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ANTICOMMUTATIVITY AND SPIN 1/2 SCHRÖDINGER OPERATORS WITH MAGNETIC FIELDS

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ABSTRACT. It is proven that the spin 1/2 Schrödinger operator $\widetilde{H}$ with a constant magnetic field is the square of a sum of mutually strongly anticommuting self-adjoint operators. As an application, by using this formula, the essential spectrum of $\widetilde{H}$ with a vector potential in a class is identified. The class contains a vector potential to which Shigekawa's theorem (I. Shigekawa, *J. Funct. Anal.* **101**(1991), 255–285) cannot be applied.

KEYWORDS: *Anticommutativity, Schrödinger operator, Dirac operator, spin 1/2, essential spectrum, magnetic field.*

AMS SUBJECT CLASSIFICATION: 47N50, 49R20, 81Q10.

1. INTRODUCTION

The spectral properties of the Schrödinger operators $\widetilde{H}$ with magnetic fields for a *spin* 1/2 particle were deeply studied by Shigekawa in [9]. The operator is given by

$$\widetilde{H} = \sum_{j=1}^{d}(-\mathrm{i}\partial_j - a_j(x))^2 + \sum_{j,k=1}^{d} \frac{\mathrm{i}}{2} b_{jk}(x)\gamma^j\gamma^k$$

acting in $L^2(\mathbb{R}^d) \otimes \mathbb{C}^r$ where $r = 2^l$, $l = [d/2]$ with $[\cdot]$ the Gauss symbol, $\partial_j = \partial/\partial x^j$, $\mathbf{a}(x) = \sum_{j=1}^{d} a_j(x)\mathrm{d}x^j$ is a real 1-form called a vector potential, $\mathbf{b} = \sum_{j<k} b_{jk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k = \mathrm{d}\mathbf{a}$ with $b_{jk} = \partial_j a_k - \partial_k a_j$ is called a magnetic field and γ^j's are $r \times r$-Hermitian matrices (so called the Dirac matrices) satisfying

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 2\delta^{jk} \tag{1.1}$$

where the δ^{jk}'s are the Kronecker delta. This $\widetilde{H}$ is also represented as $\widetilde{H} = \mathcal{D}^2$ where $\mathcal{D}$ is the Dirac operator defined by

$$\mathcal{D} = \sum_{j=1}^{d} \gamma^j(-\mathrm{i}\partial_j - a_j(x)).$$

For comparison, we define the Schrödinger operator H with a magnetic field for a *spinless* particle by

$$H = \sum_{j=1}^{d}(-\mathrm{i}\partial_j - a_j(x))^2$$

acting in $L^2(\mathbb{R}^d)$.

We are mainly interested in $\widetilde{H}$ with asymptotically constant magnetic fields. Assume that all a_j is C^∞ and

$$b_{jk}(x) \to \Lambda_{jk} \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad j, k = 1, \ldots, d, \tag{1.2}$$

where $\Lambda = (\Lambda_{jk})$ is a real skew-symmetric matrix. We note that Λ has eigenvalues of the form $\pm\mathrm{i}\lambda_1, \ldots, \pm\mathrm{i}\lambda_n, 0, \ldots, 0$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Without loss of generality, we can take $\lambda_j > 0$.

First, we consider the 2-dimensional constant magnetic field case, where $b(x) = \lambda\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ with a positive constant λ. We can take $\mathbf{a}(x) = \lambda(-x^2\mathrm{d}x^1 + x^1\mathrm{d}x^2)/2$. Let $\gamma^j = \sigma^j$, $j = 1, 2$. Here, σ^j, $j = 1, 2, 3$, are the Pauli matrices as follows:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With

$$A = (-\mathrm{i}\partial_1 - a_1) + (\partial_2 - \mathrm{i}a_2)$$

acting in $L^2(\mathbb{R}^2)$, we find

$$A^*A = AA^* + 2\lambda, \tag{1.3}$$

$$H = \frac{1}{2}(AA^* + A^*A) \quad \text{and} \quad \widetilde{H} = H + \lambda\sigma^3.$$

THEOREM 1.1. *Let $d = 2$ and $\mathbf{b} = \lambda\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ with $\lambda > 0$. Then*

