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RANK PRESERVING MAPS ON NEST ALGEBRAS

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Abstract. In this paper, a complete description of all weakly continuous rank-preserving linear maps on nest algebras is given. As an application, we get some results concerning local automorphisms of nest algebras.

Keywords: *Nest, nest algebra, K-rank preserving linear map.*

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1. INTRODUCTION

The linear preserving problem is one of the most active and fertile subject in matrix theory during the past one hundred years. In recent years, many authors have taken interest in the linear preserving problem on operator algebras, especially on $B(H)$, the algebra of all bounded linear operators on infinite dimentional Hilbert space, and many rich and deep results have been obtained. (see [3], [4], [5], [7], [8]). It is worth to notice that the solutions of some linear preserving problems on $B(H)$ were reduced to the description of rank-1 preserving linear maps. For example, in [3] the author recaptured and improved the related theorems in [2], [3], [4] and [8] applying the representation of rank-preserving linear maps on $B(H)$. In this paper, we will describe rank preserving maps on nest algebras.

In order to state our main results, we need some symbols and terminologies. In this paper, X will be a complex Banach space and $B(X)$ will denote the collection of all bounded linear operators on X. $\subseteq$ and $\subset$ denote inclusion and proper inclusion, respectively. For $x \in X$ and $f \in X^*$ the rank-1 operator $z \to f(z)x$ from X into X is denoted by $x \otimes f$, where X^* is the dual Banach space of X.

Definition 1.1. A nest of X is a chain $\mathcal{N}$ of closed (under norm topology) subspaces of X containing $\{0\}$ and X, which is closed under the formation of arbitrary closed linear span (denoted by $\vee$) and intersection(denoted by $\wedge$). $\mathrm{Alg}\,\mathcal{N}$ denotes the associated nest algebra, which is the set of all operators in $B(X)$ such that $TN \subseteq N$ for every element $N \in \mathcal{N}$. For $N \in \mathcal{N}$, define $N_- = \vee\{M \in \mathcal{N} : M \subset N\}$, $N_+ = \wedge\{M \in \mathcal{N} : N \subset M\}$, and $N_-^{\perp} = \{f \in X^* : f(N) = 0\}$. We also write $0_- = 0$ and $X_+ = X$.

Throughtout this paper, we always assume that $\mathcal{N}$ satisfies $0 \neq 0_+$ and $X \neq X_-$.

Definition 1.2. Let Φ be a linear map from $\mathrm{Alg}\,\mathcal{N}$ into $\mathrm{Alg}\,\mathcal{N}$. We say that Φ is a rank-k preserving map, if $\Phi(A)$ is a rank-k operator for every rank-k operator $A \in \mathrm{Alg}\,\mathcal{N}$, where k is a positive integer. We say that Φ is a rank preserving map, if Φ is a rank-k preserving map for every positive integer k.

In Section 2, a complete description of all weakly continuous rank preserving linear maps on $\mathrm{Alg}\,\mathcal{N}$ is given. We obtain that Φ is either of the form $\Phi(T) = ATB$, $A, B \in B(X)$ or $\Phi(T) = AT^*B$, $A \in B(X^*, X)$, $B \in B(X, X^*)$. It is a natural question that whether A and B are in $\mathrm{Alg}\,\mathcal{N}$ when $\Phi(T) = ATB$. In general case, the answer is negative (see Remark 3.3). But when Φ is surjective, the answer is positive. In Section 3, we show that if Φ is a weakly continuous rank preserving surjective linear map, then Φ is of the form $\Phi(T) = ATB$, $A, B \in \mathrm{Alg}\,\mathcal{N}$. In Section 4, applying Theorem 3.1, we show that all weakly continuous surjective local automorphisms on $\mathrm{Alg}\,\mathcal{N}$ are inner automorphisms. When $\dim X < \infty$ we get that the set of automorphisms on $\mathrm{Alg}\,\mathcal{N}$ is algebraically reflexive.

