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# NOTE ON NORM CONVERGENCE IN THE SPACE OF WEAK TYPE MULTIPLIERS

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ABSTRACT. Suppose that $1 \leqslant p < \infty$, and $G$ is a locally compact abelian group with dual group $\Gamma$. Denote by $M_p^{(\mathrm{w})}(\Gamma)$ the space of weak type $(p, p)$ multipliers for $L^p(G)$. We show that the injection mapping of $M_p^{(\mathrm{w})}(\Gamma)$ into $L^\infty(\Gamma)$ is bounded. This affords a short proof that $M_p^{(\mathrm{w})}(\Gamma)$ is complete with respect to the weak type $(p, p)$ multiplier norm. When $1 < p < \infty$, the completeness of $M_p^{(\mathrm{w})}(\Gamma)$ is further demonstrated by characterizing the transforms of the weak type $(p, p)$ multipliers as the translation-invariant continuous linear mappings of $L^p(G)$ into the weak $L^p$ space of $G$. This result permits $M_p^{(\mathrm{w})}(\Gamma)$ to be supplied with a Banach space structure when $1 < p < \infty$.

KEYWORDS: *Weak type multiplier.*

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## 1. INTRODUCTION AND NOTATION

Let $G$ be a locally compact abelian group with dual group $\Gamma$ and given Haar measure $\mu_G$. For $\psi \in L^\infty(G)$, we denote by $T_\psi$ the multiplier transform on $L^2(G)$ corresponding to $\psi$: $(T_\psi f)\widehat{\ } = \psi \widehat{f}$, for all $f \in L^2(G)$. If $1 \leqslant p < \infty$, we say that $\psi$ is a multiplier of weak type $(p, p)$ provided there is a real constant $\alpha$ such that

$$\mu_G(\{x \in G : |(T_\psi f)(x)| > y\}) \leqslant \frac{\alpha^p}{y^p} \|f\|_p^p, \tag{1.1}$$

for all $f \in L^2(G) \cap L^p(G)$, and all $y > 0$. In this case, the operator $T_\psi$ extends uniquely from $L^2(G) \cap L^p(G)$ to a linear mapping $\mathcal{T}_\psi^{(p)}$ of $L^p(G)$ into the $\mu_G$-measurable functions (identified modulo equality a.e.) such that whenever $\{f_n\}_{n=1}^\infty$

converges to $f$ in $L^p(G)$, then $\{\mathcal{T}_\psi^{(p)} f_n\}_{n=1}^\infty$ converges in (Haar) measure to $\mathcal{T}_\psi^{(p)} f$. Clearly $\mathcal{T}_\psi^{(p)}$ is translation-invariant, and the inequality in (1.1) remains valid (with the same constant $\alpha$) for $\mathcal{T}_\psi^{(p)}$, all $f \in L^p(G)$, and all $y > 0$. For $1 \leqslant p < \infty$, the linear space consisting of all multipliers of weak type $(p,p)$ (identified modulo equality locally a.e.) will be denoted by $M_p^{(\mathrm{w})}(\Gamma)$, and, for $\psi \in M_p^{(\mathrm{w})}(\Gamma)$, we symbolize by $N_p^{(\mathrm{w})}(\psi)$ the smallest constant $\alpha \geqslant 0$ such that (1.1) holds. Hence $N_p^{(\mathrm{w})}(\psi)$ is, in the usual parlance, the weak type $(p,p)$ norm of $\mathcal{T}_\psi^{(p)}$ on $L^p(G)$ (which is not a true operator norm). Elementary reasoning shows that the mapping $\psi \in M_p^{(\mathrm{w})}(\Gamma) \mapsto \mathcal{T}_\psi^{(p)}$ is linear and injective, and that, for all $\varphi \in M_p^{(\mathrm{w})}(\Gamma)$, $\psi \in M_p^{(\mathrm{w})}(\Gamma)$, and $z \in \mathbb{C}$, we have $N_p^{(\mathrm{w})}(z\varphi) = |z| N_p^{(\mathrm{w})}(\varphi)$, and $N_p^{(\mathrm{w})}(\varphi + \psi) \leqslant 2[N_p^{(\mathrm{w})}(\varphi) + N_p^{(\mathrm{w})}(\psi)]$. Consequently for $1 \leqslant p < \infty$, $N_p^{(\mathrm{w})}(\cdot)$ is a quasi-norm on $M_p^{(\mathrm{w})}(\Gamma)$.

