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THE CENTRAL HAAGERUP TENSOR PRODUCT OF A C^*-ALGEBRA

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Abstract. Let A be a C^*-algebra with an identity and let θ_Z be the canonical map from $A \otimes_Z A$, the central Haagerup tensor product of A, to $CB(A)$, the algebra of completely bounded operators on A. It is shown that if every Glimm ideal of A is primal then θ_Z is an isometry. This covers unital quasi-standard C^*-algebras and quotients of AW*-algebras.

Keywords: *C^*-algebra, Haagerup tensor product, primal ideal.*

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1. INTRODUCTION

If A is a C^*-algebra the *Haagerup norm* $\|\cdot\|_h$ is defined on an element x in the algebraic tensor product $A \otimes A$ by

$$\|x\|_h = \inf \Big\| \sum_{i=1}^{n} a_i a_i^* \Big\|^{1/2} \Big\| \sum_{i=1}^{n} b_i^* b_i \Big\|^{1/2},$$

where the infimum is taken over all possible representations of x as a finite sum $x = \sum\limits_{i=1}^{n} a_i \otimes b_i$, $a_i, b_i \in A$. The completion of $A \otimes A$ in this norm is called the *Haagerup tensor product* of A with itself. There is a natural contraction $\theta : A \otimes_h A \to CB(A)$ (where $CB(A)$ is the algebra of completely bounded operators on A with the completely bounded norm $\|\cdot\|_{\mathrm{cb}}$) given by $\theta\Big(\sum\limits_{i=1}^{n} a_i \otimes b_i\Big)(c) = \sum\limits_{i=1}^{n} a_i c b_i$, $c \in A$. It is clear that θ is not injective if A is not a prime C^*-algebra, but if A is prime then θ is an isometry ([3], 3.9).

Suppose that A is unital and that $z \in Z(A)$, the centre of A. Then it is easy to see that the element $az \otimes b - a \otimes zb$, $a, b \in A$, belongs to $\ker\theta$. Thus if J_A is the closed ideal of $A \otimes_h A$ generated by such elements, one can consider the induced map $\theta_Z : A \otimes_h A/J_A \to CB(A)$, and ask whether it is injective or isometric. The Banach algebra $A \otimes_h A/J_A$, with the quotient norm $\|\cdot\|_Z$, is called the *central Haagerup tensor product* of A, and denoted $A \otimes_Z A$. It is known that θ_Z is isometric if A is a von Neumann algebra or if A has Hausdorff primitive ideal space ([10]), or if A is boundedly centrally closed ([3]). On the other hand, if $Z(A) \cong \mathbf{C}$ then θ_Z is θ, so θ_Z is not injective in this case, unless A is prime. One could try factoring by $\ker\theta$, but an example in [10] shows that even this can fail to produce an isometry.

Von Neumann algebras and C^*-algebras with Hausdorff primitive ideal space and boundedly centrally closed C^*-algebras are all prominent examples of *quasi-standard* C^*-algebras, that is, C^*-algebras A for which Glimm(A) and MinPrimal(A) (defined below) coincide as topological spaces. This makes it natural to wonder if θ_Z is isometric whenever A is a unital quasi-standard C^*-algebra. The main result of this paper is that this is indeed so, and in fact we only require that Glimm(A) and MinPrimal(A) should coincide as sets. This weaker condition is always satisfied by quotients of von Neumann algebras, which need not necessarily be quasi-standard.

We also characterize the injectivity of θ_Z (every Glimm ideal of A must be 2-primal), and show that a necessary condition for θ_Z to be an isometry is that every Glimm ideal of A should be 3-primal. Thus the exact characterization of θ_Z being an isometry lies somewhere between the conditions that every Glimm ideal be 3-primal, and that every Glimm ideal be primal.

2. PRELIMINARIES

Let A be a C^*-algebra and let Id(A) denote the set of all ideals of A (ideal means closed, two-sided ideal in this paper). Then Id(A) has a natural topology τ_w obtained by taking as a sub-base all sets of the form $\{I \in \mathrm{Id}(A) : I \not\supseteq J\}$, where J is allowed to vary through Id(A). When restricted to Prim(A), the set of primitive ideals of A, τ_w is simply the hull-kernel topology. A second topology τ_s is defined on Id(A) as the weakest topology making the functions $I \to \|a + I\|$, $I \in \mathrm{Id}(A)$, continuous for all $a \in A$. This topology is stronger than τ_w, and $(\mathrm{Id}(A), \tau_s)$ is a compact, Hausdorff space (see [4] for a discussion of the history and properties of τ_w and τ_s).