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = \{(2n+1)\lambda;\ n \in \mathbb{Z}_+\},$$

$$\sigma(\widetilde{H}) = \sigma_{\mathrm{ess}}(\widetilde{H}) = \{2n\lambda;\ n \in \mathbb{Z}_+\},$$

where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\sigma(\cdot)$ *and* $\sigma_{\text{ess}}(\cdot)$ *denote spectrum and essential spectrum, respectively. Moreover,*

$$\ker \widetilde{H} \subset \ker(\sigma^3 + 1).$$

It is well known that by virtue of the relation (1.3) we can prove this theorem in the same way as in the harmonic oscillator case (see, e.g., [5], [10]). All the eigenvectors of H are created by repeatedly acting A and A^* on the eigenvectors with the lowest eigenvalue.

In the higher dimensional case, Shigekawa has proved the following theorem.

THEOREM 1.2. (Shigekawa, [9]) *Assume the condition* (1.2).

(i) *Assume that* 0 *is an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \ldots, \pm i\lambda_n, 0, \ldots, 0$, $(\lambda_j > 0)$ *be eigenvalues of* Λ. *Then*

$$\sigma_{\text{ess}}(H) = [\lambda_1 + \cdots + \lambda_n, \infty),$$

$$\sigma(\widetilde{H}) = \sigma_{\text{ess}}(\widetilde{H}) = [0, \infty).$$

(ii) *Assume that* 0 *is not an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \ldots, \pm i\lambda_m$, $(\lambda_j > 0$, $m = d/2)$ *be eigenvalues of* Λ. *Then*

$$\sigma_{\text{ess}}(H) = \Big\{\sum_{j=1}^{m}(2k_j + 1)\lambda_j;\ k_j \in \mathbb{Z}_+\Big\},$$

$$\sigma_{\text{ess}}(\widetilde{H}) = \Big\{\sum_{j=1}^{m} 2k_j\lambda_j;\ k_j \in \mathbb{Z}_+\Big\}.$$

Moreover 0 *is an isolated point spectrum of* $\widetilde{H}$.

Shigekawa proved this theorem using relations between the essential spectrum of $\widetilde{H}$ and H such as the following:

$$\sigma_{\text{ess}}(\widetilde{H}) = \sigma_{\text{ess}}\Big(H - \sum_{j=1}^{n} i\lambda_j\gamma^{2j-1}\gamma^{2j}\Big),$$

$$\sigma_{\text{ess}}(\widetilde{H}) = \bigcup_{\varepsilon_1\cdots\varepsilon_m=\pm 1} \sigma_{\text{ess}}\Big(H + \sum_{j=1}^{m}\varepsilon_j\lambda_j\Big),$$

$$\bigcup_{\varepsilon_1\cdots\varepsilon_m=1} \sigma_{\text{ess}}\Big(H + \sum_{j=1}^{m}\varepsilon_j\lambda_j\Big) \setminus \{0\} = \bigcup_{\varepsilon_1\cdots\varepsilon_m=-1} \sigma_{\text{ess}}\Big(H + \sum_{j=1}^{m}\varepsilon_j\lambda_j\Big) \setminus \{0\}.$$

These equations are derived from the Weyl theorem, (1.1) and the fact that H and all $i\gamma^{2j-1}\gamma^{2j}$ mutually strongly commute. In particular, in the proof of the part

(i) he constructed concrete orthonormal functions in $L^2(\mathbb{R}^d)$ in order to use the Weyl criterion in a slightly strengthened version (see, e.g., [4]). Suppose that A is self-adjoint and $A \geqslant 0$. If there exists an orthonormal sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset D(A)$ such that $\|(A+1)^{-1}(A-\alpha)\varphi_k\| \to 0$ as $k \to \infty$, then $\alpha \in \sigma_{\mathrm{ess}}(A)$.

Comparing Shigekawa's proof and that Theorem 1.1 can be proved by creating all eigenvectors of H using the relation (1.3), we feel that the inner structure of $\widetilde{H}$ is not sufficiently clear in the higher dimensional case. Why does whether 0 is an eigenvalue of Λ or not cause the difference of $\sigma_{\mathrm{ess}}(\widetilde{H})$ in each case? The aim of this paper is to clarify the inner structure of $\widetilde{H}$ and to identify the spectrum of $\widetilde{H}$.

