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ORTHOGONAL DECOMPOSITIONS OF ISOMETRIES IN HILBERT C^*-MODULES

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ABSTRACT. We give a necessary and sufficient condition on a Hilbert C^*-module isometry in order to obtain a Wold-type decomposition. A characterization of shift operators is obtained as a consequence.

KEYWORDS: *Hilbert C^*-module, adjointable isometry, shift, Wold decomposition.*

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1. INTRODUCTION

Hilbert modules over a C^*-algebra A were first introduced by I. Kaplansky in 1953 in [2] (only in the particular case when A is abelian). These objects generalize, in a certain sense, the notion of Hilbert space, replacing the scalar product with an A-valued inner product.

In 1973, W.L. Paschke presented in his thesis [6], in the form used today, the main properties of Hilbert C^*-modules and of the operators on such objects. The manner of presentation incited the interest of mathematicians all over the world for this extremely fertile domain. A year later, in the paper [9] in *Advances in Mathematics* (which circulated since 1972 as a preprint), M.A. Rieffel proved, in a manner of presentation different from the one of Paschke, the utility of such generalizations.

The following important step in the development of this theory was made by G.G. Kasparov in [3] by proving the famous stabilization theorem. This theorem shows that the standard Hilbert A-module

$$\ell^2(A) := \Big\{(x_n)_n \in \prod_1^\infty A \,\Big|\, \sum_{n=1}^\infty x_n^* x_n \text{ converges in norm in } A\Big\}$$

absorbs every countably generated Hilbert A-module E, that is

$$E \oplus \ell^2(A) \cong \ell^2(A).$$

In the paper [4], G.G. Kasparov introduced a general K-theory, today called KK-theory, in which Hilbert C^*-modules represent an important instrument of study. The appearence of this theory determinated a considerable increase of the number of papers studying or using Hilbert modules.

It is a well known fact that an isometry on a Hilbert space decomposes as the direct sum of a unitary operator and a unilateral translation, a result obtained by H. Wold in [11]. This decomposition has many applications in the description of the structure of isometric and unitary dilation spaces for a Hilbert space contraction. Also, it permits the reduction of the study of a general-type isometry to the two particular classes enumerated above. It is our aim, in this paper, to find necessary and sufficient conditions on an isometry on a Hilbert C^*-module in order to obtain a decomposition of such type.

The results obtained in [7] are completed and presented in a different manner in this paper.

2. NOTATIONS AND PRELIMINARIES

2.1. HILBERT MODULES. Let A be a C^*-algebra. We shall suppose that each module E studied below has a complex linear space structure. Also we shall suppose that the right A-module structure is compatible with that of the linear space, that is

$$\lambda(xa) = (\lambda x)a = x(\lambda a), \quad \lambda \in \mathbb{C},\ a \in A,\ x \in E.$$

DEFINITION 2.1. A *pre-Hilbert A-module* is a right A-module E equipped with an A-valued inner product, that is a map $\langle\cdot,\cdot\rangle_E : E \times E \to A$ satisfying:

(i) $\langle x, y+z\rangle_E = \langle x,y\rangle_E + \langle x,z\rangle_E$, and

$$\langle x, \lambda y\rangle_E = \lambda\langle x,y\rangle_E, \quad x,y,z \in E,\ \lambda \in \mathbb{C};$$

(ii) $\langle x, ya\rangle_E = \langle x, y\rangle_E a, \quad x, y \in E,\ a \in A$;
(iii) $\langle x, y\rangle_E^* = \langle y, x\rangle_E, \quad x, y \in E$;
(iv) $\langle x, x\rangle_E \geqslant 0, \quad x \in E$, and

$$\langle x, x\rangle_E = 0 \Leftrightarrow x = 0.$$

For a pre-Hilbert A-module E we define a norm on E by

$$\|x\|_E := \|\langle x, x\rangle_E\|^{\frac{1}{2}}, \quad x \in E.$$

A *Hilbert A-module* is a pre-Hilbert A-module E which is complete with respect to the norm $\|\cdot\|_E$.

