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FLAG MANIFOLDS AND THE COWEN-DOUGLAS THEORY

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ABSTRACT. The present article focuses on the congruence problem for holomorphic maps into flag manifolds associated with C^*-algebras and the equivalence problem for tuples of elements of a C^*-algebra in the Cowen-Douglas class. The former problem is formulated and solved for a quite large class of holomorphic maps that includes the kind of maps needed to address and solve the latter problem. Along the way towards manageable answers to both these problems we also study in detail the behavior of holomorphic families of elements with closed range of C^*-algebras.

KEYWORDS: *C^*-algebra, Cowen-Douglas theory, flag manifolds.*

AMS SUBJECT CLASSIFICATION: Primary 47C15, 47B20; Secondary 47A13.

INTRODUCTION

In the present note we propose a generalization of the Cowen-Douglas class of Hilbert space operators that includes the classes previously introduced in [8], [9] and [7]. It must be said, however, that this objective is somewhat peripheral to the main issue we will elaborate on. The primary purpose of our investigations is to classify holomorphic maps into flag manifolds of a C^*-algebra up to an equivalence relation called congruence. The specific way we address this problem relies on [17], [18] where preliminary steps towards a manageable description of the geometry underlying any flag manifold of a C^*-algebra were taken. Nevertheless, we should point out that the congruence problem also originates in [8], the influential article that set the stage for nearly all subsequent developments of the Cowen-Douglas theory.

One hallmark of the Cowen-Douglas theory is the considerable evidence it provides in support of the view that a great deal of concepts and techniques customarily encountered in classical complex geometry may be naturally defined and fruitfully handled in the framework of operator theory. In [17], [18], as well as in this note, we tried to promote the same view. As a highlight of our approach we could mention that many of the forthcoming definitions and results are formulated in the setting of general C^*-algebras. We pause to outline the features of each section in this article.

In Section 1 we introduce the kernel and range projections of elements with closed range of a C^*-algebra. The main goal is to investigate the kernel or range projections of holomorphic families of elements having closed range. Under favorable circumstances we prove that the projection-valued maps defined in this way satisfy a non-linear first-order differential equation. The precise meaning of that equation is explained in Section 2, where we briefly describe the canonical complex structure of flag manifolds associated to C^*-algebras. Section 2 is mostly concerned with various criteria for holomorphic maps into flag manifolds and some of their consequences. In Section 3 we explore the relationship between tuples of elements of a C^*-algebra in the Cowen-Douglas class and projection-valued holomorphic maps. We explain how the equivalence problem for tuples in the Cowen-Douglas class can be reduced to the congruence problem for projection-valued holomorphic maps. We also exhibit a test for the congruence of holomorphic maps in terms of their order of contact. In Section 4, the congruence theorem is generalized for holomorphic maps into the flag manifold of a C^*-algebra. The article ends with a section on a class of tuples of Hilbert space operators that are quite tractable along the lines of the previous sections, and that generalizes the Cowen-Douglas class.

Before going on, we want to make a final general observation about the subject matter of this article. It was motivated by and grew out of prior contributions aimed to widen the scope of the Cowen-Douglas theory. Precise references, hints, and clues will be indicated at the right place and time.

1. KERNEL AND RANGE PROJECTIONS

1.1. Throughout this article we will be dealing with idempotents in unital C^*-algebras. Recall that if $\mathfrak{A}$ is a C^*-algebra and $e \in \mathfrak{A}$, then e is an idempotent whenever $e^2 = e$. A self-adjoint idempotent is briefly referred to as a projection. In case $\mathfrak{A}$ is fixed, we will let $\mathcal{E}(\mathfrak{A})$ and $\mathcal{P}(\mathfrak{A})$ denote the set of all idempotents and projections in $\mathfrak{A}$, respectively.

Although the sets $\mathcal{E}(\mathfrak{A})$ and $\mathcal{P}(\mathfrak{A})$ could be quite small, however there are classes of C^*-algebras that contain an abundance of projections, as for instance the class of von Neumann algebras. The simplest example is by far provided by the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$. Any projection in $\mathcal{L}(\mathcal{H})$ occurs as a projection operator onto a closed subspace of $\mathcal{H}$, so indeed $\mathcal{L}(\mathcal{H})$ has a lot of projections. The next definition and the conventions we are going to adopt are basically motivated by this geometric situation.

DEFINITION. Let a be an element of a C^*-algebra $\mathfrak{A}$.

(i) An idempotent $e \in \mathcal{E}(\mathfrak{A})$ is called a *kernel idempotent* of a provided that $ae = 0$ and any x in $\mathfrak{A}$ satisfying $ax = 0$ is such that $x = ex$. If, in addition, e is self-adjoint, then it is called a *kernel projection* of a.

(ii) A projection $p \in \mathcal{P}(\mathfrak{A})$ is said to be a *range projection* of a provided that $pa = a$ and any x in $\mathfrak{A}$ satisfying $xa = 0$ is such that $xp = 0$.

For an excellent account of various existence of projections axioms for C^*-algebras we refer to [2].

1.2. It should be mentioned that an element a of an arbitrary C^*-algebra $\mathfrak{A}$ may have none or plenty of kernel idempotents but no more than one kernel, or one range projection. In case the kernel and the range projections of a given element $a \in \mathfrak{A}$ exist, they will be denoted by $\kappa(a)$ and $\rho(a)$, respectively. If a^* stands for the adjoint of a, then the existence of $\kappa(a)$, or $\rho(a)$, is equivalent to the existence of $\rho(a^*)$, or $\kappa(a^*)$. More precisely, one easily shows that

$$\rho(a^*) = 1 - \kappa(a), \tag{1.1}$$

and

$$\kappa(a^*) = 1 - \rho(a). \tag{1.2}$$

A particularly neat condition that ensures the existence of $\kappa(a)$ and $\kappa(a^*)$ may be formulated in terms of the spectrum $\sigma(a^*a)$ of a^*a as follows.

DEFINITION. We say that an element $a \in \mathfrak{A}$ has *closed range* if zero is an isolated point in $\sigma(a^*a) \cup \{0\}$.

The next two remarks illustrate the point.

(i) Suppose $a \in \mathfrak{A}$ has closed range and let $E_0(a^*a)$ be the spectral projection of a^*a associated with $\{0\}$. Then $E_0(a^*a)$ is the kernel projection of a and, consequently, by means of the analytic functional calculus we have

$$\kappa(a) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=\varepsilon} (\zeta - a^*a)^{-1} \, \mathrm{d}\zeta, \tag{1.3}$$

where $0 < \varepsilon < \inf\{\sigma(a^*a) \setminus \{0\}\}$.

(ii) Since $\sigma(a^*a) \cup \{0\} = \sigma(aa^*) \cup \{0\}$ for any $a \in \mathfrak{A}$, we clearly get that a has closed range if and only if a^* has closed range. Thus we conclude that when an element a has closed range both the kernel projections $\kappa(a)$ and $\kappa(a^*)$, as well as the range projections $\rho(a)$ and $\rho(a^*)$, exist.

For some other details concerned with the class of elements having closed range we refer to [21], Section 3.

1.3. The next lemma simply asserts that any idempotent $e \in \mathcal{E}(\mathfrak{A})$ has a range projection. Although we could prove this result based on the above comments, a straightforward approach is at hand ([21], Lemma 2.15).

LEMMA. *If $e \in \mathcal{E}(\mathfrak{A})$, then there is a unique projection $p \in \mathcal{P}(\mathfrak{A})$ such that*

$$ep = p, \quad pe = e. \tag{1.4}$$

Proof. If (1.4) holds, then $pe^* = p$ and $pe = e$, hence $p[1-(e^*-e)] = e$. Since $e^* - e$ is skew-hermitian, we get that $1 \notin \sigma(e^* - e)$. Therefore, $p = e(1 - e^* + e)^{-1}$, and so the uniqueness of p follows. On the other hand, if $e \in \mathcal{E}(\mathfrak{A})$, then an easy verification shows that $p = e(1 - e^* + e)^{-1}$ is a projection satisfying (1.4). ∎

The previous proof prompts us to introduce the map

$$\pi : \mathcal{E}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{A}), \quad \pi(e) = e(1 - e^* + e)^{-1}, \quad e \in \mathcal{E}(\mathfrak{A}), \tag{1.5}$$

which will play an important role in our further discussions. For the time being let us just observe that $p = \pi(e)$ is the range projection of any given $e \in \mathcal{E}(\mathfrak{A})$. Indeed, by (1.4) we know that $pe = e$, and clearly $xe = 0$ implies $xp = xep = 0$. We also notice that in case e is a kernel idempotent of an element $a \in \mathfrak{A}$, its range projection $\pi(e)$ is the kernel projection of a. In particular, because obviously e is a kernel idempotent of $e^\perp = 1 - e$, we get that $\pi(e)$ is the kernel projection of $e^\perp$.

1.4. We now return to the class of elements having closed range. Suppose $a \in \mathfrak{A}$ is such an element. We first observe that

$$[a^*a + \kappa(a)]a^* = a^*[aa^* + \kappa(a^*)].$$

On the other hand, by Subsection 1.2 we know that $\kappa(a) = E_0(a^*a)$ and $\kappa(a^*) = E_0(aa^*)$. Therefore, both $a^*a+\kappa(a)$ and $aa^*+\kappa(a^*)$ are invertible. Thus, it makes sense to define

$$\iota(a) = a^*[aa^* + \kappa(a^*)]^{-1} = [a^*a + \kappa(a)]^{-1}a^*. \tag{1.6}$$

The next two properties of $\iota(a)$ are readily verifiable (see also (1.1) and (1.2) above):

$$\iota(a)a = 1 - \kappa(a) = \rho(a^*), \tag{1.7}$$

$$a\iota(a) = 1 - \kappa(a^*) = \rho(a). \tag{1.8}$$

The element $\iota(a)$ will be referred to as the generalized inverse of a. From (1.6) we also obtain

$$\kappa(a)\iota(a) = 0, \quad \iota(a)\kappa(a^*) = 0. \tag{1.9}$$

1.5. Our main goal in this section is to investigate families of elements having closed range. Specifically, we are going to consider continuous, differentiable, or holomorphic families, and to study the behavior of the corresponding kernel and range projections.

To begin with, let Ω be a topological space, and let $A : \Omega \to \mathfrak{A}$ be a continuous map such that $A(\omega)$ has closed range for every $\omega \in \Omega$. Furthermore, let K, R, and $A^\wedge$ be the $\mathfrak{A}$-valued maps defined by

$$K(\omega) = \kappa(A(\omega)), \quad R(\omega) = \rho(A(\omega)), \quad A^\wedge(\omega) = \iota(A(\omega)), \tag{1.10}$$

for all $\omega \in \Omega$.

From (1.6), (1.7), and (1.8) above we clearly get the next result.

PROPOSITION. *The following conditions are equivalent:*

(i) K *is continuous on* Ω.

(ii) R *is continuous on* Ω.

(iii) $A^\wedge$ *is continuous on* Ω.

It should be mentioned that without some other additional assumptions none of the above conditions holds. In order to state and prove some continuity criteria we need a new tool.

1.6. In the same setting as before, assume that the point $\omega \in \Omega$ is fixed and let $B_\omega : \Omega \to \mathfrak{A}$ be the map defined by

$$B_\omega(\lambda) = K(\omega) + A^\wedge(\omega)A(\lambda), \quad \lambda \in \Omega. \tag{1.11}$$

The map B_ω is continuous, and by (1.7) we clearly have $B_\omega(\omega) = 1$. Therefore, there exists an open neighborhood $\Lambda_\omega \subset \Omega$ of ω such that $B_\omega(\lambda)$ is invertible for any $\lambda \in \Lambda_\omega$. Thus, we may define another map $F_\omega : \Lambda_\omega \to \mathfrak{A}$ by

$$F_\omega(\lambda) = [B_\omega(\lambda)]^{-1}K(\omega), \quad \lambda \in \Lambda_\omega. \tag{1.12}$$

Below we summarize a few nice properties of this map.

LEMMA. *Formula* (1.12) *defines an idempotent-valued continuous map, such that*

$$F_\omega(\lambda)K(\omega) = F_\omega(\omega), \tag{1.13}$$

$$K(\omega)F_\omega(\lambda) = K(\omega), \tag{1.14}$$

$$F_\omega(\lambda)K(\lambda) = K(\lambda), \tag{1.15}$$

for every $\lambda \in \Lambda_\omega$.

Proof. The continuity of F_ω and equality (1.13) are obvious. From (1.9) it follows that $K(\omega)B_\omega(\lambda) = K(\omega)$, hence

$$K(\omega)[B_\omega(\lambda)]^{-1} = K(\omega). \tag{1.16}$$

This equality clearly implies (1.14), as well as the desired equation $F_\omega(\lambda)^2 = F_\omega(\lambda)$.