2. Rank Preserving Linear Maps

For $x \in X$ and $f \in X^*$, we define $L_x = \{x \otimes g : g \in X^*\}$ and $R_f = \{y \otimes f : y \in X\}$. For $N \in \mathcal{N}$, $x \in N$ and $f \in N_-^{\perp}$, we define:

$$L_x^N = \{x \otimes g : g \in N_-^{\perp}\} \quad \text{and} \quad R_f^N = \{y \otimes f : y \in N\}.$$

We begin with some lemmas.

Lemma 2.1. *If Φ is a rank-1 preserving linear map on $\mathrm{Alg}\,\mathcal{N}$, then one of the following holds:*

(i) *For every $N \in \mathcal{N}$ and every $x \in X$, there exists $y(x) \in X$ such that $\Phi(L_x^N) \subseteq L_{y(x)}$.*

(ii) *For every* $N \in \mathcal{N}$ *and every* $x \in X$, *there exists* $g(x) \in X^*$ *such that* $\Phi(L_x^N) \subseteq R_{g(x)}$.

Proof. We will devide the proof into two steps.

Step 1. Let M be a fixed element in $\mathcal{N}$ and $x_0 \in M$. Then $\Phi(L_{x_0}^M) \subseteq L_{y_0}$ for some $y_0 \in X$ or $\Phi(L_{x_0}^M) \subseteq R_{g_0}$ for some $g_0 \in X^*$.

First we claim that $\dim \Phi(L_{x_0}^M) = 1$ if and only if $\dim M_-^\perp = 1$. The sufficiency is obvious, so we need only to show that the condition is neccesary. Since $\dim \Phi(L_{x_0}^M) = 1$, $\Phi(x_0 \otimes f) = \alpha(f) y_0 \otimes g_0$ for every $f \in M_-^\perp$. If $\dim M_-^\perp > 1$, then there exist two linear independent elements $f_1, f_2 \in M_-^\perp$ such that $x \otimes [\alpha(f_1) f_2 - \alpha(f_2) f_1]$ is a rank one operator, but $\Phi[x \otimes (\alpha(f_1) f_2 - \alpha(f_2) f_1)] = 0$, a contradiction.

Thus we may assume that $\dim M_-^\perp \geqslant 2$. If neither $\Phi(L_{x_0}^M) \subseteq L_{y_0}$ nor $\Phi(L_{x_0}^M) \subseteq R_{g_0}$, then there exist $f_1, f_2 \in M_-^\perp$ such that $\Phi(x_0 \otimes f_1) = x_1 \otimes g_1$ and $\Phi(x_0 \otimes f_2) = x_2 \otimes g_2$, where x_1 and x_2 are linearly independent, g_1 and g_2 also are. It follows that $x_1 \otimes g_1 + x_2 \otimes g_2$ is rank-2 and $\Phi[x_0 \otimes (f_1 + f_2)] = x_1 \otimes g_1 + x_2 \otimes g_2$. But $\Phi[x_0 \otimes (f_1 + f_2)]$ is rank-1. This is a contradiction.

Step 2. We will show that the case (i) of Lemma 2.1 holds in the case $\Phi(L_{x_0}^M) \subseteq L_{y_0}$.

We first show $\Phi(L_x^M) \subseteq L_{y(x)}$ for every $x \in M$. To do this, let $M_1 = \{x \in M : \Phi(L_x^M) \subseteq L_{y(x)}\}$. $M_2 = \{x \in M : \Phi(L_x^M) \subseteq R_{g(x)}; \dim \Phi(L_x^M) \geqslant 2\}$. It is easy to see that $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \emptyset$. So it is enough to show $M_2 = \emptyset$. Otherwise, there exists a nonzero element $x_1 \in M_2$ such that $\Phi(L_{x_1}^M) \subseteq R_{g_1}$ for some $g_1 \in X^*$. Obviously $x_0 + x_1 \in M$, so $x_0 + x_1 \in M_2$ or $x_0 + x_1 \in M_1$. If $x_0 + x_1 \in M_2$, then $\Phi[(x_0 + x_1) \otimes f] = y_2(f) \otimes g_2$ for some $g_2 \in X^*$ and every $f \in M_-^\perp$. Since $\Phi(x_0 \otimes f) = y_0 \otimes g_0(f)$ and $\Phi(x_1 \otimes f) = y_1(f) \otimes g_1$ for every $f \in M_-^\perp$, it follows that $y_0 \otimes g_0(f) + y_1(f) \otimes g_1 = y_2(f) \otimes g_2$. Since $x_1 \in M_2$ then $y_1(f)$ and y_0 are linearly independent for some $f \in M_-^\perp$, hence $g_2 = b g_1$ for some $b \in \mathbb{C}$. Furthermore $g_0(f) \in \{b g_1 : b \in \mathbb{C}\}$, that is $\dim \Phi(L_{x_0}^M) = 1$. This is a contradiction. If $x_0 + x_1 \in M_1$, similarly one can get that $\dim \Phi(L_{x_1}^M) = 1$. This is also a contradiction. So we have shown that $M_2 = \emptyset$.