Example 2.2. There exist numerous examples of Hilbert A-modules among which we mention:

(i) if $A = \mathbb{C}$ then E is exactly the usual Hilbert space, the scalar product being defined by

$$(x|y) := \langle y, x\rangle_E, \quad x, y \in E;$$

(ii) $E = A$ is a Hilbert A-module, the A-valued inner product being

$$\langle x, y\rangle_E := x^* y, \quad x, y \in A;$$

(iii) If $\{E_n\}_n$ is a sequence of Hilbert A-modules, we shall define their *direct sum*

$$E = \bigoplus_{n=1}^{\infty} E_n := \Big\{(x_n)_n \in \prod_{n=1}^{\infty} E_n \,\Big|\, \sum_n \langle x_n, y_n\rangle_{E_n} \text{ converges in norm in } A\Big\}$$

which, with the inner product

$$\langle (x_n), (y_n)\rangle_E := \sum_{n=1}^{\infty} \langle x_n, y_n\rangle_{E_n}, \quad (x_n)_n, (y_n)_n \in E,$$

forms a Hilbert A-module. A particular case is the Hilbert A-module $\ell^2(A)$ defined in the first paragraph of this paper.

2.2. Orthogonality in Hilbert modules. Elements x, y in a Hilbert module E are said to be *orthogonal,* denoted $x \perp y$, if $\langle x, y\rangle_E = 0$.

If F is a submodule of E its *orthogonal complement* is $F^\perp = \{x \in E : x \perp y, \forall\, y \in F\}$.

The sum $F_1 + F_2$ of two submodules F_1, F_2 of E is said to be *direct* if $F_1 \cap F_2 = \{0\}$ and *orthogonal* if $F_1 \perp F_2$. If the sum is orthogonal we shall use the notation $F_1 \oplus F_2$.

A submodule F of E is said to be *complementable* if there exists a submodule $G \subset E$ with $E = F \oplus G$.

EXAMPLE 2.3. In a Hilbert space every closed subspace is complementable in the sense of the definition above. So the definition "complement" is justified. This property is false in general in arbitrary Hilbert modules. For example, we can consider $E = A = \mathcal{C}([0,1])$ the C^*-algebra of all continuous functions on $[0,1]$ and $F = \mathcal{C}_0((0,1]) \subset E$. It is simple to observe that F is a closed ideal of A, so a closed submodule in E, and $F^\perp = \{0\}$. Consequently F is not complementable.

REMARK 2.4. Finally, we mention two properties:
(i) if $E = F_1 \oplus F_2$ then F_1, F_2 are closed and $F_1^\perp = F_2, F_2^\perp = F_1$;
(ii) if $\{F_n\}_n$ is a parwise orthogonal sequence of closed submodules of E then

$$\bigoplus_{n=1}^{\infty} F_n := \Big\{ x = \sum_{n=1}^{\infty} x_n (\text{convergence in } E) \Big| \; x_n \in F_n, \sum_n \langle x_n, x_n \rangle_E \text{converges in norm in } A \Big\}$$

is a closed submodule of E.

2.3. ADJOINTABLE OPERATORS ON HILBERT MODULES. Let E, F be Hilbert A-modules. A map $T : E \to F$ is said to be *adjointable* if there exists $T^* : F \to E$ (called the *adjoint of* T) with the property

$$\langle x, Ty \rangle_F = \langle T^* x, y \rangle_E, \quad x \in F, \; y \in E.$$

We shall denote by $\mathcal{L}_A(E, F)$ the set of all adjointable maps $T : E \to F$. For an adjointable map $T : E \to F$ we shall use the notation $[E, F, T]$, and if $E = F$, $[E, T]$.

$T : E \to F$ is a bounded A-module map if and only if there exists $k > 0$ such that $\langle Tx, Tx \rangle_F \leqslant k \langle x, x \rangle_E$, for each $x \in E$ ([6]). In particular, if T is adjointable then

$$\langle Tx, Tx \rangle_F \leqslant \|T\|^2 \langle x, x \rangle_E, \quad x \in E. \tag{2.1}$$

Furthermore, if $[E, T]$ is adjointable then $\operatorname{Ker} T^* = T(E)^\perp$, $\operatorname{Ker} T^*$ being the kernel of T^*.

DEFINITION 2.5. A submodule $E_0 \subset E$ is said to be
(i) *invariant* for $[E, T]$ if $TE_0 \subset E_0$;
(ii) *reducing* for $[E, T]$ if it is invariant for T and T^*.