Finally, by (1.11) we have that $K(\omega) = B_\omega(\lambda) - A^\wedge(\omega)A(\lambda)$, and (1.12) yields an alternative expression for $F_\omega(\lambda)$, namely,

$$F_\omega(\lambda) = 1 - [B_\omega(\lambda)]^{-1}A^\wedge(\omega)A(\lambda), \tag{1.17}$$

which takes care of (1.15). ∎

1.7. We are now in a position to formulate two tests for continuity.

PROPOSITION. *Each of the conditions* (i), (ii), *and* (iii) *in Proposition* 1.5 *is equivalent to:*

(iv) *for every* $\omega \in \Omega$ *there exists an open neighborhood* $\Lambda'_\omega \subset \Lambda_\omega$ *of* ω *such that* $F_\omega(\lambda)$ *is a kernel idempotent of* $A(\lambda)$ *for all* $\lambda \in \Lambda'_\omega$.

Moreover, if there exists a faithful tracial linear functional τ *on a* $*$*-ideal* $\mathfrak{J}$ *of* $\mathfrak{A}$ *and* $K(\omega) \in \mathfrak{J}$ *for all* $\omega \in \Omega$, *then any of the conditions* (i), (ii), (iii), *or* (iv) *is equivalent to:*

(v) *the function* $\tau \circ K$ *is locally constant on* Ω.

Proof. The fact that (iv) implies (i) follows from a remark at the end of 1.3. More precisely, we conclude that $K(\lambda) = \pi(F_\omega(\lambda))$ for every $\lambda \in \Lambda'_\omega$. Formula (1.5) and the continuity of F_ω on Λ'_ω clearly prove that K is continuous on Λ'_ω, and therefore, K is continuous everywhere.

Suppose next that (ii) holds true, and for a given $\omega \in \Omega$ let $\Lambda'_\omega \subset \Lambda_\omega$ be an open neighborhood of ω such that

$$\|R(\lambda) - R(\omega)\| < 1, \tag{1.18}$$

for any $\lambda \in \Lambda'_\omega$. As a first step towards the proof of (iv) we claim that

$$A(\lambda)F_\omega(\lambda) = 0, \quad \lambda \in \Lambda'_\omega. \tag{1.19}$$

The proof of (1.19) goes as follows. Since $R(\lambda)A(\lambda) = A(\lambda)$ we get that

$$[1 - R(\lambda)]A(\lambda)F_\omega(\lambda) = 0, \quad \lambda \in \Lambda_\omega. \tag{1.20}$$

On the other hand, let us observe that (1.8), (1.11), (1.12), and (1.14) lead successively to

$$\begin{aligned} R(\omega)A(\lambda)F_\omega(\lambda) = A(\omega)A^\wedge(\omega)A(\lambda)F_\omega(\lambda) &= A(\omega)[B_\omega(\lambda) - K(\omega)]F_\omega(\lambda) \\ &= A(\omega)[K(\omega) - K(\omega)] = 0. \end{aligned}$$

The last relation and (1.20) show that

$$\{1 - [R(\lambda) - R(\omega)]\}A(\lambda)F_\omega(\lambda) = 0, \tag{1.21}$$

and now (1.19) follows easily since (1.18) implies that $1 - [R(\lambda) - R(\omega)]$ is invertible.

To conclude the proof of (iv) all we need is to show that $A(\lambda)x = 0$ implies $x = F_\omega(\lambda)x$. To this end it is enough to take into account that $A(\lambda)x = 0$ implies $x = K(\lambda)x$, and then, based on (1.15), to observe that indeed $x = F_\omega(\lambda)x$.

It remains to prove that under the additional assumptions stated in the second half of our proposition, condition (v) is equivalent to any of the other four conditions. The fact that (i) implies (v) is obvious, so let us suppose that (v) holds. Let ω be a point in Ω, and let $\Lambda_\omega \subset \Omega$ and $F_\omega : \Lambda_\omega \to \mathfrak{A}$ be defined as in 1.6. Without any loss of generality we may assume that Λ_ω is connected. Define next $P : \Lambda_\omega \to \mathfrak{A}$ by $P = \pi \circ F_\omega$, where π is the map given by (1.5). The continuity of F_ω clearly implies the continuity of P on Λ_ω. By Lemma 1.3 we know that

$$P(\lambda)F_\omega(\lambda) = F_\omega(\lambda), \quad F_\omega(\lambda)P(\lambda) = P(\lambda), \quad \lambda \in \Lambda_\omega. \tag{1.22}$$

On the other hand, based on (1.15) and (1.22) we get

$$P(\lambda)K(\lambda) = P(\lambda)F_\omega(\lambda)K(\lambda) = F_\omega(\lambda)K(\lambda) = K(\lambda), \tag{1.23}$$

hence $K(\lambda) \leqslant P(\lambda)$ for all $\lambda \in \Lambda_\omega$. By (1.12) and (1.22) we also have that $P(\lambda) \in \mathfrak{J}$ for any $\lambda \in \Lambda_\omega$. Since P is projection-valued it follows that $\tau \circ P$ is constant on Λ_ω. But $P(\omega) = K(\omega)$, so $\tau(P(\lambda)) = \tau(K(\lambda))$ for all $\lambda \in \Lambda_\omega$. Finally, using the faithfulness of τ and the fact that $K(\lambda) \leqslant P(\lambda)$ we conclude that $K(\lambda) = P(\lambda)$, for every $\lambda \in \Lambda_\omega$, so that K is continuous on Λ_ω.

The proof is complete. ∎

1.8. Proposition 1.7 could be equally well used as a test for differentiability. Specifically, assume that Ω is a differentiable manifold, and let $A : \Omega \to \mathfrak{A}$ be a differentiable map such that $A(\omega)$ has closed range for every $\omega \in \Omega$. By (1.11) and (1.12) we deduce that any map $F_\omega : \Lambda_\omega \to \mathfrak{A}$ is differentiable. A brief inspection of the proofs above suffices to conclude that the continuity of the kernel projection map $K : \Omega \to \mathfrak{A}$ ensures its differentiability on Ω.

We encounter a more interesting situation in case Ω is a complex manifold and $A : \Omega \to \mathfrak{A}$ is a holomorphic map. Again the continuity of K implies a stronger property, namely, K turns out to be real-analytic, but of course it is out of question to hope that K is a holomorphic $\mathfrak{A}$-valued map. The special property inherited by K in this case has an interesting geometric significance that will be fully explained in Section 2. All we can do right now is to derive that property. So, let us assume that $A : \Omega \to \mathfrak{A}$ is holomorphic. By applying the $\overline{\partial}$ operator to

$$A(\omega)K(\omega) = 0, \quad \omega \in \Omega, \tag{1.24}$$

we get

$$A(\omega)\overline{\partial}K(\omega) = 0, \quad \omega \in \Omega, \tag{1.25}$$

where $\overline{\partial}K$ is an $\mathfrak{A}$-valued differential $(0,1)$-form. On the other hand, (1.25) and Definition 1.1 clearly imply

$$\overline{\partial}K(\omega) = K(\omega)\overline{\partial}K(\omega), \quad \omega \in \Omega, \tag{1.26}$$

where the product of $\mathfrak{A}$-valued maps and $\mathfrak{A}$-valued forms is defined in the standard fashion.

It is worth noticing that under the same assumptions the range projection map $R : \Omega \to \mathfrak{A}$ behaves in a very similar fashion. If $A : \Omega \to \mathfrak{A}$ is holomorphic, then R is a real-analytic map, and from

$$R(\omega)A(\omega) = A(\omega), \quad \omega \in \Omega, \tag{1.27}$$

we get

$$[\overline{\partial}R(\omega)]A(\omega) = 0, \quad \omega \in \Omega, \tag{1.28}$$

hence

$$[\overline{\partial}R(\omega)]R(\omega) = 0. \tag{1.29}$$

On the other hand, since $R(\omega) = R(\omega)^2$, we have

$$\overline{\partial}R(\omega) = [\overline{\partial}R(\omega)]R(\omega) + R(\omega)\overline{\partial}R(\omega),$$

so (1.29) is equivalent to

$$\overline{\partial}R(\omega) = R(\omega)\overline{\partial}R(\omega), \quad \omega \in \Omega. \tag{1.30}$$

A more detailed discussion centered on equations (1.26) and (1.30) is carried out in the next section. Before concluding the present section we want to point out that Proposition 1.7 and the comments made at 1.8 extend [11], Theorem 2.2 and [21], Theorem 3.4.

2. FLAG MANIFOLDS OF C^*-ALGEBRAS

2.1. Our goal in the sequel is to outline a few geometric properties of the space of increasing n-tuples of projections in a C^*-algebra $\mathfrak{A}$. Such spaces will be called *flag manifolds* of the given algebra $\mathfrak{A}$, for the obvious reason that the classical flag manifolds might be alternatively described in this way.

We first recall that $\mathcal{E}(\mathfrak{A})$ and, consequently, $\mathcal{P}(\mathfrak{A})$ are partially ordered sets; if e and f are two idempotents in $\mathfrak{A}$, then $e \leqslant f$ whenever $ef = e$ and $fe = e$.

DEFINITION. Let $n \geqslant 1$ be an integer.

(i) An n-tuple $\mathfrak{e} = (e_1, e_2, \ldots, e_n)$ of idempotents in $\mathfrak{A}$ is called an *extended n-flag* of $\mathfrak{A}$ provided that

$$e_1 \leqslant e_2 \leqslant \cdots \leqslant e_n. \tag{2.1}$$

(ii) An extended n-flag $\mathfrak{p} = (p_1, p_2, \ldots, p_n)$ is said to be an *n-flag* of $\mathfrak{A}$ in case every entry p_i, $1 \leqslant i \leqslant n$, is a projection in $\mathfrak{A}$.

The sets of all extended n-flags of $\mathfrak{A}$ and of all n-flags of $\mathfrak{A}$ will be denoted by $\mathcal{E}_n(\mathfrak{A})$ and $\mathcal{P}_n(\mathfrak{A})$, respectively.

2.2. Both $\mathcal{E}_n(\mathfrak{A})$ and $\mathcal{P}_n(\mathfrak{A})$ are closed subsets of $\mathfrak{A}^n$, the n-fold direct product of $\mathfrak{A}$. Actually, even more is true, namely, $\mathcal{E}_n(\mathfrak{A})$ is a complex submanifold of $\mathfrak{A}^n$, whereas $\mathcal{P}_n(\mathfrak{A})$ is merely a real-analytic submanifold of $\mathfrak{A}^n$. The interested reader can find plenty of details concerning the differential geometry of $\mathcal{E}_n(\mathfrak{A})$ and $\mathcal{P}_n(\mathfrak{A})$ in [5], [16], [17] and [18]. From now on we will refer to the spaces $\mathcal{E}_n(\mathfrak{A})$ and $\mathcal{P}_n(\mathfrak{A})$ as the extended n-flag manifold, and the n-flag manifold of $\mathfrak{A}$, respectively. In the simplest case $n = 1$ we get the spaces $\mathcal{E}(\mathfrak{A})$ and $\mathcal{P}(\mathfrak{A})$, alternatively called the extended Grassmann manifold, and the Grassmann manifold of $\mathfrak{A}$. Specific properties of these spaces are discussed in [6], [15], [19], [21], [25] and [26].

2.3. Although $\mathcal{P}_n(\mathfrak{A})$ fails to be a complex submanifold of $\mathcal{E}_n(\mathfrak{A})$, however $\mathcal{P}_n(\mathfrak{A})$ inherits a canonical complex structure from the complex manifold $\mathcal{E}_n(\mathfrak{A})$. The point is that $\mathcal{E}_n(\mathfrak{A})$ turns out to be a covering space of $\mathcal{P}_n(\mathfrak{A})$. The covering map $\Pi : \mathcal{E}_n(\mathfrak{A}) \to \mathcal{P}_n(\mathfrak{A})$ is given by

$$\Pi(e_1, e_2, \ldots, e_n) = (\pi(e_1), \pi(e_2), \ldots, \pi(e_n)), \quad (e_1, e_2, \ldots, e_n) \in \mathcal{E}_n(\mathfrak{A}), \tag{2.2}$$

where $\pi : \mathcal{E}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{A})$ is the map already defined by (1.5). Since the latter is an order preserving map, the former is well-defined.

The just introduced map $\Pi : \mathcal{E}_n(\mathfrak{A}) \to \mathcal{P}_n(\mathfrak{A})$ will be referred to as the *Gram-Schmidt map.* This map supplies the main device we need in order to prove the existence of a natural complex structure on $\mathcal{P}_n(\mathfrak{A})$. The quickest argument one might invoke was suggested to us by Harald Upmeier, and it amounts to applying a theorem of Godement to the appropriate manifolds and maps (see [4], 5.9.5 and [22], Theorem 8.14). In effect, we get the next result.

PROPOSITION. *The Gram-Schmidt map is a real-analytic submersion.*

2.4. The relevance of this result comes from the fact that the kernel of the tangent map

$$\text{(2.3)} \qquad d\Pi(\mathfrak{e}) : \mathrm{T}_{\mathfrak{e}}\mathcal{E}_n(\mathfrak{A}) \to \mathrm{T}_{\Pi(\mathfrak{e})}\mathcal{P}_n(\mathfrak{A})$$

is a *complex* subspace of the tangent space $\mathrm{T}_{\mathfrak{e}}\mathcal{E}_n(\mathfrak{A})$, for every $\mathfrak{e} \in \mathcal{E}_n(\mathfrak{A})$, and, moreover, the tangent bundle $\mathrm{T}\mathcal{E}_n(\mathfrak{A})$ splits into a direct sum of two *complex* subbundles

$$\text{(2.4)} \qquad \mathrm{T}\mathcal{E}_n(\mathfrak{A}) = \mathcal{K} \oplus \mathcal{K}',$$

where the fiber $\mathcal{K}_{\mathfrak{e}}$ of $\mathcal{K}$ at any point $\mathfrak{e} \in \mathcal{E}_n(\mathfrak{A})$ is the kernel of $d\Pi(\mathfrak{e})$.