Recall from [8], p. 351 that if A is a unital C^*-algebra then the *Glimm* ideals are the closed ideals of A generated by the maximal ideals of the centre of A. The set of Glimm ideals of A is denoted Glimm(A), and is equipped with the topology from the maximal ideal space of the centre of A, so that Glimm(A) is a compact, Hausdorff space, homeomorphic to the maximal ideal space of the centre of A. Thus we can identify the centre of A with the algebra of continuous complex-valued functions on Glimm(A). Furthermore, for each $a \in A$ the map $G \to \|a+G\|$ ($G \in$ Glimm(A)) is upper semi-continuous on Glimm(A) ([15], Theorem 1; [12], Lemma 9).

Let us say that an ideal I of A is *n-primal* ($n \geqslant 2$) if whenever $J_1, \ldots, J_n$ are n ideals of A with $J_1 \cdots J_n = 0$ then $J_i \subseteq I$ for at least one value of i. If I is n-primal for all n then I is *primal*. Note that prime (and hence primitive) ideals are primal. Let n-Primal(A), respectively Primal(A), denote the set of n-primal, respectively primal ideals of A. It is not difficult to see, using [5], 3.2, that a 2-primal ideal must contain a unique Glimm ideal. An ideal is n-primal if and only if the intersection of any n primitive ideals containing it is primal ([7], 1.3). It is shown in [5], p. 59 that for any n there is a C^*-algebra with an n-primal ideal which is not primal. An argument involving Zorn's Lemma shows that every primal ideal contains a minimal primal ideal. Let MinPrimal(A) denote the set of minimal closed primal ideals. Primal(A) is a τ_w-closed subset of Id(A), hence a compact Hausdorff space in the τ_s-topology, and the topologies τ_s and τ_w coincide on MinPrimal(A) ([4]).

A C^*-algebra A is said to be *quasi-standard* if MinPrimal(A) and Glimm(A) coincide, both as sets and as topological spaces. This is equivalent, for separable C^*-algebras, to A being isomorphic to a continuous field of C^*-algebras in which the set of primitive fibres is dense ([8], 3.5). Examples include AW*-algebras and C^*-algebras with Hausdorff primitive ideal space ([8]), boundedly centrally closed C^*-algebras ([19]), and the C^*-algebras of various groups, such as discrete amenable groups, see [13]. If A is a quotient of an AW*-algebra then MinPrimal(A) and Glimm(A) coincide as sets, but not necessarily as topological spaces ([18], 2.8).

3. RESULTS

We begin with a description of the central Haagerup norm, along the lines of [18], 2.3. For an ideal I in a C^*-algebra A, and for $u \in A \otimes_h A$, we shall use u^I to denote the image of u in the quotient algebra $A \otimes_h A/(I \otimes_h A + A \otimes_h I)$ (which is isometrically isomorphic to $A/I \otimes_h A/I$ by [2], 2.6).

Theorem 1. *Let A be a C^*-algebra with an identity and let $u \in A \otimes_h A$. Then*

$$\|u\|_Z = \sup\{\|u^G\|_h : G \in \mathrm{Glimm}(A)\}.$$

Hence $J_A = \bigcap\{G \otimes_h A + A \otimes_h G : G \in \mathrm{Glimm}(A)\}$.

Proof. It is enough to prove equality when u has the form $u = \sum_{i=1}^{n} a_i \otimes b_i$, with $a_i, b_i \in A$. Set $\alpha = \sup\{\|u^G\|_h : G \in \mathrm{Glimm}(A)\}$. Since $J_A \subseteq G \otimes_h A + A \otimes_h G$ for all $G \in \mathrm{Glimm}(A)$ it is clear that $\|u\|_Z \geqslant \alpha$. Suppose that $\varepsilon > 0$ is given. For each $G \in \mathrm{Glimm}(A)$ there exists, by [10], Lemma 2.3, an invertible $n \times n$ matrix S such that if $(a_i') = (a_i)S^{-1}$ and $(b_i') = S(b_i)$ then

$$\Big\|\sum_{i=1}^{n}(a_i' a_i'^* + G)\Big\|,\ \Big\|\sum_{i=1}^{n}(b_i'^* b_i' + G)\Big\| < \alpha + \varepsilon.$$