Now let N be an arbitrary element in $\mathcal{N}$. We also assume that $\dim N_-^\perp \geqslant 2$. We will prove that $\Phi(L_x^N) \subseteq L_{y(x)}$ for every $x \in N$. In fact, if $M \subseteq N$, then $\Phi(L_{x_0}^N) \subseteq \Phi(L_{x_0}^M) \subseteq L_{y_0}$. Thus $\Phi(L_x^N) \subseteq L_{y(x)}$ for every $x \in N$ by the previous proof. If $N \subseteq M$, then $\Phi(L_x^M) \subseteq \Phi(L_x^N)$ for every $x \in N$. Since $\Phi(L_x^M) \subseteq L_{y(x)}$ and $\dim \Phi(L_x^M) \geqslant 2$ for every $x \in N$, we have $\Phi(L_x^N) \subseteq L_{y(x)}$ for every $x \in N$.

Similarly one can show that the case (ii) of Lemma 2.1 holds in the case that $\Phi(L_{x_0}^M) \subseteq R_{g_0}$. ∎

Lemma 2.2. *If Φ is a rank-1 preserving linear map on* $\mathrm{Alg}\,\mathcal{N}$*, then one of the following holds:*

(i) *There exist a linear map A from $\bigcup\{N \in \mathcal{N} : N_- \neq X\}$ into X and a linear map B from $\bigcup\{N_-^\perp : N \neq 0\}$ into X^*, such that $\Phi(x \otimes f) = Ax \otimes Bf$ for every rank-1 operator $x \otimes f \in \mathrm{Alg}\,\mathcal{N}$.*

(ii) *There exist a linear map A from $\bigcup\{N_-^\perp : N \neq 0\}$ into X and a linear map B from $\bigcup\{N \in \mathcal{N} : N_- \neq X\}$ into X^*, such that $\Phi(x \otimes f) = Ax \otimes Bf$ for every rank-1 operator $x \otimes f \in \mathrm{Alg}\,\mathcal{N}$.*

(iii) *There exist $y_0 \in X$ and a linear map $\lambda(\cdot)$ from $\bigcup\{L_x^N : N \in \mathcal{N},\ x \in N\}$ into X^*, such that $\Phi(x\otimes f) = y_0 \otimes \lambda(x\otimes f)$ for every rank-1 operator $x\otimes f \in \mathrm{Alg}\,\mathcal{N}$.*

(iv) *There exist $g_0 \in X^*$ and a linear map $\delta(\cdot)$ from $\bigcup\{L_x^N : N \in \mathcal{N},\ x \in N\}$ into X, such that $\Phi(x\otimes f) = \delta(x\otimes f)\otimes g_0$ for every rank-1 operator $x\otimes f \in \mathrm{Alg}\,\mathcal{N}$.*

Proof. We will prove that the case (i) or (iii) of Lemma 2.2 holds in the case (i) of Lemma 2.1.

Claim 1. If $\dim\{y(x) : x \in N,\ N \in \mathcal{N}\} = 1$, then the case (iii) of Lemma 2.2 holds.