The purpose of this note is to demonstrate that $M_p^{(\mathrm{w})}(\Gamma)$ is a quasi-Banach space by establishing appropriate parallels between weak type and strong type multipliers. More specifically, the fact that the space $M_p^{(\mathrm{w})}(\Gamma)$ is complete with respect to $N_p^{(\mathrm{w})}(\cdot)$ is an immediate consequence of the following theorem (shown in Section 2). Here and henceforth, the symbol "$K$" with a (possibly empty) set of subscripts will denote a non-negative real constant which depends only on its subscripts, and which can change in value from one occurrence to another.

THEOREM 1.1. *If* $1 \leqslant p < \infty$, *and* $\psi \in M_p^{(\mathrm{w})}(\Gamma)$, *then*

$$\|\psi\|_{L^\infty(\Gamma)} \leqslant K_p N_p^{(\mathrm{w})}(\psi).$$

In Section 3 we use Lorentz space methods to show that for $1 < p < \infty$, $\{\mathcal{T}_\psi^{(p)} : \psi \in M_p^{(\mathrm{w})}(\Gamma)\}$ consists of the translation-invariant continuous linear mappings of $L^p(G)$ into the weak $L^p$ space of $G$ $(= L^{(p,\infty)}(G, \mu_G))$. It follows that when $1 < p < \infty$, $M_p^{(\mathrm{w})}(\Gamma)$ has a Banach space norm which is equivalent to $N_p^{(\mathrm{w})}(\cdot)$.

## 2. PROOF OF THEOREM 1.1

We denote by $\mathcal{C}(\Gamma)$ the Banach algebra of bounded, continuous, complex-valued functions on $\Gamma$, equipped with the norm $\|f\|_{\mathrm{u}} = \sup\{|f(\gamma)| : \gamma \in \Gamma\}$.

LEMMA 2.1. *If* $1 \leqslant p < \infty$, *and* $\psi \in M_p^{(\mathrm{w})}(\Gamma) \cap \mathcal{C}(\Gamma)$, *then* $\|\psi\|_{\mathrm{u}} \leqslant K_p N_p^{(\mathrm{w})}(\psi)$.

*Proof.* The case $1 < p < \infty$ was established in [3], Theorem (2.6). So we need only consider the case $p = 1$. Let $\Gamma_d$ denote the group $\Gamma$ endowed with the discrete topology, and let $\iota : \Gamma_d \to \Gamma$ be the identity mapping. By either [4], Theorem 1.3 or [4], Theorem 1.5, we see that there is an absolute constant $c$ such that the composite function $\psi \circ \iota \in M_1^{(\mathrm{w})}(\Gamma_d)$, with

$$N_1^{(\mathrm{w})}(\psi \circ \iota) \leqslant c N_1^{(\mathrm{w})}(\psi). \tag{2.1}$$

(If we use [4], Theorem 1.3, then the constant $c$ can be taken to be 1.) Let $\nu$ be normalized Haar measure on $b(G)$, the Bohr compactification of $G$. For $\varepsilon > 0$, and $\gamma \in \Gamma$, we see with the aid of (2.1) that

$$\nu\{x \in b(G) : |\big(T_{\psi\circ\iota}\gamma\big)(x)| > cN_1^{(\mathrm{w})}(\psi) + \varepsilon\} \leqslant \frac{cN_1^{(\mathrm{w})}(\psi)}{cN_1^{(\mathrm{w})}(\psi) + \varepsilon} < 1. \tag{2.2}$$

However, for each $x \in b(G)$, $|\big(T_{\psi\circ\iota}\gamma\big)(x)| = |\psi(\gamma)|$, and so the left member of (2.2) must be 0. It follows that $|\psi(\gamma)| \leqslant cN_1^{(\mathrm{w})}(\psi) + \varepsilon$, for all $\gamma \in \Gamma$, and $\varepsilon > 0$. Letting $\varepsilon \to 0$ completes the proof. ∎