In this paper, we investigate $\mathcal{D}$ instead of $\widetilde{H}$ itself, since $\mathcal{D}$ has more rich structures inherited from the Clifford algebra generated by γ^j's than $\widetilde{H}$. In particular, in the constant magnetic field case, it is proven that $\mathcal{D}$ is a sum of operators which mutually strongly anticommute. We remark that the anticommutativity of self-adjoint operators restricts strongly themselves. Hence this property is very useful (see [11], [7], [1], [2] and references therein). Therefore, it is very interesting to investigate the properties of $\mathcal{D}$ which are derived from the anticommutativity.

The plan of this paper is the following. In Section 2, we consider the constant magnetic field case. We prove that $\mathcal{D}$ is a sum of mutually strongly anticommuting self-adjoint operators. Using this, we identify the spectrum and the essential spectrum of $\mathcal{D}$ and $\widetilde{H}$. In Section 3, we consider perturbations of $\mathcal{D}$ and $\widetilde{H}$. We define a new class of vector potentials $\mathbf{a}$, each in which implies the same essential spectrum for $\widetilde{H}$ as in the constant magnetic field case (Theorem 3.2). This class contains vector potentials to which Theorem 1.2 cannot be applied (see Example 3.4 in Section 3).

2. CONSTANT MAGNETIC FIELD CASE

In this section, we investigate the inner structure and the spectrum of $\mathcal{D}$ and $\widetilde{H}$ with a constant magnetic field. We recall the definition of the anticommutativity of self-adjoint operators: two (non-zero) self-adjoint operators A and B in a Hilbert space are said to *strongly anticommute* if

$$\exp(\mathrm{i}tA)B \subset B\exp(-\mathrm{i}tA)$$

for all $t \in \mathbb{R}$ (see [11], [7], [1]).

First of all, we prove a proposition and a lemma.

PROPOSITION 2.1. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with a grading operator γ such that $\gamma^* = \gamma$, $\gamma^2 = 1$, and A and γ strongly anticommute. Let B be a self-adjoint operator in a Hilbert space $\mathcal{K}$. Then $A \otimes \mathbf{1}$ and $\gamma \otimes B$ are self-adjoint in $\mathcal{H} \otimes \mathcal{K}$ and strongly anticommute.*

Proof. The self-adjointness follows from general theory on tensor product of self-adjoint operators ([8]). The strongly anticommutativity follows from an application of Corollary 4.5 in [7]. ∎

We shall use the following lemma to answer the question, "Why does whether 0 is an eigenvalue of A or not cause the difference of $\sigma_{\mathrm{ess}}(\widetilde{H})$ in each case?"

LEMMA 2.2. *Let A and B be as in Proposition 2.1. Assume that there exists a unitary operator U on $\mathcal{K}$ such that $U^*BU = -B$,*

$$\sigma(B) = \mathbb{R}, \tag{2.1}$$

and

$$\sigma_{\mathrm{p}}(B) = \emptyset, \tag{2.2}$$

where $\sigma_{\mathrm{p}}(\cdot)$ denotes point spectrum. Let

$$T = A \otimes \mathbf{1} + \gamma \otimes B \quad \textit{with} \quad D(T) = D(A \otimes 1) \cap D(\gamma \otimes B),$$

where $D(T)$ denotes the domain of the operator T. Then T is self-adjoint, $\sigma_{\mathrm{p}}(T) = \emptyset$ and

$$\sigma(T) = (-\infty, -\delta] \cup [\delta, \infty) \quad \textit{with} \quad \delta = \inf\{|x|;\ x \in \sigma(A)\}. \tag{2.3}$$

Proof. The self-adjointness of T follows from Lemma 2.1 in [1] and Proposition 2.1. By Lemma 2.4 in [1],

$$T^2 = A^2 \otimes \mathbf{1} + \mathbf{1} \otimes B^2$$

holds as operator equality. Thus, we have $\sigma_{\mathrm{p}}(T^2) = \emptyset$ by (2.3) and

$$\sigma(T^2) = \overline{\{a + b;\ a \in \sigma(A^2),\ b \in \sigma(B^2)\}} = [\inf \sigma(A^2), \infty)$$

by (2.1). Since $(\gamma \otimes U)^* T (\gamma \otimes U) = -T$ and $\gamma \otimes U$ is unitary, we have $\sigma(T) = \sigma(-T)$. Therefore, we obtain (2.3). ∎