By relying once again on Godement's theorem we arrive at the next conclusion.

THEOREM. *There exists a unique complex structure on the flag manifold $\mathcal{P}_n(\mathfrak{A})$ such that the Gram-Schmidt map $\Pi : \mathcal{E}_n(\mathfrak{A}) \to \mathcal{P}_n(\mathfrak{A})$ is a holomorphic map.*

2.5. The above developed reasoning has an inherent drawback. Theorem 2.4 gives the ground for the existence of a complex structure on the flag manifold $\mathcal{P}_n(\mathfrak{A})$, but it shortly fails to make this structure a practical tool. For a complete understanding of the complex structure on $\mathcal{P}_n(\mathfrak{A})$ we have to describe the subbundles $\mathcal{K}$ and $\mathcal{K}'$ in a manageable way. This specific goal has been accomplished in [18] where a straightforward proof of Theorem 2.4 was given. More precisely, the proof is based on an explicit description of the canonical complex structure of the flag manifold, very much in the spirit — if not the letter — of the approach initiated by Wang ([24]) and by Borel and Hirzebruch [3] in their studies on classical flag manifolds. An excellent account on the subject can be found in [23].

2.6. In view of our further purposes we need a criterion for determining the holomorphic maps from a given complex manifold Ω to the flag manifold $\mathcal{P}_n(\mathfrak{A})$.

Specifically, we assume that Ω is a finite or infinite dimensional complex manifold and $\mathcal{P}_n(\mathfrak{A})$ is equipped with the canonical complex structure described above. For the sake of completeness we will next indicate two criteria for holomorphic $\mathcal{P}_n(\mathfrak{A})$-valued maps. The first one follows from Theorem 2.4 and from the general fact that a holomorphic submersion has a local holomorphic right-inverse about any given point, with a prescribed value at that point (see, for instance, [22], Corollary 8.3). The holomorphic submersion we are interested in is the Gram-Schmidt map $\Pi : \mathcal{E}_n(\mathfrak{A}) \to \mathcal{P}_n(\mathfrak{A})$.

THEOREM. *Suppose $P : \Omega \to \mathcal{P}_n(\mathfrak{A})$ is a smooth map. Then P is holomorphic if and only if for every point $\omega \in \Omega$ there exists an open neighborhood $\Lambda_\omega \subset \Omega$ of ω and a holomorphic map $F_\omega : \Lambda_\omega \to \mathcal{E}_n(\mathfrak{A})$ such that $F_\omega(\omega) = P(\omega)$, and*

$$P(\lambda) = \Pi \circ F_\omega(\lambda), \quad \lambda \in \Lambda_\omega. \tag{2.5}$$

In particular, this theorem shows that a holomorphic $\mathcal{P}_n(\mathfrak{A})$-valued map is a real-analytic $\mathfrak{A}^n$-valued map.

2.7. A more reliable criterion has been proved in [18], Theorem 3.3 and Corollary 3.4. It is exclusively stated in terms of the $\overline{\partial}$ operator acting on smooth $\mathfrak{A}$-valued maps. Quite expectably, a characterization of anti-holomorphic $\mathcal{P}_n(\mathfrak{A})$-valued maps is also available by means of the ∂ operator.

THEOREM. *Let $P = (P_1, P_2, \ldots, P_n) : \Omega \to \mathcal{P}_n(\mathfrak{A})$ be a smooth map. Then P is holomorphic if and only if*

$$\overline{\partial} P_i(\omega) = P_i(\omega) \overline{\partial} P_i(\omega), \tag{2.6}$$

for all $\omega \in \Omega$ and every $1 \leqslant i \leqslant n$, and P is anti-holomorphic if and only if

$$\partial P_i(\omega) = P_i(\omega) \partial P_i(\omega), \tag{2.7}$$

for all $\omega \in \Omega$ and every $1 \leqslant i \leqslant n$.

2.8. The last theorem generalizes some results from [15], [21], and [25], where holomorphic maps into the Grassmann manifold $\mathcal{P}(\mathfrak{A})$ are considered. An important feature of Theorem 2.7 is that it provides a componentwise test for holomorphic or anti-holomorphic maps, and, consequently, it enables us to reduce the general study of such maps into flag manifolds to the study of holomorphic or anti-holomorphic maps with values in a Grassmann manifold.

In this regard let us assume for a while that $n = 1$, and let $P : \Omega \to \mathcal{P}(\mathfrak{A})$ be a holomorphic map, that is,

$$\overline{\partial}P(\omega) = P(\omega)\overline{\partial}P(\omega), \quad \omega \in \Omega. \tag{2.8}$$

The kernel projection map K and the range projection map R defined in Section 1 provide two examples of projection-valued maps satisfying Equation (2.8).

Actually, these two examples are quite typical, in the sense that every holomorphic map $P : \Omega \to \mathcal{P}(\mathfrak{A})$ is locally given either as the kernel projection map, or as the range projection map, of a holomorphic family of elements with closed range. The simple proof of this remark follows from Theorem 2.6. Specifically, for any given $P : \Omega \to \mathcal{P}(\mathfrak{A})$ satisfying (2.8) and for any $\omega \in \Omega$ we can find a holomorphic map $F_\omega : \Lambda_\omega \to \mathcal{E}(\mathfrak{A})$ defined on a neighborhood $\Lambda_\omega \subset \Omega$ of ω, such that

$$P(\lambda) = \pi(F_\omega(\lambda)), \quad \lambda \in \Lambda_\omega. \tag{2.9}$$

Therefore, $P|\Lambda_\omega$ equals the range projection map corresponding to $F_\omega : \Lambda_\omega \to \mathcal{E}(\mathfrak{A})$. On the other hand, if we let $F_\omega^\perp : \Lambda_\omega \to \mathcal{E}(\mathfrak{A})$ be the holomorphic idempotent-valued map defined by

$$F_\omega^\perp(\lambda) = 1 - F_\omega(\lambda), \quad \lambda \in \Lambda_\omega, \tag{2.10}$$

then a straightforward computation based on Lemma 1.3 easily implies that $P(\lambda)$ is the kernel projection of $F_\omega^\perp(\lambda)$ for every $\lambda \in \Lambda_\omega$.

2.9. We next want to find a few other conditions equivalent to (2.8) above, as well as some consequences of these conditions. First, we notice that $P(\omega) = P(\omega)^2$ implies

$$\overline{\partial}P(\omega) = [\overline{\partial}P(\omega)]P(\omega) + P(\omega)\overline{\partial}P(\omega), \quad \omega \in \Omega,$$

hence (2.8) is equivalent to

$$[\overline{\partial}P(\omega)]P(\omega) = 0, \quad \omega \in \Omega. \tag{2.11}$$

By taking now the adjoints, from (2.8) and (2.11) we get two new equivalent conditions, namely,

$$\partial P(\omega) = [\partial P(\omega)]P(\omega), \quad \omega \in \Omega, \tag{2.12}$$

and, respectively,

$$P(\omega)\partial P(\omega) = 0, \quad \omega \in \Omega. \tag{2.13}$$

Equation (2.13) and Theorem 2.7 can be employed to check that the map

$$P^{\perp} : \Omega \to (\mathfrak{A}), \quad \mathcal{P}^{\perp}(\omega) = 1 - \mathcal{P}(\omega), \quad \omega \in \Omega, \tag{2.14}$$

is an anti-holomorphic projection-valued map. Indeed, by (2.13) we get

$$\begin{aligned}\partial P^{\perp}(\omega) - P^{\perp}(\omega)\partial P^{\perp}(\omega) &= -\partial P(\omega) + [1 - P(\omega)]\partial P(\omega)\\ &= -P(\omega)\partial P(\omega) = 0,\end{aligned}$$

for all $\omega \in \Omega$, hence $P^{\perp}$ is a solution of Equation (2.7). This striking behavior of $P^{\perp}$ could also be explained by relying on formula (1.1) and the comments made at the end of Subsection 2.8. Since at least locally we have

$$P(\,\cdot\,) = \kappa[A(\,\cdot\,)], \tag{2.15}$$

where $A(\,\cdot\,)$ is a holomorphic family of elements with closed range, it follows that

$$P^{\perp}(\,\cdot\,) = \rho[A(\,\cdot\,)^*], \tag{2.16}$$

and clearly $A(\,\cdot\,)^*$ is an anti-holomorphic $\mathfrak{A}$-valued map.

2.10. In order to derive some consequences of (2.8), until the end of this section we will suppose that Ω is *finite-dimensional* and *connected.* Moreover, we assume that $\omega \in \Omega$ is fixed, and let $\Lambda_\omega \subset \Omega$ be a coordinate neighborhood centered at ω with the coordinate functions $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Let $\mathbb{Z}_+^m$ denote the set of all m-tuples $I = (i_1, i_2, \ldots, i_m)$ of non-negative integers, and in case $I \in \mathbb{Z}_+^m$ is given, let

$$|I| = i_1 + i_2 + \cdots + i_m, \quad I! = i_1!i_2!\cdots i_m!.$$

In addition, we set

$$\mathrm{D}^I = (\partial/\partial\lambda_1)^{i_1}(\partial/\partial\lambda_2)^{i_2}\cdots(\partial/\partial\lambda_m)^{i_m},$$

and

$$\overline{\mathrm{D}}^I = (\partial/\partial\overline{\lambda}_1)^{i_1}(\partial/\partial\overline{\lambda}_2)^{i_2}\cdots(\partial/\partial\overline{\lambda}_m)^{i_m}.$$

The next result points out an important property of every given holomorphic map $P : \Omega \to \mathcal{P}(\mathfrak{A})$, that clearly resembles the behavior of an arbitrary $\mathfrak{A}$-valued complex-analytic map.

THEOREM. *The set $\mathcal{D}_\omega(P) = \{\mathrm{D}^I P(\omega) : I \in \mathbb{Z}_+^m\}$ of all the derivatives of P at ω with respect to the complex coordinate functions completely determines the map $P : \Omega \to \mathcal{P}(\mathfrak{A})$.*

Proof. Since P is real-analytic and Ω is connected, P is uniquely determined by its restriction to Λ_ω.

Without any loss of generality we may assume that Λ_ω is sufficiently small such that the power series expansion

$$P(\lambda) = \sum_{I,J} (1/I!J!)\mathrm{D}^I\overline{\mathrm{D}}^J P(\omega)(\lambda - \omega)^I(\overline{\lambda} - \overline{\omega})^J \tag{2.17}$$

holds for every $\lambda \in \Lambda_\omega$ and, moreover, the series above converges absolutely in $\mathfrak{A}$. Therefore, all we have to prove is that the set $\mathcal{D}_\omega(\mathcal{P})$ suffices to recover all the possible derivatives of P at ω. To this end we first notice the obvious relations

$$\overline{\mathrm{D}}^J P(\omega) = [\mathrm{D}^J P(\omega)]^*, \quad J \in \mathbb{Z}_+^m. \tag{2.18}$$

Furthermore, from equations (2.8), (2.11), (2.12) and (2.13) we get the less obvious, but important relations

$$[1 - P(\omega)]\overline{\mathrm{D}}^J P(\omega) = 0 = [\overline{D}^J P(\omega)]P(\omega), \quad |J| \geqslant 1, \tag{2.19}$$

(2.20
$$P(\omega)\mathrm{D}^I P(\omega) = 0 = [D^I P(\omega)][1 - P(\omega)], \quad |I| \geqslant 1,$$

as well as

$$\mathrm{D}^I\overline{\mathrm{D}}^J P(\omega) = [\mathrm{D}^I P(\omega)][\overline{D}^J P(\omega)] - [\overline{D}^J P(\omega)][\mathrm{D}^I P(\omega)], \quad |I| = 1 = |J|. \tag{2.21}$$

The reader can either supply the elementary proofs of these relations, or find complete arguments in [15], Section 1 or [21], Section 3.