Lemma 2.2. *Suppose that* $1 \leqslant p < \infty$, $k \in L^1(\Gamma)$, *and* $\psi \in M_p^{(\mathrm{w})}(\Gamma)$. *Then the convolution* $k * \psi \in M_p^{(\mathrm{w})}(\Gamma)$ *and*

$$N_p^{(\mathrm{w})}(k * \psi) \leqslant K_p \|k\|_{L^1(\Gamma)} N_p^{(\mathrm{w})}(\psi).$$

*Proof.* For the case $1 < p < \infty$, see [2], Lemma 2.1. The case $p = 1$ and $\psi \in M_1^{(\mathrm{w})}(\Gamma) \cap \mathcal{C}(\Gamma)$ is treated in [4], Theorem 1.1. The general case, when $p = 1$ and $\psi \in M_1^{(\mathrm{w})}(\Gamma)$ has recently been shown in [7], Theorem 1.2. (The convolution theorems for $p = 1$ in [4] and [7] are motivated by the methods of [1].) ∎

The proof of Theorem 1.1 is now readily carried out as follows. Given $\psi \in M_p^{(\mathrm{w})}(\Gamma)$ and $k \in L^1(\Gamma)$, let $k_-(\gamma) \equiv k(-\gamma)$, and use Lemmas 2.1 and 2.2 to infer that $(k_-) * \psi \in M_p^{(\mathrm{w})}(\Gamma) \cap \mathcal{C}(\Gamma)$, and

$$\Big| \int_\Gamma k(\gamma)\psi(\gamma)\,\mathrm{d}\gamma \Big| \leqslant K_p \|k\|_{L^1(\Gamma)} N_p^{(\mathrm{w})}(\psi). \tag{2.3}$$

The conclusion of Theorem 1.1 is apparent, since (2.3) holds for all $k \in L^1(\Gamma)$. Theorem 1.1 has the following consequence.

Corollary 2.3. *For* $1 \leqslant p < \infty$, *the space* $M_p^{(\mathrm{w})}(\Gamma)$ *is complete with respect to the quasi-norm* $N_p^{(\mathrm{w})}(\cdot)$.

*Proof.* Suppose that $\{\psi_n\}_{n=1}^\infty \subseteq M_p^{(\mathrm{w})}(\Gamma)$, and $N_p^{(\mathrm{w})}(\psi_m - \psi_n) \to 0$, as $m, n \to \infty$. By Theorem 1.1 there is $\psi \in L^\infty(\Gamma)$ such that $\|\psi_n - \psi\|_{L^\infty(\Gamma)} \to 0$. It follows that for each $f \in L^2(G) \cap L^p(G)$, $\{T_{\psi_n} f\}_{n=1}^\infty$ converges to $T_\psi f$ in $L^2(G)$

and hence in measure. Suppose next that $\varepsilon > 0$. Use the Cauchy condition to choose $J \in \mathbb{N}$ such that for $m \geqslant J$, $n \geqslant J$, $f \in L^2(G) \cap L^p(G)$, and $y > 0$, we have

$$\mu_G\{x \in G : |\big(T_{\psi_m} f - T_{\psi_n} f\big)(x)| > y\} \leqslant \varepsilon^p \|f\|_p^p y^{-p}. \tag{2.4}$$

For $m \geqslant J$ held fixed, we know that $(T_{\psi_m} f - T_{\psi_n} f) \to (T_{\psi_m} f - T_\psi f)$ in measure, as $n \to \infty$. Applying this fact to (2.4) we see that $N_p^{(\mathrm{w})}(\psi_m - \psi) \leqslant \varepsilon$ for all $m \geqslant J$. ∎

REMARK 2.4. Obviously $M_2^{(\mathrm{w})}(\Gamma)$ and $L^\infty(\Gamma)$ are identical as linear spaces, and, for each $\psi \in L^\infty(\Gamma)$, we have $N_2^{(\mathrm{w})}(\psi) \leqslant \|\psi\|_{L^\infty(\Gamma)}$. So in the case $p = 2$, the statement of Theorem 1.1 asserts the existence of an absolute constant $\delta$ such that

$$N_2^{(\mathrm{w})}(\psi) \leqslant \|\psi\|_{L^\infty(\Gamma)} \leqslant \delta N_2^{(\mathrm{w})}(\psi), \quad \text{for all } \psi \in L^\infty(\Gamma).$$

