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UNITARY DILATIONS AND NUMERICAL RANGES

PEI YUAN WU

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ABSTRACT. We prove that any algebraic contraction T on a (separable) Hilbert space can be dilated to an operator of the form $T_1 \oplus T_1 \oplus \cdots$, where T_1 is a cyclic contraction on a finite-dimensional space with the same minimal polynomial as T and rank $(1 - T_1^* T_1) \leqslant 1$. As applications, we use this to determine the "most economical" unitary dilations of finite-dimensional contractions and also the spatial matricial ranges of the unilateral shift.

Generalizing an example of Durszt, we give a necessary and sufficient condition on a normal contraction T such that its numerical range equals the intersection of the numerical ranges of unitary dilatons of T.

KEYWORDS: *Unitary dilation, numerical range, spatial matricial range, completely positive map.*

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0. INTRODUCTION

Let A and B be bounded linear operators on the complex Hilbert spaces H and K, respectively. A is said to be *dilated* to B (or B is a *dilation* of A) if B is unitarily equivalent to a 2×2 operator matrix of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$. This is equivalent to requiring the existence of an isometry V from H to K such that $A = V^* B V$. If T is a contraction ($\|T\| \leqslant 1$) on H, then a classical result of Halmos ([7], Problem 222 (a)) says that T can always be dilated to a unitary operator: T appears in the upper-left corner of the unitary 2×2 operator matrix

$$\begin{bmatrix} T & (1 - TT^*)^{\frac{1}{2}} \\ (1 - T^*T)^{\frac{1}{2}} & -T^* \end{bmatrix}$$

on $H \oplus H$. Further developments of this idea lead to a profound theory — the Sz.-Nagy–Foiaş dilation theory as codified in the monograph ([18]).

A contraction T is *of class* C_0 if T is completely nonunitary and there is a nonzero function f in H^∞ such that $f(T) = 0$. For every such T, there exists a unique (up to a constant factor of modulus 1) nonconstant inner function φ, called the *minimal function* of T, such that $\varphi(T) = 0$ and every other function f in H^∞ with $f(T) = 0$ is a multiple of φ (cf. [18], p. 123). One important example of such operators is the compression of the shift $S(\varphi)$ on $H(\varphi)$. Here φ is any nonconstant inner function and $S(\varphi)$ denotes the operator

$$S(\varphi)f = P(zf(z))$$

for f in $H(\varphi) = H^2 \ominus \varphi H^2$, where P denotes the orthogonal projection onto $H(\varphi)$ and $zf(z)$ denotes the function $z \mapsto zf(z)$ in H^2. This operator was initially investigated in [17] and later in the broader context of the Sz.-Nagy–Foiaş theory ([18]). Note that the minimal function of $S(\varphi)$ is φ. Another example of C_0 contractions is any contraction on a finite-dimensional space with all eigenvalues in the open unit disc. Indeed, if T is such an operator with minimal polynomial p and $p(z) = \prod_{j=1}^{n} (z - \lambda_j)$, then T is of class C_0 with minimal function

$$\varphi(z) = \prod_{j=1}^{n} \frac{z - \lambda_j}{1 - \bar{\lambda}_j z}.$$

An operator T is *algebraic* if $p(T) = 0$ for some polynomial p; T is *quadratic* if p can be taken to be of degree 2.

One of the main results of this paper concerns the dilation of algebraic contractions. In Section 1 below, we prove that any algebraic contraction with minimal polynomial p can be dilated to an operator of the form $T_1 \oplus T_1 \oplus \cdots$, where T_1 is a cyclic contraction on a finite-dimensional space with the same minimal polynomial p and rank $(1 - T_1^* T_1) \leqslant 1$ (Theorem 1.4). The proof involves (finite-dimensional versions of) results of Sarason, Arveson and Stinespring concerning completely isometric and completely positive maps between operator algebras. It is plausible that the more general assertion that any C_0 contraction with minimal function φ can be dilated to $S(\varphi) \oplus S(\varphi) \oplus \cdots$ should be true. However, at the present stage, we are not able to affirm this.

As applications, we derive in Section 2 two interesting results concerning, respectively, the unitary dilations of finite-dimensional contractions and the spatial matricial ranges of the unilateral shift. For the former, we show that any contraction on an n-dimensional space can be dilated to a unitary operator with spectrum consisting of no more than $n+1$ points and, moreover, the number $n+1$ cannot be further reduced in general (Theorem 2.2). This result has some implications on the shape of the numerical ranges of finite-dimensional contractions. Using our main result, we can also determine the spatial matricial ranges of the unilateral shift: for every $n \geqslant 1$, the nth spatial matricial range of the unilateral shift of any multiplicity consists of those n-dimensional contractions whose eigenvalues are all in the open unit disc (Theorem 2.7). Very few classes of operators have their matricial ranges completely determined. Our result adds to the short list of the known ones (cf. [3], Section 9).

In Section 3, we consider another problem, which originates from an old question of Halmos ([6]). He asked over thirty years ago whether the numerical range of a contraction equals the intersection of the numerical ranges of all its unitary dilations. This turns out not to be the case. Durszt ([2]) gave an example of a normal contraction without this property. Our contribution in this respect (Theorem 3.2) is a strengthening of this example yielding a complete characterization of those normal contractions for which such a relation on the numerical ranges holds. The modified question with "numerical range(s)" replaced throughout by "closure(s) of numerical range(s)" remains open.

Throughout this paper we consider only operators on separable Hilbert spaces. For such a space H, let $\mathcal{B}(H)$ denote the C^*-algebra of all operators on H. For an operator $T, \sigma(T)$ denotes its spectrum. Let $\mathbf{C}$ be the space of complex numbers and $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$.