In this section, we deal with the constant magnetic field case. Hence, assume that

$$b_{jk}(x) = \Lambda_{jk} \quad \text{for} \quad j,k = 1,\dots,d, \tag{2.4}$$

with a constant matrix $\Lambda = (\Lambda_{jk})$. By an orthogonal transformation, we assume that Λ is of the form

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & \mathbf{0} & \\ & & \ddots & & & & & \\ & & & 0 & \lambda_n & & & \\ & & & -\lambda_n & 0 & & & \\ & & & & & 0 & & \\ & \mathbf{0} & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix},$$

where $\lambda_j > 0$, $j = 1,\dots,n$. Moreover, we can take a vector potential a as follows:

$$\begin{aligned} &a_{2j-1}(x) = \frac{\lambda_j}{2}x^{2j}, \quad a_{2j}(x) = -\frac{\lambda_j}{2}x^{2j-1} \quad \text{for} \quad j = 1,\dots,n, \\ &a_j(x) = 0 \quad \text{for} \quad j = 2n+1,\dots,d. \end{aligned} \tag{2.5}$$

We prove that $\mathcal{D}$ is a sum of operators which mutually anticommute. Let

$$\widehat{d_j} = \sigma^1(-\mathrm{i}\partial_{2j-1} + a_{2j-1}) + \sigma^2(-\mathrm{i}\partial_{2j} + a_{2j})$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ for $j = 1,\dots,[d/2]$. Since a_{2j-1} and a_{2j} contain only the variables x^{2j-1} and x^{2j}, these operators are well-defined. Moreover, $\widehat{d_j}$ are essentially self-adjoint on the domain $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$. We denote the closure of $\widehat{d_j}$ by the same symbol. We can easily check the following proposition.

PROPOSITION 2.3. *For each $j = 1,\dots,[d/2]$, the operators σ^3 and $\widehat{d_j}$ strongly anticommute.*

Using $\widehat{d_j}$, we construct self-adjoint operators D_j whose sum is $\mathcal{D}$ in each cases where $d = 2m$ and $d = 2m+1$. First, consider the case where $d = 2m$. For $j = 1,\dots,m$, define

$$D_j = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1 \text{ times}} \otimes \widehat{d_j} \otimes \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{m-j \text{ times}}$$

acting in $\otimes^m L^2(\mathbb{R}^2;\mathbb{C}^2) \simeq L^2(\mathbb{R}^{2m};\mathbb{C}^r)$, $r = 2^m$. In the case $d = 2m+1$, define

$$D_j = \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{j-1 \text{ times}}\otimes\widehat{d_j}\otimes\underbrace{\sigma^3\otimes\cdots\otimes\sigma^3}_{m-j \text{ times}}\otimes\mathbf{1}$$

for $j = 1, \ldots, m$, and

$$D_{m+1} = \underbrace{\sigma^3\otimes\cdots\otimes\sigma^3}_{m \text{ times}}\otimes(-\mathrm{i}\partial_{2m+1})$$

acting in $\otimes^m L^2(\mathbb{R}^2;\mathbb{C}^2)\otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^{2m+1};\mathbb{C}^r)$, $r = 2^m$. Since $\widehat{d_j}$ are self-adjoint, each D_j is self-adjoint. If $d = 2m$, we define a grading operator Γ_{2m} by

$$\Gamma_{2m} = \otimes^m \sigma^3$$

acting in $\otimes^m L^2(\mathbb{R}^2;\mathbb{C}^2)$. Then Γ_{2m} and D_j strongly anticommute for $j = 1, \ldots, m$. We remark that $D_{m+1} = \Gamma_{2m}\otimes(-\mathrm{i}\partial_{2m+1})$.

