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ON THE DISCRETE SPECTRUM OF SOME SELFADJOINT OPERATOR MATRICES

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Abstract. This paper is devoted to the study of the discrete spectrum of selfadjoint operators, which are generated by symmetric operator matrices of the form

$$\mathbf{L}_0 = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

in the product Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$, where the entries A, B and C are not necessarily bounded operators in the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ or between them, respectively. Under some assumptions all selfadjoint extensions of $\mathbf{L}_0$ in $\mathcal{H}_1 \times \mathcal{H}_2$ are described and the extension $\mathbf{L}$ defined by the given selfadjoint operator C is singled out. General statements on the discrete spectrum of $\mathbf{L}$ and its accumulation points are proved. Special attention is paid to the case that C is bounded.

Keywords: *Selfadjoint operator matrices, discrete spectrum, accumulation points, matrix differential operators.*

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0. INTRODUCTION

The recent paper [4] is devoted to the investigation of linear operators $\mathbf{L}_0$ defined by 2×2 block operator matrices

$$\mathbf{L}_0 = \begin{pmatrix} A & B_1 \\ B_2 & C \end{pmatrix},$$

the entries of which are not necessarily bounded operators in Banach spaces X_1, X_2 or between them, respectively. In [4] sufficient conditions are presented which yield the closability of the operator $\mathbf{L}_0$. Further, it was shown there that under suitable assumptions, e.g. that A has a compact resolvent, the essential spectrum of $\overline{\mathbf{L}_0}$, the closure of $\mathbf{L}_0$, coincides with the essential spectrum of the operator

$$C - B_2\,(A - \mu\, I)^{-1} B_1,$$

acting in the smaller space X_2, where μ is any number in the resolvent set of A.

The investigations in [4] were motivated by various spectral problems in the dynamics of fluid and plasmas; an example of which is treated in [4] as an application of the general theory. In the forthcoming paper [5] this theory is successfully applied to a two-dimensional problem from magnetohydrodynamics, which is described by a nonelliptic system of partial differential operators of mixed order, which is not elliptic even in weaker Douglis–Nirenberg sense.

The present paper as well as [1], [2], [3], [9] and [11] continue the investigations of the [4] article in the case when X_1, X_2 are Hilbert spaces and $\mathbf{L}_0$ is a symmetric operator in the product Hilbert space $X_1 \times X_2$. Its main part deals with the question whether the discrete spectrum of a selfadjoint extension of $\overline{\mathbf{L}_0}$, if this extension exists, has accumulation points at the boundary of its essential spectrum or not.

In Section 1 we state the general assumptions on the operators A, B and C which define the symmetric operator

$$\mathbf{L}_0 = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Then we describe $\overline{\mathbf{L}_0}$, the closure of $\mathbf{L}_0$. Under the condition that $\overline{\mathbf{L}_0}$ is a symmetric operator with non-zero, equal defect numbers, all selfadjoint extensions $\mathbf{L}$ of $\overline{\mathbf{L}_0}$ are characterized. Let $\mathbf{L}$ be such a selfadjoint extension. Section 2 contains some general statements on the discrete spectrum of $\mathbf{L}$ and its accumulation points. The existence of joint eigenvalues of $\mathbf{L}$ and A is also discussed here. In Section 3 we additionally assume that C is a bounded (selfadjoint) operator which implies that the essential spectrum of $\mathbf{L}$ is a bounded (compact) subset of $\mathbb{R}$. Let δ be the upper (lower) bound of some interval which does not contain points of the essential spectrum of $\mathbf{L}$. Necessary and sufficient conditions are presented under which the point δ is an accumulation point from the left (from the right) of the discrete spectrum of $\mathbf{L}$. Finally, in Section 4 the foregoing stated results are applied to an $(n+1)\times(n+1)$-system of ordinary differential operators of mixed orders occuring in magnetohydrodynamics.

1. THE CLOSURE OF L_0 AND ITS SELFADJOINT EXTENSIONS

First of all we introduce some notations. For a densely defined closed operator S in a Hilbert space $\mathcal{H}$ we denote its kernel by $\mathcal{N}(S)$ and its range by $\mathcal{R}(S)$. We define

$$\operatorname{nul}(S) := \dim \mathcal{N}(S), \quad \operatorname{def}(S) := \operatorname{codim} \mathcal{R}(S),$$

these being finite numbers or ∞, and, if both are finite,

$$\operatorname{ind}(S) := \operatorname{nul}(S) - \operatorname{def}(S)$$

(see [6], [8]). Recall that S is called *Fredholm* if $\operatorname{nul}(S) < \infty$ and $\operatorname{def}(S) < \infty$. If S is bijective, i.e., $\mathcal{N}(S) = \{0\}$ and $\mathcal{R}(S) = \mathcal{H}$, and if S^{-1} is bounded, we say that S has a bounded inverse. Let T be an operator function, which is defined on the open subset $\Omega \subset \mathbb{C}$ and has values in the set of linear closed operators in a Hilbert space. We call

$$\rho(T) := \{z \in \Omega \mid T(z) \text{ has a bounded inverse}\}$$

the *resolvent set* of T and

$$\sigma(T) := \Omega \setminus \rho(T)$$

the *spectrum* of T. The *point spectrum* of T is defined by

$$\sigma_{\mathrm{p}}(T) := \{z \in \Omega \mid T(z) \text{ not injective}\}.$$

An element $z \in \sigma_{\mathrm{p}}(T)$ is called an *eigenvalue* of T and a vector $x \neq 0$ which belongs to the kernel of the operator $T(z)$ is called an *eigenvector* of T for the eigenvalue z. The dimension of the kernel of the operator $T(z)$ is called the *(geometric) multiplicity* of the eigenvalue z. The set

$$\sigma_{\mathrm{ess}}(T) := \{z \in \Omega \mid T(z) \text{ not Fredholm}\}$$

is called the *essential spectrum* of T. If S is a linear closed operator in a Hilbert space, we associate with S the operator function $S - z\,I$ which is defined on $\mathbb{C}$. By $\rho(S)$, $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$ and $\sigma_{\mathrm{ess}}(S)$, respectively, we denote the resolvent set, the spectrum, point spectrum and the essential spectrum of this operator function, respectively. Accordingly, we can speak of eigenvalues, eigenvectors and multiplicities of eigenvalues of the operator S.

An operator function T of the form $T(z) = T_0 + T_1(z)$, $z \in \Omega$, where T_0 is a linear closed operator and the operators $T_1(z)$, $z \in \Omega$, are (everywhere defined) bounded operators, is called holomorphic if the operator function T_1 is holomorphic.

If T is a selfadjoint operator in a Hilbert space and α is a real number, then $T > \alpha$ means that $\langle T\,x\,, x\,\rangle > \alpha\,\langle\, x\,, x\,\rangle$ for $x \in \mathcal{D}(T)$, $x \neq 0$. Finally, the notation $T \gg \beta$ means that there exists a real number $\gamma > \beta$ such that $T \geqslant \gamma$, i.e., $\langle T\,x\,, x\,\rangle \geqslant \gamma\,\langle\, x\,, x\,\rangle$ for $x \in \mathcal{D}(T)$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. In the product space $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$ (endowed with the scalar product $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}} = \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_1} + \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_2}$) we consider a symmetric operator which is formally defined by a 2×2 block operator matrix of the form

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{1.1}$$

In the following it is always assumed that the entries of the block operator matrix satisfy the following conditions:

(a) A is a selfadjoint, strictly positive operator in $\mathcal{H}_1$ (i.e., $A \gg 0$) such that A^{-1} is compact;

(b) B is a densely defined closed operator from $\mathcal{H}_2$ into $\mathcal{H}_1$;

(c) the adjoint operator B^* of B is densely defined and $\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B^*)$;

(d) C is a linear operator in $\mathcal{H}_2$ such that $\mathcal{D}(C) \supset \mathcal{D}(B)$ and $C_B := C|\mathcal{D}(B)$ is a symmetric operator.

The assumptions on A imply that the spectrum of A, $\sigma(A)$, consists of positive isolated eigenvalues with finite multiplicities. The only possible accumulation point of $\sigma(A)$ is $+\infty$.

With the matrix (1.1) we associate the following operator $\mathbf{L}_0$:

$$\mathbf{L}_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} A\,x_1 + B\,x_2 \\ B^* x_1 + C\,x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathbf{L}_0) := \mathcal{D}(A) \times \mathcal{D}(B). \tag{1.2}$$

Evidently, $\mathbf{L}_0$ is a densely defined symmetric operator in $\mathcal{H}$.

We start with a description of the closure $\overline{\mathbf{L}_0}$ of $\mathbf{L}_0$. The assumption ($\mathcal{D}(A) \subset$) $\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B^*)$ implies that for each $z \in \rho(A)$ the operator

$$G(z) := B^*(A - z\,I)^{-1} \tag{1.3}$$

is defined and bounded on $\mathcal{H}_1$ (which follows immediately from the Closed Graph Theorem). The resolvent equation for A yields that G is a holomorphic operator function on $\rho(A)$, and $G'(z) = G(z)\,(A - z\,I)^{-1}$. The adjoint operator of $G(\bar z)$ is the continuous extension of the operator $(A - z\,I)^{-1}B$, i.e.,

$$G(\bar z)^* = \overline{(A - z\,I)^{-1}B}.$$

For convenience we introduce the operator

$$J := B^* A^{-1/2}, \tag{1.4}$$

which is defined and bounded on $\mathcal{H}_1$. Its adjoint operator is the continuous extension of $A^{-1/2}B$, i.e.,

$$J^* = \overline{A^{-1/2}B}.$$

Lemma 1.1. *Let the assumptions* (a–c) *be satisfied. Then for $z \in \rho(A)$ the operator $B^*(A - z\,I)^{-1}B$, which is defined on $\mathcal{D}(B)$, has a continuous extension $H(z) = \overline{B^*(A - z\,I)^{-1}B}$. It has the form*

$$H(z) = J\,J^* + z\,J\,(A - z\,I)^{-1}J^* = J\,A\,(A - z\,I)^{-1}J^*. \tag{1.5}$$

H is a holomorphic operator function on $\rho(A)$, which satisfies

$$H(z)^* = H(\bar z), \quad z \in \rho(A); \quad H'(z) \geqslant 0, \quad z \in \rho(A) \cap \mathbb{R}.$$

Moreover,

$$\big\|H(z)\big\| \leqslant \big\|J\big\|^2 \left(1 + \frac{|\,z\,|}{|\,\mathrm{Im}\,(z)\,|}\right), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. The resolvent equation for A implies that

$$\begin{aligned} B^*(A - z\,I)^{-1}B &= B^*A^{-1}B + z\,B^*(A - z\,I)^{-1}A^{-1}B \\ &= (B^*A^{-1/2})(A^{-1/2}B) + z\,(B^*A^{-1/2})(A - z\,I)^{-1}(A^{-1/2}B) \end{aligned}$$

for $z \in \rho(A)$. Taking the closures on both sides we obtain

$$H(z) = \overline{B^*(A - z\,I)^{-1}B} = J\,J^* + z\,J\,(A - z\,I)^{-1}J^*, \quad z \in \rho(A). \quad ∎$$

By assumption (d) C_B is closable and its closure $\overline{C_B}$ is symmetric.

For $z \in \rho(A)$ the operator $C_B - B^*(A - z\,I)^{-1}B$, which is defined on $\mathcal{D}(B)$, is closable, and its closure is given by $\overline{C_B} - H(z)$.

In the sequel the closed operators

$$W(z) := -\overline{C_B} + z\,I + H(z), \quad z \in \rho(A), \tag{1.6}$$

which are defined on $\mathcal{D}(\overline{C_B})$, play an essential role.

PROPOSITION 1.2. *Let the assumptions* (a–d) *be fulfilled and the operator function W be defined by* (1.6). *Then for each $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $W(z)$ is injective, its range is closed, its inverse $W(z)^{-1}$ is bounded and*

$$\big\| W(z)^{-1} \big\| \leqslant \big| \operatorname{Im}(z) \big|^{-1}. \tag{1.7}$$

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$ be fixed. Using the resolvent equation for A we obtain

$$H(z) = H(\bar{z}) + 2\,\mathrm{i} \operatorname{Im}(z)\; J\,(A - \bar{z}\,I)^{-1} A\,(A - z\,I)^{-1} J^*.$$

Taking into account that $H(z)^* = H(\bar{z})$ it follows for $x \in \mathcal{H}_2$ that

$$\begin{aligned} \operatorname{Im} \langle\, H(z)x, x \,\rangle &= \frac{1}{2} \operatorname{Im} \langle\, (H(z) - H(\bar{z}))x, x \,\rangle \\ &= \operatorname{Im}(z)\, \langle\, A\,(A - z\,I)^{-1} J^* x, (A - z\,I)^{-1} J^* x \,\rangle. \end{aligned}$$

We conclude that

$$\operatorname{Im} \langle\, W(z)x, x \,\rangle = \operatorname{Im}(z)\, \{\, \|x\|^2 + \langle\, A\,(A - z\,I)^{-1} J^* x, (A - z\,I)^{-1} J^* x \,\rangle \,\},$$

and

$$\big| \operatorname{Im} \langle\, W(z)x, x \,\rangle \big| \geqslant \big| \operatorname{Im}(z) \big| \|x\|^2, \quad \big\| W(z)\,x \big\| \geqslant \big| \operatorname{Im}(z) \big| \|x\|$$

for $x \in \mathcal{D}(\overline{C_B})$. Hence $W(z)$ is injective, its range is closed and $W(z)^{-1}$ is bounded on its domain $\mathcal{R}(W(z))$. ∎

The *defect numbers* of a densely defined closed and symmetric operator S in a Hilbert space are defined by

$$n_\pm(S) := \operatorname{def}(S \mp \mathrm{i}\,I).$$

Remember that

$$\begin{aligned} n_+(S) &= \operatorname{def}(S - z\,I), \quad \operatorname{Im}(z) > 0, \\ n_-(S) &= \operatorname{def}(S - z\,I), \quad \operatorname{Im}(z) < 0. \end{aligned}$$

LEMMA 1.3. *Let the assumptions* (a–d) *be satisfied and W be defined as in* (1.6). *Then*

$$\operatorname{def}(W(z)) = \begin{cases} n_+(\overline{C_B}) & \textit{if}\ \operatorname{Im}(z) > 0, \\ n_-(\overline{C_B}) & \textit{if}\ \operatorname{Im}(z) < 0. \end{cases}$$

Proof. Consider the operator function

$$W_0(t, z) := -\overline{C_B} + z\,I + t\,H(z), \quad (t, z) \in [0, 1] \times (\mathbb{C} \setminus \mathbb{R}).$$