1. DILATION OF ALGEBRAIC CONTRACTIONS

A linear map Φ from a Banach algebra $\mathbf{A}$ to another Banach algebra $\mathbf{B}$ is *completely contractive* (resp. *completely isometric*) if the map Φ_n from $M_n(\mathbf{A})$ to $M_n(\mathbf{B})$ defined by

$$\Phi_n([a_{ij}]_{i,j=1}^n) = [\Phi(a_{ij})]_{i,j=1}^n$$

is contractive (resp. isometric) for every $n \geqslant 1$, where $M_n(\mathbf{A})$ (resp. $M_n(\mathbf{B})$) denotes the Banach algebra of $n \times n$ matrices with entries from $\mathbf{A}$ (resp. $\mathbf{B}$) endowed with the usual algebraic operations and the matrix norm

$$\|[a_{ij}]\| = \sup\left\{\left(\sum_i \left\|\sum_j a_{ij}x_j\right\|^2\right)^{\frac{1}{2}} : x_j \in \mathbf{A} \text{ (resp. } \mathbf{B}) \text{ with } \sum_j \|x_j\|^2 \leqslant 1\right\}$$

for $[a_{ij}] \in M_n(\mathbf{A})$ (resp. $M_n(\mathbf{B})$). For C^*-algebras $\mathbf{A}$ and $\mathbf{B}, \Phi$ is *completely positive* if every Φ_n is positive. Our reference for general properties of such maps is [15].

We start with the following proposition concerning the Sz.-Nagy–Foiaş functional calculus of C_0 contractions. For any operator T, let $\operatorname{Alg} T$ denote the weakly closed algebra generated by T and 1.

PROPOSITION 1.1. *For any C_0 contraction T with minimal function φ, the map $\Phi : H^\infty/\varphi H^\infty \to \operatorname{Alg} T$ defined by*

$$\Phi(\widehat{f}) = f(T),$$

where $\widehat{f}$ is the coset in $H^\infty/\varphi H^\infty$ determined by $f \in H^\infty$, is a completely contractive isomorphism from $H^\infty/\varphi H^\infty$ into $\operatorname{Alg} T$. If $T = S(\varphi)$, then Φ is a completely isometric isomorphism from $H^\infty/\varphi H^\infty$ onto $\operatorname{Alg} S(\varphi)$.

Proof. Assume that the C_0 contraction T acts on the space H. We need to show that for every $n \geqslant 1$ and $[\widehat{f}_{ij}]_{i,j=1}^n$ in $M_n(H^\infty/\varphi H^\infty)$, the inequality $\|[f_{ij}(T)]\| \leqslant \|[\widehat{f}_{ij}]\|$ holds. To achieve this, we have only to prove $\|[f_{ij}(T)]\| \leqslant \|[f_{ij}]\|$ for any $[f_{ij}]_{i,j=1}^n$ in $M_n(H^\infty)$. Let U on $K(\supseteq H)$ be the minimal unitary power dilation of T. It is easily seen that $[f_{ij}(U)]_{i,j=1}^n$ on $K \oplus \cdots \oplus K$ is a dilation of $[f_{ij}(T)]_{i,j=1}^n$ on $H \oplus \cdots \oplus H$. Hence $\|[f_{ij}(T)]\| \leqslant \|[f_{ij}(U)]\|$. To complete the proof, we use the spectral theorem for normal operators to obtain sequences of operators $\{D_{ij}^{(m)}\}_{m=1}^\infty$ on K which are such that with respect to some fixed orthonormal basis of K every $D_{ij}^{(m)}$ can be represented as a diagonal matrix and, for every i and j, $D_{ij}^{(m)}$ converges to $f_{ij}(U)$ in norm. In particular, each $[D_{ij}^{(m)}]_{i,j=1}^n$ is a direct sum of $n \times n$ scalar-entry matrices. Let $[a_{ij}]_{i,j=1}^n$ be one of such summand matrices. Since each a_{ij} may be assumed to belong to the essential range of f_{ij}, it is easily seen that $\|[a_{ij}]\| \leqslant \|[f_{ij}]\|$. Hence we have $\|[D_{ij}^{(m)}]\| \leqslant \|[f_{ij}]\|$ for every m. Letting m approach infinity, we obtain $\|[f_{ij}(U)]\| \leqslant \|[f_{ij}]\|$. Our assertion follows immediately.

For $T = S(\varphi)$, the assertion is a special case of [17], Theorem 3 (by letting V_1 and V_2 there have dimensions n and 1, respectively). ∎

COROLLARY 1.2. *For any C_0 contraction T with minimal function φ, the map $\Psi : \operatorname{Alg} S(\varphi) \to \operatorname{Alg} T$ defined by*

$$\Psi(f(S(\varphi))) = f(T)$$

for $f \in H^\infty$, is a completely contractive isomorphism from $\operatorname{Alg} S(\varphi)$ into $\operatorname{Alg} T$.

Proof. Since $\Psi = \Phi_1 \circ \Phi_2^{-1}$, where $\Phi_1 : H^\infty/\varphi H^\infty \to \operatorname{Alg} T$ and $\Phi_2 : H^\infty/\varphi H^\infty \to \operatorname{Alg} S(\varphi)$ are the maps given by $\Phi_1(\widehat{f}) = f(T)$ and $\Phi_2(\widehat{f}) = f(S(\varphi))$, respectively, the assertion is an easy consequence of Proposition 1.1. ∎

The next proposition gives a concrete dilation form for completely positive maps between certain operator algebras. Although this exact form does not seem to have appeared in the literature, its basic idea should be well-known among experts.

PROPOSITION 1.3. *Let H and K be Hilbert spaces with H finite-dimensional and K separable. If $\Omega : \mathcal{B}(H) \to \mathcal{B}(K)$ is a unital completely positive map, then there exists an isometry V from K to $H \oplus H \oplus \cdots$ such that*

$$\Omega(A) = V^*(A \oplus A \oplus \cdots)V$$

for any operator A in $\mathcal{B}(H)$. Moreover, if K is also finite-dimensional, then V can be taken to map from K to a direct sum of finitely many H's.