Since $\dim\{y(x) : x \in N,\ N \in \mathcal{N}\} = 1$, there exists $y_0 \in X$ such that $y(x) = \alpha(x)y_0$ and $\Phi(x \otimes f) = y(x) \otimes g_x(f) = y_0 \otimes \alpha(x)g_x(f)$, where $\alpha(x)$ is a complex number which is dependent on x. Hence we can define a linear map $\lambda(\cdot)$ from $\bigcup\{L_x^N : N \in \mathcal{N},\ x \in N\}$ into X^* such that $\lambda(x \otimes f) = \alpha(x)g_x(f)$. So we get $\Phi(x \otimes f) = y_0 \otimes \lambda(x \otimes f)$ for every rank-1 operator $x \otimes f \in \mathrm{Alg}\,\mathcal{N}$.

Claim 2. If $\dim\{y(x) : x \in N,\ N \in \mathcal{N}\} > 1$, then the case (i) of Lemma 2.2 holds.

First we observe that for every $N \in \mathcal{N}$ and arbitrary two linearly independent elements $x_1, x_2 \in N$, $y(x_1)$ and $y(x_2)$ are linearly independent. Otherwise, there exist $M \in \mathcal{N}$ and two linearly independent elements $x_1, x_2 \in M$ such that $y(x_1)$ and $y(x_2)$ are linearly dependent, say $y(x_1) = a_1y_0$ and $y(x_2) = a_2y_0$, thus we have $\Phi(x_1 \otimes f) = y_0 \otimes a_1g_1(f)$ and $\Phi(x_2 \otimes f) = y_0 \otimes a_2g_2(f)$. Hence $\Phi[(x_1 + ax_2) \otimes f] = y_0 \otimes [a_1g_1(f) + aa_2g_2(f)]$ for arbitrary $a \in \mathbb{C}$. Since $y_0 \otimes [a_1g_1(f) + aa_2g_2(f)]$ is rank-1, $g_1(f)$ and $g_2(f)$ are linearly independent for every $f \in M_-^\perp$. Now for every $N \in \mathcal{N}$ and $x \in N$, we have $\Phi(x \otimes f) = y(x) \otimes g_x(f)$. So $\Phi[(x_1 + x) \otimes f] = y_0 \otimes a_1g_1(f) + y(x) \otimes g_x(f)$ and $\Phi[(x_2 + x) \otimes f] = y_0 \otimes a_2g_2(f) + y(x) \otimes g_x(f)$ for every $f \in M_-^\perp \cap N_-^\perp$. Since Φ preserves rank-1, $y(x) \in \{ay_0 : a \in \mathbb{C}\}$, that is $\dim\{y(x) : x \in N,\ N \in \mathcal{N}\} = 1$, which is a contradiction.

Now for every $N \in \mathcal{N}$ and $x \in N$, we can define a linear map g_x from $N_-^\perp$ into X^* by $\Phi(x \otimes f) = y(x) \otimes g_x(f)$. We claim that $\dim\{g_x : x \in N\} = 1$. In fact, we may assume that $\dim N \geqslant 2$, then we can find two linearly independent elements

$x_1, x_2 \in N$ such that $y(x_1)$ and $y(x_2)$ are independent. Since $\Phi[(x_1+x_2)\otimes f] = y(x_1+x_2)\otimes g_{12}(f) = y(x_1)\otimes g_1(f) + y(x_2)\otimes g_2(f)$ is a rank-1 operator, where g_1, g_2 and g_{12} denote the maps defined by x_1, x_2 and x_1+x_2 respectively, we have $g_1(f)$ and $g_2(f)$ are linearly dependent for every $f \in N_-^\perp$. Hence $g_1, g_2 \in \{ag_{12} : a \in \mathbb{C}\}$. It is obvious that $y(x_1)$ and $y(x)$, $y(x_2)$ and $y(x)$ must not be dependent simultaneously for any $x \in N$. We may assume that $y(x_1)$ and $y(x)$ are linearly independent, then $g_x \in \{ag_{12} : a \in \mathbb{C}\}$. Let g_x absorb a suitable constant and denote it by $B_{N_-^\perp}$ which is a linear map from $N_-^\perp$ into X^*. We also get a linear map A_N from N into X such that $\Phi(x\otimes f) = A_N x \otimes B_{N_-^\perp} f$ for every $x \in N$ and $f \in N_-^\perp$.