By the upper semi-continuity of the norm functions on $\mathrm{Glimm}(A)$ there is a neighbourhood N of G such that

$$\Big\|\sum_{i=1}^{n}(a_i' a_i'^* + G')\Big\|,\ \Big\|\sum_{i=1}^{n}(b_i'^* b_i' + G')\Big\| < \alpha + \varepsilon$$

for all $G' \in N$. Thus by the compactness of $\mathrm{Glimm}(A)$ there exist open subsets $\{N_j\}_{j=1}^m$ of $\mathrm{Glimm}(A)$ and invertible $n \times n$ matrices $\{S_j\}_{j=1}^m$ such that the N_j's cover $\mathrm{Glimm}(A)$ and such that if $G \in N_j$ then

$$\Big\|\sum_{i=1}^{n}(a_i^j a_i^{j*} + G)\Big\|,\ \Big\|\sum_{i=1}^{n}(b_i^{j*} b_i^j + G)\Big\| < \alpha + \varepsilon,$$

where $(a_i^j) = (a_i)S_j^{-1}$ and $(b_i^j) = S_j(b_i)$. Let $\{z_j\}_{j=1}^m$ be a partition of the identity on $\mathrm{Glimm}(A)$ subordinate to the cover $\{N_j\}_{j=1}^m$, and set

$$v = \sum_{j=1}^{m}\sum_{i=1}^{n} a_i^j z_j^{1/2} \otimes z_j^{1/2} b_i^j.$$

Then

$$v = \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} a_i^j \otimes b_i^j \Big)(z_j^{1/2} \otimes z_j^{1/2}) = \sum_{j=1}^{m} u(z_j^{1/2} \otimes z_j^{1/2}),$$

so

$$u - v = u\Big(1 - \sum_{j=1}^{m}(z_j^{1/2} \otimes z_j^{1/2})\Big) = u\Big(\sum_{j=1}^{m} z_j \otimes 1 - z_j^{1/2} \otimes z_j^{1/2}\Big)$$

$$= u\Big(\sum_{j=1}^{m}(z_j^{1/2} \otimes 1)(z_j^{1/2} \otimes 1 - 1 \otimes z_j^{1/2})\Big).$$

Hence $u - v \in J_A$. But for $G \in \mathrm{Glimm}(A)$

$$\Big\| \sum_{j=1}^{m} \sum_{i=1}^{n} \; z_j a_i^j a_i^{j*} + G \Big\| = \Big\| \sum_{j=1}^{m} (z_j + G) \Big(\sum_{i=1}^{n} a_i^j a_i^{j*} + G \Big) \Big\| < \alpha + \varepsilon,$$

and similarly for $G' \in \mathrm{Glimm}(A)$

$$\Big\| \sum_{j=1}^{m} \sum_{i=1}^{n} \; z_j b_i^{j*} b_i^j + G' \Big\| < \alpha + \varepsilon.$$

Since $\bigcap\{G : G \in \mathrm{Glimm}(A)\} = \{0\}$ it follows that

$$\|u\|_Z \leqslant \|v\|_h \leqslant \Big\| \sum_{j=1}^{m} \sum_{i=1}^{n} \; z_j a_i^j a_i^{j*} \Big\|^{1/2} \Big\| \sum_{j=1}^{m} \sum_{i=1}^{n} \; z_j b_i^{j*} b_i^j \Big\|^{1/2} < \alpha + \varepsilon,$$

as required. ∎

REMARKS. (i) A subspace X of a Banach space Y is said to be *proximinal* if every element of Y attains its distance to X. Ideals in C^*-algebras are proximinal ([1], 4.3), and so too is the centre of a unital C^*-algebra ([20]). This makes it natural to wonder if J_A is proximinal in $A \otimes_h A$.

