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α-LIPSCHITZ ALGEBRAS ON THE NONCOMMUTATIVE TORUS

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Abstract. We define deformed, noncommutative versions of the Lipschitz algebras $\mathrm{Lip}^\alpha(\mathbb{T}^2)$ and $\mathrm{lip}^\alpha(\mathbb{T}^2)$. Deformation preserves the property that the former is isometrically isomorphic to the second dual of the latter.

Keywords: *Noncommutative torus, Lipschitz algebras, von Neumann algebras.*

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The algebra $\mathrm{Lip}(X)$ of Lipschitz functions on a complete metric space X plays a role in noncommutative metric theory similar to that played by the algebra $C(K)$ in noncommutative topology. For instance, there is a robust duality between metric properties of X and algebraic properties of $\mathrm{Lip}(X)$ ([24]) which matches closed subsets with weak*-closed ideals etc. Furthermore, one has an abstract characterization of Lipschitz algebras in terms of derivations of abelian von Neumann algebras into abelian operator bimodules ([26]) which admits a natural extension to the noncommutative setting. For more on noncommutative metrics see [4], [5], [6], [7], [15], [17] and for more on the particular approach described above see [26], [27], [28]. The abstract commutative theory of Lipschitz algebras is considered in [1], [2], [10], [12], [19], [20], [21], [22], [23], [24], [25], [29], among other places.

For $0 < \alpha \leqslant 1$ one calls a function $f : X \to \mathbb{C}$ *α-Lipschitz* (or *Hölder*) if it is Lipschitz with respect to the original metric on X raised to the power α. The space of α-Lipschitz functions on X is denoted $\mathrm{Lip}^\alpha(X)$. This concept is of interest in connection with little Lipschitz functions. A Lipschitz function on X is *little* if its slopes are locally null, i.e. every point has neighborhoods the restrictions of f to which have arbitrarily small Lipschitz number. The space of little Lipschitz functions (respectively, little α-Lipschitz functions) is denoted $\mathrm{lip}(X)$ (resp. $\mathrm{lip}^\alpha(X)$). In general, there may be no nonconstant little Lipschitz functions, but for $\alpha < 1$ little α-Lipschitz functions always exist in abundance. These notions have long been important in harmonic analysis, and have also played a special role in the abstract theory of Lipschitz algebras, going back to the seminal paper [8] which initiated this theory.

At the moment we have no general noncommutative versions of α-Lipschitz or little Lipschitz functions. However, we wish to show here that there are reasonable versions of both concepts in relation to the noncommutative torus ([16]). Our definitions are based on an approach to α-Lipschitz functions on the unit circle developed in [13]. Thus, we define and study deformed, noncommutative versions $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ and $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ of the classical algebras $\mathrm{Lip}^\alpha(\mathbb{T}^2)$ and $\mathrm{lip}^\alpha(\mathbb{T}^2)$. Among our results is the fact that for $\alpha < 1$ the space $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is isometrically isomorphic to the second dual of $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$. This holds in the commutative case by [2].

Our main interest in this material is that it provides a class of examples of noncommutative metrics which are not differential geometric in nature. For instance, the operator bimodule in Theorem 2.3 (ii) is not a Hilbert module; also, the derivation discussed there is not an actual differentiation. Much of what is done here generalizes immediately to the setting of an arbitrary Lie group acting on a von Neumann algebra. Another class of noncommutative metrics which are not Riemannian was given in [28].

Lipschitz functions on the noncommutative torus were discussed in [26] and some of our results here generalize work done there in the $\alpha = 1$ case.

1. THE NONCOMMUTATIVE TORUS

We begin with a review of the noncommutative torus, as described in [16] (we use different notation here). Fix a real number $\theta \in [0, 1)$ and define unitary operators $U, V \in B(l^2(\mathbb{Z}^2))$ by setting

$$Uv_{mn} = v_{(m+1)n} \qquad \text{and} \qquad Vv_{mn} = \mathrm{e}^{2\pi \mathrm{i}\theta m} v_{m(n+1)},$$

where v_{mn} is the canonical basis of $l^2(\mathbb{Z}^2)$. Let $C_\theta(\mathbb{T}^2)$ and $L^\infty_\theta(\mathbb{T}^2)$ respectively be the C^*-algebra and von Neumann algebra generated by U and V. In the $\theta = 0$ case the Fourier transform identifies $C_\theta(\mathbb{T}^2)$ and $L^\infty_\theta(\mathbb{T}^2)$ with $C(\mathbb{T}^2)$ and $L^\infty(\mathbb{T}^2)$, respectively. However, for $\theta \neq 0$ these algebras are noncommutative and our "function space" notation is merely symbolic.

For $x \in L^\infty_\theta(\mathbb{T}^2)$ and $N \geqslant 0$ define

$$s_N(x) = \sum_{|m|,\,|n| \leqslant N} a_{mn} U^m V^n$$

where $a_{mn} = \langle x v_{00}, v_{mn} \rangle$, and set

$$\sigma_N(x) = \frac{s_0 + \cdots + s_N}{N+1}.$$

These are respectively the partial sums and Cesaro means of the Fourier series of x. (For basic material on harmonic analysis see [9], [11], or [30].)

Define unbounded self-adjoint operators D_1, D_2 on $l^2(\mathbb{Z}^2)$ by

$$D_1 v_{mn} = m v_{mn} \qquad \text{and} \qquad D_2 v_{mn} = n v_{mn}.$$

For $\theta = 0$ these correspond via the Fourier transform to $\mathrm{i}\partial/\partial x$ and $\mathrm{i}\partial/\partial y$. Then we have two actions γ^1, γ^2 of $\mathbb{R}$ by automorphisms of $L^\infty_\theta(\mathbb{T}^2)$, given by

$$\gamma^k_t(x) = \mathrm{e}^{-\mathrm{i}tD_k} x \mathrm{e}^{\mathrm{i}tD_k}$$

for $k = 1, 2$. For $\theta = 0$ these correspond to translations of $L^\infty(\mathbb{T}^2)$ in the two variables.