PROPOSITION 2.6. *Let $[E, T]$ be an adjointable operator and $E_0 \subset E$ a closed submodule, reducing for T. Then*

(i) $T|E_0$ *is adjointable and* $(T|E_0)^* = T^*|E_0$;

(ii) $E_0^\perp$ *is reducing for* T.

Proof. Observe that (i) is obtained from

$$\langle T|E_0 x, y\rangle_{E_0} = \langle Tx, y\rangle_E = \langle x, T^*y\rangle_E = \langle x, T^*|E_0 y\rangle_{E_0}, \quad x, y \in E_0.$$

For (ii), it is sufficient to prove that if E_0 is invariant for T then $E_0^\perp$ is invariant for T^*. Indeed

$$\langle T^*x, y\rangle_E = \langle x, Ty\rangle_E = 0, \quad \text{for all } x \in E_0^\perp,\ y \in E_0. \quad ∎$$

2.4. Isometries on Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and $[\mathcal{H}, V]$ an isometry.

A closed subspace $\mathcal{L} \subset \mathcal{H}$ is said to be *wandering* for V if $V^n\mathcal{L} \perp V^m\mathcal{L}, n, m \in \mathbb{N}, n \neq m$. $[\mathcal{H}, V]$ is called a *shift* if there exists a wandering subspace $\mathcal{L}$ such that

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} V^n\mathcal{L}.$$

Theorem 2.7. (Wold, see [11], [10]) *Let* $[\mathcal{H}, V]$ *be an isometry. Then we have a uniquely determinated orthogonal decomposition*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

where $\mathcal{H}_0, \mathcal{H}_1$ *are reducing for* V, $V|\mathcal{H}_0$ *is unitary and* $V|\mathcal{H}_1$ *is a shift. Furthermore*

$$\mathcal{H}_0 = \bigcap_{n=0}^{\infty} V^n\mathcal{H}, \quad \mathcal{H}_1 = \bigoplus_{n=0}^{\infty} V^n\mathcal{L}, \quad \mathcal{L} = \mathcal{H} \ominus V\mathcal{H}.$$

3. ISOMETRIES ON HILBERT MODULES

The results contained in Propositions 3.1 and 3.3 are a consequence of those obtained by E.C. Lance in [5]. We prefer here other direct proofs.

PROPOSITION 3.1. *Let $[E, F, V]$ be an adjointable operator. The following assertions are equivalent:*

(i) *V is an isometry between the Banach spaces E and F (that is $\|Vx\|_F = \|x\|_E, x \in E$);*

(ii) *$\langle Vx, Vx\rangle_F = \langle x, x\rangle_E, \quad x \in E$;*

(iii) *$\langle Vx, Vy\rangle_F = \langle x, y\rangle_E, \quad x, y \in E$;*

(iv) *$V^*V = I_E$.*

Proof. For (ii) ⇔ (iii) the polarization identity is used, and for (iii) ⇔ (iv) the definition of the adjoint. By passing to norm, one obtains (ii) ⇒ (i). For the converse, observe firstly that $VE \subset E$ is a closed submodule. The operator

$$V_0 : E \to VE, \quad V_0x := Vx, \quad x \in E$$

is bijective and, furthermore, V_0^{-1} is a bounded A-module map.

Using (2.1) for V and a similar argument for V_0^{-1} we obtain for $x \in E$,

$$\langle x, x\rangle_E = \langle V_0^{-1}Vx, V_0^{-1}Vx\rangle_E \leqslant \|V_0^{-1}\|^2 \langle Vx, Vx\rangle_{VE} = \langle Vx, Vx\rangle_F,$$

and respectively

$$\langle Vx, Vx\rangle_F \leqslant \|V\|^2 \langle x, x\rangle_E = \langle x, x\rangle_E,$$

that is $\langle Vx, Vx\rangle_F = \langle x, x\rangle_E$. ∎

DEFINITION 3.2. $[E, F, U]$ is said to be a *unitary operator* if

$$U^*U = I_E \quad \text{and} \quad UU^* = I_F.$$

PROPOSITION 3.3. *Let $[E, F, U]$ be an adjointable operator. The following assertions are equivalent:*