Repeating the proof of Proposition 1.2 for $W_0(t, z)$ we infer that

$$\big\| W_0(t, z)\,x \big\| \geqslant \big| \operatorname{Im}(z) \big|\, \|x\|, \quad x \in \mathcal{D}(\overline{C_B}),\ (t, z) \in [0, 1] \times (\mathbb{C} \setminus \mathbb{R}).$$

Since the operator $W_0(t, z)$ is injective and the index of the operator function W_0, $\operatorname{ind}(W_0(\cdot\,,\cdot))$, is (locally) constant with respect to $t \in [0, 1]$ for fixed $z \in \mathbb{C} \setminus \mathbb{R}$ (see [8], Chapter 4, Theorem 5.22), the function $\operatorname{def}(W_0(\cdot\,,\cdot))$ is locally constant. Hence, the assertion follows. ∎

THEOREM 1.4. *Let the assumptions* (a–d) *be fulfilled and the operator* $\mathbf{L}_0$ *be defined according to* (1.2). *Let the operator functions* G *and* W *be given by* (1.3) *and* (1.6). *Then:*

(i) *The operator* $\mathbf{L}_0$ *is closable. For* $z \in \rho(A)$ *its closure* $\overline{\mathbf{L}_0}$ *is given by*

$$\overline{\mathbf{L}_0} = z\,I + \begin{pmatrix} I & 0 \\ G(z) & I \end{pmatrix} \begin{pmatrix} A - z\,I & 0 \\ 0 & -W(z) \end{pmatrix} \begin{pmatrix} I & G(\bar{z})^* \\ 0 & I \end{pmatrix}, \tag{1.8}$$

or, spelled out,

$$\begin{aligned} \mathcal{D}(\overline{\mathbf{L}_0}) &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} \mid x_1 + G(\bar{z})^* x_2 \in \mathcal{D}(A),\ x_2 \in \mathcal{D}(\overline{C_B}) \right\}, \\ \overline{\mathbf{L}_0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} A(x_1 + G(\bar{z})^* x_2) - z\,G(\bar{z})^* x_2 \\ B^*(x_1 + G(\bar{z})^* x_2) - (W(z) - z\,I)\,x_2 \end{pmatrix}. \end{aligned} \tag{1.9}$$

The operators which are given by the right-hand side of (1.8), (1.9) *do not depend on the choice of the point* $z \in \rho(A)$. *The operator* $\overline{\mathbf{L}_0}$ *is symmetric and the defect numbers of* $\overline{\mathbf{L}_0}$ *and* $\overline{C_B}$ *coincide, i.e.,* $n_\pm(\overline{\mathbf{L}_0}) = n_\pm(\overline{C_B})$.

(ii) *If* $n_+(\overline{C_B}) = n_-(\overline{C_B})$, *then all selfadjoint extensions of* $\overline{\mathbf{L}_0}$ *in* $\mathcal{H}$ *are given by the formulas*

$$\mathbf{L}_{\widehat{C}} = z\,I + \begin{pmatrix} I & 0 \\ G(z) & I \end{pmatrix} \begin{pmatrix} A - z\,I & 0 \\ 0 & -W_{\widehat{C}}(z) \end{pmatrix} \begin{pmatrix} I & G(\bar{z})^* \\ 0 & I \end{pmatrix}, \tag{1.10}$$

$$W_{\widehat{C}}(z) = -\widehat{C} + z\,I + H(z), \tag{1.11}$$

or, spelled out,

$$\begin{aligned} \mathcal{D}(\mathbf{L}_{\widehat{C}}) &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} \,\middle|\, x_1 + G(\bar{z})^* x_2 \in \mathcal{D}(A),\ x_2 \in \mathcal{D}(\widehat{C}) \right\}, \\ \mathbf{L}_{\widehat{C}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} A(x_1 + G(\bar{z})^* x_2) - z\,G(\bar{z})^* x_2 \\ B^*(x_1 + G(\bar{z})^* x_2) - (W_{\widehat{C}}(z) - z\,I)\,x_2 \end{pmatrix}, \end{aligned} \tag{1.12}$$

where $\widehat{C}$ *varies in the set of all selfadjoint extensions of* $\overline{C_B}$ *in* $\mathcal{H}_2$, *and* $z \in \mathbb{C} \setminus \mathbb{R}$. *The selfadjoint operators which are given by the right-hand side of* (1.10), (1.12) *do not depend on the choice of the point* $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. For $z \in \rho(A)$ we denote

$$K_z := \begin{pmatrix} I & 0 \\ G(z) & I \end{pmatrix}, \quad K_z^+ := \begin{pmatrix} I & G(\bar{z})^* \\ 0 & I \end{pmatrix}.$$

(i) Since $\mathbf{L}_0$ is a densely defined symmetric operator, it is closable and its closure $\overline{\mathbf{L}_0}$ is symmetric. It follows from [4] that $\overline{\mathbf{L}_0}$ is given by (1.8) and (1.9). Taking into account that the operators K_z and K_z^+ have a bounded inverse, we infer

$$\operatorname{def}(\overline{\mathbf{L}_0} - z\,I) = \operatorname{def}(A - z\,I) + \operatorname{def}(W(z)) = \operatorname{def}(W(z)), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Hence, by Lemma 1.3 the assertion follows.

(ii) Assume that $n_+(\overline{C_B}) = n_-(\overline{C_B})$. If $\widehat{C}$ is any selfadjoint extension of $\overline{C_B}$ in $\mathcal{H}_2$, it is easy to verify that the operator $\mathbf{L}_{\widehat{C}}$ is a selfadjoint extension of $\overline{\mathbf{L}_0}$.

Conversely, let $\widetilde{\mathbf{L}}$ be a selfadjoint extension of $\overline{\mathbf{L}_0}$ in $\mathcal{H}$. Since K_z and K_z^+ have a bounded inverse, we can write the resolvent of $\widetilde{\mathbf{L}}$ in the form

$$(\widetilde{\mathbf{L}} - z\,I)^{-1} = (K_z^+)^{-1} \begin{pmatrix} P(z) & Q_1(z) \\ Q_2(z) & R(z) \end{pmatrix} K_z^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where $P(z)$, $Q_1(z)$, $Q_2(z)$ and $R(z)$ are bounded operators, which are defined everywhere. Now take any selfadjoint extension $\widehat{C}$ of $\overline{C_B}$ in $\mathcal{H}_2$.

Using the relations

$$(\widetilde{\mathbf{L}} - z\,I)^{-1} y = (\mathbf{L}_{\widehat{C}} - z\,I)^{-1} y, \quad y \in \mathcal{R}(\overline{\mathbf{L}_0} - z\,I),$$

and

$$\mathcal{R}(\overline{\mathbf{L}_0} - z\,I) \supset \left\{ K_z \begin{pmatrix} (A - z\,I)\,x \\ 0 \end{pmatrix} \,\Big|\, x \in \mathcal{D}(A) \right\} = \left\{ K_z \begin{pmatrix} y \\ 0 \end{pmatrix} \,\Big|\, y \in \mathcal{H}_1 \right\},$$

we infer that

$$P(z) = (A - z\,I)^{-1}, \quad Q_2(z) = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Taking into account that

$$(\widetilde{\mathbf{L}} - z\,I)^{-1*} = (\widetilde{\mathbf{L}} - \bar{z}\,I)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

we obtain that $Q_1(z) = 0$, $z \in \mathbb{C} \setminus \mathbb{R}$. The Hilbert identity for $\widetilde{\mathbf{L}}$ yields that

$$R(z_2) - R(z_1) = (z_2 - z_1)\big(R(z_1)\,R(z_2) + R(z_1)\,G(z_1)\,G(\bar{z}_2)^*\,R(z_2)\big) \tag{1.13}$$

for $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$. It follows that the range of the operator $R(z)$ is independent of $z \in \mathbb{C} \setminus \mathbb{R}$. Let us denote this range by M. From the invertibility of $(\widetilde{\mathbf{L}} - z\,I)^{-1}$ we conclude that the operator $R(z)$, acting from $\mathcal{H}_2$ to M, is invertible for $z \in \mathbb{C} \setminus \mathbb{R}$. Put

$$-R(z)^{-1} = \widetilde{W}(z) = -\widetilde{C}(z) + z\,I + H(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Since the operators $H(z)$, $z \in \rho(A)$, are bounded, all operators $\widetilde{C}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, have the same domain M. A straightforward calculation yields that

$$\widetilde{C}(z_2) - \widetilde{C}(z_1) = (z_2 - z_1)\left(I + G(z_1)\,G(\overline{z_2})^*\right) + \left(I - R(z_1)^{-1} R(z_2)\right) R(z_2)^{-1}.$$

Using (1.13) we infer that all operators $\widetilde{C}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, coincide. Let us denote their general value by $\widetilde{C}_0$. Taking into account that

$$R(z)^* = R(\bar{z}), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

we obtain that $\widetilde{C}_0^* = \widetilde{C}_0$. Hence, $\widetilde{C}_0$ is a selfadjoint extension of $\overline{C_B}$. ∎

Remark 1.5. If A is not a strictly positive operator, but if there exists a real number γ such that $A \gg \gamma$, then Theorem 1.4 holds also.

2. On the spectrum of L and its accumulation points

In addition to the conditions (a–c) we assume in the following that

(e) C is a selfadjoint operator in $\mathcal{H}_2$ such that $\mathcal{D}(C) \supset \mathcal{D}(B)$.

Obviously, the condition (d) is satisfied. It follows that the operator $\mathbf{L} := \mathbf{L}_C$, defined according to Theorem 1.4, is selfadjoint. The corresponding operator function $W := W_C$ is defined by

$$W(z) := -C + z\,I + H(z), \quad z \in \rho(A). \tag{2.1}$$

Relation (1.10) yields

Theorem 2.1. *Let the assumptions* (a–c, e) *be fulfilled and* $\mathbf{L} := \mathbf{L}_C$, W *and* G *be given by* (1.12), (2.1) *and* (1.3). *Then the spectra of* $\mathbf{L}$ *and* W *are connected by the relations*

$$\sigma(\mathbf{L}) \cap \rho(A) = \sigma(W), \quad \sigma_{\mathrm{p}}(\mathbf{L}) \cap \rho(A) = \sigma_{\mathrm{p}}(W), \quad \sigma_{\mathrm{ess}}(\mathbf{L}) \cap \rho(A) = \sigma_{\mathrm{ess}}(W).$$

If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *is an eigenvector of* $\mathbf{L}$ *for the eigenvalue* $z \in \rho(A)$, *then* x_2 *is an eigenvector of* W *for the eigenvalue* z *and* $x_1 = -G(\bar{z})^* x_2$. *Conversely, if* x_2 *is an eigenvector of* W *for the eigenvalue* z, *then* $x = \begin{pmatrix} -G(\bar{z})^* x_2 \\ x_2 \end{pmatrix}$ *is an eigenvector of* $\mathbf{L}$ *for the eigenvalue* z. *In particular, the multiplicities of the eigenvalues of* $\mathbf{L}$ *and* W *in* $\rho(A)$ *coincide.*

Remark 2.2. Let the assumptions (a–c, e) be fulfilled and the operator function W be given by (2.1). Then the spectrum of the operator function W is a subset of $\mathbb{R}$ and $W(z)^* = W(\bar{z})$, $z \in \rho(A)$. For each $z \in \rho(A) \cap \mathbb{R}$ the operator $W(z)$ is selfadjoint. The partial multiplicities of each eigenvalue of W are equal to 1, i.e., the mapping $z \to W(z)^{-1}$ has only simple poles at each eigenvalue of W.

Proof. The first two assertions follow by Lemma 1.1 and Proposition 1.2. The third one follows from estimate (1.7), which yields that $z \to W(z)^{-1}$ has only simple poles. ∎

In the following the selfadjoint operator

$$Q := C - J\,J^* = C - (B^* A^{-1/2})\,(B^* A^{-1/2})^*, \tag{2.2}$$

which is defined on $\mathcal{D}(C)$, plays an important role. Obviously, $Q \leqslant C$.

PROPOSITION 2.3. *The operator function* W*, defined by* (2.1)*, has a representation of the form*

$$W(z) = -Q + z\,I + V(z), \quad z \in \rho(A), \tag{2.3}$$

where the operator function

$$V(z) := -J\,J^* + J\,A\,(A - z\,I)^{-1}J^* = z\,J\,(A - z\,I)^{-1}J^*, \quad z \in \rho(A), \tag{2.4}$$

is holomorphic, each operator $V(z)$ *is compact, and*

$$V'(z) = J\,(A - z\,I)^{-1}A\,(A - z\,I)^{-1}J^* \geqslant 0, \quad z \in \rho(A) \cap \mathbb{R}.$$

In particular, W *is a holomorphic operator function, and*

$$W'(z) \geqslant I, \quad z \in \rho(A) \cap \mathbb{R}.$$

Furthermore, $\sigma_{\mathrm{ess}}(W) = \sigma_{\mathrm{ess}}(Q) \cap \rho(A)$ *and*

$$\sigma_{\mathrm{ess}}(W(z)) = \{z - \lambda \mid \lambda \in \sigma_{\mathrm{ess}}(Q)\}, \quad z \in \rho(A).$$

Proof. Observe that the resolvent equation for A implies that for each $z \in \rho(A)$ the operator $(A - z\,I)^{-1}$ is compact, since by assumption (a) A^{-1} is compact. ∎

We are now ready to state Theorem 2.2 in [4], p. 9 in the case that the underlying spaces are Hilbert spaces. To avoid some technical argumentation and for completeness we reprove the result in this special case.

THEOREM 2.4. ([4]) *Let the assumptions* (a–c, e) *be fulfilled and* $\mathbf{L} := \mathbf{L}_C$*,* Q *be given by* (1.12) *and* (2.2)*. Then:*

$$\sigma_{\mathrm{ess}}(\mathbf{L}) = \sigma_{\mathrm{ess}}(Q). \tag{2.5}$$

Proof. By Theorem 2.1 and Proposition 2.3

$$\sigma_{\mathrm{ess}}(\mathbf{L}) \cap \rho(A) = \sigma_{\mathrm{ess}}(W) = \sigma_{\mathrm{ess}}(Q) \cap \rho(A).$$

It remains to show that any point $\mu \in \sigma(A)$ belongs to $\sigma_{\mathrm{ess}}(\mathbf{L})$ if and only if it belongs to $\sigma_{\mathrm{ess}}(Q)$. To this end fix $\mu \in \sigma(A)$, take the orthogonal projection P_μ onto the eigenspace M of A with respect to the eigenvalue μ, choose a number $0 < \delta < \mu$ such that $\mu - \delta \in \rho(A)$, change the upper left block A in (1.1) to $A_{\mu-\delta} = A - (\mu - \delta)\,P_\mu$ and consider the new selfadjoint operator $\mathbf{L}_{\mu-\delta}$ $(:= \mathbf{L}_{\mu-\delta,C})$ defined in the same manner as $\mathbf{L}$.