Proof. By Stinespring's dilation theorem ([15], Theorem 4.1), there exist a (separable) Hilbert space L, a unital $*$-homomorphism π from $\mathcal{B}(H)$ to $\mathcal{B}(L)$ and an isometry V_1 from K to L such that

$$\Omega(A) = V_1^* \pi(A) V_1$$

for any A in $\mathcal{B}(H)$. For the $*$-homomorphism π, there is an isometry V_2 from L onto $H \oplus \cdots \oplus H$, where the number of summands equals $\dim L$, such that

$$\pi(A) = V_2^*(A \oplus \cdots \oplus A)V_2$$

for A in $\mathcal{B}(H)$ (cf. [9], Corollary 10.4.14). Therefore,

$$\Omega(A) = (V_2V_1)^*(A \oplus \cdots \oplus A)(V_2V_1)$$

for A in $\mathcal{B}(H)$. This proves our first assertion.

If K is finite-dimensional, then, following the proof of Stinespring's theorem, L can also be taken to be finite-dimensional and hence a direct sum of finitely many H's would suffice. ∎

Note that the converse of the preceding proposition is obviously true, namely, for any isometry V from K to $H \oplus H \oplus \cdots$, the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ defined by

$$\Psi(A) = V^*(A \oplus A \oplus \cdots)V$$

for A in $\mathcal{B}(H)$ is completely positive.

We are now ready for the main result of this section.

THEOREM 1.4. *If T is an algebraic contraction on H, then there are a finite-dimensional space K, a cyclic contraction T_1 on K with the same minimal polynomial as T and* $\operatorname{rank}(1-T_1^*T_1) \leqslant 1$, *and an isometry V from H to $K \oplus K \oplus \cdots$ such that*

$$q(T) = V^*(q(T_1) \oplus q(T_1) \oplus \cdots)V$$

for any polynomial q. In particular, T dilates to $T_1 \oplus T_1 \oplus \cdots$. Moreover, if H is finite-dimensional, then a finite direct sum of K's suffices.

Proof. Let $T = U \oplus A$ on $L \oplus M$ be the decomposition of T into its unitary part U and completely nonunitary part A. Since the spectrum of U consists of a finite number of (distinct) eigenvalues, say, $\alpha_1, \ldots, \alpha_m$, in $\partial\mathbf{D}$, U dilates to $U_1 \oplus U_1 \oplus \cdots$, where U_1 is the cyclic unitary operator

$$\begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_m \end{bmatrix}$$

on $\mathbf{C}^m$, and hence there is an isometry V_1 from L to $\mathbf{C}^m \oplus \mathbf{C}^m \oplus \cdots$ such that

$$q(U) = V_1^*(q(U_1) \oplus q(U_1) \oplus \cdots)V_1$$

for any polynomial q.

On the other hand, if the minimal polynomial of A is p and $p(z) = \prod_{j=1}^{n}(z-\lambda_j)$ with λ_j in $\mathbf{D}$, then the map Ψ from $\operatorname{Alg} A_1$ to $\operatorname{Alg} A$ given by $\Psi(q(A_1)) = q(A)$ for polynomial q, where $A_1 = S(\varphi)$ and φ is the inner function

$$\varphi(z) = \prod_{j=1}^{n} \frac{z-\lambda_j}{1-\bar{\lambda}_j z},$$

is completely contractive by Corollary 1.2. Arveson's extension theorem ([13], Corollary 6.6) implies that Ψ can be extended to a completely positive map Ω from $\mathcal{B}(H(\varphi))$ to $\mathcal{B}(M)$. An application of Proposition 1.3 yields an isometry V_2 from M to $H(\varphi) \oplus H(\varphi) \oplus \cdots$ such that

$$q(A) = V_2^*(q(A_1) \oplus q(A_1) \oplus \cdots)V_2$$

for polynomial q.

Finally, $T_1 = U_1 \oplus A_1$ on $K = \mathbf{C}^m \oplus H(\varphi)$ and $V = V_1 \oplus V_2$ satisfy all our required properties. ∎

In the preceding theorem, if T is a nilpotent contraction with $T^n = 0$, then the cyclic contraction T_1 is unitarily equivalent to the $n \times n$ nilpotent Jordan block

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & \ddots & & 1 \\ & & & 0 \end{bmatrix}$$

on $\mathbb{C}^n$. Hence this yields the following corollary, which can also be deduced from [1], Theorem 1.3.1.

COROLLARY 1.5. *Let T be a nilpotent contraction on H with $T^n = 0$ $(n \geqslant 1)$. Then there exists an isometry V from H to $\mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots$ such that $T^k = V^*(J_n^k \oplus J_n^k \oplus \cdots)V$ for any $k \geqslant 1$. In particular, T can be dilated to $J_n \oplus J_n \oplus \cdots$.*

2. NUMERICAL RANGE

In this section, we consider implications of the results in Section 1 on numerical ranges and the more general spatial matricial ranges of certain contractions.

The *numerical range* of an operator T on H is the subset

$$W(T) = \{\langle Tx, x\rangle : x \in H \text{ and } \|x\| = 1\}$$

of $\mathbb{C}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H; the *numerical radius* of T is

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

The notion of numerical range is related to that of dilation by its very definition: a complex number z belongs to $W(T)$ if and only if the 1×1 matrix $[z]$ dilates to T. It follows that if A dilates to B, then $W(A)$ is contained in $W(B)$. For general properties of the numerical range, consult [7], Chapter 22. One easy consequence of Corollary 1.5 is the following.