LEMMA 2.4. *The operators D_j mutually strongly anticommute.*

Proof. By Propositions 2.1 and 2.3, $\widehat{d_1}\otimes\sigma^3$ and $\mathbf{1}\otimes\widehat{d_2}$ strongly anticommute. Since the other components in D_1 and D_2 strongly commute, we can prove that D_1 and D_2 strongly anticommute with limit argument. In the same way, we can see that all $D_1, D_2, \ldots, D_m$ strongly anticommute.

In the case $d = 2m+1$, let $A = D_j$, $\gamma = \Gamma_{2m}$, $\mathcal{H} = \otimes^m L^2(\mathbb{R}^2;\mathbb{C}^2)$, $B = -\mathrm{i}\partial_{2m+1}$ and $\mathcal{K} = L^2(\mathbb{R})$. Then, by Proposition 2.1, we obtain the desired results. ∎

The followings are the main theorems in this paper.

THEOREM 2.5. *Assume* (2.4). *Let $k = [(d+1)/2]$. Then*

$$\mathcal{D} = D_1 + D_2 + \cdots + D_k \quad \textit{with} \quad D(\mathcal{D}) = \bigcap_{j=1}^{k} D(D_j), \tag{2.6}$$

$$\widetilde{H} = \mathcal{D}^2 = D_1^2 + D_2^2 + \cdots + D_k^2 \quad \textit{with} \quad D(\widetilde{H}) = \bigcap_{j=1}^{k} D(D_j^2), \tag{2.7}$$

hold as operator equality.

REMARK 2.6. In Theorem 2.5, we take a representation of Dirac matrices γ^j as follows: $\gamma^1 = \sigma^1\otimes[\otimes^{m-1}\sigma^3]$, $\gamma^2 = \sigma^2\otimes[\otimes^{m-1}\sigma^3]$, $\gamma^3 = \mathbf{1}\otimes\sigma^1\otimes[\otimes^{m-2}\sigma^3]$, $\gamma^4 = \mathbf{1}\otimes\sigma^2\otimes[\otimes^{m-2}\sigma^3]$, and so on.

Proof of Theorem 2.5. By direct computations, (2.6) holds on $C_0^\infty(\mathbb{R}^d;\mathbb{C}^r)$, $r = 2^{[d/2]}$. Since $\mathcal{D}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d;\mathbb{C}^r)$, by Lemma 2.1 in [1] and Lemma 2.4 we obtain (2.6) as operator equality. Moreover, by Lemma 2.4 and Lemma 2.4 in [1] we obtain (2.7) as operator equality. ∎

By this theorem and Theorem 1.1, we can obtain the following spectral properties of $\mathcal{D}$ and $\widetilde{H}$.

THEOREM 2.7. *Assume* (2.4).

(i) *Assume that* 0 *is an eigenvalue of* Λ. *Then*

$$\sigma(\widetilde{H}) = \sigma_{\mathrm{ess}}(\widetilde{H}) = [0,\infty), \quad \sigma_{\mathrm{p}}(\widetilde{H}) = \emptyset,$$
$$\sigma(\mathcal{D}) = \sigma_{\mathrm{ess}}(\mathcal{D}) = \mathbf{R}, \quad \sigma_{\mathrm{p}}(\mathcal{D}) = \emptyset.$$

(ii) *Assume that* 0 *is not an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \dots, \pm i\lambda_m$, $(\lambda_j > 0$, $m = d/2)$ *be eigenvalues of* Λ. *Then*

$$\sigma(\widetilde{H}) = \sigma_{\mathrm{ess}}(\widetilde{H}) = \Big\{ \sum_{j=1}^{m} 2k_j\lambda_j;\ k_j \in \mathbf{Z}_+ \Big\}, \tag{2.8}$$

$$\sigma(\mathcal{D}) = \sigma_{\mathrm{ess}}(\mathcal{D}) = \{\pm\sqrt{\alpha};\ \alpha \in \sigma(\widetilde{H})\}. \tag{2.9}$$

Moreover, we have

$$\ker \mathcal{D} \subset \ker(\sigma^3 + 1) \otimes \cdots \otimes \ker(\sigma^3 + 1). \tag{2.10}$$