Observe that $\mathcal{D}(A_{\mu-\delta}) = \mathcal{D}(A)$. In view of the compactness of A we know that $\operatorname{rank}(P_\mu) < \infty$. Since

$$\mathbf{L}_{\mu-\delta} - \mathbf{L} = \begin{pmatrix} (\delta-\mu)\,P_\mu & 0 \\ 0 & 0 \end{pmatrix} \tag{2.6}$$

on $\mathcal{D}(A) \times \mathcal{D}(B)$ and $(\delta-\mu)\,P_\mu$ is a bounded operator, it follows that $\mathcal{D}(\mathbf{L}_{\mu-\delta}) = \mathcal{D}(\mathbf{L})$ and (2.6) remains true on $\mathcal{D}(\mathbf{L})$. Hence, $\operatorname{rank}(\mathbf{L}_{\mu-\delta} - \mathbf{L}) = \operatorname{rank}(P_\mu) < \infty$. By Theorem 2.1, Remark 1.5 and Proposition 2.3 we infer that

$$\sigma_{\text{ess}}(\mathbf{L}) \cap \rho(A_{\mu-\delta}) = \sigma_{\text{ess}}(\mathbf{L}_{\mu-\delta}) \cap \rho(A_{\mu-\delta}) = \sigma_{\text{ess}}(Q_{\mu-\delta}) \cap \rho(A_{\mu-\delta}), \tag{2.7}$$

where

$$\begin{aligned} Q_{\mu-\delta} &= C - \overline{B^*(A_{\mu-\delta})^{-1}B} = C - \overline{B^*A^{-1}B} + \left(\frac{1}{\mu} - \frac{1}{\delta}\right) \overline{B^*P_\mu B} \\ &= Q + \left(\frac{1}{\mu} - \frac{1}{\delta}\right)(B^*P_\mu)(B^*P_\mu)^*. \end{aligned} \tag{2.8}$$

Since $M \subset \mathcal{D}(A) \subset \mathcal{D}(B^*)$ and $\operatorname{rank}(P_\mu) < \infty$, we obtain $\operatorname{rank}(B^*P_\mu) < \infty$ and therefore $\operatorname{rank}(Q_{\mu-\delta} - Q) < \infty$. Hence, $\sigma_{\text{ess}}(Q_{\mu-\delta}) = \sigma_{\text{ess}}(Q)$. Since by construction $\mu \in \rho(A_{\mu-\delta})$, the theorem is proved. ∎

Corollary 2.5. *Let the assumptions* (a–c, e) *be fulfilled and* $\mathbf{L} := \mathbf{L}_C$, W *and* Q *be given by* (1.12), (2.1) *and* (2.2). *Then each point* $z_0 \in (\mathbb{R} \setminus \sigma_{\text{ess}}(Q)) \cap \sigma(W) \cap \rho(A)$ *is an isolated eigenvalue of* $\mathbf{L}$ *of finite multiplicity. All accumulation points of* $\sigma(W)$ *belong to* $\sigma_{\text{ess}}(Q)$.

Proof. By Theorem 2.4 each point of the set $(\mathbb{R} \setminus \sigma_{\text{ess}}(Q)) \cap \sigma(\mathbf{L}) \cap \rho(A)$ is an isolated eigenvalue of $\mathbf{L}$ of finite multiplicity. By Theorem 2.1 the same is true for the set $(\mathbb{R} \setminus \sigma_{\text{ess}}(Q)) \cap \sigma(W) \cap \rho(A)$ and accumulation points of $\sigma(W)$ are at the same time accumulation points of $\sigma(\mathbf{L})$. Since accumulation points of $\sigma(\mathbf{L})$ belong to $\sigma_{\text{ess}}(\mathbf{L})$, Theorem 2.4 yields that there are none of those in the set $\mathbb{R} \setminus \sigma_{\text{ess}}(Q)$. ∎

Remark 2.6. The last assertion of Corollary 2.5 means in particular that each point $z_0 \in \sigma(A) \setminus \sigma_{\text{ess}}(Q)$ is not an accumulation point of $\sigma(W)$ (or $\sigma(\mathbf{L})$).

Suppose that $\mu \in \sigma(A) \cap (\mathbb{R} \setminus \sigma_{\text{ess}}(Q))$ is fixed. Since A^{-1} is a compact positive operator, μ is a positive isolated eigenvalue of A with finite multiplicity. By Theorem 2.4 there are only two possibilities: $\mu \in \rho(\mathbf{L})$ or μ is also an isolated eigenvalue of $\mathbf{L}$ with finite multiplicity. Denote as above by P_μ the (finite dimensional) orthogonal projection onto the eigenspace M of A with respect to μ.

There exists a number $\varepsilon > 0$ such that the punctured disc around μ with radius ε belongs to $\rho(A)$. The resolvent of A has a representation of the form

$$(A - z\,I)^{-1} = \frac{1}{\mu - z}\,P_\mu + R_\mu(z), \quad 0 < \bigl|\, z - \mu \,\bigr| < \varepsilon, \tag{2.9}$$

where R_μ is a holomorphic operator function on $\bigl\{z \in \mathbb{C} \mid \bigl|\, z - \mu \,\bigr| < \varepsilon\bigr\}$, satisfying $R_\mu(z)^* = R_\mu(\bar{z})$. R_μ is called the *reduced resolvent* (or the *regular part* of the resolvent) of A at the point μ. The operator $S_\mu := B^* P_\mu$ is defined and bounded on $\mathcal{H}_1$, and the operator $B^* P_\mu\, B$, which is defined on $\mathcal{D}(B)$, has a bounded closure, which is given by $S_\mu\, S_\mu^*$. Finally, let us introduce the holomorphic operator function

$$\begin{aligned}\widetilde{W}_\mu(z) &:= -C + z\,I + \overline{B^* R_\mu(z)\,B} \\ &= -C + z\,I + J\,A\,R_\mu(z)\,J^*, \quad \bigl|\, z - \mu \,\bigr| < \varepsilon.\end{aligned} \tag{2.10}$$

We call $\widetilde{W}_\mu$ the *regular part* of W at the point μ. Observe that

$$\begin{aligned}W(z) &= \widetilde{W}_\mu(z) + \frac{1}{\mu - z}\,S_\mu\,S_\mu^* \\ &= \widetilde{W}_\mu(z) + \frac{1}{\mu - z}\,J\,A\,P_\mu\,J^*, \quad 0 < \bigl|\, z - \mu \,\bigr| < \varepsilon,\end{aligned} \tag{2.11}$$

and that $\widetilde{W}_\mu'(z) \geqslant I$ for $z \in \mathbb{R}$, $\bigl|\, z - \mu \,\bigr| < \varepsilon$.

According to (1.3) we can write

$$G(z) = \frac{1}{\mu - z}\,S_\mu + \widetilde{G}_\mu(z),$$

where $\widetilde{G}_\mu(z)$ is the regular part of $G(z)$. Observe that

$$G(\bar{z})^* = \frac{1}{\mu - z}\,S_\mu^* + \widetilde{G}_\mu(\bar{z})^*$$

and

$$G(z)\,P_\mu = \frac{1}{\mu - z}\,S_\mu, \quad P_\mu\,G(\bar{z})^* = \frac{1}{\mu - z}\,S_\mu^*. \tag{2.12}$$

Theorem 2.7. *Let the assumptions* (a–c, e) *be satisfied and* $\mathbf{L} := \mathbf{L}_C$ *be defined by* (1.12). *Assume that* $\mu \in \sigma(A)$ *and* $\mu \notin \sigma_{\mathrm{ess}}(\mathbf{L})$. *Let us denote by* P_μ *the orthogonal projection onto the eigenspace* M *of* A *with respect to* μ. *Let* S_μ *be defined by* $B^* P_\mu$ *and* $\widetilde{W}_\mu$ *be given by* (2.10). *Finally, let us denote by* D_μ *the* *orthogonal projection onto the (finite dimensional) subspace* $N_\mu := \mathcal{R}(B^*|M)$ *of* $\mathcal{H}_2$*. Then the following conditions are equivalent:*

(i) $\mu \in \rho(\mathbf{L})$*;*

(ii) $\dim(M) = \dim(N_\mu)$ *and* $\mathcal{N}(S_\mu^*) \cap \mathcal{N}((I - D_\mu)\,\widetilde{W}_\mu(\mu)) = \{0\}$*.*

Proof. Suppose that μ is an eigenvalue of $\mathbf{L}$ and also (ii) is fulfilled. By the assumptions μ is an isolated point of $\sigma(\mathbf{L})$. There exists a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(\mathbf{L})$, $x \neq 0$, such that $\lim\limits_{z\to\mu} (\mathbf{L} - z)\, x = 0$. Applying (1.12) we obtain

$$\lim_{z\to\mu} (A - z\,I)\,(x_1 + G(\bar z)^* x_2) = 0 \tag{2.13}$$

and

$$\lim_{z\to\mu} \big(B^*(x_1 + G(\bar z)^* x_2) - W(z)\,x_2\big) = 0. \tag{2.14}$$

The relations (2.12) and (2.13) imply that

$$\begin{aligned} 0 &= \lim_{z\to\mu} P_\mu\,(A - z\,I)\,(x_1 + G(\bar z)^* x_2) \\ &= \lim_{z\to\mu} (\mu - z)\left[P_\mu\, x_1 + \frac{1}{\mu - z}\, S_\mu^* x_2\right] = S_\mu^* x_2. \end{aligned} \tag{2.15}$$

It follows from (2.14) that $x_1 + \widetilde{G}_\mu(\mu)^* x_2 \in M$ and

$$\begin{aligned} S_\mu\, x_1 = B^* P_\mu\, x_1 &= \lim_{z\to\mu} B^* P_\mu\,(x_1 + G(\bar z)^* x_2) \\ &= \lim_{z\to\mu} B^*(x_1 + G(\bar z)^* x_2) = \lim_{z\to\mu} W(z)\,x_2 = \widetilde{W}_\mu(\mu)\,x_2. \end{aligned} \tag{2.16}$$

In view of (2.15) and (2.16) we obtain

$$x_2 \in \mathcal{N}(S_\mu^*) \cap \mathcal{N}((I - D_\mu)\,\widetilde{W}_\mu(\mu)).$$

Thus, by (ii), $x_2 = 0$. Hence, $x_1 \in M$ and $S_\mu\, x_1 = 0$. Since $\dim(M) = \dim(N_\mu)$, the mapping $B^*|M : M \to N_\mu$ is surjective and hence bijective. Consequently, $x_1 = 0$ and $x = 0$ which is a contradiction.

If on the other hand one of the conditions in (ii) is violated, then there exists a $h_1 \in M$ and $h_2 \in \mathcal{H}_2$ such that $h_1 \neq 0$ or $h_2 \neq 0$ and

$$S_\mu^* h_2 = 0, \quad S_\mu\, h_1 = \widetilde{W}_\mu(\mu)\, h_2.$$

Observe that in a punctured neighbourhood of μ we have that

$$\widetilde{W}_\mu(z)\, h_2 = W(z)\, h_2, \quad \widetilde{G}_\mu(z)^* h_2 = G(\bar z)^* h_2.$$

It is easy to see that the vector

$$x = \begin{pmatrix} h_1 - \widetilde{G}_\mu(\mu)^* h_2 \\ h_2 \end{pmatrix},$$

which is different from zero, satisfies the conditions (2.13) and (2.14). Hence, x is an eigenvector of $\mathbf{L}$ for the eigenvalue μ. ∎

3. EDGES OF THE ESSENTIAL SPECTRUM OF L AS ACCUMULATION POINTS OF THE DISCRETE SPECTRUM OF L

In the subsequent considerations we use the following lemmas on eigenvalues of monotonous operator functions.

Lemma 3.1. *Let G be an operator function which is defined on the open interval $I := (\gamma, \delta)$ with $-\infty \leqslant \gamma < \delta < \infty$. Assume that for $\lambda \in I$ the operators $G(\lambda)$ are bounded and selfadjoint operators on a Hilbert space into itself such that*

$$G(\lambda_1) \ll G(\lambda_2), \quad \lambda_1 < \lambda_2 \in I.$$

Suppose that the operator function G is continuous with respect to the operator norm, and

$$\sigma_{\text{ess}}(G(\lambda)) \subset (-\infty, 0), \quad \lambda \in I.$$

For $\lambda \in I$, we denote by $\nu_+(\lambda)$ the total number of eigenvalues of the operator function $G|(\gamma, \lambda)$ counted according to their multiplicities and by $n_+(\lambda)$ the number of all positive eigenvalues of the operator $G(\lambda)$ counted according to their multiplicities. If $n_0 := \min_{\lambda \in I} n_+(\lambda)$, then

$$\nu_+(\lambda) = n_+(\lambda) - n_0 < \infty, \quad \lambda \in I.$$

Proof. The assumptions imply that for $\lambda \in I$ the number $n_+(\lambda)$ is finite. Each nonnegative eigenvalue of $G(\lambda)$ is an eigenvalue of finite multiplicity. By taking into account minimal-maximal properties of eigenvalues of semibounded operators we infer from the strict monotonicity of the operator function G that

$$n_+(\lambda_1) \leqslant n_+(\lambda_2), \quad \lambda_1 < \lambda_2 \in I.$$

Suppose that $\lambda_0 \in I$ is not an eigenvalue of the operator function G. From the assumption on the essential spectra it follows that $G(\lambda_0)$ has a bounded inverse. Hence, we can choose a positive number $\eta > 0$ such that $(-\eta, \eta) \subset \rho(G(\lambda_0))$. Let us denote by $H_+(\lambda_0)$ the invariant subspace of $G(\lambda_0)$ generated by its positive spectrum (eigenvalues). Obviously,

$$\begin{aligned} \langle G(\lambda_0)\, x, x \rangle &\geqslant \eta \left\|x\right\|^2, \qquad x \in H_+(\lambda_0), \\ \langle G(\lambda_0)\, x, x \rangle &\leqslant -\eta \left\|x\right\|^2, \qquad x \in H_+(\lambda_0)^\perp. \end{aligned}$$

The continuity of the operator function G with respect to the operator norm implies that there exists a number $\varepsilon > 0$ such that, for $\lambda \in I$ with $\left| \lambda - \lambda_0 \right| < \varepsilon$,

$$\langle G(\lambda)\, x, x \rangle \geqslant \frac{\eta}{2} \left\|x\right\|^2, \quad x \in H_+(\lambda_0),$$

$$\langle\, G(\lambda)\, x, x\,\rangle \leqslant -\frac{\eta}{2}\, \|x\|^2, \quad x \in H_+(\lambda_0)^\perp.$$

Observe that $n_+(\lambda)$ coincides with the dimension of the maximal positive subspace with respect to the indefinite scalar product $\{f, g\}_\lambda := \langle\, G(\lambda)\, f, g\,\rangle$, $f, g \in \mathcal{H}$. It follows that $n_+(\lambda) = n_+(\lambda_0)$ for $\left|\, \lambda - \lambda_0 \,\right| < \varepsilon$.