PROPOSITION 2.1. *If T is a nilpotent operator with $T^n = 0$, then $W(T)$ is contained in the closed disc $\{z : |z| \leqslant \|T\| \cdot \cos \frac{\pi}{n+1}\}$.*

Proof. It is well-known that $W(J_n) = \{z : |z| \leqslant \cos \frac{\pi}{n+1}\}$ (cf. [5], Proposition 1). Hence the assertion follows from Corollary 1.5. ∎

This proposition was also proved in [5], Theorem 1 using more down-to-earth arguments.

Let T be a contraction on an n-dimensional space. We may ask what the "most economical" unitary dilation of T should be. There are different ways to interpret this. One interpretation requires that the dimension of the space on which the unitary dilation acts be a minimum. In this sense, the problem has been solved completely in [19]: the minimal dimension is $n + \text{rank}\,(1 - T^*T)$. Here we consider another criterion by requiring that the cardinality of the spectrum of the unitary dilation be a minimum. In the Halmos dilation, this cardinality is $2n$. We show next that this number can be reduced to the optimal $n + 1$.

THEOREM 2.2. *Every contraction on an n-dimensional space can be dilated to a unitary operator (on some finite-dimensional space) with spectrum consisting of no more than $n + 1$ points. Moreover, the number $n + 1$ is optimal in the sense that there is an n-dimensional contraction, namely J_n, of which every unitary dilation has at least $n + 1$ points in its spectrum.*

Proof. By Theorem 1.4, every n-dimensional contraction T can be dilated to a finite direct sum $T_1 \oplus \cdots \oplus T_1$ of some cyclic contraction T_1 on a space K of dimension no bigger than n with rank $(1 - T_1^*T_1) \leqslant 1$. Each T_1 can be dilated to a unitary operator U_1 on a space of dimension no bigger than $1 + \dim K$ (cf. [19], Theorem 2). Hence T dilates to $U_1 \oplus \cdots \oplus U_1$, the latter having at most $n + 1$ points in its spectrum. This proves the first assertion.

For the second, assume that J_n has a unitary dilation U with no more than n points in its spectrum. Then $W(J_n)$ is contained in $W(U)$. But the former is the closed disc $\{z : |z| \leqslant \cos \frac{\pi}{n+1}\}$ (cf. [5], Proposition 1) and the latter is a closed polygonal region with no more than n vertices on the unit circle (cf. [7], Problem 216). If r_j denotes the distance from the origin to the jth side of the polygon, then $\sum\limits_j r_j \leqslant n \cdot \cos \frac{\pi}{n}$ (cf. [10]). This implies that $\cos \frac{\pi}{n+1} \leqslant \min\limits_j r_j \leqslant \cos \frac{\pi}{n}$, which is absurd. Thus every unitary dilation of J_n has at least $n + 1$ points in its spectrum. ∎

The next two corollaries say something about the numerical range of an n-dimensional contraction.

COROLLARY 2.3. *Every contraction on an n-dimensional space has its numerical range contained in an $(n + 1)$-gon inscribed in the unit circle. Moreover, the number $n + 1$ is optimal in the sense that there is an n-dimensional contraction, namely J_n, whose numerical range is not contained in any n-gon inscribed in the unit circle.*

We remark that by Proposition 2.1 any nilpotent contraction T with $T^n = 0$ has its numerical range contained in every regular $(n+1)$-gon inscribed in the unit circle.

COROLLARY 2.4. *If T is an operator on an n-dimensional space whose numerical range is a disc centered at the origin, then $w(T) \leqslant \|T\| \cdot \cos \frac{\pi}{n+1}$.*

Proof. We may assume that $\|T\| = 1$. Since $W(T)$ is contained in an $(n+1)$-gon inscribed in the unit circle by Corollary 2.3, we have $\sum_j r_j \leqslant (n+1) \cdot \cos \frac{\pi}{n+1}$ as before, where r_j is the distance from the origin to the jth side of the polygon. It follows that $w(T) \leqslant \min_j r_j \leqslant \cos \frac{\pi}{n+1}$ as asserted. ∎

In Theorem 2.2, the number $n+1$ can be further reduced if the n-dimensional contraction is normal. This is achieved via the Halmos dilation and the next result.

PROPOSITION 2.5. *For any operator T on H, the following conditions are equivalent:*

(i) *$W(T)$ is contained inside a triangle Δabc;*

(ii) *$T = aT_1 + bT_2 + cT_3$ for some positive operators T_1, T_2 and T_3 with $T_1 + T_2 + T_3 = 1$;*

(iii) *T can be dilated to a normal operator N with $\sigma(N) = \{a, b, c\}$.*

Proof. The equivalence of (i) and (ii) was established in [13] in proving a result of Mirman ([12]) that if $W(T)$ is contained in Δabc, then $\|T\| \leqslant \max\{|a|, |b|, |c|\}$. Since (iii) $\Rightarrow$ (i) is trivial, we need only check (ii) $\Rightarrow$ (iii). Assume that $T = aT_1 + bT_2 + cT_3$ as above. Let $N = aI \oplus bI \oplus cI$ on $H \oplus H \oplus H$ and $V = [T_1^{\frac{1}{2}} T_2^{\frac{1}{2}} T_3^{\frac{1}{2}}]^t$. It is easily that V is an isometry and $T = V^* N V$. This shows that T dilates to N. ∎

PROPOSITION 2.6. *Any normal contraction on an n-dimensional space ($n \geqslant 2$) can be dilated to a unitary operator (on some finite-dimensional space) with no more than n points in its spectrum.*

Proof. Assume that n is odd and T is an n-dimensional normal contraction with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicity). Enclose λ_1, λ_2 and λ_3 in a triangle Δabc with vertices a, b and c on the unit circle. Proposition 2.5 implies that the 3×3 matrix