Proof. First, we prove the part (ii). By Theorem 2.5, we can rewrite $\widetilde{H}$ as

$$\widetilde{H} = \widehat{d}_1^2 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{m-1 \text{ times}} + \mathbf{1} \otimes \widehat{d}_2^2 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{m-2 \text{ times}} + \cdots + \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{m-1 \text{ times}} \otimes \widehat{d}_m^2.$$

Therefore, we have

$$\sigma(\widetilde{H}) = \overline{\{\alpha_1 + \cdots + \alpha_m;\ \alpha_j \in \sigma(\widehat{d}_j^2),\ j = 1, \dots, m\}},$$
$$\sigma_{\mathrm{ess}}(\widetilde{H}) = \overline{\{\alpha_1 + \cdots + \alpha_m;\ \alpha_j \in \sigma_{\mathrm{ess}}(\widehat{d}_j^2),\ j = 1, \dots, m\}}.$$

Since $\sigma(\widehat{d}_j^2) = \sigma_{\mathrm{ess}}(\widehat{d}_j^2) = \{2n_j\lambda_j;\ n_j \in \mathbf{Z}_+\}$ by Theorem 1.1, we have (2.8). By the supersymmetry with the grading operator Γ_d, we obtain (2.9) (see Proposition 2.5 in [9]). By the self-adjointness, we have

$$\ker \mathcal{D} = \ker \widetilde{H} = \ker \widehat{d}_1 \otimes \cdots \otimes \ker \widehat{d}_m.$$

Thus, we obtain (2.10) by Theorem 1.1.

We prove the part (i). Decompose $\mathcal{D}$ into two operators as follows. Let A be the Dirac operator in the case where $d = 2n$, the vector potential $\mathbf{a} = \sum_{j=1}^{2n} a_j(x)dx^j$ with a_j in (2.5), and the grading operator $\gamma = \Gamma_{2n}$ on $\mathcal{H} = \otimes^n L^2(\mathbf{R}^2; \mathbf{C}^2)$. Let B be the $(d-2n)$-dimensional Dirac operator with $\mathbf{a} = 0$ in $\mathcal{K} = L^2(\mathbf{R}^{d-2n}; \mathbf{C}^r)$, $r = 2^{[(d-2n)/2]}$. Then, we have $\mathcal{D} = A \otimes \mathbf{1} + \gamma \otimes B$. Moreover, let U be a unitary operator on $\mathcal{K}$ by

$$(Uf)(x) = f(-x) \quad \text{for} \quad f \in \mathcal{K},\ x \in \mathbf{R}^{d-2n}.$$

Then, the set $\{A, \gamma, \mathcal{H}, B, U, \mathcal{K}\}$ satisfies the assumptions in Lemma 2.2. With the part (ii), we obtain the desired results. ∎

In the rest of this section, we consider the *spinless* case. We can rewrite H as follows. Let

$$\widehat{h}_j = (-\mathrm{i}\partial_{2j-1} - a_{2j-1})^2 + (-\mathrm{i}\partial_{2j} - a_{2j})^2$$

in $L^2(\mathbf{R}^2)$ for $j = 1, \dots, m$. If $d = 2m+1$, let

$$\widehat{h}_{m+1} = (-\mathrm{i}\partial_{2m+1})^2$$

acting in $L^2(\mathbf{R})$. Then,

$$H = \overline{\sum_{j=1}^{k} [\otimes^{j-1}\mathbf{1}] \otimes \widehat{h}_j \otimes [\otimes^{k-j}\mathbf{1}]},$$

in $L^2(\mathbf{R}^d)$, $k = [(d+1)/2]$. Thus, we can prove the following theorem.

THEOREM 2.8. *Assume* (2.4).

(i) *Assume that* 0 *is an eigenvalue of* Λ. *Let* $\pm\mathrm{i}\lambda_1, \dots, \pm\mathrm{i}\lambda_n$, $(\lambda_j > 0)$, *be eigenvalues of* Λ. *Then*

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = \Big[\sum_{j=1}^{n} \lambda_j, \infty\Big), \quad \sigma_{\mathrm{p}}(H) = \emptyset.$$

(ii) *Assume that* 0 *is not an eigenvalue of* Λ. *Let* $\pm\mathrm{i}\lambda_1, \dots, \pm\mathrm{i}\lambda_m$, $(\lambda_j > 0$, $m = d/2)$ *be eigenvalues of* Λ. *Then*

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = \Big\{\sum_{j=1}^{m} (2k_j+1)\lambda_j;\ k_j \in \mathbb{Z}_+\Big\}.$$

Proof. This theorem follows from the general theory of tensor product of self-adjoint operators and Theorem 1.1. ∎

We can find far more discussions on this H in [6].