Suppose now that $\lambda_0 \in I$ is an eigenvalue of the operator function G with multiplicity $s := \dim \mathcal{N}(G(\lambda_0)) > 0$. Let us denote as above by $H_+(\lambda_0)$ the invariant subspace of $G(\lambda_0)$ generated by its positive spectrum (eigenvalues). It follows that, for $x \in H_+(\lambda_0) \oplus \mathcal{N}(G(\lambda_0))$, $x \neq 0$, and $\lambda > \lambda_0$, we have $\langle\, G(\lambda)\, x, x\,\rangle > 0$. On the other hand a similar argumentation as before shows that there exist numbers $\eta > 0$ and $\varepsilon > 0$ such that, for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$,

$$\langle\, G(\lambda)\, x, x\,\rangle \geqslant \frac{\eta}{2}\, \|x\|^2, \quad x \in H_+(\lambda_0),$$

$$\langle\, G(\lambda)\, x, x\,\rangle < 0, \quad x \in \Big(H_+(\lambda_0) \oplus \mathcal{N}(G(\lambda_0))\Big)^\perp, \neq 0.$$

In particular, for $x \in \mathcal{N}(G(\lambda_0))$, $x \neq 0$, and $\lambda < \lambda_0$, we have $\langle\, G(\lambda)\, x, x\,\rangle < 0$.

We have proved that the growth points of n_+ coincide with the eigenvalues of G and that a jump of n_+ at such a point is equal to the multiplicity of the corresponding eigenvalue. Hence, the lemma is proved. ∎

Lemma 3.2. *Let G be an operator function as in Lemma* 3.1. *Assume additionally that G is also defined and continuous at δ with respect to the operator norm. Then the following conditions are equivalent:*

(i) *δ is an accumulation point of eigenvalues of G in the interval (γ, δ);*

(ii) 0 *is an accumulation point of positive eigenvalues of $G(\delta)$.*

The proof starts with

Proposition 3.3. *Under the assumptions of the Lemmas* 3.1 *and* 3.2 *we have*

$$\sigma_{\mathrm{ess}}(G(\delta)) \subset (-\infty, 0]. \tag{3.1}$$

Proof. Let us fix some $\xi > 0$. Denote by $H_{\xi/2}(\delta)$ the spectral subspace of $G(\delta)$ corresponding to the spectral interval $(\xi/2, \infty)$. Since G is continuous at δ, there exists a point $\gamma < \lambda_\xi < \delta$ such that

$$\big\|G(\delta) - G(\lambda_\xi)\big\| < \frac{\xi}{4}.$$

Evidently, for each $x \in H_{\xi/2}(\delta)$, $x \neq 0$, we obtain

$$\begin{aligned}\langle\, G(\lambda_\xi)\, x, x\,\rangle &= \langle\, G(\delta)x, x\,\rangle - \langle\, (G(\delta) - G(\lambda_\xi))\, x, x\,\rangle \\ &\geqslant \xi\frac{\|x\|^2}{2} - \xi\frac{\|x\|^2}{4} = \xi\frac{\|x\|^2}{4} > 0.\end{aligned}$$

Therefore $H_{\xi/2}(\delta)$ is a positive subspace with respect to the indefinite scalar product $\{f, g\}_{\lambda_\xi} := \langle\, G(\lambda_\xi)\, f, g\,\rangle$, $f, g \in \mathcal{H}$. Hence,

$$\dim(H_{\xi/2}(\delta)) \leqslant \dim n_+(\lambda_\xi) < \infty. \tag{3.2}$$

Suppose now that some $\xi > 0$ belongs to $\sigma_{\mathrm{ess}}(G(\delta))$. From $\xi \in \sigma_{\mathrm{ess}}(G(\delta))$ we infer that

$$\dim(H_{\xi/2}(\delta)) = +\infty,$$

contrary to (3.2). ∎

After this preliminary step we can now return to the proof of Lemma 3.2. Observe that both of the functions ν_+ and n_+ are nondecreasing on (γ, δ). By Lemma 3.1 we obtain

$$\lim_{\lambda\uparrow\delta} \nu_+(\lambda) = \lim_{\lambda\uparrow\delta} [n_+(\lambda) - n_0] = \lim_{\lambda\uparrow\delta} n_+(\lambda) - n_0.$$

Therefore $\lim\limits_{\lambda\uparrow\delta} n_+(\lambda) = +\infty$ if $\lim\limits_{\lambda\uparrow\delta} \nu_+(\lambda) = +\infty$. Taking into account (3.1) and the fact that $G(\delta)$ is a bounded operator, we have proved that (i) implies (ii).

Suppose that (ii) holds and that $0 < \xi < \delta$. Denote by $H_\xi(\delta)$ the spectral subspace of $G(\delta)$ corresponding to the spectral interval $(\xi, +\infty)$ and let $N(\xi)$ be the number of eigenvalues of $G(\delta)$ in the interval $(\xi, +\infty)$ counted according to their multiplicities, i.e., $N(\xi) = \dim(H_\xi(\delta))$. By assumption (ii) we have $\lim\limits_{\xi\downarrow 0} N(\xi) = +\infty$. An argumentation similar to that in the proof of Proposition 3.3 shows that due to the continuity of G at δ there exists a number $\delta - \xi < \lambda_\xi < \delta$ such that $n_+(\lambda_\xi) \geqslant N(\xi)$. Since n_+ is a nondecreasing function, we infer that $\lim\limits_{\lambda\uparrow\delta} n_+(\lambda) = +\infty$ and $\lim\limits_{\lambda\uparrow\delta} \nu_+(\lambda) = \lim\limits_{\lambda\uparrow\delta} [n_+(\lambda) - n_0] = +\infty$. Hence, (ii) implies (i). ∎

The counterparts of the Lemmas 3.1 and 3.2 are

Lemma 3.4. *Let G be an operator function which is defined on the open interval $I := (\gamma, \delta)$ with $-\infty < \gamma < \delta \leqslant \infty$. Assume that the operators $G(\lambda)$ are bounded and selfadjoint operators on a Hilbert space into itself such that*

$$G(\lambda_1) \ll G(\lambda_2), \quad \lambda_1 < \lambda_2 \in I.$$

Suppose that the operator function G is continuous with respect to the operator norm, and

$$\sigma_{\mathrm{ess}}(G(\lambda)) \subset (0, \infty), \quad \lambda \in I.$$

For $\lambda \in I$, we denote by $\nu_-(\lambda)$ the total number of eigenvalues of the operator function $G|(\lambda, \delta)$ counted according to their multiplicities and by $n_-(\lambda)$ the number of all negative eigenvalues of the operator $G(\lambda)$ counted according to their multiplicities. If $n_0 := \min\limits_{\lambda \in I} n_-(\lambda)$, then

$$\nu_-(\lambda) = n_-(\lambda) - n_0 < \infty, \quad \lambda \in I.$$

Lemma 3.5. *Let G be an operator function as in Lemma* 3.4*. Assume additionally that G is also defined and continuous at γ with respect to the operator norm. Then the following conditions are equivalent:*

(i) *γ is an accumulation point of eigenvalues of G in the interval (γ, δ);*

(ii) *0 is an accumulation point of negative eigenvalues of $G(\gamma)$.*

In addition to the conditions (a–c) we assume in the following that

(f) C is a bounded and selfadjoint operator on $\mathcal{H}_2$;

(g) $\sigma_{\mathrm{ess}}(Q) \neq \emptyset$.

Now the operator Q, defined by (2.2), is bounded on $\mathcal{H}_2$. We introduce the real numbers

$$\alpha := \min \sigma_{\mathrm{ess}}(Q), \quad \beta := \max \sigma_{\mathrm{ess}}(Q). \tag{3.3}$$

Let γ_α be the greatest eigenvalue of A less than α or, if all eigenvalues of A are greater or equal than α, then $\gamma_\alpha := -\infty$. Observe that $(\gamma_\alpha, \alpha) \subset \rho(A)$. By the definition of α and Proposition 2.3 it follows that

$$\sigma_{\mathrm{ess}}(W(z)) = \{z - \lambda \mid \lambda \in \sigma_{\mathrm{ess}}(Q)\} \subset (-\infty, 0), \quad z \in (\gamma_\alpha, \alpha).$$

Denote by $n_+(z)$ the number of positive eigenvalues of $W(z)$, $z \in (\gamma_\alpha, \alpha)$, counted according to their multiplicities. Since the operator function W is increasing on (γ_α, α) ($W'(z) \geqslant I$, $z \in \rho(A) \cap \mathbb{R}$), n_+ is a nondecreasing function. Denote by $n_{0,\alpha}$ the minimal value of n_+ in the interval (γ_α, α), i.e.,

$$n_{0,\alpha} := \min_{z \in (\gamma_\alpha, \alpha)} n_+(z). \tag{3.4}$$

Proposition 3.6. *Suppose that the assumptions* (a–c, f–g) *are fulfilled. Let as above γ_α be the greatest eigenvalue of A less than α or, if all eigenvalues of*

A are greater or equal than α, then $\gamma_\alpha := -\infty$. Let $n_{0,\alpha}$ be given by (3.4). *If $\gamma_\alpha > -\infty$, then*

$$\widetilde{n}_+(\gamma_\alpha) - \kappa(\gamma_\alpha) \leqslant n_{0,\alpha} \leqslant \widetilde{n}_+(\gamma_\alpha), \tag{3.5}$$

where $\widetilde{n}_+(\gamma_\alpha)$ is the number of positive eigenvalues of the operator $\widetilde{W}_{\gamma_\alpha}(\gamma_\alpha)$ defined according to (2.10) *and $\kappa(\gamma_\alpha)$ is the multiplicity of γ_α as an eigenvalue of A. If $\gamma_\alpha = -\infty$, then $n_{0,\alpha} = 0$.*

Proof. Suppose that $\gamma_\alpha > -\infty$. According to (2.11) the operator function W admits in a punctured neighbourhood of the eigenvalue γ_α of A a representation of the form

$$W(z) = \widetilde{W}_{\gamma_\alpha}(z) + \frac{1}{\gamma_\alpha - z}\, S_{\gamma_\alpha}\, S^*_{\gamma_\alpha}, \tag{3.6}$$

where $\widetilde{W}_{\gamma_\alpha}$ is the regular part of W at γ_α and $\operatorname{rank}(S_{\gamma_\alpha}) \leqslant \kappa(\gamma_\alpha)$. Denote by $\widetilde{n}_+(z)$ the number of positive eigenvalues of the operator $\widetilde{W}_{\gamma_\alpha}(z)$ in the interval (γ_α, α). Since the second term on the right hand side of (3.6) is a nonpositive operator, we see that $n_+(z) \leqslant \widetilde{n}_+(z)$, $z \in (\gamma_\alpha, \alpha)$. Hence,

$$n_{0,\alpha} = \min_{z\in(\gamma_\alpha,\alpha)} n_+(z) \leqslant \min_{z\in(\gamma_\alpha,\alpha)} \widetilde{n}_+(z) = \widetilde{n}_+(\gamma_\alpha). \tag{3.7}$$

On the other hand since the rank of the second term on the right hand side of (3.6) is less or equal than $\kappa(\gamma_\alpha)$, we obtain $\widetilde{n}_+(z) - \kappa(\gamma_\alpha) \leqslant n_+(z)$, $z \in (\gamma_\alpha, \alpha)$. Consequently,

$$\widetilde{n}_+(\gamma_\alpha) - \kappa(\gamma_\alpha) = \min_{z\in(\gamma_\alpha,\alpha)} \widetilde{n}_+(z) - \kappa(\gamma_\alpha) \leqslant \min_{z\in(\gamma_\alpha,\alpha)} n_+(z) = n_{0,\alpha}. \tag{3.8}$$

Suppose now that there are no eigenvalues of A less than α, i.e., $\gamma_\alpha = -\infty$. Since $A \gg 0$, we know that

$$\big\|z\,(A - z\,I)^{-1}\big\| \leqslant 1, \quad z \in (-\infty, 0),$$

and, by (2.3) and (2.4),

$$W(z) \leqslant z\,I + \big(\|Q\| + \|z\,J\,(A - z\,I)^{-1}J^*\|\big)I \leqslant \big(z + \|Q\| + \|J\,J^*\|\big)I, \quad z \in (-\infty, 0).$$

Hence, $W(z) \ll 0$ and $n_+(z) = 0$ for $z \in (-\infty, -\|Q\| - \|J\,J^*\|)$. ∎

Theorem 3.7. *Let the assumptions* (a–c, f–g) *be fulfilled and* $\mathbf{L} := \mathbf{L}_C$ *and* W *be given by* (1.12) *and* (2.1). *Let the number* α *be defined by* (3.3). *Let us assume that* γ_α *is the greatest eigenvalue of* A *less than* α *or, if all eigenvalues of* A *are greater or equal than* α*, then* $\gamma_\alpha := -\infty$. *Let* $n_{0,\alpha}$ *be defined by* (3.4).

(a) *Suppose that* $\alpha \in \rho(A)$ *and let* $n_+(\alpha)$ *be the number of positive eigenvalues of* $W(\alpha)$ *counted according to their multiplicities.*

(i) *If* $n_+(\alpha) < \infty$*, then the number of eigenvalues of* $\mathbf{L}$ *in the interval* (γ_α, α) *counted according to their multiplicities is equal to* $n_+(\alpha) - n_{0,\alpha}$*; the number of all eigenvalues of* $\mathbf{L}$ *in* $(-\infty, \alpha)$ *counted according to their multiplicities is equal to* $n_+(\alpha)$ *provided that there are no eigenvalues of* A *less than* α.

(ii) α *is an accumulation point of the set* $(-\infty, \alpha) \cap \sigma(\mathbf{L})$ *if and only if* 0 *is an accumulation point of the set* $\sigma(W(\alpha)) \cap (0, \infty)$.