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

can be dilated to the unitary operator $aI_3 \oplus bI_3 \oplus cI_3$, where I_3 denotes the 3×3 identity matrix. For each pair of the remaining eigenvalues λ_j and λ_{j+1}, let θ be a real number satisfying $\operatorname{Im}(\lambda_j e^{-i\theta}) = \operatorname{Im}(\lambda_{j+1}e^{-i\theta})$. Then

$$\begin{bmatrix} \lambda_j & 0 \\ 0 & \lambda_{j+1} \end{bmatrix}$$

can be dilated to the unitary operator

$$\begin{bmatrix} \lambda_j & e^{i\theta}(1-|\lambda_j|^2)^{\frac{1}{2}} \\ e^{i\theta}(1-|\lambda_j|^2)^{\frac{1}{2}} & -e^{2i\theta}\bar{\lambda}_j \end{bmatrix} \oplus \begin{bmatrix} \lambda_{j+1} & e^{i\theta}(1-|\lambda_{j+1}|^2)^{\frac{1}{2}} \\ e^{i\theta}(1-|\lambda_{j+1}|^2)^{\frac{1}{2}} & -e^{2i\theta}\bar{\lambda}_{j+1} \end{bmatrix}.$$

The eigenvalues of this latter operator are the two intersections of the unit circle with the straight line connecting λ_j and λ_{j+1}. Taking the direct sum of the above unitary operators, we obtain a unitary dilation of T with no more than n points in its spectrum. If n is even, the second part of the above arguments yields the required unitary dilation. ∎

Two remarks are in order:

(1) if T is any Hermitian contraction (even on an infinite-dimensional space), then T can be dilated to a unitary operator, namely,

$$\begin{bmatrix} T & (1-T^2)^{\frac{1}{2}} \\ (1-T^2)^{\frac{1}{2}} & -T \end{bmatrix}.$$

with 2 points (± 1) in its spectrum;

(2) if T is any operator with $w(T) \leqslant \frac{1}{2}$, then $w(T)$ is contained in any regular triangle inscribed in the unit circle and hence, by Proposition 2.5, T can be dilated to a unitary operator with 3 points in its spectrum.

We now move to another application of Theorem 1.4: the determination of the spatial matricial ranges of the unilateral shift. Let S denote the simple unilateral shift on ℓ^2:

$$S(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots)$$

for $(x_0, x_1, x_2, \cdots)$ in ℓ^2. For $k = 1, 2, \ldots, \infty$, let $S_k = \underbrace{S \oplus \cdots \oplus S}_{k}$ be the unilateral shift of multiplicity k. For an operator T on H and an integer n, $1 \leqslant n \leqslant \dim H$, the *nth spatial matricial range* $W_s^n(T)$ is the set of those $n \times n$ matrices which can be dilated to T. Note that $W_s^1(T)$ coincides with the (classical) numerical range $W(T)$. A nice survey of matricial ranges and related topics is [3]. It is known that the closure of $W_s^n(S)$ consists of all n-dimensional contractions (cf. [3], Example 9.3 (b)) and that $W_s^1(S) = W(S) = \mathbf{D}$. We now proceed to determine $W_s^n(S)$ for all n.

THEOREM 2.7. *For $n \geqslant 1$ and $1 \leqslant k \leqslant \infty, W_s^n(S_k)$ consists of those n-dimensional contractions whose eigenvalues are all in* $\mathbf{D}$.

Proof. Let T be an n-dimensional contraction which can be dilated to S_k. Since S_k is completely nonunitary, the same is true for T. Hence the eigenvalues of T can only be in $\mathbf{D}$.

Conversely, since every n-dimensional contraction with eigenvalues in $\mathbf{D}$ dilates to a direct sum of finitely many copies of some $S(\varphi)$ by Theorem 1.4, we need only to prove that the latter can be dilated to S. For this purpose, let $A = S(\varphi)$ and $K = \ker(1 - A^*A)$. Then K is a subspace of $H(\varphi)$ with codimension 1. Let $\{f_1, f_2, \ldots, f_0\}$ be an orthonormal basis of $H(\varphi)$ such that $\{f_1, f_2, \cdots\}$ forms a basis for K, and let A have the matrix representation $[g_1 g_2 \cdots g_0]$ with respect to this basis, where each g_j represents a column vector. Since $K = \{f \in H(\varphi) : \|Af\| = \|f\|\}$, we have $\|g_j\| = 1$ for all $j \geqslant 1$ and $\|g_0\| < 1$. Let $a = (1 - \|g_0\|^2)^{\frac{1}{2}}$ and let S_1 be the infinite matrix

$$\begin{bmatrix} g_1 & g_2 & \cdots & g_0 & & & \\ 0 & 0 & \cdots & a & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 1 & 0 \\ & & & & & & \ddots & \ddots \end{bmatrix},$$

where the unspecified entries are all zero. Since

$$\langle g_j, g_k \rangle = \langle Af_j, Af_k \rangle = \langle A^*Af_j, f_k \rangle = \langle f_j, f_k \rangle = 0$$

for any $j \geqslant 1, k \geqslant 0$ and $j \neq k$, a simple computation shows that S_1 can be considered as the operator

$$\begin{bmatrix} A & & & \\ (1 - A^*A)^{\frac{1}{2}} & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots \end{bmatrix}$$

on $H(\varphi) \oplus R \oplus R \oplus \cdots$, where $R = \overline{\operatorname{ran}(1 - A^*A)^{\frac{1}{2}}}$. This is exactly the construction for the minimal isometric power dilation of $S(\varphi)$ (cf. [18], Section I.5), which is known to be unitarily equivalent to S. On the other hand, by permuting rows and columns, S_1 is also easily seen to be unitarily equivalent to

$$\begin{bmatrix} S(\varphi) & 0 & 0 \\ 0 & S & e \\ a' & 0 & 0 \end{bmatrix},$$

where $a' = [0\ 0 \cdots a]$ and $e = [1\ 0\ 0 \cdots]^t$. We can now again apply the above process to the unilateral shift S in the middle of the above matrix to infer that S_1 is unitarily equivalent to

$$\begin{bmatrix} S(\varphi) & 0 & 0 \\ 0 & \begin{bmatrix} S(\varphi) & 0 & 0 \\ 0 & S & e \\ a' & 0 & 0 \end{bmatrix} & * \\ * & 0 & 0 \end{bmatrix}.$$