(ii) An ideal in $A \otimes_h A$ is said to be *upper*, see [2], 6.7 (ii), if it is the intersection of the primitive ideals containing it. If J is a proper ideal of A then $J \otimes_h A + A \otimes_h J$ is upper; in fact $J \otimes_h A + A \otimes_h J = \bigcap\{P \otimes_h A + A \otimes_h Q : P, Q \in \mathrm{Prim}(A/J)\}$ ([6], 1.3). Thus Theorem 1 shows that J_A is an upper ideal, or in other words, that $A \otimes_Z A$ is a semisimple Banach algebra.

An ideal in $A \otimes_h A$ is *lower* ([2], Section 6) if it is generated by the elementary tensors that it contains, and J_A looks a good candidate, being generated by differences of elementary tensors. Since J_A is generated by elements of the form $z \otimes 1 - 1 \otimes z$, $z \in Z(A)$, it is enough to consider the case when A is an abelian C^*-algebra, but even here the answer seems to be unknown.

(iii) Let $I(A,A)$ denote the ideal in $A \otimes A$ (the algebraic tensor product) generated by elements of the form $az \otimes b - a \otimes zb$, $a, b \in A$, $z \in Z(A)$, and let $J(A,A)$ be the ideal $\bigcap\{G \otimes A + A \otimes G : G \in \mathrm{Glimm}(A)\} \subseteq A \otimes A$. Clearly $I(A,A) \subseteq J(A,A)$. It is known that $I(A,A) = J(A,A)$ if A is a continuous field of C^*-algebras over $\mathrm{Glimm}(A)$, see [9]. Theorem 1 implies that for any unital C^*-algebra $I(A,A)$ and $J(A,A)$ have the same closure, namely J_A, in $A \otimes_h A$ (and hence the same closure, namely the closure of J_A, in $A \otimes_{\min} A$, the minimal C^*-tensor product).

The next result is a combination of Lemma 3.1 and Theorem 3.4 of [6].

Proposition 2. *Let A be a C^*-algebra. For each $u \in A \otimes_h A$ the map*

$$(I,J) \to \|u + (I \otimes_h A + A \otimes_h J)\|_h, \ (I,J) \in \mathrm{Id}(A) \times \mathrm{Id}(A),$$

is continuous for the product τ_s-topology on $\mathrm{Id}(A) \times \mathrm{Id}(A)$.

The next result generalizes [18], 2.6, using the same method of proof.

Proposition 3. *Let A be a C^*-algebra and let $u \in A \otimes_h A$. Then*

$$\|\theta(u)\|_{\mathrm{cb}} = \sup\{\|u^P\|_h : P \in \mathrm{MinPrimal}(A)\}.$$

Proof. Let D denote the diagonal of $\mathrm{Primal}(A) \times \mathrm{Primal}(A)$, in the product τ_s-topology. Then D is a compact set, and the norm function $(P,P) \to \|u^P\|_h$ $((P,P) \in D)$ is continuous on D, by Proposition 2, so it attains its supremum, clearly at some (R,R) with $R \in \mathrm{MinPrimal}(A)$. But R is in the τ_s-closure of $\mathrm{Prim}(A)$ ([4] , 4.3), so $\|u^R\|_h = \sup\{\|u^P\|_h : P \in \mathrm{Prim}(A)\}$. But $\|\theta(u)\|_{\mathrm{cb}} = \sup\{\|u^P\|_h : P \in \mathrm{Prim}(A)\}$, by [3], 3.6, and the result follows. ∎

For $a \in A$, let D_a denote the inner derivation induced by a. Then $D_a = \theta(a \otimes 1 - 1 \otimes a)$, and $\|D_a\| = \|D_a\|_{\mathrm{cb}}$, see [11], 4.1. Now $\|a \otimes 1 - 1 \otimes a\|_h$ is equal to twice the distance from a to the scalars [14], 3.3, so it follows from Theorem 1 and [18], 2.3 that $\|a \otimes 1 - 1 \otimes a\|_Z = 2\, d(a, Z(A))$, where $d(a, Z(A))$ is the distance from a to the centre of A. But it was shown in [18], 3.2, 3.3 that a necessary and sufficient condition for $\|D_a\|$ to equal $2\, d(a, Z(A))$ for all $a \in A$ is that every Glimm ideal of A should be 3-primal. Thus a necessary condition for θ_Z to be an isometry is that every Glimm ideal of A should be 3-primal. Whether this is also a sufficient condition, we do not know. Our main result, however, is a partial converse. It follows from Theorem 1 and Proposition 3.