3. PERTURBATION

In this section, we consider perturbations of $\mathcal{D}_0$ which is the Dirac operator with a constant magnetic field considered in the previous section. Though Shigekawa proved Theorem 1.2 under conditions on the asymptotic behavior of the magnetic field $\mathbf{b} = \mathrm{d}\mathbf{a}$ as (1.2), we shall give assumptions on the asymptotic behavior of the vector potential $\mathbf{a}$, up to gauge transformation. One of the reasons is that we investigate $\mathcal{D}$ instead of $\widetilde{H}$ itself and $\mathcal{D}$ contains explicitly $\mathbf{a}$ and no $\mathbf{b}$. Therefore, this seems natural at least from the mathematical point of view. We will give a theorem with assumptions on $\mathbf{b}$, too.

We start with the following abstract lemma.

LEMMA 3.1. *Assume that* $\mathbf{a}_0 = \sum_{j=1}^{d} a_{0j}\, dx^j$ *and* $\mathbf{a} = \sum_{j=1}^{d} a_j\, dx^j$ *are real* 1-*forms such that* a_{0j} *and* a_j *are in* C^∞, $\mathbf{b}_0 = d\mathbf{a}_0$ *is a bounded* 2-*form and* $\mathbf{a} \to 0$ *as* $|x| \to \infty$ *(i.e.,* $a_j(x) \to 0$ *as* $|x| \to \infty$ *for all* j*). Let*

$$\mathcal{D} = \sum_{j=1}^{d} \gamma^j(-\mathrm{i}\partial_j + a_{0j}(x)) \quad \textit{and} \quad \mathcal{A} = \sum_{j=1}^{d} \gamma^j a_j(x)$$

acting in $L^2(\mathbb{R}^d; \mathbb{C}^r)$, $r = 2^{[d/2]}$. *Then* $\mathcal{A}$ *is* $\mathcal{D}$-*compact.*

Proof. Let $\mathbf{b}_0 = d\mathbf{a}_0 = \sum_{j<k} b_{0jk}\, dx^j \wedge dx^k$ with $b_{0jk} = \partial_j a_{0k} - \partial_k a_{0j}$. Then

$$\mathcal{D}^2 = H + \sum_{j<k} i\, b_{0jk} \gamma^j \gamma^k \quad \text{with} \quad H = \sum_{j=1}^{d} (-\mathrm{i}\partial_j + a_{0j})^2.$$

Since $a_j(x) \to 0$ as $|x| \to \infty$, a_j is $-\Delta = -\left(\sum_{j=1}^{d} \partial_j^2\right)$-compact. Thus, a_j is H-compact by Lemma 2.3 in [3]. Since $\sum_{j<k} \mathrm{i}\, b_{0jk}\gamma^j\gamma^k$ is bounded, a_j is $\mathcal{D}^2$-compact and thus $\mathcal{A}$ is so. Since $\mathcal{A}$ is $\mathcal{D}$-bounded with $\mathcal{D}$-bound 0, $\mathcal{A}$ is $\mathcal{D}$-compact by Theorem 9.11 in [12]. ∎

The following is the main theorem in this section.

THEOREM 3.2. *Assume that the given vector potential* $\mathbf{A}$ *can be rewritten as the sum of* 1-*forms* $\mathbf{a}_0$ *and* $\mathbf{a}$ *such that* $d\mathbf{a}_0$ *is a constant magnetic field and* $\mathbf{a}$ *tends to* 0 *as* $|x| \to \infty$. *Define* $\mathcal{D}$ *and* $\widetilde{H}$ *as the Dirac and Schrödinger operators with* $\mathbf{A}$, *respectively. Put* Λ *for* $d\mathbf{a}_0$ *as same as* (2.4).