## 3. CHARACTERIZATION OF $M_p^{(w)}(\Gamma)$, $1 < p < \infty$, BY TRANSLATION-INVARIANT OPERATORS

In this section we shall consider $M_p^{(\mathrm{w})}(\Gamma)$ when $1 < p < \infty$. In order to place the discussion in proper perspective, we shall require some aspects of the theory of Lorentz spaces. For the general theory of Lorentz spaces associated with arbitrary measures, we refer the reader to the standard reference works in [6] and [8], Section V.3. One cautionary remark is in order here: the discussion of Lorentz spaces in [6] and [8] takes place under the blanket assumption that the underlying measure space is sigma-finite. Consequently we shall only use results from [6] and [8] which remain valid for our treatment of the general locally compact abelian group $G$ and its Haar measure $\mu_G$.

Fix $p$ in the range $1 < p < \infty$, and let $p'$ be the index conjugate to $p$. For structural considerations associated with weak type estimates, we shall utilize the Lorentz spaces $L^{(p,\infty)}(G, \mu_G)$ and $L^{(p',1)}(G, \mu_G)$. If $f$ is a measurable complex-valued function on $G$, we denote by $\varphi(f, \cdot)$ the distribution function of $f$ specified by $\varphi(f, y) = \mu_G\{x \in G : |f(x)| > y\}$, for $0 < y < \infty$. The decreasing rearrangement $f^*$ of $f$ is defined by writing, for $0 < t < \infty$, $f^*(t) = \inf\{y > 0 : \varphi(f, y) \leqslant t\}$. We further define

$$\|f\|_{p,\infty}^* = \sup\{y[\varphi(f, y)]^{\frac{1}{p}} : 0 < y < \infty\}; \quad \|f\|_{p',1}^* = (p')^{-1} \int_0^{+\infty} t^{-\frac{1}{p}} f^*(t)\, \mathrm{d}t.$$

After the usual identification of measurable functions modulo equality $\mu_G$-a.e., the Lorentz space $L^{(p,\infty)}(G, \mu_G)$ (respectively, $L^{(p',1)}(G, \mu_G)$) consists of all $f$ such that $\|f\|^*_{p,\infty} < \infty$ (respectively, $\|f\|^*_{p',1} < \infty$). In particular, each of $L^{(p,\infty)}(G, \mu_G)$ and $L^{(p',1)}(G, \mu_G)$ is a linear space containing all integrable simple functions. Moreover, as described in [6], Section 2, there are Banach space norms $\|\cdot\|_{p,\infty}$ and $\|\cdot\|_{p',1}$ on $L^{(p,\infty)}(G, \mu_G)$ and $L^{(p',1)}(G, \mu_G)$, respectively, which satisfy the relations

$$\begin{aligned} \|f\|^*_{p,\infty} &\leqslant \|f\|_{p,\infty} \leqslant p' \, \|f\|^*_{p,\infty}, \quad \text{for all } f \in L^{(p,\infty)}(G, \mu_G); \\ \|g\|^*_{p',1} &\leqslant \|g\|_{p',1} \leqslant p \|g\|^*_{p',1}, \quad \text{for all } g \in L^{(p',1)}(G, \mu_G). \end{aligned} \tag{3.1}$$

We shall be concerned with the first chain of inequalities in (3.1), since these inequalities permit us to use the quasi-norm $\|\cdot\|^*_{p,\infty}$ for weak type inequalities, while at the same time regarding $L^{(p,\infty)}(G, \mu_G)$ as a Banach space. In particular, for each $\psi \in M_p^{(\mathrm{w})}(\Gamma)$, $\mathcal{T}_\psi^{(p)}$ can be viewed as a continuous linear operator from $L^p(G, \mu_G)$ to the Banach space $L^{(p,\infty)}(G, \mu_G)$, with corresponding Banach space operator norm $\|\mathcal{T}_\psi^{(p)}\|$ satisfying

$$N_p^{(\mathrm{w})}(\psi) \leqslant \|\mathcal{T}_\psi^{(p)}\| \leqslant p' \, N_p^{(\mathrm{w})}(\psi). \tag{3.2}$$

Since $\mu_G$ need not be sigma-finite, we shall require the following variant of [6], pp. 261, 262 for duality relations.