(i) *U is a unitary operator;*

(ii) *U is an isometry and $UE = F$;*

(iii) *U, U^* are isometries.*

Proof. Using Proposition 3.1 it is sufficient to prove (ii) ⇒ (i). Being isometric and surjective, U is bijective. Furthermore

$$U^{-1} = (U^*U)U^{-1} = U^*(UU^{-1}) = U^*,$$

whence

$$UU^* = UU^{-1} = I_F,$$

that is U is unitary. ∎

DEFINITION 3.4. Let $[E, V]$ be an isometry. A closed submodule $L \subset E$ is said to be *wandering for V* if

$$V^n L \perp V^m L, \quad \text{for all } m, n \in \mathbb{N},\ m \neq n.$$

V is said to be a *shift* if there exists a wandering submodule $L \subset E$ such that

$$E = \bigoplus_{n=0}^{\infty} V^n L.$$

REMARK 3.5. (i) $L \subset E$ is a submodule wandering for the isometry $[E, V]$ if and only if

$$L \perp V^n L, \quad \text{for all } n \in \mathbb{N}^*.$$

(ii) $L = \operatorname{Ker}(V^*)$ is wandering for V because, for $l, l' \in L, n \in \mathbb{N}^*$

$$\langle l, V^n l' \rangle_E = \langle V^* l, V^{n-1} l' \rangle_E = 0.$$

REMARK 3.6. If $[E, V]$ is a shift then $V^{*n} x \to 0$, for all $x \in E$.

Indeed, let $x = \sum\limits_{n=0}^{\infty} V^n l_n \in E$, where $l_n \in L, n \in \mathbb{N}$ have the property that $\sum\limits_{n=0}^{\infty} \langle l_n, l_n \rangle_E = \sum\limits_{n=0}^{\infty} \langle V^n l_n, V^n l_n \rangle_E$ is norm convergent in A. Since $VE = \bigoplus\limits_{n=1}^{\infty} V^n L$ we obtain $E = L \oplus VE$. Using Subsections 2.2 and 2.3 $L = VE^{\perp} = \operatorname{Ker} V^*$. Acting by induction,

$$V^{*k} x = \sum_{n=0}^{\infty} V^n l_{n+k}, \quad k \in \mathbb{N}^*.$$

A simple calculation shows that

$$\|V^{*k} x\|_E = \left\| \sum_{n=k}^{\infty} \langle l_n, l_n \rangle_E \right\| \xrightarrow{k} 0.$$

REMARK 3.7. If $[E, V]$ is an isometry then $E = \operatorname{Ker} V^* \oplus VE$ and $VE = \{x \in E \mid \langle V^* x, V^* x \rangle_E = \langle x, x \rangle_E\}$. It is sufficient to observe that $I_E = (I_E - VV^*) + VV^*$ and consequently $E = (I_E - VV^*)E + VV^* E$.

If $x \in (I_E - VV^*)E$ then $V^* x \in (V^* - V^* V V^*)E = \{0\}$. Conversely if $V^* x = 0$ then $x = (I_E - VV^*)x$ and consequently $(I_E - VV^*)E = \operatorname{Ker} V^*$. Furthermore $VV^* E = VE$ because for $x \in E, Vx = VV^*(Vx) \in VV^* E$. The conclusion is obtained.

Also

$$\begin{aligned} VE &= (I_E - VV^*)E^{\perp} = \operatorname{Ker}(I_E - VV^*) = \{x \in E \mid x = VV^* x\} \\ &= \{x \in E \mid \langle (I_E - VV^*)x, x \rangle_E = 0\} = \{x \in E \mid \langle V^* x, V^* x \rangle_E = \langle x, x \rangle_E\}. \end{aligned}$$

4. THE WOLD-TYPE DECOMPOSITION

Definition 4.1. We say that an isometry $[E,V]$ admits a *Wold-type decomposition* if there exist two submodules $E_0, E_1 \subset E$ with the properties:

(i) $E = E_0 \oplus E_1$;

(ii) E_0 reduces V and $V|E_0$ is unitary;

(iii) $V|E_1$ is a shift.