The following was noted in [26], and is probably well-known.

PROPOSITION 1.1. (i) *γ^1 and γ^2 are ultraweakly continuous actions of $\mathbb{R}$ on $L^\infty_\theta(\mathbb{T}^2)$.*

(ii) *$C_\theta(\mathbb{T}^2)$ is stable for the actions of γ^1 and γ^2, and consists of precisely those elements of $L^\infty_\theta(\mathbb{T}^2)$ for which both actions are continuous in operator norm.*

(iii) *For any $x \in L^\infty_\theta(\mathbb{T}^2)$, $s_N(x) \to x$ ultraweakly.*

(iv) *For any $x \in C_\theta(\mathbb{T}^2)$, $\sigma_N(x) \to x$ in operator norm.*

In [26] we defined a θ-deformed version of the algebra of Lipschitz functions on $\mathbb{T}^2$ by $\mathrm{Lip}_\theta(\mathbb{T}^2) = \mathrm{dom}(\delta_1) \cap \mathrm{dom}(\delta_2)$, where δ_k $(k = 1, 2)$ is the generator of the flow γ^k, i.e. $\delta_k(x) = \mathrm{i}[D_k, x]$. This is a variation on a definition in [4]. In the $\theta = 0$ case it corresponds to precisely the algebra of Lipschitz functions on $\mathbb{T}^2$.

The following is also from [26].

THEOREM 1.2. (i) *$\mathrm{Lip}_\theta(\mathbb{T}^2)$ is a dual Banach space.*

(ii) *$\mathrm{Lip}_\theta(\mathbb{T}^2) \subset C_\theta(\mathbb{T}^2)$, densely in operator norm.*

(iii) *For any $x \in \mathrm{Lip}_\theta(\mathbb{T}^2)$, $s_N(x) \to x$ in operator norm.*

$\mathrm{Lip}_\theta(\mathbb{T}^2)$ can also be viewed in the following way. Consider $E = L^\infty_\theta(\mathbb{T}^2) \oplus L^\infty_\theta(\mathbb{T}^2)$ as a Hilbert $L^\infty_\theta(\mathbb{T}^2)$-bimodule in the natural way. Then one has an unbounded derivation $\delta : L^\infty_\theta(\mathbb{T}^2) \to E$ defined by $\delta(x) = \delta_1(x) \oplus \delta_2(x)$. This exhibits $\mathrm{Lip}_\theta(\mathbb{T}^2)$ as the domain of a natural "exterior derivative" on the noncommutative torus.

2. NONCOMMUTATIVE α-LIPSCHITZ ALGEBRAS

We retain the notation of the previous section.

Definition 2.1. Let $0 < \alpha \leqslant 1$. Then we define $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ to be the set of $x \in L^\infty_\theta(\mathbb{T}^2)$ for which there exists a constant $C \geqslant 0$ such that

$$\|x - \gamma^k_t(x)\| \leqslant Ct^\alpha$$

for $k = 1, 2$ and all $t > 0$. We let $L^\alpha(x)$ be the least such value of C and norm $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ by

$$\|x\|_\alpha = \max(\|x\|, L^\alpha(x)),$$

which we call the *Lipschitz norm*. We define $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ to be the set of $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ such that

$$\frac{\|x - \gamma^k_t(x)\|}{t^\alpha} \to 0$$

for $k = 1, 2$ as $t \to 0$.

Proposition 2.2. (i) $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ *and* $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ *are involutive Banach algebras for the Lipschitz norm* $\|\cdot\|_\alpha$.

(ii) *For* $\theta = 0$, $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ *and* $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ *are identified by means of the Fourier transform with the classical* α*-Lipschitz and little* α*-Lipschitz algebras on* $\mathbb{T}^2$*, respectively.*

Proof. (i) Checking that $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ and $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ are involutive algebras is a straightforward calculation. For instance, if x and y belong to $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ then

$$\begin{aligned}\|xy - \gamma^k_t(xy)\| &\leqslant \|xy - x\gamma^k_t(y)\| + \|x\gamma^k_t(y) - \gamma^k_t(x)\gamma^k_t(y)\|\\ &\leqslant \|x\|\,\|y - \gamma^k_t(y)\| + \|x - \gamma^k_t(x)\|\,\|\gamma^k_t(y)\|\\ &\leqslant (\|x\|L^\alpha(y) + \|y\|L^\alpha(x))t^\alpha\\ &\leqslant 2\|x\|_\alpha\|y\|_\alpha t^\alpha\end{aligned}$$

shows that $xy \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$. This also shows that $\|xy\|_\alpha \leqslant 2\|x\|_\alpha\|y\|_\alpha$, hence multiplication is continuous for the Lipschitz norm, although note that $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is not a Banach algebra in the stricter sense of satisfying $\|xy\| \leqslant \|x\|\,\|y\|$.