(b) *Suppose that* α *is an eigenvalue of* A *of multiplicity* $\kappa(\alpha)$ *and let* $\widehat{n}_+(\alpha)$ *be the number of positive eigenvalues of* $\widetilde{W}_\alpha(\alpha)$ *counted according to their multiplicities, where* $\widetilde{W}_\alpha(\alpha)$ *is given by* (2.10).

(i) *If* $\widehat{n}_+(\alpha) < \infty$*, then the number of eigenvalues of* $\mathbf{L}$ *in* (γ_α, α) *counted according to their multiplicities is less or equal than* $\widehat{n}_+(\alpha) + \kappa(\alpha) - n_{0,\alpha}$ *and greater or equal than* $\widehat{n}_+(\alpha) - n_{0,\alpha}$.

(ii) α *is an accumulation point of the set* $(-\infty, \alpha) \cap \sigma(\mathbf{L})$ *if and only if* 0 *is an accumulation point of the set* $\sigma(\widetilde{W}_\alpha(\alpha)) \cap (0, \infty)$.

Proof. In the case (a) the operator function W satisfies on (γ_α, α) all conditions of the Lemmas 3.1 and 3.2. Applying these lemmas, Theorem 2.1 and Proposition 3.6 yield immediately statement (a).

In the case (b) according to (2.11) the operator function W has a representation of the form

$$W(z) = \widetilde{W}_\alpha(z) + \frac{1}{\alpha - z} S_\alpha S_\alpha^*, \quad z \in (\gamma_\alpha, \alpha). \tag{3.9}$$

Denote by $\widehat{n}_+(z)$ the number of positive eigenvalues of the operator $\widetilde{W}_\alpha(z)$ in the interval (γ_α, α). Since the second term on the right hand side of (3.9) is nonnegative on (γ_α, α) and since its rank is not greater than $\kappa(\alpha)$, we obtain that

$$\widehat{n}_+(z) \leqslant n_+(z) \leqslant \widehat{n}_+(z) + \kappa(\alpha), \quad z \in (\gamma_\alpha, \alpha). \tag{3.10}$$

As $\widetilde{W}_\alpha$ as well as W are increasing operator functions on (γ_α, α), it follows by (3.10) that

$$\widehat{n}_+(\alpha) \leqslant \max_{z \in (\gamma_\alpha, \alpha)} n_+(z) \leqslant \widehat{n}_+(\alpha) + \kappa(\alpha). \tag{3.11}$$

Taking into account that W and $\widetilde{W}_\alpha$ satisfy on the interval (γ_α, α) the conditions of Lemma 3.1 and $\widetilde{W}_\alpha$ satisfies on the interval (γ_α, α) also the conditions of Lemma 3.2, we obtain by applying Theorem 2.1 and the inequality (3.11) the desired statements (b) (i) and (ii). ∎

The following theorem specifies conditions under which the relation $(-\infty, \alpha) \subset \rho(\mathbf{L})$ is fulfilled.

Theorem 3.8. *Let the assumptions* (a–c, f–g) *be satisfied and* $\mathbf{L} := \mathbf{L}_C$, W *and* H *be given by* (1.12), (2.1) *and* (1.5). *Suppose that* α *is defined by* (3.3) *and that* $\alpha < \min \sigma(A)$. *Then the following three conditions are equivalent:*

(i) $\mathbf{L}$ *has no eigenvalues in the interval* $(-\infty, \alpha)$*;*

(ii) *the operator* $W(\alpha)$ *is nonpositive, i.e.,* $W(\alpha) \leqslant 0$*;*

(iii) *for* $z < \alpha$ *we have* $C - z\,I \gg 0$ *and*

$$\lim_{z \uparrow \alpha} \left\| (C - z\,I)^{-1/2} H(z)\,(C - z\,I)^{-1/2} \right\| \leqslant 1. \tag{3.12}$$

Proof. Since by Proposition 2.3

$$\sigma_{\mathrm{ess}}(W(\alpha)) = \{\alpha - z \mid z \in \sigma_{\mathrm{ess}}(Q)\},$$

the conditions (i) and (ii) are equivalent by Theorem 3.7 (a) (i). As $W'(z) \geqslant I$, $z \in \rho(A) \cap \mathbb{R}$, the condition (ii) is equivalent to

$$W(z) \ll 0, \quad z \in (-\infty, \alpha). \tag{3.13}$$

Suppose that (3.13) holds. Taking into account that

$$H(z) = \overline{B^*(A - z\,I)^{-1} B} \geqslant 0, \quad z \in (-\infty, \alpha),$$

we infer that

$$C - z\,I = -W(z) + H(z) \gg 0, \quad z \in (-\infty, \alpha). \tag{3.14}$$

Since

$$\begin{aligned} I = &-(C - z\,I)^{-1/2} W(z)\,(C - z\,I)^{-1/2} \\ &+ (C - z\,I)^{-1/2} H(z)\,(C - z\,I)^{-1/2}, \quad z \in (-\infty, \alpha), \end{aligned} \tag{3.15}$$

and since both terms on the right hand side of (3.15) are nonnegative, we obtain that

$$0 \leqslant (C - z\,I)^{-1/2} H(z)\,(C - z\,I)^{-1/2} \leqslant I, \quad z \in (-\infty, \alpha),$$

and consequently

$$\big\|(C - z\,I)^{-1/2} H(z)(C - z\,I)^{-1/2}\big\| \leqslant 1, \quad z \in (-\infty, \alpha). \tag{3.16}$$

Observe that the operator functions $(C - \cdot I)^{-1}$ and H are increasing and nonnegative on $(-\infty, \alpha)$. It follows for $z_1 < z_2 < \alpha$ that

$$0 \leqslant (C - z_1\,I)^{-1/2} H(z_1)\,(C - z_1\,I)^{-1/2} \leqslant (C - z_1\,I)^{-1/2} H(z_2)\,(C - z_1\,I)^{-1/2}$$

and

$$0 \leqslant H(z_2)^{1/2}(C - z_1\,I)^{-1} H(z_2)^{1/2} \leqslant H(z_2)^{1/2}(C - z_2\,I)^{-1} H(z_2)^{1/2}.$$

Using the equality $\|X^* X\| = \|X\,X^*\|$, which is valid for any bounded operator X, we conclude for $z_1 < z_2 < \alpha$ that

$$\begin{aligned}
&\big\|(C - z_1\,I)^{-1/2} H(z_1)(C - z_1\,I)^{-1/2}\big\| \\
&\qquad \leqslant \big\|(C - z_1\,I)^{-1/2} H(z_2)^{1/2} H(z_2)^{1/2}(C - z_1\,I)^{-1/2}\big\| \\
&\qquad = \big\|H(z_2)^{1/2}(C - z_1\,I)^{-1} H(z_2)^{1/2}\big\| \\
&\qquad \leqslant \big\|H(z_2)^{1/2}(C - z_2\,I)^{-1/2}(C - z_2\,I)^{-1/2} H(z_2)^{1/2}\big\| \\
&\qquad = \big\|(C - z_2\,I)^{-1/2} H(z_2)(C - z_2\,I)^{-1/2}\big\| \leqslant 1.
\end{aligned}$$

Since the function

$$\varphi(z) := \big\|(C - z\,I)^{-1/2} H(z)(C - z\,I)^{-1/2}\big\|, \quad z \in (-\infty, \alpha),$$

is nondecreasing, we have proved that

$$\varphi(z) \leqslant \lim_{z' \uparrow \alpha} \varphi(z') \leqslant 1, \quad z \in (-\infty, \alpha).$$

Hence, (ii) implies (iii).

If (iii) holds, a similar argumentation as before yields that the function φ defined above is nondecreasing. It follows that

$$0 \leqslant (C - z\,I)^{-1/2} H(z)\,(C - z\,I)^{-1/2} \leqslant I, \quad z \in (-\infty, \alpha),$$

i.e., $H(z) \leqslant C - z\,I$, $z \in (-\infty, \alpha)$. Hence, $W(\alpha) = -C + \alpha\,I + H(\alpha) \leqslant 0$. ∎

The discrete spectrum of $\mathbf{L}$ in the right neighbourhood of the upper bound β of $\sigma_{\mathrm{ess}}(Q)$ can be investigated in a quite similar way using the Lemmas 3.4 and 3.5.

Let δ_β be the smallest eigenvalue of A greater than β or, if all eigenvalues of A are less or equal than β, then $\delta_\beta := +\infty$.

Observe that $(\beta, \delta_\beta) \subset \rho(A)$. By the definition of β and Proposition 2.3 it follows that

$$\sigma_{\mathrm{ess}}(W(z)) = \{z - \lambda \mid \lambda \in \sigma_{\mathrm{ess}}(Q)\} \subset (0, +\infty), \quad z \in (\beta, \delta_\beta).$$

Denote by $n_-(z)$ the number of negative eigenvalues of $W(z)$, $z \in (\beta, \delta_\beta)$, counted according to their multiplicities. Since W is increasing on (β, δ_β) ($W'(z) \geqslant I$, $z \in \rho(A) \cap \mathbb{R}$), n_- is a nondecreasing function. Denote by $n_{0,\beta}$ the minimal value of n_- in the interval (β, δ_β), i.e.,

$$n_{0,\beta} := \min_{z \in (\beta, \delta_\beta)} n_-(z). \tag{3.17}$$

Proposition 3.9. *Suppose that the assumptions* (a–c, f–g) *are satisfied. Let δ_β be the smallest eigenvalue of A greater than β or, if all eigenvalues of A are less or equal than β, then $\delta_\beta := +\infty$. Let $n_{0,\beta}$ be given by* (3.17). *If $\delta_\beta < +\infty$, then*

$$\widetilde{n}_-(\delta_\beta) - \kappa(\delta_\beta) \leqslant n_{0,\beta} \leqslant \widetilde{n}_-(\delta_\beta), \tag{3.18}$$

where $\widetilde{n}_-(\delta_\beta)$ is the number of negative eigenvalues of the operator $\widetilde{W}_{\delta_\beta}(\delta_\beta)$ defined according to (2.10) *and $\kappa(\delta_\beta)$ is the multiplicity of δ_β as an eigenvalue of A.*

Theorem 3.10. *Let the assumptions* (a–c, f–g) *be fulfilled and $\mathbf{L} := \mathbf{L}_C$ and W be given by* (1.12) *and* (2.1). *Let the number β be defined by* (3.3). *Assume that δ_β is the smallest eigenvalue of A greater than β or, if all eigenvalues of A are less or equal than β, then $\delta_\beta := +\infty$. Let $n_{0,\beta}$ be defined by* (3.17).

(a) *Suppose that $\beta \in \rho(A)$ and let $n_-(\beta)$ be the number of negative eigenvalues of $W(\beta)$ counted according to their multiplicities.*

(i) *If $n_-(\beta) < \infty$, then the number of eigenvalues of $\mathbf{L}$ in the interval (β, δ_β) counted according to their multiplicities is equal to $n_-(\beta) - n_{0,\beta}$.*

(ii) *β is an accumulation point of the set $(\beta, +\infty) \cap \sigma(\mathbf{L})$ if and only if 0 is an accumulation point of the set $\sigma(W(\beta)) \cap (-\infty, 0)$.*

(b) *Suppose that β is an eigenvalue of A of multiplicity $\kappa(\beta)$ and let $\widehat{n}_-(\beta)$ be the number of negative eigenvalues of $\widetilde{W}_\beta(\beta)$ counted according to their multiplicities, where $\widetilde{W}_\beta(\beta)$ is given by* (2.10).

(i) *If* $\widehat{n}_-(\beta) < \infty$*, then the number of eigenvalues of* $\mathbf{L}$ *in* (β, δ_β) *counted according to their multiplicities is less or equal than* $\widehat{n}_-(\beta) + \kappa(\beta) - n_{0,\beta}$ *and greater or equal than* $\widehat{n}_-(\beta) - n_{0,\beta}$*.*

(ii) β *is an accumulation point of the set* $(\beta, \infty) \cap \sigma(\mathbf{L})$ *if and only if* 0 *is an accumulation point of the set* $\sigma(\widetilde{W}_\beta(\beta)) \cap (-\infty, 0)$*.*

The following theorem deals with accumulation points of isolated eigenvalues of $\mathbf{L}$ inside the interval (α, β).

Theorem 3.11. *Suppose that the assumptions* (a–c, f–g) *are satisfied and that* $\mathbf{L} := \mathbf{L}_C$ *and* W *are given by* (1.12) *and* (2.1)*. Let the numbers* α *and* β *be defined by* (3.3)*. Assume that* $\alpha \leqslant \tau < \omega \leqslant \beta$ *and* $(\tau, \omega) \cap \sigma_{\mathrm{ess}}(Q) = \emptyset$*. Suppose that* $\omega \in \rho(A)$ *(*$\tau \in \rho(A)$*). Then* ω *(*τ*) is an accumulation point of* $\sigma(\mathbf{L}) \cap (\tau, \omega)$ *if and only if* 0 *is an accumulation point of positive (negative) eigenvalues of* $W(\omega)$ *(*$W(\tau)$*).*

If $\omega \in \sigma(A)$ *(*$\tau \in \sigma(A)$*), the statements remain true with* $\widetilde{W}_\omega(\omega)$ *(*$\widetilde{W}_\tau(\tau)$*) instead of* $W(\omega)$ *(*$W(\tau)$*), where* $\widetilde{W}_\omega(\omega)$ *(*$\widetilde{W}_\tau(\tau)$*) is defined by* (2.10)*.*

Proof. We show the theorem only for the point ω. The proof for the point τ can be obtained in a similar way using Lemma 3.5.