Theorem 4.2. *Let $[E,V]$ be an isometry. V admits a Wold-type decomposition if and only if for all $x \in E$,*

$$(\langle V^{*n}x, V^{*n}x\rangle_E)_n \text{ is norm convergent in } A.$$

Proof. Suppose first that for every $x \in E$, $(\langle V^{*n}x, V^{*n}x\rangle_E)_n$ is norm convergent in A. Since, for $n,m \in \mathbb{N}, n > m$,

$$\begin{aligned}\|V^nV^{*n}x - V^mV^{*m}x\|_E^2 &= \Big\| \langle V^{*n}x, V^{*n}x\rangle_E - \langle V^nV^{*n}x, V^mV^{*m}x\rangle_E \\ &\quad - \langle V^mV^{*m}x, V^nV^{*n}x\rangle_E + \langle V^{*m}x, V^{*m}\rangle_E \Big\| \\ &= \|\langle V^{*m}x, V^{*m}x\rangle_E - \langle V^{*n}x, V^{*n}x\rangle_E\| \overset{m,n}{\longrightarrow} 0,\end{aligned}$$

$(V^nV^{*n}x)_n$ is Cauchy in E, so it is convergent with the limit $x_0 \in E$. Furthermore, $x_0 \in \bigcap\limits_{n\geqslant 0} V^nE$ because $V^nV^{*n}x \in \bigcap\limits_{k=0}^{n} V^kE$, for each $n \in \mathbb{N}$.

Let $L = \operatorname{Ker} V^*$. Using Remark 3.5 (ii) we could write the following sequence of equalities

$$E = L \oplus VE = L \oplus VL \oplus V^2E = \cdots = L \oplus VL \oplus \cdots \oplus V^nL \oplus V^{n+1}E, \quad n \in \mathbb{N}.$$

Consequently $x = \sum\limits_{k=0}^{n} V^kl_k + V^{n+1}z_{n+1}$, with $\{l_k\}_{k=0}^n \subset L$ and $z_{n+1} \in E$. Furthermore, a simple calculus shows that

$$l_0 = (I_E - VV^*)x, \quad l_1 = (I_E - VV^*)V^*x, \quad l_2 = (I_E - VV^*)V^{*2}x, \ldots$$

and so

$$\begin{aligned}\langle l_0, l_0\rangle_E + \langle l_1, l_1\rangle_E + \cdots + \langle l_n, l_n\rangle_E &= \sum_{k=0}^{n} \langle (I_E - VV^*)V^{*k}x, V^{*k}x\rangle_E \\ &= \sum_{k=0}^{n} (\langle V^{*k}x, V^{*k}x\rangle_E - \langle V^{*(k+1)}x, V^{*(k+1)}x\rangle_E) \\ &= \langle x, x\rangle_E - \langle V^{*(n+1)}x, V^{*(n+1)}x\rangle_E.\end{aligned}$$

We have shown that there exists $\sum_{n=0}^{\infty} V^n l_n \in \bigoplus_{n=0}^{\infty} V^n L$.

Because $z_{n+1} = V^{*(n+1)}x$ and $V^{(n+1)}V^{*(n+1)}x \xrightarrow{n} x_0$, $x - x_0 \in \bigoplus_{n=0}^{\infty} V^n L$. We have proved that

$$E = E_0 \oplus E_1, \text{ where } E_0 = \bigcap_{n \geqslant 0} V^n E, E_1 = \bigoplus_{n=0}^{\infty} V^n L$$

(the orthogonality $E_0 \perp E_1$ is immediate).

$\bigoplus_{n=0}^{\infty} V^n L$ reduces V and, using the Proposition 2.6, E_0 is also reducing for V.

Let $V_1 := V|E_1$. V_1 is a shift because $E_1 = \bigoplus_{n=0}^{\infty} V_1^n L$. Also $V_0 := V|E_0$ is unitary operator because, for $x \in E_0$,

$$V_0 V_0^* x = V_0 V_0^* V x_1 = V(V^* V) x_1 = V x_1 = x,$$

and so, V_0^* is an isometry, like V_0.

According to Definition 4.1, V admits a Wold-type decomposition.