To see that $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is complete for the Lipschitz norm, let $(x_n) \subset \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ be Cauchy. It follows that (x_n) is Cauchy in operator norm, hence converges in this sense to some $x \in L^\infty_\theta(\mathbb{T}^2)$. For any $t > 0$ choose n such that $\|x - x_n\| \leqslant t^\alpha$; then

$$\begin{aligned}\|x - \gamma^k_t(x)\| &\leqslant \|x - x_n\| + \|x_n - \gamma^k_t(x_n)\| + \|\gamma^k_t(x_n - x)\|\\ &\leqslant t^\alpha + Ct^\alpha + t^\alpha = (C+2)t^\alpha\end{aligned}$$

where $C = \sup \|x_n\|_\alpha$. This shows that $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$. Furthermore, given $\varepsilon > 0$ choose n large enough that $\|x_m - x_n\|_\alpha \leqslant \varepsilon$ for all $m > n$. Then for any $t > 0$ we can find $m > n$ so that $\|x - x_m\| \leqslant \varepsilon t^\alpha$, and then

$$\begin{aligned}\|(x - x_n) - \gamma^k_t(x - x_n)\| &\leqslant \|(x - x_m) - \gamma^k_t(x - x_m)\| + \|(x_m - x_n) - \gamma^k_t(x_m - x_n)\| \\ &\leqslant 2\varepsilon t^\alpha + \varepsilon t^\alpha = 3\varepsilon t^\alpha.\end{aligned}$$

This shows that $L^\alpha(x_n - x) \to 0$, and as we already know $\|x_n - x\| \to 0$, it follows that $\|x_n - x\|_\alpha \to 0$. Thus, $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is complete for the Lipschitz norm.

For completeness of $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ let $(x_n) \subset \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ be Cauchy, so that by the above x_n converges in Lipschitz norm to some $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$. We must show $x \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$. Given $\varepsilon > 0$ choose n such that $\|x_m - x_n\|_\alpha \leqslant \varepsilon$ for $m > n$. Then since $x_n \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ there exists $\delta > 0$ such that $t \leqslant \delta$ implies $\|x_n - \gamma^k_t(x_n)\| \leqslant \varepsilon t^\alpha$. For any $t \leqslant \delta$ we can find $m > n$ so that $\|x - x_m\|_\alpha \leqslant \varepsilon t^\alpha$, and then

$$\begin{aligned}\|x - \gamma^k_t(x)\| &\leqslant \|x - x_m\| + \|x_n - \gamma^k_t(x_n)\| \\ &\quad + \|(x_m - x_n) - \gamma^k_t(x_m - x_n)\| + \|\gamma^k_t(x_m - x)\| \\ &\leqslant \varepsilon t^\alpha + \varepsilon t^\alpha + \varepsilon t^\alpha + \varepsilon t^\alpha = 4\varepsilon t^\alpha.\end{aligned}$$

This shows that $\|x - \gamma^k_t(x)\|/t^\alpha \to 0$ as $t \to 0$, so $x \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$.

(ii) In the $\theta = 0$ case, $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is identified with the set of functions $f \in L^\infty(\mathbb{T}^2)$ which satisfy

$$\|f - \gamma^k_t(f)\|_\infty \leqslant C t^\alpha$$

for $k = 1, 2$ and all t. That is, these are the functions which satisfy

$$\sup\{|f(x, y) - f(x + t, y)|, |f(x, y) - f(x, y + t)| : (x, y) \in \mathbb{T}^2\} \leqslant C t^\alpha$$

for all $t > 0$. This condition is automatically satisfied by any α-Lipschitz function on $\mathbb{T}^2$; conversely, for any function f which satisfies this condition we have

$$\begin{aligned}|f(x_1, y_2) - f(x_2, y_2)| &\leqslant |f(x_1, y_1) - f(x_2, y_1)| + |f(x_2, y_1) + f(x_2, y_2)| \\ &\leqslant C(d^\alpha(x_1, x_2) + d^\alpha(y_1, y_2)) \\ &\leqslant 2C d^\alpha((x_1, y_1), (x_2, y_2))\end{aligned}$$

where d denotes the ordinary Euclidean distance on $\mathbb{T}$ and $\mathbb{T}^2$, hence f is α-Lipschitz. Thus, for $\theta = 0$ we may identify $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ with the α-Lipschitz functions on $\mathbb{T}^2$.

To see that $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ is identified with the little α-Lipschitz functions, suppose that $t \leqslant \delta$ implies

$$|f(x, y) - f(x + t, y)|, |f(x, y) - f(x, y + t)| \leqslant \varepsilon t^\alpha$$

for all $(x, y) \in \mathbb{T}^2$; then $d((x_1, y_1), (x_2, y_2)) \leqslant \delta$ implies

$$\begin{aligned}|f(x_1, y_1) - f(x_2, y_2)| &\leqslant |f(x_1, y_1) - f(x_2, y_1)| + |f(x_2, y_1) - f(x_2, y_2)| \\ &\leqslant \varepsilon d^\alpha(x_1, x_2) + \varepsilon d^\alpha(y_1, y_2) \\ &\leqslant 2\varepsilon d^\alpha((x_1, y_1), (x_2, y_2)).\end{aligned}$$

Conversely, if f is a little α-Lipschitz function then for every $\varepsilon > 0$ we can find $\delta > 0$ such that for all $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$, $d((x_1, y_1), (x_2, y_2)) \leqslant \delta$ implies

$$|f(x_1, y_1) - f(x_2, y_2)| \leqslant \varepsilon d^\alpha((x_1, y_1), (x_2, y_2)).$$

(Each point has a neighborhood in which this is true, and then by compactness we can take δ to be the Lebesgue number of the resulting covering of $\mathbb{T}^2$.) In particular,

$$|f(x, y) - f(x + t, y)|, |f(x, y) - f(x, y + t)| \leqslant \varepsilon t^\alpha$$

for $t \leqslant \delta$, i.e. $\|f - \gamma_t^k(f)\|_\infty \leqslant \varepsilon t^\alpha$ for $t \leqslant \delta$. ∎