Repeating this procedure finitely many times, we obtain that any finite direct sum $S(\varphi) \oplus \cdots \oplus S(\varphi)$ can be dilated to S. This completes the proof. ∎

Let U be the simple bilateral shift:

$$U(\cdots, x_{-1}, \boxed{x_0}, x_1, \cdots) = (\cdots, x_{-2}, \boxed{x_{-1}}, x_0, x_1, \cdots)$$

for $(\cdots, x_{-1}, \boxed{x_0}, x_1, \cdots)$ in $\ell^2_{\mathbf{Z}}$, where the component on the position zero is marked by a square. For $k = 1, 2, \ldots, \infty$, let $U_k = \underbrace{U \oplus \cdots \oplus U}_{k}$ be the bilateral shift of multiplicity k. It is well-known that $W_s^1(U) = W(U) = \mathbf{D}$ (this also follows from Lemma 3.3 below). The next corollary determines $W_s^n(U)$ for all n.

COROLLARY 2.8. *For $n \geqslant 1$ and $1 \leqslant k \leqslant \infty$, $W_s^n(U_k)$ consists of those n-dimensional contractions whose eigenvalues are all in $\mathbf{D}$.*

Proof. Since S_k dilates to U_k, we obviously have $W_s^n(S_k) \subseteq W_s^n(U_k)$. On the other hand, if T is an n-dimensional contraction in $W_s^n(U_k)$ which has an eigenvalue, say λ, in $\partial\mathbf{D}$, then λ is a reducing eigenvalue for T and hence the same for U_k. This contradicts the fact that U_k has no eigenvalue. Thus all eigenvalues of T are in $\mathbf{D}$. This completes the proof. ∎

A subset $\mathbf{A}$ of $\mathcal{B}(H)$ is C^*-convex if for any finitely many $T_1, \ldots, T_n$ in $\mathbf{A}$ and $A_1, \ldots, A_n$ in $\mathcal{B}(H)$ with $\sum_{j=1}^{n} A_j^* A_j = 1$, the C^*-convex combination $\sum_{j=1}^{n} A_j^* T_j A_j$ is in $\mathbf{A}$. For general properties of C^*-convexity, consult [11].

COROLLARY 2.9. *$W_s^n(S_k)$ and $W_s^n(U_k)$ are C^*-convex for any $n \geqslant 1$ and $1 \leqslant k \leqslant \infty$.*

This follows from Theorem 2.7, Corollary 2.8, Lemma 2.10 below and the fact that $\sigma(T) \subseteq \mathbf{D}$ if and only if $W(T) \subseteq \mathbf{D}$ for any finite-dimensional contraction T (cf. [7], Solution 212).

LEMMA 2.10. *If T is a C^*-convex combination of $T_1, \ldots, T_n$, then*
(i) $\|T\| \leqslant \max\{\|T_j\| : j = 1, 2, \ldots, n\}$ *and*
(ii) $W(T)$ *is contained in the convex hull of* $W(T_1) \cup \cdots \cup W(T_n)$.

Proof. (i) can be proved as in [11], Example 3 and (ii) is proved in [8], Lemma 2.3. ∎

3. UNITARY DILATION

In this section, we explore the relation between the numerical range of a contraction and those of its unitary dilations. The original problem asked by Halmos in [6] is whether

(∗) the numerical range of a contraction T is equal to the intersection of the numerical ranges of unitary dilations of T.

The problem has very interesting geometrical manifestations.

If the space H on which T acts is finite-dimensinoal, then there are some cases under which (∗) has an affirmative answer. For example, if $\dim H = 1$, then since the 1×1 matrix $[\lambda]$ ($|\lambda| \leqslant 1$) can be dilated to the unitary matrix

$$\begin{bmatrix} \lambda & e^{i\theta}(1-|\lambda|^2)^{\frac{1}{2}} \\ e^{i\theta}(1-|\lambda|^2)^{\frac{1}{2}} & -e^{2i\theta}\bar{\lambda} \end{bmatrix}$$

for any real θ, (∗) is true. Geometrically, this means that the point λ is the intersection of all the chords of the unit circle which pass through λ. The operator version of this argument yields the validity of (∗) for any finite-dimensional normal contractions. This was carried out in [6]. When $\dim H = 2$, again (∗) is true by the next proposition.

PROPOSITION 3.1. *If T is a contraction on a 2-dimensional space, then* $W(T) = \bigcap\{W(U) : U$ *is a unitary dilation of* $T\}$ *holds.*

Proof. We may assume that $W(T)$ is a (nondegenerate) elliptical disc. By [4], Corollary, there is a triangle inscribed in the circle $\{z : |z| = \|T\|\}$ and circumscribed about $\partial W(T)$. Poncelet's closure theorem of 1822 ([16]) implies that in this case for any point p on the circle there is such a triangle with p as one of its vertices. Since $\|T\| \leqslant 1$, this implies that, for any chord $[a, b]$ of the unit circle which is tangent to the boundary of $W(T)$, there is a triangle Δabc which has this chord as its one side, is inscribed in the unit circle and contains $W(T)$. By Proposition 2.5, T can be dilated to a unitary operator whose numerical range is the closed triangular region enclosed by Δabc. This implies that $W(T) = \bigcap\{W(U) : U$ is a unitary dilation of $T\}$. ∎

Another instance for which $(*)$ holds is when T is the $n \times n$ nilpotent Jordan block J_n since in this case $W(J_n) = \{z : |z| \leqslant \cos \frac{\pi}{n+1}\}$ is the intersection of all the polygonal regions enclosed by the regular $(n+1)$-gons inscribed in the unit circle and the latter are the numerical ranges of the $(n+1) \times (n+1)$ unitary dilations

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & 1 \\ \lambda & & & & 0 \end{bmatrix}$$

$(|\lambda| = 1)$ of J_n. (The case $n = 2$ was already pointed out in [6].) It is in general not known whether $(*)$ is true for all contractions on a finite-dimensinoal space. However, on infinite-dimensional spaces $(*)$ is false even for normal contractions. An example was constructed in [2]. Our next result, which characterizes all normal contractions for which $(*)$ holds, is a strengthening of this example.