Conversely, let $E = E_0 \oplus E_1$ be a Wold-type decomposition for V. Since $V_1 := V|E_1$ is a shift, E_1 is of the form $\bigoplus_{n=0}^{\infty} V^n L, L \subset E_1$ being a submodule wandering for V_1. Furthermore, $V_0 := V|E_0$ being unitary, for each $x \in E_0$ and $n \in \mathbb{N}$, we have $x = V^n V^{*n} x$.

Using the continuity of the inner product and the fact that L is wandering for V_1 we obtain

$$\begin{aligned} \langle V_1^* l, x \rangle_{E_1} &= \langle V_1^* l, \sum_{n=0}^{\infty} V_1^n l_n \rangle_{E_1} \\ &= \sum_{n=0}^{\infty} \langle V_1^* l, V_1^n l_n \rangle_{E_1} \\ &= \sum_{n=0}^{\infty} \langle l, V_1^{n+1} l_n \rangle_{E_1} = 0, \end{aligned} \tag{4.1}$$

for all $l \in L$ and $x \in E_1$, that is $V^* l = V_1^* l = 0$ $(l \in L)$.

Let $x \in E$. Then $x = x_0 + \sum_{n=0}^{\infty} V^n l_n$, where $x_0 \in E_0$, and $\sum_{n \geqslant 0} \langle l_n, l_n \rangle_E$

converges in norm in A. Because

$$\begin{aligned}
&(I_E - VV^*)x = x_0 - VV^*x_0 + \sum_{n=0}^{\infty}(I_E - VV^*)V^n l_n = l_0,\\
&(I_E - VV^*)V^*x = V^*x_0 - VV^{*2}x_0 + \sum_{n=0}^{\infty}(I_E - VV^*)V^n l_{n+1} = l_1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&(I_E - VV^*)V^{*n}x = l_n,
\end{aligned}$$

$n \in \mathbb{N}$, a simple calculation shows that

$$\sum_{n=0}^{n} \langle l_k, l_k\rangle_E = \langle x, x\rangle_E - \langle V^{*(n+1)}x, V^{*(n+1)}x\rangle_E, \quad n \in \mathbb{N}.$$

Consequently $(\langle V^{*n}x, V^{*n}x\rangle_E)_n$ converges for all $x \in E$.

REMARK 4.3. Because every decreasing sequence of positive numbers is convergent, as a particular case of Theorem 4.2 we obtain the classical theorem of Wold mentioned in Subsection 2.4.

REMARK 4.4. If an isometry $[E, V]$ admits a Wold-type decomposition then this decomposition is unique. Indeed, let $E = E_0 \oplus E_1$ be a Wold-type decomposition for V. Since $E_1 = \bigoplus_{n=0}^{\infty} V^n L$, L being a submodule of E_1 wandering for $V_1 = V|E_1$ and $E_0 \subset \bigcap_{n\geqslant 0} V^n E$ then, using (4.1), $E_0 \oplus E_1 \subset L + VE \subset \operatorname{Ker} V^* \oplus VE$. So $E = L \oplus VE$ and, with Subsections 2.2 and 2.3, $L = VE^{\perp} = \operatorname{Ker} V^*$. Furthermore $E_0 = E_1^{\perp} = \Big(\bigoplus_{n=0}^{\infty} V^n L\Big)^{\perp} = \bigcap_{n\geqslant 0} V^n E$. In conclusion, the Wold-type decomposition for V is

$$E = \bigcap_{n\geqslant 0} V^n E \oplus \bigoplus_{n=0}^{\infty} V^n L,$$

where $L = \operatorname{Ker} V^*$.

REMARK 4.5. If $E = E_0 \oplus E_1$ is a Wold-type decomposition for the isometry $[E, V]$ then

$$E_0 = \{x \in E \mid \langle V^{*n}x, V^{*n}x\rangle_E = \langle x, x\rangle_E, \text{ for all } n \in \mathbb{N}\}$$

and

$$E_1 = \{x \in E \mid V^{*n}x \xrightarrow{n} 0\}.$$

The structure of E_0 is obtained using Remark 3.5 (ii) and the observation above. Since V_1 is a shift, using again Remark 3.5 (ii), for $x \in E_1, V^{*n}x \xrightarrow{n} 0$. Conversely let $x \in E$ with $V^{*n}x \xrightarrow{n} 0$. Writing $x = x_0 + x_1, x_0 \in E_0, x_1 \in E_1$ we obtain immediately

$$\langle V^{*n}x, V^{*n}x\rangle_E = \langle V^{*n}x_0, V^{*n}x_0\rangle_E + \langle V^{*n}x_1, V^{*n}x_1\rangle_E, \quad n \in \mathbb{N},$$

that is

$$\langle V^{*n}x, V^{*n}x\rangle_E = \langle x_0, x_0\rangle_E + \langle V^{*n}x_1, V^{*n}x_1\rangle_E, \quad n \in \mathbb{N}. \tag{4.2}$$