(i) *Assume that* 0 *is an eigenvalue of* Λ. *Then*

$$\sigma_{\mathrm{ess}}(\widetilde{H}) = [0, \infty), \quad \sigma_{\mathrm{ess}}(\mathcal{D}) = \mathbb{R}.$$

(ii) *Assume that* 0 *is not an eigenvalue of* Λ. *Let* $\pm\mathrm{i}\lambda_1, \ldots, \pm\mathrm{i}\lambda_m$, ($\lambda_j > 0$, $m = d/2$) *be eigenvalues of* Λ. *Then*

$$\sigma_{\mathrm{ess}}(\widetilde{H}) = \left\{ \sum_{j=1}^{m} 2k_j\lambda_j;\ k_j \in \mathbb{Z}_+ \right\},$$

$$\sigma_{\mathrm{ess}}(\mathcal{D}) = \{\pm\sqrt{\alpha};\ \alpha \in \sigma(\widetilde{H})\}.$$

Proof. Define $\mathcal{D}_0$ as the Dirac operator with $\mathbf{a}_0$. Since $a_j(x) \to 0$ as $|x| \to \infty$, $\mathcal{D} - \mathcal{D}_0$ is $\mathcal{D}_0$-compact by Lemma 3.1. Therefore, by Theorem 2.7, we obtain the desired results. ∎

The following theorem gives a condition on magnetic field $\mathbf{b}$ which implies the same essential spectra of $\mathcal{D}$ and $\widetilde{H}$ as in Theorem 3.2.

THEOREM 3.3. *Assume that* $\mathbf{b} = \sum_{j<k} b_{jk}\, dx^j \wedge dx^k$ *is a real* C^∞ *2-form such that for any* j *and* k,

$$|x|(b_{jk}(x) - \Lambda_{jk}) \to 0 \quad as \quad |x| \to \infty$$

with a constant matrix $\Lambda = (\Lambda_{jk})$. *Then, the statements* (i) *and* (ii) *in Theorem* 3.2 *hold.*

Proof. For the 2-form $\widetilde{\mathbf{b}} = \sum_{j<k}(b_{jk} - \Lambda_{jk})dx^j \wedge dx^k$ we can choose a 1-form $\mathbf{a}$ such that $\widetilde{\mathbf{b}} = d\mathbf{a}$ and $\mathbf{a} \to 0$ as $|x| \to \infty$ by taking

$$\mathbf{a}(x) = \sum_{j<k} \int_0^1 t(b_{jk} - \Lambda_{jk})(tx)\, dt\, (x^j dx^k - x^k dx^j)$$

as in the proof of Poincaré's Lemma. Since the 2-form $\sum_{j<k} \Lambda_{jk}\, dx^j \wedge dx^k$ is a constant magnetic field, by Theorem 3.2 we obtain the desired results. ∎

Of course, above Theorem 3.3 is weaker than Theorem 1.2. However, Theorem 3.2 is not weaker than Theorem 1.2 as we see in the following example.

EXAMPLE 3.4. Let $d = 2$ and $\mathbf{a} = a_1\, dx^1 + a_2\, dx^2$ be a C^∞ 1-form such that

$$a_1(x) = \frac{\lambda}{2}x^2 + \frac{\sin|x^2|^r}{|x|} \quad \text{and} \quad a_2(x) = -\frac{\lambda}{2}x^1$$

near $|x| = \infty$ with a constant $r > 2$ and a positive constant λ. Then, $\mathbf{a}$ satisfies the assumptions in Theorem 3.2. Therefore, we have $\sigma_{\rm ess}(\widetilde{H}) = \{2n\lambda;\ n \in \mathbb{Z}_+\}$. However, $\mathbf{b} = d\mathbf{a}$ does not converge as $|x| \to \infty$. Therefore, we can not apply Theorem 1.2.

We remark on perturbations of the spinless Schrödinger operator H. Assume that magnetic field $\mathbf{b}$ satisfies the conditions in Theorem 3.3. Then, using the general theory of perturbations of differential operators (see, e.g., [12]) and using the vector potential $\mathbf{a}$ in the proof of Theorem 3.3 we can prove that the perturbed H has the same essential spectrum of the unperturbed H as in Theorem 2.8. However, this result is evidently weaker than the Shigekawa's results in [9]. This difference is due to the difference between the unperturbed operators taken in each proof.

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