By a repeated use of (2.21), in conjunction with (2.19) and (2.20), we conclude that every derivative $\mathrm{D}^I\overline{\mathrm{D}}^J P(\omega)$, for arbitrary $I, J \in \mathbb{Z}_+^m$, may be expressed as a sum of monomials of the form

(i) $\pm[\mathrm{D}^{I_1} P(\omega)][\overline{\mathrm{D}}^{J_1} P(\omega)] \cdots [\mathrm{D}^{I_k} P(\omega)][\overline{\mathrm{D}}^{J_k} P(\omega)]$,

or

(ii) $\pm[\overline{\mathrm{D}}^{J_1} P(\omega)][\mathrm{D}^{I_1} P(\omega)] \cdots [\overline{\mathrm{D}}^{J_k} P(\omega)][\mathrm{D}^{I_k} P(\omega)]$,

where

$$I_1 + I_2 + \cdots + I_k = I \quad \text{and} \quad J_1 + J_2 + \cdots + J_k = J.$$

This remark and (2.18) complete the proof of our theorem. ∎

2.11. There are at least two consequences of Theorem 2.10, and of its proof, that deserve a special attention. First, we want to notice that the complex-analytic part of the power series (2.17), namely, the map F_ω from Λ_ω to $\mathfrak{A}$ given by

$$F_\omega(\lambda) = \sum_I (1/I!)\mathrm{D}^I P(\omega)(\lambda - \omega)^I, \quad \lambda \in \Lambda_\omega, \tag{2.22}$$

is an $\mathcal{E}(\Omega)$-valued map satisfying the following conditions:

(i) $F_\omega(\lambda)P(\lambda) = P(\lambda)$,
(ii) $P(\Lambda)F_\omega(\lambda) = F_\omega(\lambda)$,
(iii) $F_\omega(\lambda)P(\omega) = F_\omega(\lambda)$,
(iv) $P(\omega)F_\omega(\lambda) = P(\omega)$,

for any $\lambda \in \Lambda_\omega$. In particular, F_ω also satisfies (2.9). Borrowing a term introduced in [21] we will refer to any holomorphic map $F_\omega : \Lambda_\omega \to \mathcal{E}(\Omega)$ subject to conditions (i)–(iv) above as a *frame* of P about the given point ω. This terminology is aimed to stress on the geometric significance of Equation (2.8) that becomes transparent in case $\mathfrak{A} = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. As a matter of fact, right now we are very close to one of the major themes of the Cowen-Douglas theory. Specifically, let us suppose that $P : \Omega \to \mathcal{P}(\mathcal{L}(\mathcal{H}))$ is merely a continuous map such that

$$\dim[P(\omega)\mathcal{H}] = k, \quad \omega \in \Omega, \tag{2.23}$$

for a given integer k. Any such map P determines a continuous vector bundle $\mathcal{K}$ over Ω, where the fiber $\mathcal{K}_\omega$ of $\mathcal{K}$ at every point $\omega \in \Omega$ is given by

$$\mathcal{K}_\omega = P(\omega)\mathcal{H}. \tag{2.24}$$

With the aid of the generalized Grothendieck's lemma proved by Malgrange ([13]), one may show that $\mathcal{K}$ is a holomorphic vector bundle if and only if P is a smooth map satisfying (2.8) (for details, see [15], Proposition 1.1).

2.12. The next result summarizes a second consequence of Theorem 2.10.

PROPOSITION. *Suppose that $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a unital C^*-algebra homomorphism, and let $P : \Omega \to \mathcal{P}(\mathfrak{A})$ and $Q : \Omega \to \mathcal{P}(\mathfrak{B})$ be two given holomorphic projection-valued maps. The following two conditions are equivalent:*

(i) $\varphi(P(\cdot)) = Q(\cdot)$;
(ii) *there exists a point $\omega \in \Omega$ such that*

$$\varphi(\mathrm{D}^I P(\omega)) = \mathrm{D}^I Q(\omega), \tag{2.25}$$

for every $I \in \mathbb{Z}_+^m$.

The easy proof is left to our reader.

3. INTERACTIONS WITH THE COWEN-DOUGLAS THEORY

3.1. The preceding sections already displayed two of the three interrelated objects encountered in the framework of the Cowen-Douglas theory, namely, the class of holomorphic maps into a Grassmann manifold and its natural companion, the class of hermitian holomorphic vector bundles. The third equally important item is provided by Hilbert space operators in the Cowen-Douglas class (see [8], [9]). Though this concept was first defined only for operators or tuples of operators on a Hilbert space, the basic features of the Cowen-Douglas class can be described in a more general setting. The initial data we need consist of a C^*-algebra $\mathfrak{A}$ and a bounded connected open set $\Omega \subset \mathbb{C}^m$.

DEFINITION. ([21], Definition 3.10). An m-tuple $\mathfrak{A} = (a_1, a_2, \ldots, a_m)$ of elements of $\mathfrak{A}$ is said to be in *the class* $\mathcal{B}(\Omega, \mathfrak{A})$ whenever the following conditions are satisfied:

(i) zero is an isolated point of the spectrum of the element

$$\Delta(\omega) = (\omega_1 - a_1)^*(\omega_1 - a_1) + (\omega_2 - a_2)^*(\omega_2 - a_2) + \cdots + (\omega_m - a_m)^*(\omega_m - a_m), \tag{3.1}$$

for every $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ in Ω;

(ii) the map $P : \Omega \to \mathcal{P}(\mathfrak{A})$ defined by

$$P(\omega) = E_0[\Delta(\omega)], \quad \omega \in \Omega, \tag{3.2}$$

is continuous, where $E_0[\Delta(\omega)]$ stands for the spectral projection of $\Delta(\omega)$ associated with $\{0\}$;

(iii) if $a \in \mathfrak{A}$ and $aP(\omega) = 0$ for every $\omega \in \Omega$, then $a = 0$.

3.2. We next point out a few properties of the class $\mathcal{B}(\Omega, \mathfrak{A})$.

First, let us notice that given $(a_1, a_2, \ldots, a_m) \in \mathcal{B}(\Omega, \mathfrak{A})$ we have

$$a_i P(\omega) = \omega_i P(\omega), \quad 1 \leqslant i \leqslant m, \tag{3.3}$$

for each $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ in Ω. In particular, from (3.3) and condition (iii) it follows that $(a_1, a_2, \ldots, a_m)$ is a commuting m-tuple. Therefore, it makes sense to consider the Taylor spectrum $\sigma(a_1, a_2, \ldots, a_m)$ of such an m-tuple. Condition (i) implies that the domain Ω is contained in $\sigma(a_1, a_2, \ldots, a_m)$.

Furthermore, based on some comments made in the first paragraph of Subsection 1.8, from the continuity condition (ii) we get that $P : \Omega \to \mathcal{P}(\mathfrak{A})$ is a real-analytic map. In fact, much more is true, namely, P is a holomorphic map from Ω to the Grassmann manifold $\mathcal{P}(\mathfrak{A})$. A quick proof goes as follows. By (3.3) we clearly have

$$[\overline{\partial}\Delta(\omega)]P(\omega) = 0, \quad \omega \in \Omega.$$

Since $0 = \overline{\partial}[\Delta(\omega)P(\omega)] = [\overline{\partial}\Delta(\omega)]P(\omega) + \Delta(\omega)[\overline{\partial}P(\omega)]$ we get $\Delta(\omega)[\overline{\partial}P(\omega)] = 0$, and because $P(\omega)$ is the kernel projection of $\Delta(\omega)$ we obtain

$$\overline{\partial}P(\omega) = P(\omega)\overline{\partial}P(\omega), \quad \omega \in \Omega. \tag{3.4}$$

Theorem 2.7 completes the proof.

The additional properties of P may be used to show that

$$\mathcal{B}(\Omega, \mathfrak{A}) \subset \mathcal{B}(\Lambda, \mathfrak{A}) \tag{3.5}$$

for every non-empty connected open set $\Lambda \subset \Omega$. Since conditions (i) and (ii) behave well with respect to restrictions, only condition (iii) remains to be checked. Specifically, assume $a \in \mathfrak{A}$ is such that $aP(\lambda) = 0$ for every $\lambda \in \Lambda$, where the map $P : \Omega \to \mathcal{P}(\mathfrak{A})$ corresponds to an m-tuple $(a_1, a_2, \ldots, a_m)$ in $\mathcal{B}(\Omega, \mathfrak{A})$. By the identity principle of real-analytic maps we get that $aP(\omega) = 0$ for all $\omega \in \Omega$, hence $a = 0$.

3.3. For the sake of completeness we next recall the original definition of the Cowen-Douglas class (see [8], [9] and [11]). One starts with a set $\Omega \subset \mathbb{C}^m$ as above, a complex separable infinite dimensional Hilbert space $\mathcal{H}$, and an integer $k \geqslant 1$.

DEFINITION. An m-tuple $\mathfrak{A} = (a_1, a_2, \ldots, a_m)$ of operators on $\mathcal{H}$ is said to be in the *Cowen-Douglas class* $\mathcal{B}_k(\Omega, \mathcal{L}(\mathcal{H}))$ if the following conditions are satisfied:

(i) the operator $D(\omega) : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^m$ defined by

$$D(\omega)\xi = (\omega_1 - a_1)\xi \otimes e_1 + (\omega_2 - a_2)\xi \otimes e_2 + \cdots + (\omega_m - a_m)\xi \otimes e_m, \quad \xi \in \mathcal{H},$$

has closed range for every $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ in Ω, where $\{e_1, e_2, \ldots, e_m\}$ is the standard basis for $\mathbb{C}^m$;

(ii) $\dim \ker D(\omega) = k$, for all $\omega \in \Omega$;

(iii) $\vee\{\ker D(\omega) : \omega \in \Omega\} = \mathcal{H}$, where $\vee$ stands for the closed linear span of a collection of sets.

The reader could easily verify that $\mathcal{B}_k(\Omega, \mathcal{L}(\mathcal{H})) \subset \mathcal{B}(\Omega, \mathcal{L}(\mathcal{H}))$. Actually, each condition in the last definition leads to the condition in Definition 3.1 carrying the same label. The only nonobvious fact is the continuity of the projection-valued map P defined as in (3.2). Since $P(\omega) = \kappa(D(\omega))$, $\omega \in \Omega$, condition (ii) above enables us to take advantage of the last test for continuity formulated in Proposition 1.7. As a $*$-ideal $\mathfrak{J}$ we may take the ideal of all trace-class operators on $\mathcal{H}$.

3.4. A pretty large part of the Cowen-Douglas theory is concerned with the search for simple and complete invariants of tuples in the Cowen-Douglas class. The relevant notions are given below.

DEFINITION. Let $\mathfrak{A}$ and Ω be as above.

(i) Two tuples $\mathfrak{A} = (a_1, a_2, \ldots, a_m)$ and $\mathfrak{b} = (b_1, b_2, \ldots, b_m)$ in the class $\mathcal{B}(\Omega, \mathfrak{A})$ are called *unitarily equivalent* if there exists a unitary $u \in \mathfrak{A}$ such that

$$ua_i u^* = b_i, \quad 1 \leqslant i \leqslant m. \tag{3.6}$$

(ii) Two holomorphic maps P and Q from Ω to $\mathcal{P}(\mathfrak{A})$ are called *congruent* if there exists a unitary $u \in \mathfrak{A}$ such that

$$uP(\omega)u^* = Q(\omega), \quad \omega \in \Omega. \tag{3.7}$$

3.5. It is not surprising that the following result holds. Once again we entrust our reader with the details of the proof. For a more general statement and a complete proof we refer to [21], Theorem 3.12.

PROPOSITION. *Two tuples in the Cowen-Douglas class $\mathcal{B}(\Omega, \mathfrak{A})$ are unitarily equivalent, if and only if the corresponding holomorphic projection-valued maps provided by Definition* 3.1 *are congruent.*

Thus, the equivalence problem for elements in the class $\mathcal{B}(\Omega, \mathfrak{A})$ reduces to the congruence problem for holomorphic maps into the Grassmann manifold $\mathcal{P}(\mathfrak{A})$. On the other hand, let us also observe that the latter problem has a natural extension to holomorphic maps into a flag manifold, an extension that apparently goes beyond the class $\mathcal{B}(\Omega, \mathfrak{A})$. We will address this topic in Section 4 of this article.

The approach we are going to pursue relies on some other ideas and results already developed in the study of holomorphic maps into a Grassmann manifold. Our subsequent presentation mostly follows ([1], [14], [15], and [21]). The main concept we will employ goes back to Griffiths ([12]), and its role in the classification of holomorphic maps has been beautifuly explained in [8].

3.6. We need a few preliminaries. Let $P : \Omega \to \mathcal{P}(\mathfrak{A})$ be a given holomorphic map, and assume $\mathfrak{X} \subseteq \mathfrak{A}$ is a fixed subset containing the unit of $\mathfrak{A}$. For every point $\omega \in \Omega$ and any $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$ we introduce the set

$$\mathfrak{G}_\omega^\alpha(P, \mathfrak{X}) = \{\overline{\mathrm{D}}^J P(\omega) y^* x \mathrm{D}^I P(\omega) : I, J \in \mathbb{Z}_+^m,\ |I|, |J| \leqslant \alpha,\ x, y \in \mathfrak{X}\}, \tag{3.8}$$

and let $\mathfrak{A}_\omega^\alpha = \mathfrak{A}_\omega^\alpha(P, \mathfrak{X})$ be the closure of the $*$-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{G}_\omega^\alpha(P, \mathfrak{X})$. Since P is a holomorphic map, from (2.19) and (2.20) we get that each $\mathfrak{A}_\omega^\alpha$ is a $*$-subalgebra of the compressed algebra $P(\omega)\mathfrak{A}P(\omega)$, for every $\omega \in \Omega$. Moreover,

$$\mathfrak{A}_\omega^0 \subseteq \mathfrak{A}_\omega^1 \subseteq \cdots \subseteq \mathfrak{A}_\omega^\infty,$$

and $P(\omega) \in \mathfrak{A}_\omega^0$ is the common unit of all these algebras.

The next definition supplies a substitute for the Cowen-Douglas class $\mathcal{B}_k(\Omega, \mathcal{L}(\mathcal{H}))$.