By passing to the limit in (4.2) $x_0 = 0$, so $x = x_1 \in E_1$.

Remark 4.6. Consider the case where A is abelian and so, using the theorem of Gelfand, A is identified with $\mathcal{C}_0(\Omega), \Omega$ being a locally compact Hausdorff space. If, in addition, the map defined pointwise by the limit

$$\lim_{n\to\infty} \langle V^{*n}x, V^{*n}x\rangle_E(\xi), \quad x \in E,\ \xi \in \Omega,$$

is continuous, with the theorem of Dini, we obtain a Wold-type decomposition for the isometry $[E, V]$.

Example 4.7. In [8] is presented in detail the following example in which we can apply the Theorem 4.2. Using the usual notation for a contraction $[E, T]$ with its minimal unitary dilation $[F, U]$, that is

$$L = \overline{(U - T)E}, \quad L^* = \overline{(U^* - T^*)E},$$
$$M(L) = \bigoplus_{n=-\infty}^{\infty} U^n L, \quad M(L^*) = \bigoplus_{n=-\infty}^{\infty} U^n L^*,$$
$$R = M(L^*)^{\perp}, \quad R_* = M(L)^{\perp},$$

we shall consider the unitary operators $\overline{R} := U|R$ called *the residual part of* U and respectively $\overline{R}_* := U|R_*$ called *the dual ($*$-residual) part of* U.

The main result which we should mention here is the following:

The minimal isometric dilation $[F_+, U_+]$ *of* $[E, T]$ *admits a Wold-type decomposition if and only if* $M(L^*)$ *is complementable in* F *if and only if* $(\langle T^{*n}x, T^{*n}x\rangle_E)_n$ *is norm convergent in* A *for all* $x \in E$.

One of the equivalent conditions above is verified, for example, by the operator $[\ell^2(A), T]$ defined by

$$T((x_n)_n) := ((1 - a^*a)^{\frac{1}{2}}x_1 - a^*x_2, ax_1 + (1 - aa^*)^{\frac{1}{2}}x_2, 0, 0, \ldots)$$

where $a \in A$ (a unital C^*-algebra) with $\|a\| \leqslant 1$.

EXAMPLE 4.8. In contrast with the Hilbert space particular case, not every adjointable isometry on a Hilbert module admits a Wold-type decomposition. To build an example let A be a unital C^*-algebra and the Hilbert A-module $E = A$ presented in Subsection 2.1. An operator $V : A \to A$ with $V(b)^*V(b) \leqslant kb^*b$ $(b \in A)$ for some constant k has the form $V = V_a$ with $a \in A$ uniquely determinated by V where

$$V_a : A \to A, \quad V_a(b) = ab, \quad b \in A,$$

a result obtained by B.E. Johnson in [1]. Furthermore $V_a^*(b) = a^*b, b \in A$ and so V_a is an isometry if and only if $a^*a = 1$.

Adding the condition $aa^* \neq 1$ (V_a non-unitary isometry) we shall show that V_a does not admits a Wold-type decomposition. Since $\langle V_a^{*n}(b), V_a^{*n}(b)\rangle_E = b^*a^na^{*n}b$, $b \in A$, V_a admits a Wold-type decomposition if and only if $(a^na^{*n})_n$ converges in norm in A.

But, for every $n \in \mathbb{N}$, a^na^{*n} is a projection in the C^*-algebra A, $a^{n+1}a^{*(n+1)} \leqslant a^na^{*n}$ and so $a^ma^{*m} - a^na^{*n}$ is a projection for every $m, n \in \mathbb{N}$.

If there exists $n \in \mathbb{N}$ such that $a^na^{*n} = a^{n+1}a^{*(n+1)}$, then $aa^* = 1$, which is a contradiction with the choice of a.