For any two points λ_1 and λ_2 in the plane, let (λ_1, λ_2) (resp. $[\lambda_1, \lambda_2]$) denote the open (resp. closed) line segment $\{t\lambda_1 + (1-t)\lambda_2 : 0 < t < 1\}$ (resp. $\{t\lambda_1 + (1-t)\lambda_2 : 0 \leqslant \lambda \leqslant 1\}$).

THEOREM 3.2. *Let T be a normal contraction. Then $W(T) = \bigcap\{W(U) : U$ is a unitary dilation of $T\}$ holds if and only if whenever $W(T)$ contains a point λ in its boundary $\partial W(T)$ it also contains every open line segment (λ, λ') in $\partial W(T)$.*

For the proof, we need the following lemma, which was proved in [1], Theorem 1.

LEMMA 3.3. *Let T be a normal operator with spectral measure $E_T(\cdot)$. Then $W(T) = \bigcap\{\alpha : \alpha$ is a convex Borel subset of $\mathbb{C}$ with $E_T(\alpha) = 1\}$.*

When proving Theorem 3.2, it is instructive to keep in mind the following two examples.

EXAMPLE 3.4. Let K_1 be the rectangle

$$\left\{z : \operatorname{Re} z = \pm\frac{1}{2} \text{ and } -\frac{1}{2} \leqslant \operatorname{Im} z \leqslant \frac{1}{2}\right\} \cup \left\{z : \operatorname{Im} z = \pm\frac{1}{2} \text{ and } -\frac{1}{2} \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\right\},$$

and μ_1 be the linear Lebesgue measure on K_1. If T_1 is the operator of multiplication by the variable z on $L^2(\mu_1)$, then $\sigma(T_1) = K_1$ and

$$W(T_1) = \left\{z : -\frac{1}{2} \leqslant \operatorname{Re} z,\ \operatorname{Im} z \leqslant \frac{1}{2}\right\} \setminus \left\{\pm\frac{1}{2} \pm \frac{1}{2}\mathrm{i}\right\}$$

by Lemma 3.3.

EXAMPLE 3.5. Let K_2 be the closed rectangular region

$$\left\{z : -\frac{1}{2} \leqslant \operatorname{Re} z,\ \operatorname{Im} z \leqslant \frac{1}{2}\right\}$$

and μ_2 be the planar Lebesgue measure on K_2. If $T = T_2 \oplus T_3$, where T_2 is the operator of multiplication by the variable z on $L^2(\mu_2)$ and T_3 is the 1×1 matrix $\left[\frac{1}{2} + \frac{1}{2}\mathrm{i}\right]$, then $\sigma(T) = K_2$ and

$$W(T) = \left\{z : -\frac{1}{2} < \operatorname{Re} z,\ \operatorname{Im} z < \frac{1}{2}\right\} \cup \left\{\frac{1}{2} + \frac{1}{2}\mathrm{i}\right\}.$$

Note that the condition of Theorem 3.2 is satisfied for T_1 but not for T, and hence the assertion on the numerical ranges holds for the former but not for the latter. We are now ready for the

Proof of Theorem 3.2. The necessity is proved following an analogous argument as in [2]. Let $\lambda \in W(T)$ be a boundary point of $W(T)$, and (λ, λ') an open line segment in $\partial W(T)$. For an arbitrary unitary dilation U of T, we check that (λ, λ') is contained in $W(U)$. To achieve this, we extend (λ, λ') on both directions until it intersects the unit circle at two points, say, λ_1 and λ_2 (so that λ_1 is closer to λ and λ_2 to λ'). If λ_2 is an eigenvalue of U, then it is in $W(U)$. Since λ is also in $W(U)$, the closed line segment $[\lambda, \lambda_2]$ is contained in $W(U)$. Hence in this case $(\lambda, \lambda') \subseteq W(U)$ as asserted. Therefore, in the following we assume that λ_2 is not an eigenvalue of U. Let L be the closed half plane determined by λ_1 and λ_2 which contains $W(T)$. Let Γ denote the open arc $\partial\mathbb{D} \setminus L$ and α denote the region $(\bar{\mathbb{D}} \cap L) \setminus [\lambda, \lambda_2]$. We claim that $\Gamma \cap \sigma(U) \neq \emptyset$. Indeed, if otherwise, then α, being a convex Borel set with $E_U(\alpha) = 1$, must contain $W(U)$ by Lemma 3.3. Thus α contains $W(T)$ and hence λ. This is a contradication. Hence there is some point λ_0 in $\sigma(U)$ which is on the arc Γ. In particular, λ_0 is in $\overline{W(U)}$. We infer from the convexity of $W(U)$ that (λ, λ') is contained in $W(U)$. Since U is an arbitrary unitary dilation of T, it follows from our assumption that (λ, λ') is contained in $W(T)$.