Finally, for arbitrary two elements $M, N \in \mathcal{N}$, we have $\Phi(x\otimes f) = A_N x \otimes B_{N_-^\perp} f = A_M x \otimes B_{M_-^\perp} f$ for every $x \in M\cap N$ and $f \in M_-^\perp \cap N_-^\perp$. We may assume $N \subset M$, it easilly follows that $A_M|N = a_{MN}A_N$ and $B_{N_-^\perp}|M_-^\perp = a_{MN}B_{M_-^\perp}$ for some $0 \neq a_{MN} \in \mathbb{C}$. Fix $M \in \mathcal{N}$; for every $N \in \mathcal{N}$, we define:

$$\begin{aligned}
&\widetilde{A}_N = A_N \quad \text{and} \quad \widetilde{B}_{N_-^\perp} = B_{N_-^\perp} && \text{for} \quad N = M,\\
&\widetilde{A}_N = a_{MN}A_N \quad \text{and} \quad \widetilde{B}_{N_-^\perp} = \frac{1}{a_{MN}}B_{N_-^\perp} && \text{for} \quad N \subset M,\\
&\widetilde{A}_N = \frac{1}{a_{NM}}A_N \quad \text{and} \quad \widetilde{B}_{N_-^\perp} = a_{NM}B_{N_-^\perp} && \text{for} \quad N \supset M.
\end{aligned}$$

It is easy to verify that $\{\widetilde{A}_N : N \in \mathcal{N},\ N_- \neq X\}$ and $\{\widetilde{B}_{N_-^\perp} : N \in \mathcal{N},\ N \neq 0\}$ are two compatible map varieties. Hence we get two linear maps $A : \bigcup\{N \in \mathcal{N},\ N_- \neq X\} \to X$ such that $A|N = \widetilde{A}_N$ and $B : \bigcup\{N_-^\perp : N \in \mathcal{N},\ N \neq 0\} \to X^*$ such that $B|N_-^\perp = \widetilde{B}_{N_-^\perp}$ for every $N \in \mathcal{N}$, which satisfy the requirement of Lemma 2.2.

Similarly one can show that the case (ii) or (iv) holds in the case (ii) of Lemma 2.1. ∎

Theorem 2.3. *Let Φ be a weakly continuous rank-1 preserving linear map on* $\operatorname{Alg}\mathcal{N}$. *Then one of the following holds:*

(i) *There exist A and C in $B(X)$ such that $\Phi(T) = ATC$.*

(ii) *There exist $A \in B(X^*, X)$ and $C \in B(X, X^*)$ such that $\Phi(T) = AT^*C$.*

(iii) *There exist a weakly-weakly star continuous linear map $\lambda(\cdot)$ from* $\operatorname{Alg}\mathcal{N}$ *into X^* and $y_0 \in X$ such that $\Phi(T) = y_0 \otimes \lambda(T)$.*

(iv) *There exist a weakly-weakly continuous linear map $\delta(\cdot)$ from* $\operatorname{Alg}\mathcal{N}$ *into X and $g_0 \in X^*$ such that $\Phi(T) = \delta(T)\otimes g_0$.*

Proof. Note that one of the four cases of Lemma 2.2 occurs under the condition of Theorem 2.3.

1° If the case (i) of Lemma 2.2 occurs, then the case (i) of Theorem 2.3 holds. We may assume that $0 \neq 0_+$, then the linear map B in Lemma 2.2 is from X^* into X^*. It is easy to show that B is a closed operator since Φ is weakly continuous. Hence B lies in $B(X^*, X^*)$ by the Closed Graph Theorem. Let $C = J^{-1}B^*J$ where J is the canonical map from X into X^{**}, then $C \in B(X)$ and $\Phi(x \otimes f) = A(x \otimes f)C$ for every rank-1 operator $x \otimes f \in \mathrm{Alg}\,\mathcal{N}$.