DEFINITION. Let $k \geqslant 1$ be an integer. A pair $(P, \mathfrak{X})$ as above is said to be in *the class* $\mathcal{A}_k(\Omega, \mathfrak{A})$ provided that the following two conditions are satisfied:

(i) $\mathfrak{A}_\omega^\infty$ is a finite-dimensional C^*-algebra for each $\omega \in \Omega$;

(ii) if k_ω denotes the cardinal of any maximal collection of mutually orthogonal minimal projections in $\mathfrak{A}_\omega^\infty$, then

$$k_\omega \leqslant k, \quad \omega \in \Omega. \tag{3.9}$$

The pair $(P, \mathfrak{X})$ is said to be in a *general position* in case the next additional condition is fulfilled:

(iii) if $a \in \mathfrak{A}$ and $aP(\omega) = 0$ for all $\omega \in \Omega$, then $a = 0$.

Before proceeding with a more detailed description of these new objects, we should explain the relationship between the classes $\mathcal{A}_k(\Omega, \mathfrak{A})$ and $\mathcal{B}_k(\Omega, \mathcal{L}(\mathcal{H}))$. Accordingly, suppose that $(a_1, a_2, \ldots, a_m)$ is an element of $\mathcal{B}_k(\Omega, \mathcal{L}(\mathcal{H}))$, and let $P : \Omega \to \mathcal{L}(\mathcal{H})$ be the holomorphic map associated with $(a_1, a_2, \ldots, a_m)$ as in Subsection 3.3 above. This map yields the first entry of a pair $(P, \mathfrak{X})$ in the class $\mathcal{A}_k(\Omega, \mathcal{L}(\mathcal{H}))$. The second entry is the simplest one we might choose, namely $\mathfrak{X} = \{1\}$. The pair $(P, \mathfrak{X})$ defined in this way is in a general position.

3.7. The finiteness condition (3.9) in Definition 3.6 has some very strong consequences. We collect them below in a form that meets our forthcoming purposes.

For the sake of convenience, we first introduce a couple of new notations. Given an open set $\Lambda \subset \Omega$ we let $C^\infty(\Lambda, \mathfrak{A})$ denote the $*$-algebra of all smooth maps from Λ to $\mathfrak{A}$. For any $(P, \mathfrak{X})$ in $\mathcal{A}_k(\Omega, \mathfrak{A})$ and every $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$ we set

$$\Gamma^\alpha(\Lambda; P, \mathfrak{X}) = \{A \in C^\infty(\Lambda, \mathfrak{A}) : A(\omega) \in \mathfrak{A}^\alpha_\omega,\ \omega \in \Omega\}.$$

Occasionally we will write $\Gamma^\alpha(\Lambda)$ instead of $\Gamma^\alpha(\Lambda; P, \mathfrak{X})$. Clearly both $C^\infty(\Lambda, \mathfrak{A})$ and $\Gamma^\alpha(\Lambda)$ are $C^\infty(\Lambda)$-modules, where $C^\infty(\Lambda)$ is the algebra of complex-valued smooth functions on Ω.

The next technical result is a strengthened version of [1], Theorem A and [15], Theorem 2.4.

THEOREM. *Suppose $(P, \mathfrak{X})$ is a pair in $\mathcal{A}_k(\Omega, \mathfrak{A})$. Then there exists a non-empty open set $\Lambda \subset \Omega$ such that:*

(i) $\mathfrak{A}^k_\lambda = \mathfrak{A}^\infty_\lambda$ *for any* $\lambda \in \Lambda$*;*

(ii) *the set* $\{(\lambda, a) : \lambda \in \Lambda,\ a \in \mathfrak{A}^\infty_\lambda\}$ *is the total space of a smooth $*$-algebra bundle over* Λ*;*

(iii) *if* $\Phi : \Gamma^\infty(\Lambda) \to C^\infty(\Lambda, \mathfrak{A})$ *is a $C^\infty(\Lambda)$-linear $*$-algebra homomorphism such that*

$$\Phi(P(\mathrm{D}^I\overline{\mathrm{D}}^J A)P) = \Phi(P)[\mathrm{D}^I\overline{\mathrm{D}}^J \Phi(A)]\Phi(P), \tag{3.10}$$

for all $A \in \Gamma^{k-1}(\Lambda)$ *and any* $I, J \in \mathbb{Z}^m_+$ *with* $|I|, |J| \leqslant 1$*, then*

$$\Phi(P(\mathrm{D}^I\overline{\mathrm{D}}^J A)P) = \Phi(P)[\mathrm{D}^I\overline{\mathrm{D}}^J \Phi(A)]\Phi(P), \tag{3.11}$$

for all $A \in \Gamma^\infty(\Lambda)$ *and any* $I, J \in \mathbb{Z}^m_+$*.*

We do not intend to present a proof of this theorem. The main idea is to carry out a parametrized version of the standard construction of a matrix unit system of finite-dimensional C^*-algebras. With a careful choice of the set Λ we can find a collection of partial isometries in $\Gamma^\infty(\Lambda)$ whose values at every point $\lambda \in \Lambda$ yield a matrix unit system for $\mathfrak{A}^\infty_\lambda$. This part of the proof takes care of assertion (ii). The point is that the algebra $\Gamma^{k-1}(\Lambda)$ contains enough elements to manufacture such a colection of partial isometries. Assertions (i) and (iii) are basically additional dividends.

3.8. The next definition introduces a global, and, respectively, an infinitesimal equivalence relation in $\mathcal{A}_k(\Omega, \mathfrak{A})$.

DEFINITION. Let $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ be two elements of $\mathcal{A}_k(\Omega, \mathfrak{A})$, and suppose that $\psi : \mathfrak{X} \to \mathfrak{Y}$ is a given bijection with $\psi(1) = 1$.

(i) The pairs $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ are called *unitarily equivalent* if there exists a unitary $u \in \mathfrak{A}$ such that

$$uP(\cdot)u^* = Q(\cdot), \tag{3.12}$$

and

$$uxu^* = \psi(x), \quad x \in \mathfrak{X}. \tag{3.13}$$

(ii) Assume $\omega \in \Omega$ and $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$ are fixed. We say that $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ have *order of contact* α *at* ω if there exists a unitary $v \in \mathfrak{A}$ such that

$$v\overline{\mathrm{D}}^J P(\omega)y^* x\mathrm{D}^I P(\omega)v^* = \overline{\mathrm{D}}^J Q(\omega)\psi(y)^*\psi(x)\mathrm{D}^I Q(\omega), \tag{3.14}$$

for any $x, y \in \mathfrak{X}$ and all $I, J \in \mathbb{Z}_+^m$ satisfying $|I|, |J| \leqslant \alpha$.

Clearly two unitarily equivalent pairs have order of contact α at ω for any α and every ω. A very nice feature of the Cowen-Douglas theory is that the finite order of contact $\alpha = k$ at every point suffices to reach the unitary equivalence, in case both $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ are in a general position and the algebra $\mathfrak{A}$ shares a specific property with algebras of Hilbert space operators. In this regard we may conclude that the set $\mathfrak{G}_\omega^k(P, \mathfrak{X})$ defined as in (3.8) yields a complete collection of unitary invariants of an element $(P, \mathfrak{X})$ of $\mathcal{A}_k(\Omega, \mathfrak{A})$.

3.9. It remains of course to introduce the required property of $\mathfrak{A}$ and to explain how it works. We begin with some conventions.

We will say that a set $\mathfrak{S} \subset \mathfrak{A}$ is a *separating subset* of $\mathfrak{A}$ if

$$\{a \in \mathfrak{A} : as = 0, \ s \in \mathfrak{S}\} = \{0\}. \tag{3.15}$$

Further, assume $\mathfrak{S}$ and $\mathfrak{T}$ are two separating subsets of $\mathfrak{A}$, and let $\theta : \mathfrak{S} \to \mathfrak{T}$ be a given bijection. We say that θ is *inner*, or *semi-inner*, if there exists a unitary $u \in \mathfrak{A}$ such that

$$usu^* = \theta(s), \quad s \in \mathfrak{S}, \tag{3.16}$$

or, respectively, if there exists a unitary $v \in \mathfrak{A}$ such that

$$vt^*sv^* = \theta(t)^*\theta(s), \quad s, t \in \mathfrak{S}. \tag{3.17}$$

Finally, the algebra $\mathfrak{A}$ is said to be *inner* whenever each semi-inner bijection between two separating subsets of $\mathfrak{A}$ is inner.

It is a very elementary exercise to check that the algebra $\mathcal{L}(\mathcal{H})$ is inner. At the same time the innerness of a C^*-algebra allows us to use condition (iii) in Definition 3.6 at its full extent.

3.10. We conclude this section by stating *the congruence theorem* for the class $\mathcal{A}_k(\Omega, \mathfrak{A})$. Earlier versions of this theorem, stated and proved only for the algebra $\mathcal{L}(\mathcal{H})$, may be found in [8], [10], [14] and [15].

THEOREM. *Let $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ be two elements of the class $\mathcal{A}_k(\Omega, \mathfrak{A})$ in general position, and assume that $\mathfrak{A}$ is an inner algebra. The following two conditions are equivalent:*

(i) *$(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ are unitarily equivalent;*

(ii) *$(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ have order of contact k at every point $\omega \in \Omega$.*

There is no need to proceed with a proof of this theorem, because in the next section we are going to formulate and prove a more general result. We will confine our attention for a while to a few prerequisites. All the specific technicalities are handled along the pattern already displayed in Section 3.

4. A GENERALIZATION OF THE CONGRUENCE THEOREM

4.1. The title of this section and the brief comment made just after Proposition 3.5 give a fairly good description of our next goal. We want to extent Theorem 3.10 to the case of holomorphic maps into a flag manifold. The task is not at all so obvious as one might first think. At first glance the second half of Section 2 could be somewhat misleading. In order to understand the structure of a holomorphic map $P = (P_1, P_2, \ldots, P_n) : \Omega \to \mathcal{P}_n(\mathfrak{A})$, we must study the joint behavior of its components. This will be made clear shortly. Throughout this section $\mathfrak{A}$ and Ω are the same as in Section 3.

4.2. Our first objective is to find a suitable extension of Definition 3.6. To this end we let $P = (P_1, P_2, \ldots, P_n) : \Omega \to \mathcal{P}_n(\mathfrak{A})$ be a given holomorphic map and assume that $\mathfrak{X} \subseteq \mathfrak{A}$ is a fixed subset containing the unit of $\mathfrak{A}$. We let $\langle\alpha\rangle = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote an n-tuple whose entries are from $\mathbb{Z}_+ \cup \{\infty\}$, and let $1 \leqslant l \leqslant n$ be a given integer. For every point $\omega \in \Omega$ and any $\langle\alpha\rangle$ and l as above we next introduce the set

$$ {}^l\mathfrak{S}_\omega^{\langle\alpha\rangle}(P, \mathfrak{X}) = \{\overline{\mathrm{D}}^J P_j(\omega) y^* x \mathrm{D}^I P_i(\omega) : 1 \leqslant i, j \leqslant l,\ |I| \leqslant \alpha_i,\ |J| \leqslant \alpha_j,\ x, y \in \mathfrak{X}\}, $$

and denote by ${}^l\mathfrak{A}_\omega^{\langle\alpha\rangle} = {}^l\mathfrak{A}_\omega^{\langle\alpha\rangle}(P,\mathfrak{X})$ the closure of the $*$-subalgebra of $\mathfrak{A}$ generated by ${}^l\mathfrak{G}_\omega^{\langle\alpha\rangle}(P,\mathfrak{X})$. If $\langle\infty\rangle = (\infty,\infty,\dots,\infty)$, then each ${}^l\mathfrak{A}_\omega^{\langle\alpha\rangle}$ is a $*$-subalgebra of ${}^l\mathfrak{A}_\omega^{\langle\infty\rangle}$, and $P_l(\omega)$ is the common unit of all theses algebras.

It seems to be a right moment to point out that ${}^l\mathfrak{A}_\omega^{\langle\alpha\rangle}$ is in general larger than the algebra $\mathfrak{A}_\omega^{\alpha_l}(P_l,\mathfrak{X})$. The former depends on all the first l components of P and that is exactly what we need. Moreover, let us observe that

$$
{}^1\mathfrak{A}_\omega^{\langle\infty\rangle} \subseteq {}^2\mathfrak{A}_\omega^{\langle\infty\rangle} \subseteq \cdots \subseteq {}^n\mathfrak{A}_\omega^{\langle\infty\rangle}, \quad \omega\in\Omega. \tag{4.1}
$$

Thus, the pair $(P,\mathfrak{X})$ determines at each point $\omega\in\Omega$ a chain of C^*-algebras of length n. Alternatively, we may think of this chain as a filtered C^*-algebra. Of course, the algebra we refer to is ${}^n\mathfrak{A}_\omega^{\langle\infty\rangle}$, and the filtration is provided by (4.1). It should be noticed that this specific filtration is more refined than another one that also is at hand. Specifically, ${}^l\mathfrak{A}_\omega^{\langle\infty\rangle}$ is in general smaller than the compressed algebra $P_l(\omega){}^n\mathfrak{A}_\omega^{\langle\infty\rangle}P_l(\omega)$, for any $l\leqslant n-1$.

Our search for an extension of Definition 3.6 is now over.