SCHOLIUM 3.1. (see [2], Proposition 1.10) *Suppose* $1 < p < \infty$. *If* $g \in L^{(p,\infty)}(G, \mu_G)$, *and* $f \in L^{(p',1)}(G, \mu_G)$, *then*

$$\int_G |f| \, |g| \, \mathrm{d}\mu_G \leqslant p' \|g\|^*_{p,\infty} \|f\|^*_{p',1}. \tag{3.3}$$

*On the other hand, if* $g$ *is a complex-valued measurable function on* $G$, $E$ *is a sigma-finite measurable subset of* $G$, $g$ *vanishes on the complement* $G \setminus E$ *of* $E$, *and, for some real constant* $A \geqslant 0$ *we have:*

$$\Big| \int_G f g \, \mathrm{d}\mu_G \Big| \leqslant A \|f\|^*_{p',1},$$

*for every integrable simple function* $f$ *which vanishes on* $G \setminus E$, *then* $g \in L^{(p,\infty)}(G, \mu_G)$, *and* $\|g\|^*_{p,\infty} \leqslant A$.

Having set the stage, we now take up the central result of this section, which is stated as follows.

THEOREM 3.2. *Suppose that $1 < p < \infty$, and $S$ is a translation-invariant linear mapping of weak type $(p,p)$ on $L^p(G)$. Then $S = \mathcal{T}_\psi^{(p)}$ for some $\psi \in M_p^{(\mathrm{w})}(\Gamma)$.*

*Proof.* Denote by $N_p^{(\mathrm{w})}(S)$ the usual weak type $(p,p)$ norm of $S$ on $L^p(G,\mu_G)$. In what follows, we shall regard $S$ as a continuous linear mapping of $L^p(G,\mu_G)$ into $L^{(p,\infty)}(G,\mu_G)$. For each $u \in G$, we shall signify by $\tau_{\mathrm{u}}$ the corresponding translation operator on measurable functions: $\big(\tau_{\mathrm{u}}h\big)(x) = h(x+u)$, for all $x \in G$.

To begin with, suppose that $f$ and $g$ are integrable simple functions on $G$. Then, in terms of $L^p(G)$-valued Bochner integration, we have

$$f * g = \int_G f(u)\tau_{-u}g \,\mathrm{d}\mu_G(u),$$

and consequently we can use the continuity and translation-invariance of $S$ to infer that

$$S(f * g) = \int_G f(u)\tau_{-u}Sg \,\mathrm{d}\mu_G(u). \tag{3.4}$$

Now let $h$ be another integrable simple function on $G$. Since $h \in L^{(p',1)}(G,\mu_G)$, we see by Scholium 3.1 that $h$ defines a continuous linear functional on $L^{(p,\infty)}(G,\mu_G)$. Applying this fact and Fubini's Theorem to (3.4), we obtain

$$\int_G hS(f*g)\,\mathrm{d}\mu_G = \int_G h(t)\big(f*(Sg)\big)(t)\,\mathrm{d}\mu_G(t). \tag{3.5}$$

Notice that by virtue of Scholium 3.1, together with the continuity and translation-invariance of $S$, the convolution $f * (Sg)$ is a bounded continuous function on $G$. Clearly each of the functions $Sg \in L^{(p,\infty)}(G,\mu_G)$ and $f$ vanishes outside some countable union of compact sets, and hence so does $f * (Sg)$. In view of these technical observations, we can let $h$ in (3.5) run through the characteristic functions of sets of finite measure to infer that for arbitrary integrable simple functions $f$ and $g$ on $G$:

$$\begin{aligned} &f*(Sg) \text{ is a bounded continuous function on } G \text{ such that} \\ &S(f*g) = f*(Sg), \quad \text{a.e. on } G; \\ &\big(f*(Sg)\big)(t) = \big((Sf)*g\big)(t), \quad \text{for all } t \in G. \end{aligned} \tag{3.6}$$