We now wish to demonstrate that the definitions given in this paper match up with our previous work, specifically, that $\mathrm{Lip}_\theta^1(\mathbb{T}^2)$ equals the Lipschitz algebra $\mathrm{Lip}_\theta(\mathbb{T}^2)$ defined in [26] (and above in Section 1), and that each $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ is a Lipschitz algebra in the sense of [26], i.e. is the domain of a von Neumann algebra derivation. For the latter, let

$$E = \bigoplus_{t>0}{}^\infty (L_\theta^\infty(\mathbb{T}^2) \oplus L_\theta^\infty(\mathbb{T}^2))$$

be the l^∞ direct sum of von Neumann algebras. It is a von Neumann algebra, and it is also a dual operator $L_\theta^\infty(\mathbb{T}^2)$-bimodule with left action given by the diagonal embedding of $L_\theta^\infty(\mathbb{T}^2)$ in E and right action given by the embedding

$$x \mapsto \bigoplus_{t>0}(\gamma_t^1(x) \oplus \gamma_t^2(x)).$$

Define an unbounded map $\delta : L_\theta^\infty(\mathbb{T}^2) \to E$ with domain $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ by $\delta = \bigoplus(\delta_t^1 \oplus \delta_t^2)$ with

$$\delta_t^k(x) = \frac{x - \gamma_t^k(x)}{t^\alpha}.$$

Notice that indeed $\delta(x) \in E$ if $x \in \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ since $\sup_{t,k} \|\delta_t^k(x)\| = L^\alpha(x) < \infty$.

Theorem 2.3. (i) $\mathrm{Lip}^1_\theta(\mathbb{T}^2) = \mathrm{Lip}_\theta(\mathbb{T}^2)$ *as sets.*

(ii) $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ *is the domain of an unbounded von Neumann algebra derivation with weak*-closed graph.*

Proof. (i) Let $x \in L^\infty_\theta(\mathbb{T}^2)$. Then $x \in \mathrm{Lip}^1_\theta(\mathbb{T}^2)$ if and only if

$$\sup_{t>0}\left\{\frac{\|x-\gamma^1_t(x)\|}{t},\ \frac{\|x-\gamma^2_t(x)\|}{t}\right\} < \infty,$$

while $x \in \mathrm{Lip}_\theta(\mathbb{T}^2)$ if and only if it belongs to the domains of the generators of γ^1 and γ^2. According to [3], Proposition 3.1.23, these two conditions are equivalent. (Note however that the norm $\|x\|_1$ defined here on $\mathrm{Lip}^1_\theta(\mathbb{T}^2)$ does not agree with the norm $\|x\|_L$ given in [26] on $\mathrm{Lip}_\theta(\mathbb{T}^2)$, although the two are equivalent.)

(ii) An easy calculation shows that the map δ defined before the theorem is linear and self-adjoint and satisfies the derivation identity (with respect to the bimodule structure described above), and its domain is $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ by definition. To check ultraweak closure of the graph of δ, suppose $x_\lambda \oplus \delta(x_\lambda)$ is a bounded net in the graph which converges ultraweakly to some element $x \oplus y \in L^\infty_\theta(\mathbb{T}^2) \oplus E$. (By the Krein-Smulian theorem, it is sufficient to consider bounded nets.) Write $y = \bigoplus(y^1_t \oplus y^2_t)$. Then for each $t > 0$ we have

$$y^k_t = \lim_\lambda \delta^k_t(x_\lambda) = \lim_\lambda \frac{(x_\lambda - \gamma^k_t(x_\lambda))}{t^\alpha} = \frac{(x-\gamma^k_t(x))}{t^\alpha}$$

$(k = 1, 2)$. As this holds for all t and

$$\sup_{t>0}\{\|y^1_t\|, \|y^2_t\|\} = \|y\| < \infty,$$

it follows that $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ and $\delta(x) = y$. Thus, the graph of δ is weak*-closed. ∎

Corollary 2.4. $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ *is a dual Banach space.*

Proof. For any $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ we have

$$\begin{aligned}\|x\|_\alpha = \max(\|x\|, L^\alpha(x)) &= \max\Big(\|x\|,\ \sup_{t,k}\frac{\|x-\gamma^k_t(x)\|}{t^\alpha}\Big)\\ &= \max(\|x\|, \|\delta(x)\|) = \|x \oplus \delta(x)\|.\end{aligned}$$

Thus, $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is linearly isometric to the graph of δ. But the latter is an ultraweakly closed subspace of $L^\infty_\theta(\mathbb{T}^2) \oplus E$, hence a dual Banach space. ∎

In consequence of this corollary $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ has a weak*-topology. In general it is distinct from the restriction of the ultraweak topology on $L^\infty_\theta(\mathbb{T}^2)$, which of course is itself a weak*-topology. To avoid confusion we shall always refer to the latter topology with the term "ultraweak" rather than "weak*".

3. RELATIONS BETWEEN α-LIPSCHITZ SPACES

In this section we investigate the various containments that obtain among the big and little α-Lipschitz spaces, the algebra of polynomials in U and V, $C_\theta(\mathbb{T}^2)$, and $L^\infty_\theta(\mathbb{T}^2)$. Corresponding statements for classical Lipschitz algebras were proved in [13] and [14] (for the unit circle) and [2] and [25] (for any compact metric space).