Suppose first that $[\tau, \omega] \subset \rho(A)$. Since $(\tau, \omega) \cap \sigma_{\mathrm{ess}}(Q) = \emptyset$ and

$$\sigma_{\mathrm{ess}}(W(z)) = \{z - \lambda \mid \lambda \in \sigma_{\mathrm{ess}}(Q)\}, \quad z \in [\tau, \omega],$$

we obtain that $\sigma_{\mathrm{ess}}(W(z)) \cap (z-\omega, z-\tau) = \emptyset$ for $z \in [\tau, \omega]$. Therefore $\sigma(W(\omega)) \cap (0, \omega - \tau)$ may contain only isolated eigenvalues of $W(\omega)$ of finite multiplicity. Let $\xi \in (0, \omega - \tau) \cap \rho(W(\omega))$. By the continuity of W in $[\tau, \omega]$ with respect to the operator norm there exists a number $\gamma_\omega \in (\tau, \omega)$ such that $0 < \xi < \gamma_\omega - \tau$ and $\xi \in \bigcap\limits_{z \in [\gamma_\omega, \omega]} \rho(W(z))$. Observe that by our choice $(0, \xi) \subset (z - \omega, z - \tau)$ for $z \in [\gamma_\omega, \omega]$. Hence, $\sigma_{\mathrm{ess}}(W(z)) \cap (0, \xi) = \emptyset$ for $z \in [\gamma_\omega, \omega]$. Let us introduce the analytic operator function

$$G_\omega(z) := W(z)\,(\xi\, I - W(z))^{-1}, \quad z \in [\gamma_\omega, \omega]. \tag{3.19}$$

It is easy to see that

$$0 \notin \sigma_{\mathrm{ess}}(G_\omega(z)), \quad \sigma(G_\omega(z)) = \left\{ \frac{\mu}{\xi - \mu} \,\Big|\, \mu \in \sigma(W(z)) \right\}, \quad z \in [\gamma_\omega, \omega].$$

Since the transformation

$$\varphi(\mu) = \frac{\mu}{\xi - \mu}, \quad \mu \in \mathbb{R} \setminus \{\xi\},$$

maps the interval $(0,\xi)$ onto $(0,+\infty)$ and $\sigma(W(z))\cap(0,\xi)$, $z\in(\gamma_\omega,\omega)$, consists only of a finite number of eigenvalues of finite multiplicity, we conclude that $\sigma(G_\omega(z))\cap(0,+\infty)$ consists only of a finite number of eigenvalues of finite multiplicity, i.e., $\sigma_{\mathrm{ess}}(G_\omega(z))\subset(-\infty,0)$, $z\in(\gamma_\omega,\omega)$. Furthermore, for $z\in(\gamma_\omega,\omega)$,

$$G'_\omega(z)=\xi\,(\xi\,I-W(z))^{-1}W'(z)\,(\xi\,I-W(z))^{-1}\geqslant\xi\,(\xi\,I-W(z))^{-2}\gg 0,$$

i.e., G_ω is an increasing operator function on (γ_ω,ω). Thus the operator function G_ω satisfies all conditions of Lemma 3.2. Therefore ω is an accumulation point of eigenvalues of G_ω if and only if 0 is an accumulation point of positive eigenvalues of $G_\omega(\omega)$. But by (3.19) positive eigenvalues of $G_\omega(\omega)$ accumulate at the point 0 if and only if the eigenvalues of $W(\omega)$ in the interval $(0,\xi)$ accumulate at the point 0. Furthermore, the eigenvalues of G_ω in (γ_ω,ω) coincide with the eigenvalues of W and with the corresponding eigenvalues of $\mathbf{L}$ in (γ_ω,ω).

The existence of eigenvalues of A inside of (τ,ω) involves no trouble, because in the previous argumentation the point γ_ω may be taken greater than the maximal eigenvalue of A in (τ,ω).

Suppose now that $\omega\in\sigma(A)$. This eigenvalue may be shifted to the right or may be removed at all without any changes of other eigenvalues and corresponding eigenvectors by a finite dimensional perturbation of A and consequently of $\mathbf{L}$.

The corresponding operator function W_ω constructed with respect to the perturbated operator $\mathbf{L}_\omega$ (as W was introduced with respect to $\mathbf{L}$) is a finite dimensional perturbation of $\widetilde{W}_\omega$. A selfadjoint perturbation of rank $\kappa(\omega)$ can reduce the number of eigenvalues of $\mathbf{L}$ in (γ_ω,ω) at most by $\kappa(\omega)$ and the same is true for the number of positive eigenvalues of $\widetilde{W}_\omega$ in $(0,\xi)$. Therefore the statement of the theorem for $\mathbf{L}_\omega$ and W_ω remains true with $\mathbf{L}_\omega$ replaced by $\mathbf{L}$ and $W_\omega(\omega)$ replaced by $\widetilde{W}_\omega(\omega)$, defined according to (2.10). ∎

The following example shows that α and β, respectively, can be or not be accumulation points of $\sigma(\mathbf{L})\cap(-\infty,\alpha)$ and $\sigma(\mathbf{L})\cap(\beta,+\infty)$, respectively.

Example 3.12. Suppose that $\{e_j\mid j\in\mathbb{N}\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_1=\mathcal{H}_2$. Let us consider the selfadjoint operator

$$A\,x:=\sum_{j=1}^{\infty}j\,\langle\,x,e_j\,\rangle\,e_j,\quad x\in\mathcal{D}(A),$$

where

$$\mathcal{D}(A):=\left\{x\in\mathcal{H}_1\;\middle|\;\sum_{j=1}^{\infty}j^2\,\big|\,\langle\,x,e_j\,\rangle\,\big|^2<\infty\right\}.$$

A is strictly positive and A^{-1} is compact. The spectrum of A is given by the set of natural numbers $\{j \mid j \in \mathbb{N}\}$. Assume that $a \leqslant 1$ and $b > 0$, and that

$$B := b^{1/2} A^{1/2}, \quad C := a\, I.$$

The corresponding operator function W, defined by (2.1), is given by

$$W(z) = -Q + z\, I + V(z), \quad z \in \rho(A),$$

where

$$Q := (a-b)\, I, \quad V(z) := b\, z\, (A - z\, I)^{-1} = b\, z \sum_{j=1}^{\infty} \frac{1}{j-z} \langle \cdot, e_j \rangle\, e_j, \quad z \in \rho(A).$$

We have

$$\sigma_{\mathrm{ess}}(W) = \sigma_{\mathrm{ess}}(Q) = \{a-b\}, \quad \alpha = \beta = a-b.$$

Furthermore, the real numbers

$$\lambda_j^{\pm} := \frac{j+a}{2} \pm \sqrt{\Big(\frac{j-a}{2}\Big)^2 + b\, j}, \quad j \in \mathbb{N},$$

which are (for fixed j) solutions of the quadratic equation

$$z^2 - (j+a)\, z + j\, (a-b) = 0,$$

are eigenvalues of W for the eigenvectors e_j, $j \in \mathbb{N}$. An easy calculation gives

$$\lambda_j^{+} > j, \quad j \in \mathbb{N},$$

and

$$\lambda_j^{-} = \frac{j}{\lambda_j^{+}} \cdot (a-b) = \frac{a-b}{\frac{j+a}{2\,j} + \sqrt{(\frac{j-a}{2\,j})^2 + \frac{b}{j}}}, \quad j \in \mathbb{N}.$$

Observe that

$$\begin{aligned} \lambda_j^{-} &< a-b, \quad j \in \mathbb{N}, \quad \text{if } a-b > 0,\\ \lambda_j^{-} &> a-b, \quad j \in \mathbb{N}, \quad \text{if } a-b < 0, \end{aligned}$$

and $\lim\limits_{j\to\infty} \lambda_j^{-} = a-b$. If $a = b$, then all eigenvalues λ_j^{-}, $j \in \mathbb{N}$, merge into one isolated eigenvalue of W of infinite multiplicity, which is equal to zero.

Taking orthogonal sums of operators as in Example 3.12 with different numbers a and b one can easily construct examples of selfadjoint operator matrices with gaps inside the essential spectrum such that the edges of these gaps can be or not be accumulation points of eigenvalues of $\mathbf{L}$ situated inside of the interval (α, β).

4. EXAMPLE

In this section we will apply the results to an example from ordinary differential equations. First we describe the problem and show that the assumptions (a–c, f) are fulfilled.

Let n be a natural number and I be the interval $[0,1]$. Let c be a function on I with values in the set of $n \times n$-matrices with complex coefficients, $b = (b_j)_{j=1}^n$ a vector function on I with values in $\mathbb{C}^n$, and p, w be real-valued functions on I. We assume that all these functions are continuous. Further, suppose that c is hermitian, i.e.,

$$c(x)^* = c(x), \quad x \in I,$$

$b_j(x) \neq 0$, $x \in I$, $j = 1, \ldots, n$, b, p, w are continuously differentiable and

$$p_0 := \min_{x \in I} p(x) > 0, \quad w_0 := \min_{x \in I} w(x) > 0. \tag{4.1}$$

For $k \in \mathbb{N}$ we introduce the Hilbert space

$$L^2_w(I, \mathbb{C}^k) := \left\{ f : I \to \mathbb{C}^k \,\Big|\, \int_I |\, f(x) \,|^2 w(x)\, \mathrm{d}x < \infty \right\},$$

endowed with the scalar product

$$\langle\, f, g \,\rangle_w := \int_I g(x)^* f(x)\, w(x)\, \mathrm{d}x$$

and the corresponding norm

$$\|f\|_w := \left(\int_I |\, f(x) \,|^2 w(x)\, \mathrm{d}x \right)^{1/2}.$$

Furthermore, for $k, l \in \mathbb{N}$ we define

$$H^2_{l,w}(I, \mathbb{C}^k) := \Big\{ f \in L^2_w\big(I, \mathbb{C}^k\big) \,\big|\, f,\, f', \ldots, f^{(l-1)} \text{ absolutely continuous}, \\ f^{(l)} \in L^2_w\big(I, \mathbb{C}^k\big) \Big\}.$$

By $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote the Hilbert space $L^2_w(I, \mathbb{C})$ and $L^2_w(I, \mathbb{C}^n)$, respectively, and by $\mathcal{H}$ the Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$. D denotes the differential operator $\mathrm{d}/\mathrm{d}x$.

Define the operator A in $\mathcal{H}_1$, the operator B from $\mathcal{H}_2$ into $\mathcal{H}_1$ and the operator C in $\mathcal{H}_2$ in the following way:

$$\begin{aligned} &\mathcal{D}(A) := \dot{H}^2_{2,w}(I,\mathbb{C}) := \left\{\eta \in H^2_{2,w}(I,\mathbb{C}) \mid \eta(0)=\eta(1)=0\right\}, \\ &A := -\frac{1}{w}\,D\,w\,p\,D, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\mathcal{D}(B) := \dot{H}^2_{1,w}(I,\mathbb{C}^n) := \left\{y \in H^2_{1,w}(I,\mathbb{C}^n) \mid y(0)=y(1)=0\right\}, \\ &B := -\mathrm{i}\frac{1}{w}\,D\,w\,b^*, \end{aligned} \tag{4.3}$$

and C is the operator of multiplication with the matrix valued function c on $\mathcal{H}_2$.

It is well known that A is a selfadjoint, strictly positive Sturm–Liouville operator in $\mathcal{H}_1$. The spectrum of A consists of a strictly increasing sequence $(\mu_j)_{j=1}^\infty$ of infinitely many simple, positive eigenvalues and for $z \in \mathbb{C}\setminus\{\mu_j \mid j \in \mathbb{N}\}$ the operator $(A - z\,I)^{-1}$ is compact. In particular, A^{-1} is compact.

By $\{\eta_j \mid j \in \mathbb{N}\}$ we denote an orthonormal basis of eigenfunctions of A in $\mathcal{H}_1$, i.e.,

$$-\frac{1}{w}\,D\,w\,p\,D\,\eta_j = \mu_j\,\eta_j, \quad \eta_j(0)=\eta_j(1)=0, \quad \eta_j \neq 0, \quad j \in \mathbb{N}. \tag{4.4}$$

Observe that

$$A^{-1}f = \sum_{j=1}^{\infty} \mu_j^{-1}\,c_j(f)\,\eta_j, \quad f \in \mathcal{H}_1, \tag{4.5}$$

and

$$A^{-1/2}f = \sum_{j=1}^{\infty} \mu_j^{-1/2}\,c_j(f)\,\eta_j, \quad f \in \mathcal{H}_1, \tag{4.6}$$

where

$$c_j(f) := \langle\, f, \eta_j\,\rangle_w = \int_I f(x)\,\overline{\eta_j(x)}\,w(x)\,\mathrm{d}x, \quad j \in \mathbb{N}.$$

Evidently,

$$(c_j(f))_{j=1}^\infty \in \ell_2(\mathbb{C}) := \left\{(c_j)_{j=1}^\infty \in \mathbb{C}^{\mathbb{N}} \,\Big|\, \sum_{j=1}^{\infty} \left|\,c_j\,\right|^2 < \infty\right\}.$$

Lemma 4.1. *We have:*

$$\mathcal{D}(A^{1/2}) \subset H^2_{1,w}(I,\mathbb{C}).$$

Proof. In view of the relation (4.6) we infer that

$$\mathcal{D}(A^{1/2}) = \left\{ f \in \mathcal{H}_1 \,\middle|\, \exists\, (c_j)_{j=1}^{\infty} \in \ell_2(\mathbb{C}) :\ f = \sum_{j=1}^{\infty} \mu_j^{-1/2}\, c_j\, \eta_j \right\}.$$

By $\widetilde{D}$ we denote the differential operator in $\mathcal{H}_1$, which is defined by

$$\widetilde{D} f := f', \quad \mathcal{D}(D) := H^2_{1,w}(I, \mathbb{C}).$$

It is well known that $\widetilde{D}$ is a closed operator. Let $(c_j)_{j=1}^{\infty} \in \ell_2(\mathbb{C})$ be fixed. Since

$$f_k := \sum_{j=1}^{k} \mu_j^{-1/2}\, c_j\, \eta_j \in H^2_{1,w}(I, \mathbb{C}), \quad k \in \mathbb{N},$$

and

$$\widetilde{D}\, f_k = \sum_{j=1}^{k} \mu_j^{-1/2}\, c_j\, \eta_j', \quad k \in \mathbb{N}, \tag{4.7}$$

it is sufficient to prove that the series

$$\sum_{j=1}^{\infty} \mu_j^{-1/2}\, c_j\, \eta_j'$$

converges in $\mathcal{H}_1$. It follows immediately from (4.4) that the set

$$\left\{ \mu_j^{-1/2}\, \eta_j' \mid j \in \mathbb{N} \right\}$$

is an orthonormal system in $L^2_{wp}(I, \mathbb{C})$. But by assumption (4.1) the metrics on $L^2_{wp}(I, \mathbb{C})$ and $L^2_{w}(I, \mathbb{C})$ are equivalent. Hence, the lemma is proved. ∎

The operator B is densely defined and closable. The adjoint operator B^* of B is given by

$$B^* = -\mathrm{i}\, b\, D, \quad \mathcal{D}(B^*) = H^2_{1,w}(I, \mathbb{C}). \tag{4.8}$$

An immediate consequence of Lemma 4.1 is

Lemma 4.2. *We have:*

$$\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B^*).$$

Finally, the operator C is a bounded and selfadjoint operator on $\mathcal{H}_2$.