Specializing the second equation in (3.6) to $t = 0$, we get

$$\int_G \big(Sf\big)(u)g(-u)\,\mathrm{d}\mu_G(u) = \int_G f(u)\big(Sg\big)(-u)\,\mathrm{d}\mu_G(u).$$

An application of (3.3) to the right-hand side permits us to infer that

$$\Big|\int_G (Sf)(u)g(u)\,\mathrm{d}\mu_G(u)\Big| \leqslant p'\|f\|^*_{p',1} N^{(\mathrm{w})}_p(S)\|g\|_{L^p(G)}.$$

It follows by Lebesgue space duality that for each integrable simple function $f$ on $G$, the function $Sf \in L^{p'}(G,\mu_G)$, with

$$\|Sf\|_{L^{p'}(G)} \leqslant p' N^{(\mathrm{w})}_p(S)\|f\|^*_{p',1}. \tag{3.7}$$

Suppose now that $g \in L^pG)$, and let $\{g_n\}_{n=1}^\infty$ be a sequence of integrable simple functions such that $\|g_n - g\|_{L^p(G)} \to 0$. By (3.6), we see that for an arbitrary integrable simple function $f$ and each $n \in \mathbb{N}$, we have a.e. on $G$

$$S(f * g_n) = f * (Sg_n) = (Sf) * g_n. \tag{3.8}$$

Since $S$ is of weak type $(p,p)$, the left-hand member of (3.8) obviously converges in measure to $S(f*g)$. Moreover, $Sg_n \to Sg$ in $L^{(p,\infty)}(G,\mu_G)$, and so the middle member of (3.8) converges pointwise on $G$ to $f*(Sg)$. Since $Sf \in L^{p'}(G)$, the right-hand member of (3.8) converges pointwise on $G$ to $(Sf)*g$. Clearly $(Sf)*g$ is a bounded continuous function on $G$, and it is easy to see from the translation-invariance of $S$ that $f*(Sg)$ is also bounded and continuous on $G$. These observations allow us to replace the integrable simple function $g$ in (3.6) by an arbitrary element of $L^p(G)$. Specifically, we have: for each integrable simple function $f$ on $G$ and each $g \in L^p(G)$, $f*(Sg)$ and $(Sf)*g$ are identical bounded continuous functions on $G$ such that

$$S(f*g) = f*(Sg) = (Sf)*g, \quad \text{a.e. on } G. \tag{3.9}$$

Since $\|f\|_{L^p(\mu_G)} = \|f\|^*_{L^{(p,p)}(G,\mu_G)} \leqslant \|f\|^*_{p,1}$ ([6], (1.8)), the weak type $(p,p)$ boundedness of $S$ implies directly that

$$\|Sf\|^*_{p,\infty} \leqslant N^{(\mathrm{w})}_p(S)\|f\|^*_{p,1}, \tag{3.10}$$

for every integrable simple function $f$ on $G$. Moreover, an application of Chebychev's Inequality (for $L^{p'}(\mu_G)$) to the left-hand member of (3.7) gives:

$$\|Sf\|^*_{p',\infty} \leqslant p'\, N^{(\mathrm{w})}_p(S)\|f\|^*_{p',1}, \tag{3.11}$$

for every integrable simple function $f$ on $G$.

Provided that our index $p \in (1,\infty)$ is distinct from 2, we can apply the interpolation theorem for Lorentz spaces in [8], Theorem V.3.15, p. 197 to (3.10) and (3.11), and thereby deduce that $S$ is of strong type $(2,2)$ on the linear space of integrable simple functions. It now follows from the translation-invariance of $S$ that there is $\psi \in L^\infty(G)$ such that $Sf = (\psi \widehat{f})^\vee$, for each integrable simple function $f$ on $G$. Using this and the weak type $(p,p)$ boundedness of $S$ on $L^p(G)$, we see easily that $\psi \in M_p^{(\mathrm{w})}(\Gamma)$, and $S = \mathcal{T}_\psi^{(p)}$ on $L^p(G)$.