Our first lemma provides basic tools that we will use repeatedly. It is a noncommutative version of basic facts from harmonic analysis and was proved in [26]. Let K_N be the Fejér kernel,

$$K_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \mathrm{e}^{int} = \frac{1}{N+1} \left(\frac{\sin((N+1)t/2)}{\sin(t/2)}\right)^2.$$

It has the properties that:

(1) $K_N(t) \geqslant 0$ for all $t \in [-\pi, \pi]$;

(2) $\int_{-\pi}^{\pi} K_N(t)\,\mathrm{d}t = 1$; and

(3) for any $\varepsilon > 0$, $\int_{|t|\geqslant\varepsilon} K_N(t)\,\mathrm{d}t \to 0$ as $N \to \infty$.

Lemma 3.1. *Let $x \in L^\infty_\theta(\mathbb{T}^2)$. Then*

$$\sigma_N(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \gamma^1_s(\gamma^2_t(x)) K_N(s) K_N(t)\,\mathrm{d}s\mathrm{d}t$$

and

$$x - \sigma_N(x) = \int_{-\pi}^{\pi} (x - \gamma^1_s(x)) K_N(s)\,\mathrm{d}s + \int_{-\pi}^{\pi} \gamma^1_s\Big(\int_{-\pi}^{\pi} (x - \gamma^2_t(x)) K_N(t)\,\mathrm{d}t\Big) K_N(s)\,\mathrm{d}s,$$

where all operator integrals are taken in the ultraweak sense.

Lemma 3.2. *For any $\varepsilon > 0$ there exists N large enough that $\|x - \sigma_n(x)\| \leqslant \varepsilon$ for all $x \in \mathrm{ball}(\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2))$ and $n \geqslant N$.*

Proof. Consider the second formula in Lemma 3.1. For any $x \in \mathrm{ball}(\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2))$ we have

$$\Big\|\int_{-\pi}^{\pi} (x - \gamma^1_s(x)) K_N(s)\,\mathrm{d}s\Big\| \leqslant \int_{-\pi}^{\pi} \|x - \gamma^1_s(x)\| K_N(s)\,\mathrm{d}s \leqslant \int_{-\pi}^{\pi} |s|^\alpha K_N(s)\,\mathrm{d}s$$

and

$$\Big\| \int_{-\pi}^{\pi} \gamma_s^1\Big(\int_{-\pi}^{\pi} (x-\gamma_t^2(x))K_N(t)\mathrm{d}t\Big)K_N(s)\,\mathrm{d}s\Big\|$$
$$\leqslant \int_{-\pi}^{\pi} \Big\| \int_{-\pi}^{\pi} (x-\gamma_t^2(x))K_N(t)\,\mathrm{d}t\Big\| K_N(s)\,\mathrm{d}s = \Big\| \int_{-\pi}^{\pi} (x-\gamma_t^2(x))K_N(t)\,\mathrm{d}t\|$$
$$\leqslant \int_{-\pi}^{\pi} |t|^\alpha K_N(t)\,\mathrm{d}t.$$

Since the function $t \mapsto |t|^\alpha$ is continuous on $[-\pi,\pi]$ and vanishes at $t=0$, it follows that

$$\int_{-\pi}^{\pi} |t|^\alpha K_N(t)\,\mathrm{d}t \to 0$$

as $N \to \infty$. The second formula given in Lemma 3.1 then implies that for any $\varepsilon > 0$ we can choose N large enough that $\|x-\sigma_n(x)\| \leqslant \varepsilon$ for all $x \in \mathrm{ball}(\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2))$ and $n \geqslant N$. ∎

The next lemma was proved for $\mathrm{Lip}_\theta(\mathbb{T}^2)$ in [26]. The proof for $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ given here is essentially the same. The result in [26] can also be generalized in a different direction, in the broad setting of compact groups acting on C^*-algebras ([18]).

LEMMA 3.3. *On the unit ball of* $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ *the weak*-topology agrees with the operator norm topology.*

Proof. Both topologies are Hausdorff on $\mathrm{ball}(\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2))$, and the weak*-topology is compact. Furthermore, the weak*-topology is weaker than the operator norm topology; for if $x, x_\lambda \in \mathrm{ball}(\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2))$ and $x_\lambda \to x$ in operator norm, then in the notation of Section 2 we have $\delta_t^k(x_\lambda) \to \delta_t^k(x)$ in operator norm for each $k = 1, 2$ and $t > 0$, hence (by boundedness) $x_\lambda \oplus \delta(x_\lambda) \to x \oplus \delta(x)$ ultraweakly, i.e. $x_\lambda \to x$ weak*. Thus, it will suffice to show that the unit ball of $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ is compact in operator norm.

To see this let $(x_k) \subset \mathrm{ball}(\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2))$; we will find a subsequence which converges in operator norm. (Since the topology is metric, we may use sequences rather than nets.) Recalling the representation on $l^2(\mathbb{Z}^2)$ described in Section 1, let $a_{mn}^k = \langle x_k v_{00}, v_{mn}\rangle$ be the Fourier coefficients of x_k. Since $\|x_k\| \leqslant \|x_k\|_\alpha \leqslant 1$ it follows that $|a_{mn}^k| \leqslant 1$ for all k, m, n and so we may choose a subsequence x_{j_k} such that the coefficients $(a_{mn}^{j_k})$ converge for each index m, n.

Let x be an ultraweak cluster point of (x_{j_k}) and let a_{mn} be its Fourier coefficients; then a_{mn} is a cluster point of $(a_{mn}^{j_k})$ for each m,n. But the latter sequences have been chosen to converge, so we must have $a_{mn}^{j_k} \to a_{mn}$ for each m,n. We will show that $x_{j_k} \to x$ in operator norm.