DEFINITION. Let $\langle k\rangle = (k_1,k_2,\dots,k_n)$ be an n-tuple of positive integers. A pair $(P,\mathfrak{X})$ as above is said to be in the *class* $\mathcal{A}_{\langle k\rangle}(\Omega,\mathfrak{A})$ provided that the following two conditions are satisfied:

(i) ${}^n\mathfrak{A}_\omega^{\langle\infty\rangle}$ is a finite-dimensional C^*-algebra for each $\omega\in\Omega$;

(ii) if k_ω^l denotes the cardinal of any maximal collection of mutually orthogonal minimal projections in ${}^l\mathfrak{A}_\omega^{\langle\infty\rangle}$, then

$$
k_\omega^1 \leqslant k_1 \quad\text{and}\quad k_\omega^{l+1} \leqslant k_\omega^l + k_{l+1}, \quad 1\leqslant l\leqslant n-1,\ \omega\in\Omega. \tag{4.2}
$$

The pair $(P,\mathfrak{X})$ is said to be in a *general position* in case the next additional condition is fulfilled:

(iii) if $a\in\mathfrak{A}$ and $aP_n(\omega)=0$ for all $\omega\in\Omega$, then $a=0$.

4.3. The versatility of the last definition is convincingly supported by the next straightforward generalization of Theorem 3.7. Before formulating it, we need a new notation, namely,

$$
{}^l\Gamma^{\langle\alpha\rangle}(\Lambda;P,\mathfrak{X}) = \{A\in C^\infty(\Lambda,\mathfrak{A}) : A(\omega)\in {}^l\mathfrak{A}_\omega^{\langle\alpha\rangle},\ \omega\in\Omega\}.
$$

THEOREM. *Suppose $(P,\mathfrak{X})$ is a pair in $\mathcal{A}_{\langle k\rangle}(\Omega,\mathfrak{A})$. Then there exists a non-empty open set $\Lambda\subset\Omega$ such that:*

(i) ${}^l\mathfrak{A}_\alpha^{\langle k\rangle} = {}^l\mathfrak{A}_\alpha^{\langle\infty\rangle}$ *for any $\lambda\in\Lambda$ and any $1\leqslant l\leqslant n$;*

(ii) *the set $\{(\lambda,a) : \lambda\in\Lambda,\ a\in {}^l\mathfrak{A}_\alpha^{\langle\infty\rangle}\}$ is the total space of a smooth $*$-algebra bundle over Λ, for each $1\leqslant l\leqslant n$;*

(iii) *if* $\Phi : {}^n\Gamma^{\langle\infty\rangle}(\Lambda; P, \mathfrak{X}) \to C^\infty(\Lambda, \mathfrak{A})$ *is a* $C^\infty(\Lambda)$*-linear* $*$*-algebra homomorphism such that*

$$\Phi(P_i(\mathrm{D}^I\overline{\mathrm{D}}^J A)P_j) = \Phi(P_i)[\mathrm{D}^I\overline{\mathrm{D}}^J\Phi(A)]\Phi(P_j), \tag{4.3}$$

for all $1 \leqslant i, j \leqslant n$, *any* $I, J \in \mathbb{Z}_+^m$ *with* $|I|, |J| \leqslant 1$, *and every* $A \in {}^l\Gamma^{\langle\alpha\rangle}(\Lambda; P, \mathfrak{X})$, *where* $l = \max\{i, j\}$ *and* $\langle\alpha\rangle = (k_1, \ldots, k_{l-1}, k_l - 1, 0, \ldots, 0)$, *then*

$$\Phi(P_i(\mathrm{D}^I\overline{\mathrm{D}}^J A)P_j) = \Phi(P_i)[\mathrm{D}^I\overline{\mathrm{D}}^J\Phi(A)]\Phi(P_j), \tag{4.4}$$

for all $1 \leqslant i, j \leqslant n$, *any* $I, J \in \mathbb{Z}_+^m$, *and every* $A \in {}^n\Gamma^{\langle\infty\rangle}(\Lambda; P, \mathfrak{X})$.

The proof amounts to the construction of a well-behaved matrix unit system for the chain (4.1) that depends smoothly on ω if ω is a point in a conveniently chosen subset $\Lambda \subset \Omega$. For some pertinent details on parameter-free constructions of this kind and on their role in handling chains of finite-dimensional C^*-algebras we may refer to [20], Section 1, or to some sources indicated there. However, since we deal with a field of chains, we need a parametrized version of the construction of a matrix unit system. Fortunately, because the chains under consideration have finite length, the standard approach employed in the usual situation without parameters can be pursued in its essential features, by merely shrinking the open set Ω several times.

More precisely, we first select $\Lambda_1 \subset \Omega$ and a collection of partial isometrics in ${}^1\Gamma^{\langle\infty\rangle}(\Lambda_1; P, \mathfrak{X}) = \Gamma^\infty(\Lambda_1; P_1, \mathfrak{X})$ whose values at every point $\lambda \in \Lambda_1$ yield a matrix unit system for ${}^1\mathfrak{A}_\lambda^{\langle\infty\rangle}(P, \mathfrak{X}) = \mathfrak{A}_\lambda^\infty(P_1, \mathfrak{X})$. This initial step is the same as in the proof of Theorem 3.7 for the pair $(P_1, \mathfrak{X}) \in \mathcal{A}_{k_1}(\Omega, \mathfrak{A})$. Next we select a subset $\Lambda_2 \subset \Lambda_1$ and a collection of partial isometries in ${}^2\Gamma^{\langle\infty\rangle}(\Lambda_2; P, \mathfrak{X})$ whose values at every point $\lambda \in \Lambda_2$ yield a matrix unit system for ${}^2\mathfrak{A}_\lambda^{\langle\infty\rangle}(P, \mathfrak{X})$, for each $\lambda \in \Lambda_2$. This collection is constructed such that the values at each $\lambda \in \Lambda_2$ of the previously defined partial isometries in ${}^1\Gamma^{\langle\infty\rangle}(\Lambda_1; P, \mathfrak{X})$ are sums of matrix units in this new system. The procedure has an obvious continuation that finally yields a set $\Lambda_n = \Lambda$ and a collection of partial isometries in ${}^n\Gamma^{\langle\infty\rangle}(\Lambda; P, \mathfrak{X})$. The main point is that along this recurrent process it suffices to use only the maps $\overline{\mathrm{D}}^J P_j(\,\cdot\,)y^*x\mathrm{D}^I P_i(\,\cdot\,)$, where $|I| \leqslant k_i$, $|J| \leqslant k_j$, and $x, y \in \mathfrak{X}$. To be more specific, the whole construction requires n steps, the l-th steps involves only the first l components of P, and the transition to the next step is completely accomplished by employing the component P_{l+1} and its derivatives $\mathrm{D}^I P_{l+1}$ of order $|I| \leqslant k_{l+1}$.

As soon as the sketched above process ends up, the complete proof of Theorem 4.3 is just a matter of careful bookkeeping.

4.4. Although by now everything should be clear, for convenience we include below the natural counterpart of Definition 3.8.

DEFINITION. Let $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ be two elements of $\mathcal{A}_{\langle k\rangle}(\Omega, \mathfrak{A})$, where $P = (P_1, P_2, \ldots, P_n)$ and $Q = (Q_1, Q_2, \ldots, Q_n)$, and suppose that $\psi : \mathfrak{X} \to \mathfrak{Y}$ is a given bijection with $\psi(1) = 1$.

(i) The pairs $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ are called *unitarily equivalent* if there exists a unitary $u \in \mathfrak{A}$ such that

$$uP_i(\cdot)u^* = Q_i(\cdot), \quad 1 \leqslant i \leqslant n, \tag{4.5}$$

and

$$uxu^* = \psi(x), \quad x \in \mathfrak{X}. \tag{4.6}$$

(ii) Assume $\omega \in \Omega$ and $\langle \alpha \rangle = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ are fixed. We say that $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ have *order of contact* α at ω if there exists a unitary $v \in \mathfrak{A}$ such that

$$v\overline{\mathrm{D}}^J P_j(\omega)y^*x\mathrm{D}^I P_i(\omega)v^* = \overline{\mathrm{D}}^J Q_j(\omega)\psi(y)^*\psi(x)\mathrm{D}^I Q_i(\omega), \tag{4.7}$$

for any $x, y \in \mathfrak{X}$, every $1 \leqslant i, j \leqslant n$, and all $I, J \in \mathbb{Z}_+^m$ satisfying $|I| \leqslant \alpha_i$, $|J| \leqslant \alpha_j$.

4.5. The following result ends our search for a generalization of the congruence theorem.

THEOREM. *Let $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ be two elements of the class $\mathcal{A}_{\langle k\rangle}(\Omega, \mathfrak{A})$ in general position, and assume that $\mathfrak{A}$ is an inner algebra. The following two conditions are equivalent:*

(i) *$(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ are unitarily equivalent;*

(ii) *$(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ have order of contact $\langle k \rangle$ at every point $\omega \in \Omega$.*

The main ingredient of the proof is provided by Theorem 4.3 above. In fact, that theorem lies at the very heart of the understanding of the inner structure inherent to any element of $\mathcal{A}_{\langle k\rangle}(\Omega, \mathfrak{A})$. The congruence theorem simply converts Theorem 4.3 into a tool.

Proof. All we have to show is that condition (ii) implies condition (i).

Condition (ii) asserts that for each $\omega \in \Omega$ there exists a unitary $v_\omega \in \mathfrak{A}$ such that

$$v_\omega\overline{\mathrm{D}}^J P_j(\omega)y^*x\mathrm{D}^I P_i(\omega)v_\omega^* = \overline{\mathrm{D}}^J Q_j(\omega)\psi(y)^*\psi(x)\mathrm{D}^I Q_i(\omega), \tag{4.8}$$

for any $x, y \in \mathfrak{X}$, every $1 \leqslant i, j \leqslant n$, and all $I, J \in \mathbb{Z}_+^m$ satisfying $|I| \leqslant k_i$, $|J| \leqslant k_j$. By ψ we denote a prescribed bijection from $\mathfrak{X}$ to $\mathfrak{Y}$, as in Definition 4.4.

Let $\Lambda \subset \Omega$ be the open set provided by Theorem 4.3. We define $\Phi : \Gamma^{(\infty)}(\Lambda; P, \mathfrak{X}) \to C^\infty(\Lambda, \mathfrak{A})$ by

$$\Phi(A)(\lambda) = v_\lambda A(\lambda) v_\lambda^*, \quad A \in \Gamma^{(\infty)}(\Lambda; P, \mathfrak{X}),\ \lambda \in \Lambda. \tag{4.9}$$

Although we do not assume that v_λ depends smoothly on $\lambda \in \Lambda$, however, assertions (i) and (ii) in Theorem 4.3 imply that Φ is a well-defined $C^\infty(\Lambda)$-linear $*$-algebra homomorphism. Moreover, by (4.8) and based on some remarks made in Subsection 2.10 we get that Φ satisfies (4.3). Therefore, Φ also satisfies the stronger condition (4.4). Since $\Phi(P) = Q$, from (4.4) we have

$$v_\lambda \overline{\mathrm{D}}^J P_j(\lambda) y^* x \mathrm{D}^I P_i(\lambda) v_\lambda^* = \overline{\mathrm{D}}^J Q_j(\lambda) \psi(y)^* \psi(x) \mathrm{D}^I Q_i(\lambda), \tag{4.10}$$

for each $\lambda \in \Lambda$, any $x, y \in \mathfrak{X}$, and $1 \leqslant i, j \leqslant n$, and all $I, J \in \mathbb{Z}_+^m$.

We next choose a point $\omega \in \Lambda$ and consider the sets

$$\mathfrak{S} = \{x \mathrm{D}^I P_i(\omega) : x \in \mathfrak{X},\ 1 \leqslant i \leqslant n,\ I \in \mathbb{Z}_+^m\},$$

and

$$\mathfrak{T} = \{\psi(x) \mathrm{D}^I Q_I(\omega) : x \in \mathfrak{X},\ 1 \leqslant i \leqslant n,\ I \in \mathbb{Z}_+^m\}.$$

The reasonings developed at the end of Section 2 yield the important conclusion that condition (iii) in Definition 4.2 implies that $\mathfrak{S}$ is a separating subset of $\mathfrak{A}$. This is the place where we really take advantage of the specific properties of a holomorphic map into the flag manifold $\mathcal{P}_n(\mathfrak{A})$. Since $\mathfrak{T}$ alike is a separating subset of $\mathfrak{A}$, by (4.10) we deduce that the map $\theta : \mathfrak{S} \to \mathfrak{T}$ given by

$$\theta(x \mathrm{D}^I P_i(\omega)) = \psi(x) \mathrm{D}^I Q_i(\omega),$$

is a semi-inner bijection. It is the right time to employ the innerness of $\mathfrak{A}$. Consequently, there exists a unitary $u \in \mathfrak{A}$ such that

$$u x \mathrm{D}^I P_i(\omega) u^* = \psi(x) \mathrm{D}^I Q_i(\omega), \tag{4.11}$$

for any $x \in \mathfrak{X}$, every $1 \leqslant i \leqslant n$, and all $I \in \mathbb{Z}_+^m$.