Now we will show that the map A in Lemma 2.2 is bounded. Since Φ is weakly continuous and $\Phi(x \otimes f) = Ax \otimes Bf$ for every $N \in \mathcal{N}$ and $f \in N_-^{\perp}$, then

$$\|Ax\|\,\|Bf\| = \|Ax \otimes Bf\| = \|\Phi(x \otimes f)\| \leqslant \|\Phi\|\,\|x\|\,\|f\|.$$

Since B is bounded, $\sup\{\frac{\|Ax\|}{\|x\|} : N \in \mathcal{N}, x \in N\}\|B\| \leqslant \|\Phi\|$. Hence A is bounded and can be extended to a bounded linear operator on X, still denote it by A. So we have found two operators A and C in $B(X)$ such that $\Phi(x \otimes f) = A(x \otimes f)C$ for every rank-1 operator in $\mathrm{Alg}\,\mathcal{N}$. Recall one result in [10] that any finite rank operator in $\mathrm{Alg}\,\mathcal{N}$ can be represented as a sum of rank-1 operators in $\mathrm{Alg}\,\mathcal{N}$, we have $\Phi(F) = AFC$ for every finite rank operator in $\mathrm{Alg}\,\mathcal{N}$ for Φ is linear. Recalling another result in [9] that the set of all finite rank operator in $\mathrm{Alg}\,\mathcal{N}$ is dense in the strong operator topology. Since Φ is weakly continous,then $\Phi(T) = ATC$ for every $T \in \mathrm{Alg}\,\mathcal{N}$.

2° Similar to the proof of 1°, the case (ii) of Theorem 2.3 holds in the case (ii) of Lemma 2.2.

3° If the case (iii) of Lemma 2.2 occurs, then the case (iii) of Theorem 2.3 holds. Since Φ is weakly continuous and linear, the linear map $\lambda(\cdot)$ in Lemma 2.2 can be extended to a weakly-weakly star continuous map from $\mathrm{Alg}\,\mathcal{N}$ into X^* by an argument analogous to that in 1°.

4° If the case (iv) of Lemma 2.2 occurs, similar to the proof of 3°, one can show that the case (iv) of Theorem 2.3 holds. ∎

Applying Theorem 2.3, we easily obtain the following result:

COROLLORY 2.4. *Let Φ be an weakly continuous rank-1 preserving linear map on $\mathrm{Alg}\,\mathcal{N}$. If there exists $T_0 \in \mathrm{Alg}\,\mathcal{N}$ such that $\mathrm{rank}\,\Phi(T_0) > 1$, then one of the following holds:*

(i) *There exist A and C in $B(X)$ such that $\Phi(T) = ATC$.*

(ii) *There exist A in $B(X^*, X)$ and C in $B(X, X^*)$ such that $\Phi(T) = AT^*C$.*

THEOREM 2.5. *If Φ is a linear map on $\mathrm{Alg}\,\mathcal{N}$, and there exists T_0 in $\mathrm{Alg}\,\mathcal{N}$ such that $\mathrm{rank}\,\Phi(T_0) > 1$, then Φ is a weakly continuous rank-1 preserving map if and only if one of the following holds:*

(i) *There exist an injective operator* $A \in B(X)$ *and a dense range operator* $C \in B(X)$ *such that* $\Phi(T) = ATC$.

(ii) *There exist an injective operator* $A \in B(X^*, X)$ *and a dense range operator* $C \in B(X, X^*)$ *such that* $\Phi(T) = AT^*C$.

Proof. The sufficiency is easily obtained. We only prove the necessity. Under the condition of Theorem 2.5 one of the cases of Corollary 2.4 holds. If the case (i) of Corollary 2.4 occurs, then the operator A in Corollary 2.4 is injective. Otherwise, there exists $x \in X$ with $x \neq 0$ such that $Ax = 0$. Let $f \in X^*$ such that $x \otimes f \in \operatorname{Alg}\mathcal{N}$, then $\Phi(x \otimes f) = Ax \otimes C^*f = 0$. This is a contradiction. Similarly one can show that C^* is injective. Hence the range of C is dense. The case (ii) of Theorem 2.5 holds in the case (ii) of Corollory 2.4. ∎

THEOREM 2.6. Φ *is an weakly continuous rank preserving linear map on* $\operatorname{Alg}\mathcal{N}$ *if and only if one of the following holds:*

(i) *There exist an injective operator* $A \in B(X)$ *and a dense range operator* $C \in B(X)$ *such that* $\Phi(T) = ATC$.