Given $\varepsilon > 0$, by Lemma 3.2 we can choose N so that

$$\|x - \sigma_N(x)\|, \|x_{j_k} - \sigma_N(x_{j_k})\| \leqslant \varepsilon$$

for all k. By the last paragraph we can then choose M so that $k \geqslant M$ implies

$$|a_{mn} - a_{mn}^{j_k}| \leqslant \frac{\varepsilon}{(2N+1)^2}$$

for all $|m|, |n| \leqslant N$. This implies that $\|s_n(x) - s_n(x_{j_k})\| \leqslant \varepsilon$ for $n \leqslant N$ hence $\|\sigma_N(x) - \sigma_N(x_{j_k})\| \leqslant \varepsilon$. We conclude that

$$\|x - x_{j_k}\| \leqslant \|x - \sigma_N(x)\| + \|\sigma_N(x) - \sigma_N(x_{j_k})\| + \|\sigma_N(x_{j_k}) - x_{j_k}\| \leqslant 3\varepsilon$$

for $k \geqslant M$. So $x_{j_k} \to x$ in operator norm, as desired. ∎

LEMMA 3.4. (i) *Any polynomial formed from* U *and* V *and their adjoints belongs to* $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ *for all* $\alpha \leqslant 1$ *and to* $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ *for all* $\alpha < 1$.

(ii) *Let* $x \in \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ $(\alpha \leqslant 1)$. *Then* $\|\sigma_N(x)\|_\alpha \leqslant \|x\|_\alpha$ *for all* N *and* $\sigma_N(x) \to x$ *weak**.

(iii) *Let* $x \in \mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ $(\alpha < 1)$. *Then* $\sigma_N(x) \to x$ *in Lipschitz norm.*

Proof. (i) The operators U and V were defined in Section 1. Now U belongs to $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ for $\alpha < 1$ since $\gamma_t^2(U) = U$ and

$$\frac{\|U - \gamma_t^1(U)\|}{t^\alpha} = \frac{\|U - \mathrm{e}^{-\mathrm{i}t}U\|}{t^\alpha} = \frac{|1 - \mathrm{e}^{-\mathrm{i}t}|}{t^\alpha} \to 0$$

as $t \to 0$. For $\alpha = 1$ we still have $U \in \mathrm{Lip}_\theta^1(\mathbb{T}^2)$ since $|1 - \mathrm{e}^{-\mathrm{i}t}|/t$ is bounded for $t > 0$. Similar statements hold for V, and so the polynomials formed from U and V and their adjoints belong to $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ for $\alpha < 1$ and to $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ for $\alpha \leqslant 1$ by Proposition 2.2 (i).

(ii) First of all, $\sigma_N(x) \in \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ by part (i). The sequence is bounded because, using the first formula in Lemma 3.1,

$$\begin{aligned}\|\sigma_N(x)\|_\alpha &= \Big\| \int\!\!\int \gamma_s^1(\gamma_t^2(x)) K_N(s) K_N(t)\, \mathrm{d}s\mathrm{d}t \Big\|_\alpha \\ &\leqslant \int\!\!\int \|x\|_\alpha K_N(s) K_N(t)\, \mathrm{d}s\mathrm{d}t = \|x\|_\alpha.\end{aligned}$$

Weak*-convergence then follows from Lemmas 3.2 and 3.3.

(iii) We have $\sigma_N(x) \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ by part (i). Given $\varepsilon > 0$, find $\delta > 0$ such that $t \leqslant \delta$ implies $\|x - \gamma^k_t(x)\| \leqslant \varepsilon t^\alpha$. Then choose N large enough that $n \geqslant N$ implies

$$\int\limits_{|s|\geqslant\delta} K_n(s)\,\mathrm{d}s \leqslant \frac{\varepsilon\delta^\alpha}{\|x\|}.$$

We are going to estimate $\|(x - \sigma_n(x)) - \gamma^k_t(x - \sigma_n(x))\|$ (hence $L^\alpha(x - \sigma_n(x))$) for $n \geqslant N$ by using the second formula in Lemma 3.1.

For $t \leqslant \delta$ and $n \geqslant N$, we have

$$\begin{aligned}\Big\| \int ((x - \gamma^1_s(x)) - \gamma^k_t(x - \gamma^1_s(x)))K_n(s)\,\mathrm{d}s \Big\| \\ &= \Big\| \int ((x - \gamma^k_t(x)) - \gamma^1_s(x - \gamma^k_t(x)))K_n(s)\,\mathrm{d}s \Big\| \\ &\leqslant \int (\|x - \gamma^k_t(x)\| + \|\gamma^1_s(x - \gamma^k_t(x))\|)K_n(s)\,\mathrm{d}s \leqslant 2\varepsilon t^\alpha.\end{aligned}$$

For $t \geqslant \delta$, our choice of N implies that

$$\Big\| \int\limits_{|s|\geqslant\delta} ((x - \gamma^1_s(x)) - \gamma^k_t(x - \gamma^1_s(x)))K_n(s)\,\mathrm{d}s \Big\| \leqslant \int\limits_{|s|\geqslant\delta} 4\|x\|K_n(s)\,\mathrm{d}s \leqslant 4\varepsilon\delta^\alpha \leqslant 4\varepsilon t^\alpha$$

for $n \geqslant N$, while

$$\begin{aligned}\Big\| \int\limits_{|s|\leqslant\delta} ((x - \gamma^1_s(x)) - \gamma^k_t(x - \gamma^1_s(x)))K_n(s)\,\mathrm{d}s \Big\| \\ &\leqslant \int\limits_{|s|\leqslant\delta} (\|x - \gamma^1_s(x)\| + \|\gamma^k_t(x - \gamma^1_s(x))\|)K_n(s)\,\mathrm{d}s \\ &\leqslant \int\limits_{|s|\leqslant\delta} 2\varepsilon|s|^\alpha K_n(s)\,\mathrm{d}s \leqslant 2\varepsilon\delta^\alpha \leqslant 2\varepsilon t^\alpha\end{aligned}$$