It remains now to establish the theorem in the remaining case $p = 2$. In this case, the statement of the theorem obviously takes the following form: if $S$ is a translation-invariant linear mapping which is of weak type $(2,2)$ on $L^2(G)$, then $S$ is of strong type $(2,2)$ on $L^2(G)$. This formulation of the case $p = 2$ is known to be true in the more general context of an amenable locally compact group ([5], Theorem 5.4). In order to provide a detailed discussion of the current context, we shall present a proof for the case $p = 2$ in the setting of our locally compact abelian group $G$. We are indebted to Robert Kaufman for helpful conversations regarding the approach used in this regard. Fix an integrable simple function $f$ on $G$, and put $h = Sf$. By (3.7), $h \in L^2(G)$, and by (3.9), we have:

$$S(f * F) = F * h, \quad \text{a.e. on } G, \text{ for all } F \in L^2(G). \tag{3.12}$$

We now proceed to show that for each $k \in L^1(\Gamma) \cap L^2(\Gamma)$, the bounded continuous function $k * \widehat{h}$ defined on $\Gamma$ satisfies

$$\|k * \widehat{h}\|_{\mathrm{u}} \leqslant K \|k\|_{L^1(\Gamma)}\, N_2^{(\mathrm{w})}(S) \|f\|_{L^1(\mu_G)}. \tag{3.13}$$

In view of Theorem 1.1, it suffices for the inequality in (3.13) to prove that

$$N_2^{(\mathrm{w})}(k * \widehat{h}) \leqslant 2 \|k\|_{L^1(\Gamma)}\, N_2^{(\mathrm{w})}(S) \|f\|_{L^1(\mu_G)}. \tag{3.14}$$

For the proof of (3.14), let $F \in L^1(G) \cap L^2(G)$. Since $T_{k*\widehat{h}} F \in L^2(G)$, it vanishes outside a set of sigma-finite measure. Hence by the second part of Scholium 3.1, we need only establish that

$$\Big| \int_G (T_{k*\widehat{h}} F) \overline{W} \, \mathrm{d}\mu_G \Big| \leqslant 2 \|k\|_{L^1(\Gamma)} \|F\|_{L^2(\mu_G)}\, N_2^{(\mathrm{w})}(S) \|W\|_{2,1}^* \, \|f\|_{L^1(\mu_G)}, \tag{3.15}$$

for each integrable simple function $W$ on $G$. We can obtain (3.15) by using Plancherel's Theorem, Fubini's Theorem, (3.12), and Scholium 3.1 to reason as follows:

$$\Big|\int_G (T_{k*\widehat{h}}F)\overline{W}\,\mathrm{d}\mu_G\Big| = \Big|\int_\Gamma k(\alpha)\Big\{\int_\Gamma \widehat{F}(\gamma)\overline{\widehat{W}(\gamma)}\,\widehat{h}(\gamma-\alpha)\,\mathrm{d}\gamma\Big\}\,\mathrm{d}\alpha\Big|$$
$$= \Big|\int_\Gamma k(\alpha)\Big\{\int_G \big((\overline{\alpha}F)*h\big)(u)\alpha(u)\overline{W(u)}\,\mathrm{d}\mu_G(u)\Big\}\,\mathrm{d}\alpha\Big|$$
$$= \Big|\int_\Gamma k(\alpha)\Big\{\int_G \big(S(f*(\overline{\alpha}F))\big)(u)\alpha(u)\overline{W(u)}\,\mathrm{d}\mu_G(u)\Big\}\,\mathrm{d}\alpha\Big|$$
$$\leqslant 2N_2^{(\mathrm{w})}(S)\|F\|_{L^2(\mu_G)}\|W\|_{2,1}^{*}\|k\|_{L^1(\Gamma)}\|f\|_{L^1(\mu_G)}.$$

This completes the demonstration of (3.13).