(ii) *There exist an injective operator* $A \in B(X^*, X)$ *and a dense range operator* $C \in B(X, X^*)$ *such that* $\Phi(T) = AT^*C$.

Proof. By Theorem 2.5, we need only to prove that the condition is sufficient. If the case (i) occurs, then for an arbitrary rank-n operator $F \in \operatorname{Alg}\mathcal{N}$ there exist two linearly independent sets $\{x_i : 1 \leqslant i \leqslant n\} \subseteq X$ and $\{f_i : 1 \leqslant i \leqslant n\} \subseteq X^*$, such that $F = \sum_{i=1}^{n} x_i \otimes f_i$ and $x_i \otimes f_i \in \operatorname{Alg}\mathcal{N}$ for $1 \leqslant i \leqslant n$. Since A is injective and the range C is dense, A and C^* are injective. Hence $\Phi(F) = \sum_{i=1}^{n} Ax_i \otimes C^*f_i$ also is rank-n. If the case (ii) occurs, similarly one can show that Φ is a weakly continuous rank preserving linear map. ∎

By Theorem 2.5 and Theorem 2.6 we can get the following result:

COROLLARY 2.7. *Let* Φ *be weakly continuous. Then* Φ *is a rank preserving map on* $\operatorname{Alg}\mathcal{N}$ *if and only if* Φ *is a rank-*1 *preserving map and there exists* $T_0 \in \operatorname{Alg}\mathcal{N}$ *such that* $\operatorname{rank}\Phi(T_0) > 1$.

REMARK 2.8. When $\mathcal{N}$ is trivial, the associated nest algebra $\operatorname{Alg}\mathcal{N}$ is just $B(X)$. Hence we get the main results in [3].

3. RANK PRESERVING SURJECTIVE MAPS

Lemma 3.1. *Let Φ be an injective linear map on* $\mathrm{Alg}\mathcal{N}$ *such that* $\{\Phi(x\otimes f): x\otimes f\in \mathrm{Alg}\mathcal{N}\}=\{y\otimes g: y\otimes g\in \mathrm{Alg}\mathcal{N}\}$, $\mathcal{N}_0$ *denotes* $\mathcal{N}\setminus\{N\in\mathcal{N}: N=N_-\neq N^+\}$. *Then one of the following holds:*

(i) *For every element* $N\in\mathcal{N}_0$ *and* $x\in N$, *there exists* $y(x)\in N$ *such that* $\Phi(L_x^N)=L_{y(x)}^N$.

(ii) *For every element* $N\in\mathcal{N}_0$ *and* $x\in N$, *there exists* $g(x)\in N_-^\perp$ *such that* $\Phi(L_x^N)=R_{g(x)}^N$.

The case (ii) *will not occur when N is nontrivial.*

Proof. First we verify some claims.

Claim 1. Let $N\in\mathcal{N}_0$ such that $\dim N_-^\perp\geqslant 1$. Let $x_0\in N$ such that N is the smallest element containing x_0 in $\mathcal{N}_0$. If $\Phi(L_x^N)\subseteq L_{y(x)}$ for every $x\in N$, then there exists $M\in\mathcal{N}$ such that $y(x)\in M$ and $\Phi(L_x^N)=L_{y(x)}^M$ for every $x\in N$.

Let M be the smallest element containg y_0 in $\mathcal{N}_0$. We first show that $y_0\in M$ and $\Phi(Lx_0^N)=L_{y_0}^M$. Otherwise, $\{g(f): f\in N_-^\perp\}\subset M_-^\perp$. We can find $g_0\in M_-^\perp\setminus\{g(f): f\in N_-^\perp\}$ and $y'\otimes g'\in\mathrm{Alg}\mathcal{N}$ such that $\Phi(y'\otimes g')=y_0\otimes g_0$. So $\Phi(y'\otimes g'+x_0\otimes f)=y_0\otimes[g_0+g(f)]$ for every $f\in N_-^\perp$, hence $y'\otimes g'+x_0\otimes f$ is rank-1. Since $\dim N_-^\perp>1$, we must have that x_0 and y' are linearly dependent. Let $y'=\alpha x_0$ and $g''=\alpha g'$, then $y'\in N$, $g''\in N_-^\perp$ and $\Phi(x\otimes g'')=\Phi(y'\otimes g')=y_0\otimes g_0$. Thus $g_0=g(g'')\in\{g(f): f\in N_-^\perp\}$. This is a contradiction.