for $n \geqslant N$. Thus, for any $t > 0$ we have a bound of $6\varepsilon t^\alpha$ on the first integral in the second formula in Lemma 3.1 as applied to

$$\|(x - \sigma_n(x)) - \gamma^k_t(x - \sigma_n(x))\|;$$

the second integral is bounded similarly. We conclude that $L^\alpha(x - \sigma_N(x)) \to 0$, and as we already know that $\|x - \sigma_N(x)\| \to 0$ by Lemma 3.2, it follows that $\|x - \sigma_N(x)\|_\alpha \to 0$. ∎

Theorem 3.5. (i) $\mathrm{lip}^1_\theta(\mathbb{T}^2) = \mathbb{C}$.

(ii) *The space of polynomials formed from* U *and* V *and their adjoints is Lipschitz norm dense in* $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ *for* $\alpha < 1$ *and weak*-dense in* $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ *for* $\alpha \leqslant 1$.

(iii) $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2) \subset C_\theta(\mathbb{T}^2)$ *for all* $\alpha \leqslant 1$. *If* $\alpha < 1$ *then* $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ *is operator norm (ultraweakly) dense in* $C_\theta(\mathbb{T}^2)$ $(L^\infty_\theta(\mathbb{T}^2))$*, and if* $\alpha \leqslant 1$ *then* $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ *is operator norm (ultraweakly) dense in* $C_\theta(\mathbb{T}^2)$ $(L^\infty_\theta(\mathbb{T}^2))$.

(iv) *For* $\alpha < \beta \leqslant 1$ *we have* $\mathrm{Lip}^\beta_\theta(\mathbb{T}^2) \subset \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$*, densely in Lipschitz norm.*

Proof. (i) It is clear that $\mathrm{lip}^1_\theta(\mathbb{T}^2)$ contains the constants. Conversely, for any $x \in \mathrm{Lip}^1_\theta(\mathbb{T}^2)$ we have

$$\frac{(x - \gamma^k_t(x))}{t} \to \mathrm{i}[D_k, x]$$

ultraweakly. It follows that $x \in \mathrm{lip}^1_\theta(\mathbb{T}^2)$, i.e. $\|x - \gamma^k_t(x)\|/t \to 0$, only if $[D_1, x] = [D_2, x] = 0$. But then

$$0 = \langle [D_1, x]v_{00}, v_{mn}\rangle = m\langle xv_{00}, v_{mn}\rangle$$

implies that the Fourier coefficient a_{mn} vanishes for $m \neq 0$, and similarly a_{mn} vanishes for $n \neq 0$. Thus the Fourier series of x consists of simply a constant term, and convergence of Fourier series (Lemma 3.4 (ii)) implies that x is a constant.

(ii) Containment was proved in Lemma 3.4 (i), and density follows from Lemma 3.4 (ii) and (iii).

(iii) For any $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ we have $\|x - \gamma^k_t(x)\| \leqslant L^\alpha(x)t^\alpha \to 0$ as $t \to 0$, so $x \in C_\theta(\mathbb{T}^2)$ by Proposition 1.1 (ii). This shows that $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2) \subset C_\theta(\mathbb{T}^2)$. The density assertions follow from Lemma 3.4 (i).

(iv) Suppose $x \in \mathrm{Lip}^\beta_\theta(\mathbb{T}^2)$. Then

$$\|x - \gamma^k_t(x)\| \leqslant L^\beta(x)t^\beta = (L^\beta(x)t^{\beta-\alpha})t^\alpha.$$

As $t^{\beta-\alpha} \to 0$ as $t \to 0$, this shows that $x \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$. Density follows from Lemma 3.4 (i) and (iii). ∎

4. DOUBLE DUALITY

We now aim to prove for any $\alpha < 1$ that $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is naturally isometrically isomorphic to the double dual of $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$. This was established for α-Lipschitz functions on the unit circle in [8] and later generalized to a large class of spaces by many people, most notably in [2] and [10] (see also [29]).

For $n \in \mathbb{N}$ define

$$\mathcal{A}_n = C\left(\left[\frac{\pi}{n+1}, \frac{\pi}{n}\right], C_\theta(\mathbb{T}^2)\right),$$

the C^*-algebra of continuous functions from the interval $[\pi/(n+1), \pi/n]$ into $C_\theta(\mathbb{T}^2)$. By Proposition 1.1 (ii), for any $x \in C_\theta(\mathbb{T}^2)$ the function

$$\delta^k_n : t \mapsto \frac{(x - \gamma^k_t(x))}{t^\alpha}$$

(with domain $[\pi/(n+1), \pi/n]$) belongs to $\mathcal{A}_n$, and if $x \in \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ then these functions have uniformly bounded norms. Thus, we have a map

$$\delta : \mathrm{Lip}^\alpha_\theta(\mathbb{T}^2) \to \bigoplus_n{}^\infty (\mathcal{A}_n \oplus \mathcal{A}_n)$$

into the l^∞ direct sum, defined by $\delta = \bigoplus(\delta^1_n \oplus \delta^2_n)$. Note that $x \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ precisely if $\|\delta^k_n(x)\| \to 0$ for $k = 1, 2$ as $n \to \infty$, so that δ takes $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ into the c_0 direct sum $\bigoplus_n{}^0(\mathcal{A}_n \oplus \mathcal{A}_n)$.