We have proved that the operators A, B and C fulfill the assumptions (a–c, f). Let us consider the symmetric operator $\mathbf{L}_0$, which is defined by the operator matrix

$$\mathbf{L}_0 = \begin{pmatrix} -\frac{1}{w}\, D\, w\, p\, D & -\mathrm{i}\frac{1}{w}\, D\, w\, b^* \\ -\mathrm{i}\, b\, D & c \end{pmatrix}$$

on $\dot{H}^2_{2,w}(I, \mathbb{C}) \times \dot{H}^2_{1,w}(I, \mathbb{C}^n)$. Taking into account that C is a bounded and selfadjoint operator we infer that $\mathbf{L}_0$ is essentially selfadjoint, i.e., $\overline{\mathbf{L}_0} =: \mathbf{L}$ is selfadjoint. The selfadjoint operator $\mathbf{L}$ ($= \mathbf{L}_C$) and the operator function $W := W_C$ are given according to Theorem 1.4. W is defined on $\rho(A)$ and is given by

$$W(z) = -C + z\, I + \overline{B^*(A - z\, I)^{-1} B},$$

where

$$\Big((B^*(A - z\, I)^{-1} B)\, y\Big)(x) = -b(x) \frac{\mathrm{d}}{\mathrm{d}x} \int_I G_z(x, \xi)\, (w\, \langle\, y, b\, \rangle)'(\xi)\, \mathrm{d}\xi. \tag{4.9}$$

The function

$$G_z(x, \xi) := \sum_{j=1}^{\infty} \frac{1}{\mu_j - z}\, \eta_j(x)\, \overline{\eta_j(\xi)}, \quad x,\, \xi \in I,\, z \in \rho(A),$$

is the Green function of the Sturm–Liouville operator A, i.e., G_z is the kernel of the compact operator $(A - z\, I)^{-1}$. Recall that

$$p(\xi)\, w(\xi) \left(\frac{\partial G_z(x, \xi)}{\partial x} \Big|_{x=\xi-0} - \frac{\partial G_z(x, \xi)}{\partial x} \Big|_{x=\xi+0} \right) = 1, \quad \xi \in (0, 1).$$

The function $G_z(\cdot, \xi)$ satisfies (4.4) everywhere except $x = \xi$. Let us define

$$\varphi(x) := \int_0^x \frac{\mathrm{d}s}{w(s)\, p(s)}, \quad \psi(x) := \int_x^1 \frac{\mathrm{d}s}{w(s)\, p(s)}, \quad x \in I,$$

and

$$\varepsilon := \psi(0)^{-1} = \left(\int_0^1 \frac{\mathrm{d}s}{w(s)\, p(s)} \right)^{-1}.$$

Obviously, $A\varphi = A\psi = 0$ and $\varphi(0) = 0$, $\psi(1) = 0$. It is well known that

$$G_0(x,\xi) = \begin{cases} \varepsilon\,\varphi(x)\,\psi(\xi), & 0 \leqslant x < \xi \leqslant 1, \\ \varepsilon\,\psi(x)\,\varphi(\xi), & 0 \leqslant \xi < x \leqslant 1. \end{cases}$$

On the other hand we have

$$G_0(x,\xi) = \sum_{j=1}^{\infty} \mu_j^{-1}\,\eta_j(x)\,\overline{\eta_j(\xi)}, \quad x,\xi \in I. \tag{4.10}$$

A straightforward calculation yields for any absolutely continuous function g that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\int_I G_0(x,\xi)\,g'(\xi)\,\mathrm{d}\xi \\ &= \varepsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi(x)\int_0^x \varphi(\xi)g'(\xi)\,\mathrm{d}\xi\right) + \varepsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\varphi(x)\int_x^1 \psi(\xi)g'(\xi)\,\mathrm{d}\xi\right) \\ &= -\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi(x)\int_0^x \varphi'(\xi)g(\xi)\,\mathrm{d}\xi\right) - \varepsilon\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\varphi(x)\int_x^1 \psi'(\xi)g(\xi)\,\mathrm{d}\xi\right) \\ &= -\varepsilon\,\big(\psi(x)\,\varphi'(x) - \varphi(x)\,\psi'(x)\big)g(x) + \frac{\varepsilon}{w(x)\,p(x)}\int_0^1 \frac{g(\xi)}{p(\xi)w(\xi)}\,\mathrm{d}\xi \\ &= -\frac{1}{p(x)w(x)}\,g(x) + \frac{\varepsilon}{w(x)\,p(x)}\int_0^1 \frac{g(\xi)}{p(\xi)w(\xi)}\,\mathrm{d}\xi. \end{aligned} \tag{4.11}$$

It follows that

$$W(z) = -C + z\,I + \overline{B^*A^{-1}B} + z\,\overline{B^*A^{-1}(A - z\,I)^{-1}B}$$

has a representation of the form

$$\begin{aligned} (W(z)\,y)(x) = &-\Big(c(x)y(x) - \frac{1}{p(x)}\langle\, y(x), b(x)\,\rangle b(x)\Big) \\ &+ z\,y(x) + b(x)\int_0^1 K_z(x,\xi)\langle\, y(\xi), b(\xi)\,\rangle w(\xi)\,\mathrm{d}\xi \\ &- \frac{\varepsilon}{w(x)\,p(x)}\int_0^1 \frac{1}{p(\xi)}\langle\, y(\xi), b(\xi)\,\rangle\,\mathrm{d}\xi\, b(x), \end{aligned} \tag{4.12}$$

where

$$K_z(x,\xi) := \frac{\partial}{\partial x}\frac{\partial}{\partial \xi}\,[G_z(x,\xi) - G_0(x,\xi)] = \sum_{j=1}^{\infty} \frac{z}{\mu_j - z}\frac{1}{\mu_j}\,\eta_j'(x)\,\overline{\eta_j'(\xi)}.$$

Let us denote by V the operator function, which is defined on $\rho(A)$ and is given by

$$(V(z)\,y)(x) := b(x)\int_0^1 K_z(x,\xi)\,\langle\, y(\xi), b(\xi)\,\rangle\, w(\xi)\,\mathrm{d}\xi, \quad y \in L^2_w(I,\mathbb{C}^n).$$

The operator $V(z)$ is a compact operator on $L^2_w(I,\mathbb{C}^n)$. If $z \leqslant 0$, then $V(z) \leqslant 0$. If $0 \leqslant z < \mu_1$, then $V(z) \geqslant 0$. If $\mu_s < z < \mu_{s+1}$ for some $s \in \mathbb{N}$, then $V(z)$ has at most a finite number of negative eigenvalues counted according to their multiplicities. For each $s \in \mathbb{N}$ the regular part of V at the point μ_s has at most a finite number of negative eigenvalues counted according to their multiplicities. The last term in (4.12) defines a nonpositive one-dimensional operator on $L^2_w(I,\mathbb{C}^n)$.

Let us denote by Q the multiplication operator on $L^2_w(I,\mathbb{C}^n)$ with respect to the matrix function $(Q_{jk})_{j,k=1}^n$, which is defined by

$$Q_{jk}(x) := c_{jk}(x) - \frac{1}{p(x)}\, b_j(x)\,\overline{b_k(x)}, \quad x \in I,\ j,k = 1,\dots,n, \tag{4.13}$$

where $c =: (c_{jk})_{j,k=1}^n$, $b =: (b_j)_{j=1}^n$. Furthermore, let us define

$$\alpha := \min \sigma_{\mathrm{ess}}(Q), \quad \beta := \max \sigma_{\mathrm{ess}}(Q).$$

Remember that by Theorem 2.4

$$\sigma_{\mathrm{ess}}(Q) = \sigma_{\mathrm{ess}}(\mathbf{L}).$$

An immediate consequence of Theorems 3.10 and 3.8 is

Theorem 4.3. (i) *If $\beta \geqslant 0$, then β is not an accumulation point of eigenvalues of $\mathbf{L}$ greater than β.*

(ii) *If $\alpha \leqslant 0$, then the operator $\mathbf{L}$ has no eigenvalues less than α.*

Proof. (i) If $\beta \geqslant 0$, it follows from (4.12) and above mentioned properties of $V(z)$ that the operator $W(\beta)$ or the regular part of W at the point β has only a finite number of negative eigenvalues counted according to their multiplicities. Hence, by Theorem 3.10, the point β is not an accumulation point of the set $\sigma(\mathbf{L}) \cap (\beta, +\infty)$.

(ii) The second statement is an immediate consequence of Theorem 3.8. ∎

Let

$$\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_n(x), \quad x \in I,$$

be the eigenvalues of $Q(x)$, $x \in I$, repeated according to their multiplicities. Denote

$$\omega_j := \min_{x \in I} \lambda_j(x), \quad \tau_j := \max_{x \in I} \lambda_j(x), \quad j = 1, \ldots, n.$$

Evidently, $(\tau_j, \omega_{j+1}) \cap \sigma_{\text{ess}}(\mathbf{L}) = \emptyset$ if $\tau_j < \omega_{j+1}$ for some $j \in \{1, \ldots, n-1\}$.

THEOREM 4.4. *If $\tau_j < \omega_{j+1} \leqslant 0$ for some $j \in \{1, \ldots, n-1\}$, then ω_{j+1} is not an accumulation point of the set $\sigma(\mathbf{L}) \cap (-\infty, \omega_{j+1})$.*

We start the proof with the following

PROPOSITION 4.5. *Let $\mathbf{M}$ be a selfadjoint operator and $\mathbf{T}$ be a nonpositive compact operator in the Hilbert space $\mathcal{H}$. Suppose that for some $\Delta > 0$ we have $\sigma(\mathbf{M}) \cap (0, \Delta) = \emptyset$. Then 0 is not an accumulation point of the set $\sigma(\mathbf{M} + \mathbf{T}) \cap (0, \Delta)$.*

Proof. Denote by $\mathcal{H}_-$ and $\mathcal{H}_+$ the spectral subspaces of $\mathbf{M}$ corresponding to the intervals $(-\infty, 0]$ and $[\Delta, +\infty)$, respectively, and let P_- and P_+ be the orthogonal projections on $\mathcal{H}_-$ and $\mathcal{H}_+$, respectively. Put

$$M_\pm := P_\pm \mathbf{M}|\mathcal{H}_\pm, \quad T_\pm := P_\pm \mathbf{T}|\mathcal{H}_\pm, \quad \Gamma := P_+ \mathbf{T}|\mathcal{H}_-.$$

Considering $\mathcal{H}$ as the product space of $\mathcal{H}_-$ and $\mathcal{H}_+$ the operator $\mathbf{M} + \mathbf{T}$ has a representation of the form

$$\mathbf{M} + \mathbf{T} = \begin{pmatrix} M_+ + T_+ & \Gamma \\ \Gamma^* & M_- + T_- \end{pmatrix}.$$

Since $\sigma(M_+) \subset [\Delta, +\infty)$ and T_+ is a compact operator, there are no accumulation points of $\sigma(M_+ + T_+)$ in $(-\infty, \Delta/2)$. It follows that $\sigma(M_+ + T_+) \cap (-\infty, \Delta/2)$ contains at most a finite number of eigenvalues of $M_+ + T_+$ of finite multiplicity. Remove all such eigenvalues back to the interval $[\Delta, +\infty)$ by a finite dimensional perturbation of $M_+ + T_+$ and denote by $\widetilde{M}_+$ the perturbated operator obtained in this way. Applying a similar argumentation as in Section 1 and Section 2 of this paper, it follows immediately that the spectrum of the operator

$$\widetilde{\mathbf{M}} := \begin{pmatrix} \widetilde{M}_+ & \Gamma \\ \Gamma^* & M_- + T_- \end{pmatrix}$$

in $(0, \Delta/2)$ coincides with the spectrum of the operator function

$$U(z) := (z\,I - M_- - T_-) + \Gamma^*(\widetilde{M}_+ - z\,I)^{-1}\Gamma$$

in $(0, \Delta/2)$. By our assumption M_- and $M_- + T_-$ are nonpositive operators and by construction $\sigma(\widetilde{M}_+) \subset (\Delta/2, +\infty)$. Hence, $U(z) \gg 0$ for $z \in (0, \Delta/2)$ and $(0, \Delta/2) \subset \rho(\widetilde{\mathbf{M}})$. Taking into account that

$$\operatorname{rank}(\widetilde{\mathbf{M}} - (\mathbf{M} + \mathbf{T})) < +\infty,$$

we obtain that the set $\sigma(\mathbf{M} + \mathbf{T}) \cap (0, \Delta/2)$ is empty or finite. ∎

Now we can continue with the proof of Theorem 4.4.

Consider the operator $W(\omega_{j+1})$ using the representation (4.12). By the assumptions the sum of the first two terms on the right hand side of (4.12) with $z = \omega_{j+1}$ forms a multiplication operator on a continuous operator matrix, the spectrum of which does not intersect $(0, \omega_{j+1} - \tau_j)$. The second two terms on the right hand side of (4.12) with $z = \omega_{j+1}$ are nonpositive compact operators. Hence, by Proposition 4.5 zero is not an accumulation point of $\sigma(W(\omega_{j+1})) \cap (0, +\infty)$. Theorem 3.11 completes the proof. ∎

We leave to the reader to formulate and prove the corresponding theorem for τ_j as a possible accumulation point of $\sigma(\mathbf{L}) \cap (\tau_j, +\infty)$.

Let us now consider the more general case, when the Sturm-Liouville operator is given by

$$A_q := -\frac{1}{w}\, D\, w\, p\, D + q\, I = A + q\, I \tag{4.14}$$

on

$$\mathcal{D}(A_q) := \dot{H}^2_{2,w}(I) := \left\{ \eta \in H^2_{2,w}(I, \mathbb{C}) \mid \eta(0) = \eta(1) = 0 \right\},$$

where $q : I \to \mathbb{R}$ is a continuous function and

$$q_0 := \min_{x \in I} q(x) \geqslant 0.$$

The operator A_q is a selfadjoint, strictly positive operator in $\mathcal{H}_1$ and A_q^{-1} is compact.

