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BOOK REVIEW:

Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras, **Michael Demuth, Elmar Schrohe, Bert-Wolfgang Schulze, Johannes Sjöstrand** (Editors), 406 pag., ISBN 3-05-501710-2.

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The Volume 11 in the *"Mathematical Topics"* series of Akademie Verlag, contains a new issue of the "*Advances in Partial Differential Equations*" series, entitled "*Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras*" published in 1996 and edited by Michael Demuth, Elmar Schrohe, Bert-Wolfgang Schulze (editor in chief) and Johannes Sjöstrand. This volume gathers five contributions: *"Recent Results and Open Problems on Schrödinger Operators, Laplace Integrals and Transfer Operators in Large Dimension"* by Bernard Helffer, *"Some Problems on Markov Semigroups"* by V.A. Liskevich and Yu.A. Semenov, *"Wavelet Analysis of Partial Differential Operators"* by Matthias Holschneider, *"On the Index of Elliptic Operators on a Cone"* by B.V. Fedosov and B.-W. Schulze and *"On Composition Series in Algebras of Pseudodifferential Operators and in Algebras of Wiener-Hopf Operators"* by B.A. Plamenevskij and V.N. Senichkin.

The first contribution is a survey by Bernard Helffer presenting some semiclassical and maximum principle techniques used in some problems of statistical mechanics, mainly inspired by a course by M. Kac ("Mathematical Mechanics of Phase Transitions", Brandeis lectures, 1966, Gordon and Breach). The author considers the asymptotic behaviour with respect to large dimension and a small parameter (usually the Plank constant), in three problems: Laplace integrals, Schrödinger operators and transfer operators (also known as Kac operators) in statistical mechanics. One considers the Laplace operator $\Delta^{(m)}$ in $L^2(\mathbb{R}^m)$ and a family of real C^∞ potentials $V^{(m)}$ defined on $\mathbb{R}^m$ and the asymptotic behaviour of the Laplace integral:

$$\frac{1}{m}\left[\ln\left(\left(\frac{1}{h\pi}\right)^{\frac{m}{2}}\int\exp\left(-\frac{V^{(m)}(x)}{h}\right)dx\right)\right]$$

when the dimension m tends to infinity, with a good control for the positive parameter h going to zero. Related to this problem one considers to other problems coming from physics and involving similar asymptotic analysis. First one considers the largest eigenvalue $\mu_1(m,h)$ of the operator:

$$K^{(m)}(h) := \exp\left(-\frac{V^{(m)}(x)}{2}\right)\exp\left(h^2\Delta^{(m)}\right)\exp\left(-\frac{V^{(m)}(x)}{2}\right)$$

and studies the asymptotic behaviour of $-(1/m)\ln\mu_1(m,h)$ as the dimension m tends to infinity with the control of this limit for the positive parameter h going to zero. Secondly one considers the smallest eigenvalue $\lambda_1(m,h)$ of the Schrödinger operator:

$$S^{(m)}(h) := -h^2\Delta^{(m)} + V^{(m)}(x)$$

and studies the asymptotic behaviour of $(1/m)\lambda_1(m,h)$ as the dimension m goes to infinity with the control of this limit for the positive parameter h going to zero. A list of some other related problems is given in the paragraph 1.2 of the paper. The second section of the review presents five models from statistical mechanics (the Curie-Weiss, the 1 and 2 dimensional Ising models, the Kac model A and the multiexponential model) and one simple model from quantum mechanics, all of them leading to some asymptotic estimations related to those stated above. Section 3 is devoted to a short review of the Laplace method and its relation with the Brascamp-Lieb and the FKG inequalities as well with the semiclassical expansion of the thermodynamic limit. Section 4 is devoted to the analysis of the bottom of the spectrum of the Schrödinger operator:

$$S_V(x, hD_x) := -h^2\Delta + V(x)$$

with V a real C^∞ potential, with a control on the dependence on the positive parameter h. Section 5 gives a review of the results of B. Helffer and J. Sjöstrand concerning the former Schrödinger operator in large dimension. The results are compared with many other related studies existing in the literature. Sections 6 and 7 are devoted to the study of the Kac operator.

The second contribution to the volume, by V.A. Liskevich and Yu.A. Semenov, presents a perturbation theory for generators of Markov semigroups in L^p

spaces. For the case of symmetric Markov semigroups the authors extend to the L^p case the well known quadratic form procedures from the Hilbert space situation; they characterize the domain of the perturbed generator by means of the domain of an associated quadratic form, defined with a semi-inner product in L^p (as defined in Section 1 of the paper) and they obtain an extension of the well known KLMN theorem. The characterization obtained for the domains of the generators allow the authors to derive criteria for m-accretive closability.

The third contribution, by Matthias Holschneider, gives a self-contained presentation of the principles of the analysis of partial differential operators through continuous wavelet transforms. The first section provides an overview of wavelet transform analysis. The following three sections are dealing with the problem of giving a precise description for the growth properties of the wavelet transforms of function spaces with given regularity properties and to define spaces of wavelets transforms associated to functions with different regularity properties in different regions. The following three sections are discussing operator algebras on wavelet spaces and their relation with Calderon-Zygmund operators. The last four sections present the relation of these operator algebras with the usual pseudodifferential operators and as an application the author determines the essential spectral radius for a class of Fourier integral operators.

The fourth contribution to the volume, by B.V. Fedosov and B.-W. Schulze considers a class of Mellin pseudodifferential operators on a cone $(X \times \mathbb{R}_+)/(X \times 0)$ with X a smooth compact manifold of dimension n without boundary. The authors define the family of elliptic operators of this type and prove that ellipticity implies the Fredholm property. In Section 2 a regularized trace is defined that is used in Section 3 for defining an algebraical index. One can prove that this algebraical index coincides with the analytical one. The last section applies the general construction to the case of a singular integral operator on the half-line (the cone over a point).

The fifth contribution to the volume, by B.A. Plamenevskij and V.N. Senichkin proves the solvability of two important C^*-algebras of operators: the algebra of Wiener-Hopf operators on a closed convex cone and respectively the algebra of pseudodifferential operators with discontinuous symbols on a smooth compact m-dimensional manifold without boundary. In the second paragraph of the first section the authors discuss the notion of solvability introducing the notion of solvability in the narrow sense and in the second section a sufficient condition for this property is given. The third section is devoted to Wiener-Hopf algebras; one proves a necessary and sufficient criterion for a complete classification of the irreducible representations and for the equivalence of the Jacobson topology and the r-topology; then the solving composition series is constructed. Section 4 proves the solvability of the C^*-algebra of pseudodifferential operators with discontinuous symbols on a smooth compact manifold without boundary; the structure of admissible discontinuities of the symbols is described by means of a stratification of the manifold.

The contributions included in this volume of the "*Advances in Partial Differential Equations*" series present some self-contained and detailed analysis of some very important advances in the fields of semiclassical analysis, pseudodifferential operators, index theory, perturbation of Markov semigroups and wavelet analysis; they are addressing a very large auditory among mathematicians and theoretical physicists and may provide an introduction to some very active and important domains of research.

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