To prove the sufficiency of our condition, we need only check that $\bigcap\{W(U) :$ U is a unitary dilation of $T\}$ is contained in $W(T)$. Since the former set is contained in $\overline{W(T)}$ by (a similar proof as) [6], Theorem 2, it suffices to check that for any λ in $\partial W(T) \setminus W(T)$, there exists a unitary dilation U of T such that λ is not in $W(U)$. Let λ be such a point, and let $T = V \oplus T'$, where V is unitary and T' is completely nonunitary. We consider two cases separately:

(1) Assume that λ is an extreme point of $\overline{W(T)}$. Let L be a closed half plane with the properties that L contains $W(T)$ and $L \cap \partial W(T) = \{\lambda\}$. Let

$\{\lambda_n\}$ be the sequence of points with unit length and rational arguments which are in the interior of L. Using these λ_n's as vertices, we form closed triangular regions $\{\Delta_j\}$ such that their interiors are mutually disjoint and their union satisfies $\sigma(T') \cap \operatorname{int} L \subseteq \cup_j \Delta_j \subseteq (\mathbf{D} \cap \operatorname{int} L) \cap \{\lambda_n\}$. By the spectral theorem for normal operators, we may decompose T as a direct sum $T = V \oplus \left(\sum_j {}^{\oplus} T_j\right)$ with each T_j satisfying $\sigma(T_j) \subseteq \Delta_j$ (since λ cannot be an eigenvalue of T and T' is completely nonunitary, we have $E_T(\{\lambda\}) = 0$ and $E_{T'}(\partial \mathbf{D}) = 0$). Since $W(T_j) \subseteq \Delta_j$, by Proposition 2.5 each T_j can be dilated to a unitary operator U_j with $W(U_j) = \Delta_j$. Hence T can be dilated to the unitary operator $U = V \oplus \left(\sum_j {}^{\oplus} U_j\right)$ with $W(U) =$ convex hull of $W(V) \cup (\cup_j \Delta_j)$. In particular, λ is not in $W(U)$. This proves our assertion.

(2) Assume that λ is a nonextreme point of $\overline{W(T)}$. Let $[\lambda_1, \lambda_2]$ be the longest closed line segment in $\partial W(T)$ which contains λ. Since λ is not in $W(T)$, our assumption implies that $[\lambda_1, \lambda_2]$ is disjoint from $W(T)$. Hence Lemma 3.3 implies that there exists some convex Borel set α with $E_T(\alpha) = 1$ such that $\frac{1}{2}(\lambda_1 + \lambda_2)$ is not in α. Since $\alpha \cap [\lambda_1, \lambda_2]$ is convex, it can only be contained in one of $[\lambda_1, \frac{1}{2}(\lambda_1+\lambda_2))$ and $(\frac{1}{2}(\lambda_1+\lambda_2), \lambda_2]$. Therefore, the one which is outside $\alpha \cap [\lambda_1, \lambda_2]$ has E_T measure zero. Applying this argument successively to midpoints of the remaining segments, we infer that $E_T([\lambda_1, \lambda_2]) = 0$. Now, as in (1), let L be the closed half plane determined by λ_1 and λ_2 which contains $W(T)$, and let $\{\lambda_n\}$ be the sequence of points with unit length and rational arguments which are in the interior of L. We form closed triangular regions $\{\Delta_j\}$ with these λ_n's as vertices such that their interiors are mutually disjoint and their union satisfies $\sigma(T') \cap \operatorname{int} L \subseteq \cup_j \Delta_j \subseteq (\mathbf{D} \cap \operatorname{int} L) \cup \{\lambda_n\}$. As before, we may decompose T as $V \oplus \left(\sum_j {}^{\oplus} T_j\right)$, where V is unitary and each T_j satisfies $\sigma(T_j) \subseteq \Delta_j$. Since $W(T_j) \subseteq \Delta_j$, by Proposition 2.5 each T_j can be dilated to a unitary operator U_j with $W(U_j) = \Delta_j$. Hence T has a unitary dilation $U = V \oplus \left(\sum_j {}^{\oplus} U_j\right)$ with $W(U) =$ convex hull of $W(V) \cup (\cup_j \Delta_j)$. In particular, we have $\lambda \notin W(U)$. This completes the proof. ∎

We conclude this paper with one more result. Note that the more general question as to whether $\overline{W(T)} = \bigcap \{\overline{W(U)} : U$ is a unitary dilation of $T\}$ holds for every contraction T is also open. As elaborations of previous arguments, we have the following partial result.

PROPOSITION 3.6. *If T is a quadratic contraction or T satisfies $w(T) \leqslant \frac{1}{2}$, then $\overline{W(T)} = \bigcap\{\overline{W(U)} : U$ is a unitary dilation of $T\}$ holds.*

Proof. If T is quadratic, then T has a canonical representation

$$\lambda_1 I_1 \oplus \lambda_2 I_2 \oplus \begin{bmatrix} \lambda_1 & A \\ 0 & \lambda_2 \end{bmatrix}$$

on $H_1 \oplus H_2 \oplus (K \oplus K)$, where $A > 0$ acts on K, and $\overline{W(T)}$ equals the numerical range of the 2×2 matrix

$$\begin{bmatrix} \lambda_1 & \|A\| \\ 0 & \lambda_2 \end{bmatrix}$$

(cf. [20], Theorems 1.1 and 2.1). A proof analogous to that for Proposition 3.1 yields our assertion.

On the other hand, if $w(T) \leqslant \frac{1}{2}$, then T must be a contraction and $W(T)$ is, for any chord $[a, b]$ of the unit circle which is on one side of $W(T)$, contained in a triangle Δabc inscribed in the unit circle. We infer from Proposition 2.5 that our assertion holds. ∎

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Note added in proof. By the Sz.-Nagy–Foiaş theory of contractions, it can be easily shown that a much stronger result than our Theorem 1.4 is true: every C_0 contraction T with minimal function φ can be extended to the operator $S(\varphi) \oplus \cdots \oplus S(\varphi)$, where the number of summands is $\operatorname{rank}(1 - T^*T)^{\frac{1}{2}}$. This was proved in [14], Lemma 4.

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PEI YUAN WU
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan
REPUBLIC OF CHINA
E-mail: pywu@cc.nctu.edu.tw

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