If we take $x = 1$, then (4.11) and Proposition 2.12 show that

$$u P_i(\,\cdot\,) u^* = Q_i(\,\cdot\,), \quad 1 \leqslant i \leqslant n, \tag{4.12}$$

hence condition (4.5) holds true. On the other hand, let us notice that from (4.11) and (4.12) we have

$$[ux - \psi(x) u^*] \mathrm{D}^I P_n(\omega) = 0, \tag{4.13}$$

for any $x \in \mathfrak{X}$ and all $I \in \mathbb{Z}_+^m$. Since $\{\mathrm{D}^I P_n(\omega) : I \in \mathbb{Z}_+^m\}$ is also a separating subset of $\mathfrak{A}$ we get

$$ux = \psi(x) u^*, \quad x \in \mathfrak{X}, \tag{4.14}$$

a relation equivalent to (4.6).

The proof of the congruence theorem is complete. ∎

5. OPERATORS WITH THE HOLOMORPHIC SPANNING PROPERTY

5.1. In the next section we will examine a multidimensional analog of the class of operators singled out by M.J. Cowen ([7]). More precisely, we are going to derive a generalization of [7], Theorem 0.5 from the just proved congruence theorem for holomorphic maps into a flag manifold. Our intention is to explain the meaning of the order of contact in a geometric situation. We should first establish the setting.

Throughout this section we will let $\mathfrak{A}$ denote the C^*-algebra $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a fixed separable infinite dimensional Hilbert space. In addition, we assume that Ω' and Ω'' are two bounded and connected open subsets of $\mathbb{C}^m$. Given a commuting m-tuple $(a_1, a_2, \ldots, a_m)$ of operators on $\mathcal{H}$ and two points $\omega' = (\omega_1', \omega_2', \ldots, \omega_m') \in \Omega'$ and $\omega'' = (\omega_1'', \omega_2'', \ldots, \omega_m'') \in \Omega''$, we let $D'(\omega') : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^m$ and $D''(\omega'') : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^m$ denote the operators defined by

$$D'(\omega')\xi = (\omega_1' - a_1)\xi \otimes e_1 + (\omega_2' - a_2)\xi \otimes e_2 + \cdots + (\omega_m' - a_m)\xi \otimes e_m, \quad \xi \in \mathcal{H},$$

and

$$D''(\omega'')\xi = (\omega_1'' - a_1^*)\xi \otimes e_1 + (\omega_2'' - a_2^*)\xi \otimes e_2 + \cdots + (\omega_m'' - a_m^*)\xi \otimes e_m, \quad \xi \in \mathcal{H},$$

where $\{e_1, e_2, \ldots, e_m\}$ is the standard basis for $\mathbb{C}^m$.

Finally, let k' and k'' be two given positive integers. The next definition is a straightforward extension of [7], Definition 0.2.2.

DEFINITION. A commuting m-tuple $\mathfrak{a} = (a_1, a_2, \ldots, a_m)$ of operators on $\mathcal{H}$ is said to be in the *class* $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ whenever the following condition are satisfied:

(i) the operators $D'(\omega')$ and $D''(\omega'')$ have closed range, for any $\omega' \in \Omega'$ and $\omega'' \in \Omega''$;

(ii) $\dim \ker D'(\omega') = k'$ and $\dim \ker D''(\omega'') = k''$, for every $\omega' \in \Omega'$ and $\omega'' \in \Omega''$;

(iii) $\vee[\{\ker D'(\omega') : \omega' \in \Omega'\} \cup \{\ker D''(\omega'') : \omega'' \in \Omega''\}] = \mathcal{H}$.

5.2. We follow [7] in saying that the elements of $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ are tuples with the *holomorphic spanning property*. The terminology is motivated by the very specific behavior of two projection-valued maps natually associated to such tuples. Explicitly, assume $\mathfrak{A}$ is a tuple in $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ and let $P' : \Omega' \to \mathcal{P}(\mathfrak{A})$ and $P'' : \Omega'' \to \mathcal{P}(\mathfrak{A})$ be the maps defined by

$$(5.1) \qquad P'(\omega') = \kappa(D'(\omega')), \quad P''(\omega'') = \kappa(D''(\omega'')), \quad (\omega', \omega'') \in \Omega' \times \Omega''.$$

From conditions (i) and (ii) in Definition 5.1 and based on Proposition 1.7 we easily get that both P' and P'' are continuous. Moreover, since D' and D'' are holomorphic operator-valued maps, we may use Theorem 2.7 and an obvious adjustment of the computations made in Subsection 1.8 to conclude that P' and P'' are in fact holomorphic maps into the Grassmann manifold of $\mathfrak{A}$.

Let us also notice that the operators $a_1, a_2, \ldots, a_m$ and the projections $P'(\omega')$ and $P''(\omega'')$ interact in a peculiar way. If $\omega' = (\omega'_1, \omega'_2, \ldots, \omega'_m) \in \Omega'$ and $\omega'' = (\omega''_1, \omega''_2, \ldots, \omega''_m) \in \Omega''$ are given, then

$$(5.2) \qquad a_j P'(\omega') = \omega'_j P'(\omega'), \quad 1 \leqslant j \leqslant m,$$

and

$$(5.3) \qquad a_j^* P''(\omega'') = \omega''_j P''(\omega''), \quad 1 \leqslant j \leqslant m.$$

In particular, the obvious relation $[a_j P'(\omega')]^* P''(\omega'') = P'(\omega')[a_j^* P''(\omega'')]$ yields the equality $(\overline{\omega}'_j - \omega''_j) P'(\omega') P''(\omega'') = 0$. Thus we get

$$(5.4) \qquad P'(\omega') P''(\omega'') = 0, \quad (\omega', \omega'') \in \Omega' \times \Omega'',$$

at least in case there exists $1 \leqslant j \leqslant m$ such that $\overline{\omega}'_j \neq \omega''_j$. Since the two maps P' and P'' are continuous we easily deduce that (5.4) holds for arbitrary points $\omega' \in \Omega'$ and $\omega'' \in \Omega''$.

Relation (5.4) has some nice consequences. First, it implies that the subspaces

$$(5.5) \qquad \mathcal{H}' = \vee\{\ker D'(\omega') : \omega' \in \Omega'\},$$

and

$$(5.6) \qquad \mathcal{H}'' = \vee\{\ker D''(\omega'') : \omega'' \in \Omega''\}$$

are orthogonal. This observation enables us to state the spanning property (iii) in Definition 5.1 in a more precise form, namely,

$$(5.7) \qquad \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''.$$

On the other hand, relation (5.4) provides the simple but necessary condition that makes the next construction meaningful. Specifically, let $\Omega = \Omega' \times \Omega''$ and let $P = (P_1, P_2) : \Omega \to \mathcal{P}_2(\mathfrak{A})$ be the map given by

$$(5.8) \qquad P_1(\omega) = P'(\omega'), \quad P_2(\omega) = P'(\omega') + P''(\omega''), \quad \omega = (\omega', \omega'') \in \Omega.$$

We really need (5.4) in checking that P is a well-defined map from Ω into the flag manifold $\mathcal{P}_2(\mathfrak{A})$. Moreover, since the two components P_1 and P_2 of P inherit the special properties of P' and P'', we get that P is a holomorphic map into $\mathcal{P}_2(\mathfrak{A})$.

There is an obvious asymmetry in the just carried out construction. We first employed P', and then P''. Nevertheless, the other way around seems to be legitimate as well. We get something different, of course, but not completely different. The point is that we are allowed to replace $(a_1, a_2, \ldots, a_m)$ by $(a_1^*, a_2^*, \ldots, a_m^*)$. To be more specific, we should observe that $(a_1, a_2, \ldots, a_m)$ is an element of $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ if and only if $(a_1^*, a_2^*, \ldots, a_m^*)$ is an element of $\mathcal{B}_{k'',k'}(\Omega'', \Omega')$. This simple remarks motivate an additional asymmetric condition that we are going to impose. From now on, without any other special mention, we will assume that

$$k' \geqslant k''. \tag{5.9}$$

Under this assumption, the particular choice of an order of preference in the previously done makeup of P gains a better relevance.

5.3. So far we assigned to every m-tuple $(a_1, a_2, \ldots, a_m)$ in $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ a holomorphic map $P : \Omega \to \mathcal{P}_2(\mathfrak{A})$, where, as before, $\Omega = \Omega' \times \Omega''$. A second related item is the set $\mathfrak{X} = \{1, a_1, a_2, \ldots, a_m\}$.

PROPOSITION. *The pair $(P, \mathfrak{X})$ is in the class $\mathcal{A}_{(k)}(\Omega, \mathfrak{A})$, where $(k) = (k', k'')$.*

Proof. We have to check the three conditions in Definition 4.2. To this end, assume $\omega = (\omega', \omega'') \in \Omega$ is fixed, and observe that the compressed algebras $P_1(\omega)\mathfrak{A}P_1(\omega)$ and $P_2(\omega)\mathfrak{A}P_2(\omega)$ coincide with the finite-dimensional C^*-algebras $\mathcal{L}(\operatorname{Range} P'(\omega'))$ and $\mathcal{L}(\operatorname{Range} \mathcal{P}'(\omega') \oplus \operatorname{Range} \mathcal{P}''(\omega''))$, respectively. This remark clearly yields conditions (i) and (ii) in Definition 4.2. Condition (iii) follows from the spanning property (5.7). ∎

5.4. Let us now assume that $\mathfrak{A} = (a_1, a_2, \ldots, a_m)$ and $\mathfrak{b} = (b_1, b_2, \ldots, b_m)$ are two elements of the same class $\mathcal{B}_{k',k''}(\Omega', \Omega'')$. We will say that the two tuples are *unitarily equivalent* whenever there exists a unitary $u \in \mathfrak{A}$ such that

$$u a_i u^* = b_i, \quad 1 \leqslant i \leqslant m. \tag{5.10}$$

We next let $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ denote the pairs in $\mathcal{A}_{(k)}(\Omega, \mathfrak{A})$ corresponding to $\mathfrak{A}$ and $\mathfrak{b}$, respectively. There is a natural bijection $\psi : \mathfrak{X} \to \mathfrak{Y}$, defined by

$$\psi(1) = 1 \quad \text{and} \quad \psi(a_i) = b_i, \quad 1 \leqslant i \leqslant m. \tag{5.11}$$

With all the prerequisites provided by Subsection 5.2 at hand, it is really easy to observe that in terms of these notations the following result holds.

PROPOSITION. *Two tuples* $(a_1, a_2, \ldots, a_m)$ *and* $(b_1, b_2, \ldots, b_m)$ *in* $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ *are unitarily equivalent if and only if the associated pairs* $(P, \mathfrak{X})$ *and* $(Q, \mathfrak{Y})$ *in* $\mathcal{A}_{\langle k\rangle}(\Omega, \mathfrak{A})$ *are unitarily equivalent.*

The last proposition allows us to take full advantage of the congruence theorem for holomorphic maps into a flag manifold. It remains, of course, to explain when two pairs $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ as above have order of contact $\langle k\rangle$ at every point. Some of the notations used below were previously introduced in Section 4.

5.5. We proceed with a brief examination of the $*$-algebras ${}^1\mathfrak{A}_\omega^{\langle k\rangle}$ and ${}^2\mathfrak{A}_\omega^{\langle k\rangle}$ related to the pair $(P, \mathfrak{X})$ constructed as in Subsection 5.2 from a given m-tuple $(a_1, a_2, \ldots, a_m)$ in $\mathcal{B}_{k',k''}(\Omega', \Omega'')$. The point we want to make is that these two algebras are generated by two very simple sets of operators, patently smaller than the sets employed in the general situation analyzed in Section 4. All we have to do is to take into account formulas (5.2) and (5.3).

More precisely, suppose $\omega = (\omega', \omega'') \in \Omega$ is fixed, and let $I = (i_1, i_2, \ldots, i_m)$ be a given tuple in $\mathbb{Z}_+^m$. From (5.2) and (5.3) we get

$$a_j \mathrm{D}^I P'(\omega') = \omega_j' \mathrm{D}^I P'(\omega') + i_j \mathrm{D}^{I(j)} P'(\omega'), \tag{5.12}$$

and, respectively,

$$a_j^* \mathrm{D}^I P''(\omega'') = \omega_j'' \mathrm{D}^I P''(\omega'') + i_j \mathrm{D}^{I(j)} P''(\omega''), \tag{5.13}$$

where $1 \leqslant j \leqslant m$ and $I(j) = (i_1, \ldots, i_{j-1}, \max\{i_j - 1, 0\}, i_{j+1}, \ldots, i_m)$. By taking now the adjoints, we get

$$\overline{\mathrm{D}}^I P'(\omega') a_j^* = \overline{\omega}_j' \overline{\mathrm{D}}^I P'(\omega') + i_j \overline{\mathrm{D}}^{I(j)} P'(\omega'), \tag{5.14}$$

and, respectively,

$$\overline{\mathrm{D}}^I P''(\omega'') a_j = \overline{\omega}_j'' \overline{\mathrm{D}}^I P''(\omega'') + i_j \overline{\mathrm{D}}^{I(j)} P''(\omega''). \tag{5.15}$$

A repeated manipulation of the last four formulas, in conjunction with the general constructions made in Section 4, leads to the next result.