Evaluating the convolution $k*\widehat{h}$ in (3.13) at the identity element of $\Gamma$, we infer that there is an absolute constant $\eta$ such that

$$\Big|\int_\Gamma k(\gamma)\widehat{h}(\gamma)\,\mathrm{d}\gamma\Big| \leqslant \eta\|k\|_{L^1(\Gamma)}\,N_2^{(\mathrm{w})}(S)\|f\|_{L^1(\mu_G)}, \quad \text{for all } k\in L^1(\Gamma)\cap L^2(\Gamma).$$

Recalling that $h = Sf$, we can summarize the preceding discussion as follows:

(3.16) for each integrable simple function $f$ on $G$, $(Sf)^\frown \in L^\infty(\Gamma)$, and satisfies $\|(Sf)^\frown\|_{L^\infty(\Gamma)} \leqslant \eta\, N_2^{(\mathrm{w})}(S)\|f\|_{L^1(\mu_G)}$.

It will be convenient at this juncture to introduce the following additional notation. Given a compact neighborhood $V$ of the identity element in $G$, let $\chi_V$ denote the characteristic function of $V$, and define $f_V$ on $G$ by writing $f_V = \chi_V/\mu_G(V)$. Clearly $f_V$ is an integrable simple function such that $\|f_V\|_{L^1(\mu_G)} = 1$. Now fix an integrable simple function $g$, and let $\varepsilon$ be a positive real number. Since $Sg\in L^2(G)$, standard considerations using the Generalized Minkowski Inequality show that there is a compact neighborhood $U$ of the identity in $G$ such that

$$\|Sg - f_U*(Sg)\|_{L^2(\mu_G)} \leqslant \varepsilon,$$

and consequently the triangle inequality for $L^2(G)$, followed by an application of (3.9), gives

$$\|Sg\|_{L^2(\mu_G)} \leqslant \varepsilon + \|(Sf_U)*g\|_{L^2(\mu_G)}. \tag{3.17}$$

However, $(Sf_U)*g = \big((Sf_U)^\frown\widehat{g}\big)^\vee$, and hence by (3.16) we have

$$\|(Sf_U)*g\|_{L^2(\mu_G)} \leqslant \eta\, N_2^{(\mathrm{w})}(S)\|g\|_{L^2(\mu_G)}.$$

Using this in (3.17), and letting $\varepsilon$ tend to 0, we see that

$$\|Sg\|_{L^2(\mu_G)} \leqslant \eta\, N_2^{(\mathrm{w})}(S)\|g\|_{L^2(\mu_G)},$$

for every integrable simple function $g$ on $G$. This fact, coupled with the weak type $(2,2)$ boundedness of $S$ on $L^2(G)$, shows that $S$ is of strong type $(2,2)$ on $L^2(G)$, and thereby completes the proof for the case $p = 2$. ∎

When $1 < p < \infty$, it follows from Theorem 3.2 that the injective linear mapping $\psi \in M_p^{(\mathrm{w})}(\Gamma) \mapsto \mathcal{T}_\psi^{(p)}$ sends $M_p^{(\mathrm{w})}(\Gamma)$ onto the set $\mathfrak{T}_p$ consisting of all translation-invariant continuous linear mappings of $L^p(\mu_G)$ into the Banach space $L^{(p,\infty)}(G,\mu_G)$ (furnished with the norm $\|\cdot\|_{p,\infty}$ discussed in (3.1)). Since it is clear that $\mathfrak{T}_p$ is a closed linear manifold in the space of bounded linear transformations from $L^p(\mu_G)$ into $L^{(p,\infty)}(G,\mu_G)$, we can define a Banach space norm $[\,\cdot\,]_{M_p^{(\mathrm{w})}(\Gamma)}$ on $M_p^{(\mathrm{w})}(\Gamma)$ by taking $[\psi]_{M_p^{(\mathrm{w})}(\Gamma)}$ to be the Banach space operator norm of $\mathcal{T}_\psi^{(p)}$. Hence Theorem 3.2 has the following corollary.

COROLLARY 3.3. *If $1 < p < \infty$, then $M_p^{(\mathrm{w})}(\Gamma)$ has a Banach space norm $[\,\cdot\,]_{M_p^{(\mathrm{w})}(\Gamma)}$ such that*

$$\frac{p-1}{p}\,[\psi]_{M_p^{(\mathrm{w})}(\Gamma)} \leqslant N_p^{(\mathrm{w})}(\psi) \leqslant [\psi]_{M_p^{(\mathrm{w})}(\Gamma)}, \quad \textit{for all } \psi \in M_p^{(\mathrm{w})}(\Gamma).$$

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