THEOREM 4. *Let A be a C^*-algebra with an identity. If every Glimm ideal of A is primal then the map $\theta_Z : A \otimes_Z A \to CB(A)$ is an isometry.*

Thus θ_Z is an isometry if A is a unital quasi-standard C^*-algebra, or a quotient of an AW*-algebra. It seems worth remarking that is very easy to show that every Glimm ideal of a von Neumann algebra is primal ([5], 4.1).

Since $A \otimes_Z A$ and $CB(A)$ are both not only Banach spaces but operator spaces, it would be interesting to know whether θ_Z is, in fact, a complete isometry.

Finally we show that the injectivity of θ_Z has a simple characterization in terms of Glimm and 2-primal ideals.

LEMMA 5. *Let A be a unital C^*-algebra, and let $R \in \mathrm{Id}(A)$. Then R is 2-primal if and only if $R \otimes_h A + A \otimes_h R \supseteq \ker\theta$.*

Proof. Suppose that R is not 2-primal. Then there exist orthogonal ideals I and J with $I, J \not\subseteq R$. If $a \in I \setminus R$ and $b \in J \setminus R$ then $a \otimes b \notin R \otimes_h A + A \otimes_h R$ but $\theta(a \otimes b) = 0$. Hence $R \otimes_h A + A \otimes_h R \not\supseteq \ker\theta$.

Conversely, suppose that R is 2-primal, and that $c \in A \otimes_h A$ with $c \notin R \otimes_h A + A \otimes_h R$. Then by [6], 1.3 there exist $P, Q \in \mathrm{Prim}(A/R)$ such that $c \notin P \otimes_h A + A \otimes_h Q$. But $S = P \cap Q$ is primal, since R is 2-primal, and $c \notin S \otimes_h A + A \otimes_h S$. This means, by Proposition 3, that $\theta(c)$ is non-zero. Hence $R \otimes_h A + A \otimes_h R \supseteq \ker\theta$. ∎

COROLLARY 6. *Let A be a unital C^*-algebra. Then*

(i) $\ker\theta = \bigcap\{R \otimes_h A + A \otimes_h R : R \in 2\text{-Primal}(A)\}$;

(ii) *θ_Z is injective if and only if every Glimm ideal of A is 2-primal.*

Proof. Set $I = \bigcap\{R \otimes_h A + A \otimes_h R : R \in 2\text{-Primal}(A)\}$.

(i) It is clear from Lemma 5 that $\ker\theta \subseteq I$. On the other hand, if $c \in I$ then $\theta(c) = 0$, by Proposition 3. Thus $I = \ker\theta$.

(ii) If every Glimm ideal of A is 2-primal then $I = J_A$, by Theorem 1, so θ_Z is injective. Conversely, if G is a Glimm ideal of A which is not 2-primal then $G \otimes_h A + A \otimes_h G \not\supseteq \ker\theta$ by Lemma 5, so θ_Z is not injective. ∎

The condition of every Glimm ideal being 2-primal has a number of equivalent formulations. For $P, Q \in \mathrm{Prim}(A)$, let $P \sim Q$ if P and Q cannot be separated by disjoint open sets, and $P \approx Q$ if P and Q cannot be separated by continuous, complex functions on $\mathrm{Prim}(A)$. Define a graph structure on $\mathrm{Prim}(A)$ by saying that P and Q are adjacent if $P \sim Q$, and let $\mathrm{Orc}(A)$ be the supremum of the diameters of the connected components of $\mathrm{Prim}(A)$ in this graph structure (with the convention that a singleton has diameter 1). The work in [17] shows that for a unital C^*-algebra A the following are equivalent:

(i) $\mathrm{Orc}(A) = 1$;
(ii) $\sim$ is an equivalence relation on $\mathrm{Prim}(A)$;
(iii) the relations $\sim$ and $\approx$ coincide on $\mathrm{Prim}(A)$;
(iv) every Glimm ideal of A is 2-primal.

One of the main results of [17] is that $\mathrm{Orc}(A) = 1$ if and only if $\|D_a\| = 2d(a, Z(A))$ for all self-adjoint $a \in A$.

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