PROPOSITION. *Suppose* $\omega = (\omega', \omega'') \in \Omega$ *is given. Let* $(P, \mathfrak{X})$ *be the element of* $\mathcal{A}_{\langle k\rangle}(\Omega, \mathfrak{A})$ *associated with an* m*-tuple* $(a_1, a_2, \ldots, a_m)$ *in* $\mathcal{B}_{k',k''}(\Omega', \Omega'')$, *and let* ${}^1\mathfrak{A}_\omega^{\langle k\rangle}$ *and* ${}^2\mathfrak{A}_\omega^{\langle k\rangle}$ *be the* $*$*-algebras corresponding to the pair* $(P, \mathfrak{X})$, *where* $\langle k\rangle = (k', k'')$.

(i) *The $*$-algebra ${}^1\mathfrak{A}_\omega^{\langle k\rangle}$ is generated by the set*

$$\{\overline{\mathrm{D}}^J P'(\omega')\mathrm{D}^I P'(\omega') : |I|, |J| \leqslant k'\}. \tag{5.16}$$

(ii) *The $*$-algebra ${}^2\mathfrak{A}_\omega^{\langle k\rangle}$ is generated by the set* (5.16) *above, together with the sets*

$$\{\overline{\mathrm{D}}^J P''(\omega'')\mathrm{D}^I P''(\omega'') : |I|, |J| \leqslant k''\}, \tag{5.17}$$

and

$$\{\overline{\mathrm{D}}^J P'(\omega')a_j\mathrm{D}^I P''(\omega'') : 1 \leqslant j \leqslant m,\ |I| \leqslant k'',\ |J| \leqslant k'\}. \tag{5.18}$$

5.6. We now return to the situation considered in Subsection 5.4. Besides the notations already used there, we let P' and P'' denote the projection-valued maps associated with $(a_1, a_2, \ldots, a_m)$ by formulas (5.1). Their counterparts corresponding to $(b_1, b_2, \ldots, b_m)$ are denoted by Q' and Q''. The next result is basically a consequence of Proposition 5.5. Once again, the details are trustfully left to our reader.

PROPOSITION. *Suppose $\omega = (\omega', \omega'') \in \Omega$ is given. The following two conditions are equivalent:*

(i) *$(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ have order of contact $\langle k\rangle$ at ω;*

(ii) *there exist two partial isometries v'_ω and v''_ω in $\mathfrak{A}$ such that*

$$v'^*_\omega v'_\omega = P'(\omega'), \quad v''^*_\omega v''_\omega = P''(\omega''), \tag{5.19}$$

$$v'_\omega v'^*_\omega = Q'(\omega'), \quad v''_\omega v''^*_\omega = Q''(\omega''), \tag{5.20}$$

$$v'_\omega\overline{\mathrm{D}}^J P'(\omega')\mathrm{D}^I P'(\omega')v'^*_\omega = \overline{\mathrm{D}}^J Q'(\omega')\mathrm{D}^I Q'(\omega'), \quad |I|, |J| \leqslant k', \tag{5.21}$$

$$v''_\omega\overline{\mathrm{D}}^J P''(\omega'')\mathrm{D}^I P''(\omega'')v''^*_\omega = \overline{\mathrm{D}}^J Q''(\omega'')\mathrm{D}^I Q''(\omega''), \quad |I|, |J| \leqslant k'', \tag{5.22}$$

and

$$\begin{gathered} v'_\omega\overline{\mathrm{D}}^J P'(\omega')a_j\mathrm{D}^I P''(\omega'')v''^*_\omega = \overline{\mathrm{D}}^J Q'(\omega')b_j\mathrm{D}^I Q''(\omega''), \\ 1 \leqslant j \leqslant m,\ |I| \leqslant k'',\ |J| \leqslant k'. \end{gathered} \tag{5.23}$$

An obvious advantage of this proposition despite the long list summed up in condition (ii), is that it substantially reduces the number of requirements accompanying condition (i). A less obvious advantage will be explained in a moment, after some preliminaries.

5.7. Assume once more that $\mathfrak{A} = (a_1, a_2, \ldots, a_m)$ in $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ and $\omega = (\omega', \omega'')$ in Ω are given. We next define three subspaces of $\mathcal{H}$ by

$$\mathcal{K}'_\omega(\mathfrak{A}) = \{\xi \in \mathcal{H}' : (\omega' - \mathfrak{A})^J \xi = 0,\ J \in \mathbb{Z}_+^m,\ |J| = k' + 1\}, \tag{5.24}$$

$$\mathcal{K}''_\omega(\mathfrak{A}) = \{\xi \in \mathcal{H}'' : (\omega'' - \mathfrak{A}^*)^J \xi = 0,\ J \in \mathbb{Z}_+^m,\ |J| = k'' + 1\}, \tag{5.25}$$

and

$$\mathcal{K}_\omega(\mathfrak{A}) = \mathcal{K}'_\omega(\mathfrak{A}) \oplus \mathcal{K}''_\omega(\mathfrak{a}), \tag{5.26}$$

where $\mathcal{H}'$ and $\mathcal{H}''$ are given by (5.5) and (5.6), and

$$(\omega' - \mathfrak{A})^J = (\omega'_1 - a_1)^{j_1}(\omega'_2 - a_2)^{j_2} \cdots (\omega'_m - a_m)^{j_m}, \quad J = (j_1, j_2, \ldots, j_m),$$

$$(\omega'' - \mathfrak{A}^*)^J = (\omega''_1 - a_1^*)^{j_1}(\omega''_2 - a_2^*)^{j_2} \cdots (\omega''_m - a_m^*)^{j_m}, \quad J = (j_1, j_2, \ldots, j_m).$$

For a later use, we collect below some properties of these subspaces.

First we notice that a straightforward computation based on formulas (5.12) and (5.13) yields the following relations:

$$(\omega' - \mathfrak{A})^J \mathrm{D}^I P'(\omega') = 0, \quad I, J \in \mathbb{Z}_+^m,\ |I| \leqslant k',\ |J| = k' + 1, \tag{5.27}$$

$$(\omega'' - \mathfrak{A}^*)^J \mathrm{D}^I P''(\omega'') = 0, \quad I, J \in \mathbb{Z}_+^m,\ |I| \leqslant k'',\ |J| = k'' + 1. \tag{5.28}$$

In their turn, these relations show that

$$\vee\{\mathrm{Range}\, \mathrm{D}^I P'(\omega') : I \in \mathbb{Z}_+^m,\ |I| \leqslant k'\} \subseteq \mathcal{K}'_\omega(\mathfrak{A}), \tag{5.29}$$

and, respectively,

$$\vee\{\mathrm{Range}\, \mathrm{D}^I P''(\omega'') : I \in \mathbb{Z}_+^m,\ |I| \leqslant k''\} \subseteq \mathcal{K}''_\omega(\mathfrak{A}). \tag{5.30}$$

As a matter of fact, in each of the last two relations the indicated inclusion is an equality. For a complete proof we refer to [15], Section 4 where the reader can find all the missing details.

The just mentioned facts imply that $\mathcal{K}'_\omega(\mathfrak{A})$ and $\mathcal{K}''_\omega(\mathfrak{A})$ are finite-dimensional subspaces of $\mathcal{H}$. Furthermore, $\mathcal{K}'_\omega(\mathfrak{A})$ is an invariant subspace of each a_j, $1 \leqslant j \leqslant m$, and $\mathcal{K}''_\omega(\mathfrak{A})$ is an invariant subspace of any a_j^*, $1 \leqslant j \leqslant m$. If $1 \leqslant j \leqslant m$ is fixed, we will denote by

$$a_j|\mathcal{K}'_\omega(\mathfrak{A}) : \mathcal{K}'_\omega(\mathfrak{A}) \to \mathcal{K}'_\omega(\mathfrak{A})$$

the restriction of a_j to $\mathcal{K}'_\omega(\mathfrak{A})$, and by

$$a_j^*|\mathcal{K}''_\omega(\mathfrak{A}) : \mathcal{K}''_\omega(\mathfrak{A}) \to \mathcal{K}''_\omega(\mathfrak{A})$$

the restriction of a_j^* to $\mathcal{K}''_\omega(\mathfrak{A})$. In addition, we let

$$\mathcal{K}'_\omega(\mathfrak{A})|a_j|\mathcal{K}''_\omega(\mathfrak{A}) : \mathcal{K}''_\omega(\mathfrak{A}) \to \mathcal{K}'_\omega(\mathfrak{A})$$

denote the operator given by restricting a_j to $\mathcal{K}''_\omega(\mathfrak{a})$ and then projecting onto $\mathcal{K}'_\omega(\mathfrak{A})$.

5.8. We are now ready to use Proposition 5.6 at its full extent. Before stating the next result, it is worth recalling the general setting. We start with two elements $\mathfrak{A} = (a_1, a_2, \ldots, a_m)$ and $\mathfrak{b} = (b_1, b_2, \ldots, b_m)$ of class $\mathcal{B}_{k',k''}(\Omega', \Omega'')$, and then we consider their corresponding pairs $(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ in $\mathcal{A}_{\langle k \rangle}(\Omega, \mathfrak{A})$, where $\langle k \rangle = (k', k'')$ and $\Omega = \Omega' \times \Omega''$.

PROPOSITION. *Suppose $\omega = (\omega', \omega'') \in \Omega$ is given. The following two conditions are equivalent:*

(i) *$(P, \mathfrak{X})$ and $(Q, \mathfrak{Y})$ have the order of contact $\langle k \rangle$ at ω;*

(ii) *there exist two isometries $u'_\omega : \mathcal{K}'_\omega(\mathfrak{A}) \to \mathcal{K}'_\omega(\mathfrak{b})$ and $u''_\omega : \mathcal{K}''_\omega(\mathfrak{A}) \to \mathcal{K}''_\omega(\mathfrak{b})$, such that*

$$u'_\omega[a_j|\mathcal{K}'_\omega(\mathfrak{A})]u'^*_\omega = b_j|\mathcal{K}'_\omega(\mathfrak{b}), \quad 1 \leqslant j \leqslant m, \tag{5.31}$$

$$u''_\omega[a_j^*|\mathcal{K}''_\omega(\mathfrak{A})]u''^*_\omega = b_j^*|\mathcal{K}''_\omega(\mathfrak{b}), \quad 1 \leqslant j \leqslant m, \tag{5.32}$$

and

$$u'_\omega[\mathcal{K}'_\omega(\mathfrak{A})|a_j|\mathcal{K}''_\omega(\mathfrak{A})]u''^*_\omega = \mathcal{K}'_\omega(\mathfrak{b})|b_j|\mathcal{K}''_\omega(\mathfrak{b}), \quad 1 \leqslant j \leqslant m. \tag{5.33}$$

Proof. It will be enough to show that condition (ii) in Proposition 5.6 is equivalent to condition (ii) above. In case relations (5.19)–(5.23) are satisfied, we define u'_ω and u''_ω by

$$u'_\omega[\mathrm{D}^I P'(\omega')\xi] = \mathrm{D}^I Q'(\omega')\xi, \quad \xi \in \mathcal{H},\ I \in \mathbb{Z}^m_+,\ |I| \leqslant k', \tag{5.34}$$

and, respectively,

$$u''_\omega[\mathrm{D}^I P''(\omega'')\xi] = \mathrm{D}^I Q''(\omega'')\xi, \quad \xi \in \mathcal{H},\ I \in \mathbb{Z}_+^m,\ |I| \leqslant k''. \tag{5.35}$$

Relations (5.31)–(5.33) follow by a repeated use of (5.2) and (5.3). It goes without saying that along the way we have to use all the facts mentioned in Subsection 5.7.

The converse is also handled in a straightforward manner. If u'_ω and u''_ω subject to conditions (5.31)–(5.33) are given, then we should first check that they satisfy (5.34) and (5.35) above, and then we define v'_ω and v''_ω by

$$v'_\omega = Q'(\omega')u'_\omega P'(\omega'), \tag{5.36}$$

and, respectively,

$$v''_\omega = Q''(\omega'')u''_\omega P''(\omega''). \tag{5.37}$$

The only additional observation we need in order to verify all the relations (5.19)–(5.23) is that the operators a_j and b_j in (5.23) may be replaced without any harm by $\mathcal{K}'_\omega(\mathfrak{A})|a_j|\mathcal{K}''_\omega(\mathfrak{A})$ and $\mathcal{K}'_\omega(\mathfrak{b})|b_j|\mathcal{K}''_\omega(\mathfrak{b})$, respectively. ∎

5.9. We conclude Section 5 with the following theorem concerning the unitary equivalence of tuples in $\mathcal{B}_{k',k''}(\Omega', \Omega'')$.

THEOREM. *Two m-tuples* $\mathfrak{A}$ *and* $\mathfrak{b}$ *in* $\mathcal{B}_{k',k''}(\Omega', \Omega'')$ *are unitarily equivalent if and only if condition* (ii) *in Proposition* 5.8 *is satisfied.*

Proof. All we need to do is to use Proposition 5.4, Theorem 4.5, and Proposition 5.8. ∎

As it was mentioned at the beginning of Section 5, the last theorem generalizes [7], Theorem 0.5. Yet we would like to believe that its proof sketched above makes a good advertisement for the possible applications of the congruence theorem.

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