Now define

$$\mathcal{A} = C_\theta(\mathbb{T}^2) \oplus \bigoplus_n{}^\infty (\mathcal{A}_n \oplus \mathcal{A}_n)$$

and

$$\mathcal{B} = C_\theta(\mathbb{T}^2) \oplus \bigoplus_n{}^0 (\mathcal{A}_n \oplus \mathcal{A}_n)$$

(the l^∞ and c_0 direct sums, respectively). The map $\Gamma : x \mapsto x \oplus \delta(x)$ defines an isometric embedding of $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ in $\mathcal{A}$ and of $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ in $\mathcal{B} \subset \mathcal{A}$.

Theorem 4.1. *Let* $0 < \alpha < 1$. *Then* $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2) \cong \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)^{**}$.

Proof. We already know from Corollary 2.4 that $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$ is a dual space. We begin by defining a map from the dual of $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ into the predual of $\mathrm{Lip}^\alpha_\theta(\mathbb{T}^2)$.

Given a bounded linear functional $f \in \mathrm{lip}^\alpha_\theta(\mathbb{T}^2)^*$, we can extend it to a bounded linear functional $F \in \mathcal{B}^*$ via the embedding Γ of $\mathrm{lip}^\alpha_\theta(\mathbb{T}^2)$ in $\mathcal{B}$. Since $\mathcal{B}$ is a c_0 direct sum its dual space is an l^1 direct sum of the dual summands, i.e.

$$\mathcal{B}^* = C_\theta(\mathbb{T}^2)^* \oplus \bigoplus_n{}^1 (\mathcal{A}^*_n \oplus \mathcal{A}^*_n).$$

Therefore F has a natural action on $\mathcal{A}$, i.e. we may consider $F \in \mathcal{A}^*$, hence $F \circ \Gamma \in \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)^*$. We now must show that $F \circ \Gamma$ is weak*-continuous on $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$.

It will suffice to show that $F \circ \Gamma$ is weak*-continuous on the unit ball of $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$. We will apply Lemma 3.3. Thus, let $x, x_m \in \mathrm{ball}(\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2))$ and suppose $x_m \to x$ in operator norm. Let $\varepsilon > 0$. Writing

$$F = F_0 \oplus \bigoplus_n (F_n^1 \oplus F_n^2),$$

we may choose N large enough that $\sum_{n>N} \|F_n^k\| \leqslant \varepsilon$ for $k = 1, 2$. Also, from the definition of δ_n^k we have $\delta_n^k(x_m) \to \delta_n^k(x)$ in $\mathcal{A}_n$ for each k and n, so we may then choose M large enough that $m \geqslant M$ implies

$$\|\delta_n^k(x_m) - \delta_n^k(x)\| \leqslant \frac{\varepsilon}{N\|F_n^k\|}$$

for $k = 1, 2$ and all $n \leqslant N$. We may also take M large enough that $\|x_m - x\| \leqslant \varepsilon/\|F_0\|$ for $m \geqslant M$. It follows that $m \geqslant M$ implies

$$\begin{aligned} |F(\Gamma(x_m)) - F(\Gamma(x))| &\leqslant |F_0(x_m) - F_0(x)| + \sum_{n,k} |F_n^k(\delta_n^k(x_m)) - F_n^k(\delta_n^k(x))| \\ &\leqslant \|F_0\|\,\|x_m - x\| + \sum_{n \leqslant N,k} \|F_n^k\|\,\|\delta_n^k(x_m) - \delta_n^k(x)\| \\ &\quad + \sum_{n>N,k} \|F_n^k\|\,\|\delta_n^k(x_m) - \delta_n^k(x)\| \\ &\leqslant \varepsilon + 2N\left(\frac{\varepsilon}{N}\right) + 4\varepsilon = 7\varepsilon. \end{aligned}$$

We conclude that $F(\Gamma(x_m)) \to F(\Gamma(x))$, and this completes the proof that $F \circ \Gamma$ is weak*-continuous on $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$.

We have seen that every bounded linear functional on $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ extends to a weak*-continuous linear functional on $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$. The extension is unique by weak*-density of $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ in $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ (Theorem 3.5 (ii)). Thus we may define a map $T : \mathrm{lip}_\theta^\alpha(\mathbb{T}^2)^* \to \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)_*$ by setting $Tf = F \circ \Gamma$. This map is obviously 1-1, and it is onto since every weak*-continuous linear functional on $\mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$ restricts to a bounded linear functional on $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$, of which it is then an extension. Also it is clear that $\|Tf\| \geqslant \|f\|$, since Tf is an extension of f.

To complete the proof that $\mathrm{lip}_\theta^\alpha(\mathbb{T}^2)^* \cong \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)_*$ we must show that $\|Tf\| \leqslant \|f\|$ for any $f \in \mathrm{lip}_\theta^\alpha(\mathbb{T}^2)^*$. To see this let $x \in \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$. Then for each N, $\sigma_N(x) \in \mathrm{lip}_\theta^\alpha(\mathbb{T}^2)$ by Lemma 3.4 (i) and $\|\sigma_N(x)\|_\alpha \leqslant \|x\|_\alpha$ by Lemma 3.4 (ii), so $|f(\sigma_N(x))| \leqslant \|f\|\,\|x\|_\alpha$. But $f(\sigma_N(x)) \to (Tf)(x)$ by weak*-continuity of Tf and Lemma 3.4 (ii), so we conclude that $|(Tf)(x)| \leqslant \|f\|\,\|x\|_\alpha$ for all $x \in \mathrm{Lip}_\theta^\alpha(\mathbb{T}^2)$. Thus $\|Tf\| \leqslant \|f\|$. ∎

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