tag{6.8}$$

In [5] it is shown that the resolvents of $A(\tau)$ in (6.8) are parametrized by

$$(A(\tau) - \ell)^{-1} = (A - \ell)^{-1} - \left(Q(\ell) + \frac{1}{\tau}\right)^{-1} [\cdot, \chi(\bar{\ell})]\chi(\ell), \quad \tau \in \mathbb{R} \cup \{\infty\}. \tag{6.9}$$

Comparing (6.9) with (3.10) we see that the value $\tau = \infty$ corresponds to the exceptional extension of S. In (6.7) this corresponds to the value $Q(\bar{\mu})/Q(\mu)$. ∎

In terms of the defect vector $\chi(\mu)$ and (the continuation of) the selfadjoint extension A determined by $\chi(\bar{\mu})$, the exceptional value can be written as

$$\frac{((\widetilde{A} - \mu)\chi(\mu), \chi(\mu))}{((\widetilde{A} - \bar{\mu})\chi(\mu), \chi(\mu))}.$$

A more geometric proof of Proposition 6.4 can be based on Lemma 3.3 and von Neumann's formula (6.6). Assume that $Q(\ell) \in \mathbb{N}_1 \setminus \mathbb{N}_0$ has the representation $Q(\ell) = ((\widetilde{A} - \ell)^{-1}\omega, \omega)$ and that $\chi(\ell) = (\widetilde{A} - \ell)^{-1}\omega$. It is clear that

$$\chi(\ell) \in \mathfrak{H}_{+1} \setminus \mathfrak{H}_S, \ \ell \in \mathbb{C} \setminus \mathbb{R}, \tag{6.10}$$

since otherwise $Q(\ell) = (\chi(\ell), \omega) = 0$, which is impossible. Since $\chi(\mu)$ and $\chi(\overline{\mu})$ span a two-dimensional subspace in $\mathfrak{H}_{+1}$, and since by Lemma 3.3, $\dim \mathfrak{H}_{+1}/\mathfrak{H}_S = 1$, it follows from (6.10) that there is a unique $\zeta \in \mathbb{C}$ satisfying

$$\chi(\overline{\mu}) - \zeta\chi(\mu) \in \mathfrak{H}_S.$$

This implies that $0 = (\chi(\overline{\mu}) - \zeta\chi(\mu), \omega) = Q(\overline{\mu}) - \zeta Q(\mu)$, i.e., $\zeta = Q(\overline{\mu})/Q(\mu)$. Clearly, $\zeta = Q(\overline{\mu})/Q(\mu)$ gives the only selfadjoint extension $A(\zeta)$ of S in (6.6) such that $\operatorname{dom} A(\zeta) \subset \mathfrak{H}_S$, cf. Proposition 3.5. In case $Q(\ell) \in \mathbf{N}_0$ the above arguments must be suitably interpreted.

The second author was supported by "Fonds zur Förderung der wissenschaftlichen Forschung" of Austria, Project P 09832-MAT.

REFERENCES

1. N.I. ACHIESER, I.M. GLASMAN, *Theorie der Linearen Operatoren im Hilbertraum*, 8. Auflage, Akademie Verlag, Berlin, 1981.
2. JU.M. BEREZANSKI, Expansions in eigenfunctions of selfadjoint operators [Russian], Naukova Dymka, Kiev 1965; English transl., *Transl. Math. Monographs*, vol. 17, Amer. Math. Soc., 1968.
3. E.A. CODDINGTON, Extension theory of formally normal and symmetric subspaces, *Mem. Amer. Math. Soc.* **134**(1973).
4. E.A. CODDINGTON, H.S.V. DE SNOO, Positive selfadjoint extensions of positive symmetric subspaces, *Math. Z.* **159**(1978), 203–214.
5. S. HASSI, H. LANGER, H.S.V. DE SNOO, Selfadjoint extensions for a class of symmetric operators with defect numbers (1, 1), 15th OT Conference Proceedings, 1995, 115–145.
6. S. HASSI, H.S.V. DE SNOO, One-dimensional graph perturbations of selfadjoint relations, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, to appear.
7. P. JONAS, H. LANGER, Some questions in the perturbation theory of J-nonnegative operators in Kreĭn spaces, *Math. Nachr.* **114**(1983), 205–226.
8. P. JONAS, H. LANGER, Selfadjoint extensions of a closed linear relation of defect one in a Kreĭn space, *Oper. Theory Adv. Appl.* vol. 80, 1995, pp. 176–205.
9. I.S. KAC, On integral representations of analytic functions mapping the upper halfplane onto a part of itself [Russsian], *Uspekhi Mat. Nauk* **11**(1956), 139–144.
10. I.S. KAC, M.G. KREĬN, R-functions-analytic functions mapping the upper halfplane into itself, Supplement I to the Russian edition of F.V. Atkinson, Discrete and continuous boundary problems [Russian], Mir, Moscow 1968; English transl., *Amer. Math. Soc. Transl. Ser. 2* **103**(1974), 1–18.
11. M.G. KREĬN, The theory of self-adjoint extensions of semibounded Hermitian transformations and its applications I [Russian], *Mat. Sb.* **62**(1947), 431–495.
12. M.G. KREĬN, The theory of self-adjoint extensions of semibounded Hermitian transformations and its applications. II [Russian], *Mat. Sb.* **63**(1947), 365–404.

13. M.G. KREĬN, A.A. NUDEL'MAN, The Markov moment problem and extremal problems [Russian], Nauka, Moscow, 1973; English transl., *Transl. Math. Monographs*, vol. 50, Amer. Math. Soc., 1970.
14. M.G. KREĬN, V.A. YAVRYAN, Spectral shift functions that arise in perturbations of a positive operator [Russian], *J. Operator Theory* **6**(1981), 155–191.
15. H. LANGER, B. TEXTORIUS, On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space, *Pacific J. Math.* **72**(1977), 135–165.
16. H. LANGER, B. TEXTORIUS, L-resolvent matrices of symmetric linear relations with equal defect numbers; applications to canonical differential equations, *Integral Equations Operator Theory* **5**(1982), 208–243.
17. A.G.R. MCINTOSH, Bilinear forms in Hilbert space, *J. Math. Mech.* **19**(1970), 1027-1045.
18. A.G.R. MCINTOSH, Hermitian bilinear forms which are not semibounded, *Bull. Amer. Math. Soc. (N.S)* **76**(1970), 732–737.
19. A.G.R. MCINTOSH, On the comparability of $A^{\frac{1}{2}}$ and $A^{*\frac{1}{2}}$, *Proc. Amer. Math. Soc. (N.S.)* **32**(1972), 430–434.
20. E.R. TSEKANOVSKIĬ, YU.L. SHMUL'YAN, The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions [Russian], *Uspekhi Mat. Nauk* **32**(1977), 69–124; English transl., *Russian Math. Surveys* **32**(1977), 73–131.

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Received February 16, 1996; revised June 11, 1996.