Consequently, for $m, n \in \mathbb{N}$, $m \neq n$,

$$\|a^ma^{*m} - a^na^{*n}\| = 1.$$

$(a^na^{*n})_n$ is not Cauchy and so does not converges in norm in A.

COROLLARY 4.9. *Let $[E, V]$ be an isometry. Then V is a shift if and only if*

$$V^{*n}x \xrightarrow{n} 0, \qquad \textit{for every } x \in E.$$

Proof. Taking into account, Remark 3.5 (ii) it is sufficient to prove that if V^{*n} converges pointwise to 0 then V is a shift. According to Theorem 4.2, V admits a Wold-type decomposition $E = E_0 \oplus E_1$. Furthermore, $E_0 = \{x \in E \mid \langle V^{*n}x, V^{*n}x\rangle_E = \langle x, x\rangle_E$, for all $n \in \mathbb{N}\}$. By passing to the limit, $E_0 = \{0\}$ and so $E = E_1 = \bigoplus_{n=0}^{\infty} V^nL$, where $L = \operatorname{Ker} V^*$, that is V is a shift. ∎

DEFINITION 4.10. An operator $[E, T]$ on a Hilbert module is said to be *completely non-unitary* (*c.n.u.*) if the restriction to every submodule F reducing for T is not unitary (excepting the case $F = \{0\}$).

REMARK 4.11. (i) If $[E, V]$ is an isometry on the Hilbert module E then

$$\bigcap_{n \geqslant 0} V^nE = \{0\} \quad \text{if and only if } V \text{ is c.n.u.}$$

If $F \subset E$ is a submodule of E which reduces V to an unitary operator, then for all $x \in F$ and $n \in \mathbb{N}, x = V^n V^{*n} x \in \bigcap_{n\geqslant 0} V^n E$. Furthermore, since $\bigcap_{n\geqslant 0} V^n E$ reduces V to unitary operator one obtains the conclusion.

(ii) If $[E, V]$ is a shift, then $\bigcap_{n\geqslant 0} V^n E = \{0\}$.

A converse of this result is the following

COROLLARY 4.12. *Let* $[E, V]$ *be an isometry. If*
(i) $\bigcap_{n\geqslant 0} V^n E = \{0\}$;
(ii) $(\langle V^{*n}x, V^{*n}x\rangle_E)_n$ *converges in norm in* A *for all* $x \in E$
then V *is a shift.*

EXAMPLE 4.13. If the condition (ii) from Corollary 4.12 is not verified the conclusion is not necessarily true.

Let $\mathcal{H}$ be a Hilbert space, $A = \mathcal{L}(\mathcal{H})$ and the Hilbert module $E = \mathcal{L}(\mathcal{H})$. Let S be a shift in $\mathcal{L}(\mathcal{H})$ and the isometry $V = V_S \in \mathcal{L}_{\mathcal{L}(\mathcal{H})}(\mathcal{L}(\mathcal{H}))$. We shall prove that $\bigcap_{n\geqslant 0} V^n\mathcal{L}(\mathcal{H}) = \{0\}$ although V is not a shift.

Let $X \in \bigcap_{n\geqslant 0} V^n\mathcal{L}(\mathcal{H}), X = S^n T_n, T_n \in \mathcal{L}(\mathcal{H}), n \in \mathbb{N}$. So $T_n = S^{*n}X$, that is $(I - S^n S^{*n})X = 0$, for all $n \in \mathbb{N}$. Since $I - S^n S^{*n}$ is the orthogonal projection on $\mathrm{Ker}\,(S^{*n}), X\xi \in S^n\mathcal{H}$ for all $\xi \in \mathcal{H}$ and $n \in \mathbb{N}$. Consequently $X\xi \in \bigcap_{n\geqslant 0} S^n\mathcal{H} = \{0\}, \xi \in \mathcal{H}$, that is $X = 0$.

We have obtained that $\bigcap_{n\geqslant 0} V^n\mathcal{L}(\mathcal{H}) = \{0\}$, but V is not a shift according to Example 4.5. In conclusion the condition (i) from Corollary 4.12 is necessary, but not sufficient for $[E, V]$ to be a shift.

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