Lemma 4.6. *Suppose that D is a selfadjoint, strictly positive operator in the Hilbert space $\mathcal{H}$. Then the domain of the square root of D is given by*

$$\mathcal{D}(D^{1/2}) = \{ x \in \mathcal{H} \mid \exists\, (x_n)_{n=1}^{\infty} \subset \mathcal{D}(D) : x_n \to x,\ n \to \infty; \\ \langle D(x_n - x_m), x_n - x_m \rangle \to 0,\ n, m \to \infty \}.$$

Proof. See [13], Section 4.4.3, p. 253. ∎

An immediate consequence of Lemma 4.6 is

$$\mathcal{D}(A_q^{1/2}) = \mathcal{D}(A^{1/2}) \subset \mathcal{D}(B^*).$$

Hence, the operators A_q, B and C fulfill the assumptions (a–c, f). Let us denote by $\mathbf{L}_q$ the corresponding selfadjoint operator matrix and by W_q the corresponding operator function. Let us define

$$q_1 := \max_{x\in I} q(x).$$

For the subsequent considerations we need the following well known

Lemma 4.7. *For selfadjoint operators U, V with $\mathcal{D}(U) = \mathcal{D}(V)$ the relation $U \geqslant V \gg 0$ implies $V^{-1} \geqslant U^{-1} \geqslant 0$.*

Proof. By our assumptions both operators U and V have a bounded inverse. From $U \geqslant V \geqslant 0$ it follows that $I \geqslant U^{-1/2} V U^{-1/2} \geqslant 0$, where $U^{-1/2} V U^{-1/2}$ is densely defined, since its domain is given by $\mathcal{R}(U^{1/2}|\mathcal{D}(U))$, $U^{1/2}$ is surjective and $\mathcal{D}(U)$ is a core for $U^{1/2}$. Observe that $V^{1/2}U^{-1/2}$ is bounded and

$$1 \geqslant \|U^{-1/2} V U^{-1/2}\| = \|(V^{1/2}U^{-1/2})^*(V^{1/2}U^{-1/2})\| = \|V^{1/2}U^{-1}V^{1/2}\|.$$

Since $V^{1/2}U^{-1}V^{1/2} \geqslant 0$, we conclude that

$$I \geqslant V^{1/2}U^{-1}V^{1/2}.$$

Hence, $V^{-1} \geqslant U^{-1} \geqslant 0$. ∎

Theorem 4.8. *If $\beta \geqslant q_1$, then β is not an accumulation point of $\sigma(\mathbf{L}_q) \cap (\beta, \infty)$.*

Proof. Let us consider at first the case when $q \equiv q_1$ ($\leqslant \beta$) is constant. In this case the operator function W_{q_1} is obtained by substituting the kernel $K_z(x, \xi)$ in the representation (4.12) of W by the kernel

$$K_{q_1,z}(x,\xi) := \frac{\partial}{\partial x}\frac{\partial}{\partial \xi}\left[G_{z-q_1}(x,\xi) - G_0(x,\xi)\right] = \sum_{j=1}^{\infty} \frac{z-q_1}{\mu_j - (z-q_1)} \frac{1}{\mu_j} \eta_j'(x)\,\overline{\eta_j'(\xi)}.$$

For $z \geqslant q_1$ the corresponding integral operator $V_{q_1}(z)$ or the regular part of V_{q_1} at the point β has (as in the case $q \equiv q_1 = 0$) only a finite number of negative eigenvalues counted according to their multiplicities. Hence, in the case $\beta \geqslant q_1$ the operator $W_{q_1}(\beta)$ or the regular part of W_{q_1} at the point β has only a finite number of negative eigenvalues counted according to their multiplicities.

In the general case the operator function W_q has a representation of the form

$$\begin{aligned}(4.15)\qquad &(W_q(z)\,y)(x)\\ &= (W_{q_1}(z)\,y)(x) - b(x)\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_I \Big(G_{q,z}(x,\xi) - G_{z-q_1}(x,\xi)\Big)(w\,\langle\, y, b\,\rangle)'(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where $G_{q,z}(x,\xi)$ denotes the Green function of the Sturm-Liouville operator A_q. Since

$$A + q_1\, I \geqslant A_q \gg 0$$

in $L^2_w(I,\mathbb{C}^n)$, it follows by Lemma 4.7 that the integral operator with the kernel $G_{q,z}(x,\xi)$ is (for real z) greater or equal than the integral operator with the kernel $G_{z-q_1}(x,\xi)$. Consequently, the second term on the right-hand side of (4.15) defines a nonnegative operator in $L^2_w(I,\mathbb{C}^n)$. Thus, if the operator $W_{q_1}(\beta)$ or the regular part of W_{q_1} at the point β has only a finite number of negative eigenvalues counted according to their multiplicities, the same is true for the operator $W_q(\beta)$ or the regular part of W_q at the point β. By Theorem 3.10 the assertion follows. ∎

Theorem 4.9. *If $\alpha \leqslant q_0$, then the operator $\mathbf{L}_q$ has no eigenvalues less than α.*

Proof. We use the same trick as in the proof before. Let us consider at first the case when $q \equiv q_0$ is constant. In this case the operator function W_{q_0} is obtained by substituting the kernel $K_z(x,\xi)$ in the representation (4.12) of W by the kernel

$$K_{q_0,z}(x,\xi) := \sum_{j=1}^{\infty} \frac{z-q_0}{\mu_j - (z-q_0)}\,\frac{1}{\mu_j}\,\eta_j'(x)\,\overline{\eta_j'(\xi)}.$$

If $\alpha \leqslant q_0$, then the corresponding operator $W_{q_0}(\alpha)$ is nonpositive.

Let us now consider the general case. In view of the definition of q_0 the operator function W_q has a representation of the form

$$\begin{aligned}(4.16)\qquad &(W_q(z)\,y)(x)\\ &= (W_{q_0}(z)\,y)(x) - b(x)\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_I (G_{q,z}(x,\xi) - G_{z-q_0}(x,\xi))\,(w\,\langle\, y, b\,\rangle)'(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where $G_{q,z}(x,\xi)$ denotes the Green function of the Sturm-Liouville operator A_q. Since

$$0 \ll A + q_0\, I \leqslant A_q$$

in $L^2_w(I,\mathbb{C}^n)$, it follows from Lemma 4.7 that the second term on the right-hand side of (4.16) defines a nonpositive operator in $L^2_w(I,\mathbb{C}^n)$. Hence, all terms in the representation of $W_q(\alpha)$ are nonpositive operators in $L^2_w(I,\mathbb{C}^n)$, and $W_q(\alpha) \leqslant 0$. By Theorem 3.8 the assertion follows. ∎

With a slight modification the previous argumentation can be used to prove the following

THEOREM 4.10. *If $\tau_j < \omega_{j+1}$ and $\omega_{j+1} \leqslant q_0$ for some $j \in \{1, \dots, n-1\}$, then ω_{j+1} is not an accumulation point of the set $\sigma(\mathbf{L}_q) \cap (-\infty, \omega_{j+1})$.*

Suppose in addition that the components of b and c are also continuously differentiable and for simplicity that the matrix function c is diagonal, i.e., $c_{jk} = \lambda_j \, \delta_{jk}$, $j, k = 1, \dots, n$.

As in [3] one can verify that under these assumptions a point $z_0 \in \mathbb{R}$ is an isolated eigenvalue of $\mathbf{L}_q$ if and only if z_0 is an eigenvalue of the Sturm-Liouville boundary eigenvalue problem

$$\begin{aligned} &-\frac{1}{w(x)} \frac{\mathrm{d}}{\mathrm{d}x} w(x) \, g(x,z) \frac{\mathrm{d}}{\mathrm{d}x} \eta(x) + q(x) \, \eta(x) = z \, \eta(x), \\ &\eta(0) = \eta(1) = 0 \end{aligned} \tag{4.17}$$

with

$$g(x,z) = p(x) + \sum_{j=1}^{n} \frac{\left| b_j(x) \right|^2}{z - \lambda_j(x)}. \tag{4.18}$$

Evidently, each eigenvalue problem for the Sturm-Liouville system (4.17) with g having a representation of the form (4.18), continuously differentiable functions w, p, b_j, λ_j, $j = 1, \dots, n$, a continuous function q, $p(x) \geqslant p_0 > 0$, $w(x) \geqslant w_0 > 0$, $\lambda_1(x) \leqslant \dots \leqslant \lambda_n(x)$, $x \in I$, can be transformed to an eigenvalue problem for a selfadjoint operator matrix $\mathbf{L}_q$. The results obtained above can be summarized in the following

THEOREM 4.11. *Suppose that g admits a representation of the form (4.18), w, p, b_j, λ_j, $j = 1, \dots, n$, are continuously differentiable functions, q is real-valued and continuous, $p(x) \geqslant p_0 > 0$, $w(x) \geqslant w_0 > 0$, $x \in I$. For each $x \in I$ the rational function $g(x, \cdot)$ has only real roots $\mu_1(x) \leqslant \mu_2(x) \leqslant \dots \leqslant \mu_n(x)$. Put $\omega_j := \min_{x \in I} \mu_j(x)$, $\tau_j := \max_{x \in I} \mu_j(x)$, $j = 1, \dots, n$, $q_0 := \min_{x \in I} q(x)$ and $q_1 := \max_{x \in I} q(x)$. If $\omega_1 \leqslant q_0$, then there are no eigenvalues of the boundary eigenvalue problem (4.17) less than ω_1. If $\tau_j < \omega_{j+1}$ and $\omega_{j+1} \leqslant q_0$ for some $j \in \{1, \dots, n-1\}$, then the eigenvalues of the problem (4.17) in the interval (τ_j, ω_{j+1}) do not accumulate at ω_{j+1}. If $\tau_j < \omega_{j+1}$ and $\tau_j \geqslant q_1$ for some $j \in \{1, \dots, n-1\}$, then τ_j is not an accumulation point of eigenvalues of the problem (4.17) greater than τ_j. Finally, if $\tau_n \geqslant q_1$, then τ_n is not an accumulation point of eigenvalues of the problem (4.17) greater than τ_n.*

Let us now apply the results to a problem occuring in magnetohydrodynamics.

The oscillations of a hot compressible gravitating plasma layer in an ambient magnetic field are described by boundary eigenvalue problems of the form

$$\mathbf{L}_P\,\underline{\xi} = \lambda\,\underline{\xi}, \tag{4.19}$$

where $\underline{\xi}$ is the displacement vector and $\mathbf{L}_P$ is the differential operator given by

$$\left(\begin{array}{c|cc} \rho_0^{-1}\,D\,\rho_0\,(v_a^2+v_s^2)\,D+k^2\,v_a^2 & (\rho_0^{-1}\,D\,\rho_0\,(v_a^2+v_s^2)+\mathrm{i}\,g)\,k_\perp & (\rho_0^{-1}\,D\,\rho_0\,v_s^2+\mathrm{i}\,g)\,k_\parallel \\ \hline k_\perp\,(\,(v_a^2+v_s^2)\,D-\mathrm{i}\,g) & k^2\,v_a^2+k_\perp^2\,v_s^2 & k_\perp\,k_\parallel\,v_s^2 \\ k_\parallel\,(v_s^2\,D-\mathrm{i}\,g) & k_\perp\,k_\parallel\,v_s^2 & k_\parallel^2\,v_s^2 \end{array}\right),$$

see [10], Chapter 7.3. Here D denotes the differential operator $-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}x}$, $\rho_0(x)$ the equilibrium density of the plasma, $v_a(x)$ the Alfvén speed, $v_s(x)$ the sound speed, $k_\perp(x)$ and $k_\parallel(x)$ are the coordinates of the wave vector $\mathbf{k}(x)$ with respect to the field allied orthonormal bases, $k(x)$ is the length of the vector $\mathbf{k}(x)$ and g is the gravitational constant.

The boundary value problem (4.19) is considered on the interval $[0,1]$. In the physical literature usually the first component of the displacement vector $\underline{\xi}$ is supposed to fulfill Dirichlet boundary conditions at $x=0$ and $x=1$.

Let us neglect the gravitational effects, i.e., $g=0$. Then the operator $\mathbf{L}_P$ is of the form considered in this section with

$$p(x) = v_a^2(x) + v_s^2(x),$$

$$q(x) = k^2(x)\,v_a^2(x) = (k_\perp^2(x) + k_\parallel^2(x))\,v_a^2(x),$$

$$b(x) = \begin{pmatrix} k_\perp(x)\,p(x) \\ k_\parallel(x)\,v_s^2(x) \end{pmatrix}$$

and

$$Q(x) = \begin{pmatrix} k_\parallel^2(x)\,v_a^2(x) & 0 \\ 0 & k_\parallel^2(x)\dfrac{v_s^2(x)\,v_a^2(x)}{v_s^2(x)+v_a^2(x)} \end{pmatrix},$$

see [4]. Evidently,

$$\beta_P = \sup\sigma(Q) = \max_{x\in I}\,k_\parallel^2(x)\,v_a^2(x) \leqslant \max_{x\in I}\,(k_\perp^2(x)+k_\parallel^2(x))\,v_a^2(x) =: q_1, \tag{4.20}$$

and equality in (4.20) is only possible in the case when $k_\parallel^2(x_0)\,v_a^2(x_0) = \beta_P$ implies $k_\perp(x_0) = 0$.

Theorem 4.12. *Let the differential operator* $\mathbf{L}_P$ *be given by* (4.19) *and* β_P *be defined by* (4.20). *If* $g = 0$ *and* $\beta_P = q_1$, *then* β_P *is not an accumulation point of eigenvalues of* $\mathbf{L}_P$ *greater than* β_P.

Theorem 4.12 is a direct consequence of Theorem 4.8.

On the other hand,

$$\begin{aligned}\alpha_P = \inf \sigma(Q) &= \min_{x\in I} k_\parallel^2(x) \frac{v_s^2(x)\, v_a^2(x)}{v_s^2(x) + v_a^2(x)} \\ &\leqslant \min_{x\in I} \left(k_\perp^2(x) + k_\parallel^2(x)\right) v_a^2(x) =: q_0,\end{aligned} \tag{4.21}$$

and as a direct consequence of Theorem 4.9 we obtain

Theorem 4.13. *Let the differential operator* $\mathbf{L}_P$ *be given by* (4.19) *and* α_P *be defined by* (4.21). *If* $g = 0$, *then*

$$\alpha_P = \inf \sigma_{\mathrm{ess}}(\mathbf{L}_P) = \inf \sigma(\mathbf{L}_P),$$

i.e., there are no eigenvalues of $\mathbf{L}_P$ *less than* α_P.

Finally, Theorem 4.10 yields

Theorem 4.14. *Let the differential operator* $\mathbf{L}_P$ *be given by* (4.19). *If* $g = 0$ *and*

$$\max_{x\in I} k_\parallel^2(x) \frac{v_s^2(x)\, v_a^2(x)}{v_s^2(x) + v_a^2(x)} < \min_{x\in I} k_\parallel^2(x)\, v_a^2(x) =: \omega_P,$$

then ω_P *is not an accumulation point of eigenvalues of* $\mathbf{L}_P$ *less than* ω_P.

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