Now for arbitrary $x\in N$, we may assume $x\notin\{\alpha x_0;\alpha\in\mathbb{C}\}$. For every $f\in N_-^\perp$, we have $\Phi(x\otimes f)=y(x)\otimes g_x(f)$ and $\Phi(x_0\otimes f)=y_0\otimes g_0(f)$. It follows that $y(x)\otimes g_x(f)+y_0\otimes g_0(f)$ is rank-1. So $g_0(f)$, $g_x(f)$ or $y(x)$, y_0 are linearly dependent. Since x and x_0 are linearly independent, $y(x)$ and y_0 are linearly independent. We must have $g_0(f)$, $g_x(f)$ are linearly dependent, therefore $\{g_x(f): f\in N_-^\perp\}=\{g_0(f): f\in N_-^\perp\}=M_-^\perp$. Hence $\Phi(L_x^N)=L_{y(x)}^M$. Let N_x and M_x are the smallest element containing x and $y(x)$ in $\mathcal{N}_0$ respectively, then $L_{y(x)}^M=\Phi(L_x^N)\subseteq\Phi(L_x^{N_x})=L_{y(x)}^{M_x}$. Since M_x and M are in $\mathcal{N}_0$, we have $M_x\subseteq M$, hence $y(x)\in M$.

Claim 2. $y_0\in N$.

If $y_0\notin N$, then $N\subset M$. Since $N\in\mathcal{N}_0$, we also have $M_-^\perp\subset N_-^\perp$. By Claim 1, we can get $\{\Phi(x\otimes f): x\in N, f\in N_-^\perp\}\not\subseteq\{y\otimes g: y\in N, g\in N_-^\perp\}$. Thus we can find $z\in N$, $g\in N_-^\perp\setminus M_-^\perp$ and $y\notin N$, $f\in L_-^\perp\subseteq N_-^\perp$ (here L is the smallest element containing y in $\mathcal{N}$, it is easy to see $N\subset L$ and $L_-^\perp\subseteq N_-^\perp$) such that $\Phi(y\otimes f)=z\otimes g$. Thus $\Phi(x_0\otimes f+y\otimes f)=y_0\otimes g(f)+z\otimes g$ is rank-2. This is a contradiction.

Claim 3. Not only $y_0 \in N$, but also $\Phi(L_{x_0}^N) = L_{y_0}^M = L_{y_0}^N$. Furthermore $y(x) \in N$ and $\Phi(L_x^N) = L_{y(x)}^N$ for every $x \in N$.

If $L_{y_0}^M \neq L_{y_0}^N$, then $M \subset N$ and $N_-^\perp \subset M_-^\perp$. Since $\Phi^{-1}(L_{y_0}^M) = L_{x_0}^N$ and Φ^{-1} has the same property as Φ, using Claim 1 and Claim 2, we have $x_0 \in M$. Hence $N \subset M$. This is a contradiction. By Claim 1 again, we get $y(x) \in N$ and $\Phi(L_x^N) = L_{y(x)}^N$ for every $x \in N$.

When $\dim N_-^\perp = 1$ and $\Phi(L_x^N) \subseteq L_{y(x)}$, it is easy to see $N = X$, hence $y(x) \in N$ and $\Phi(L_x^N) = L_{y(x)}^N$. Now we have proved the case (i) of Lemma 3.1 holds in the case (i) of Lemma 2.1.

If the case (ii) of Lemma 2.1 occurs, similarly one can show the case (ii) of Lemma 3.1 holds.

Finally, we need to show that the case (ii) of Lemma 3.1 does not occur when $\mathcal{N}$ is nontrivial. Since $\mathcal{N}$ is nontrivial, there exists $N \in \mathcal{N}_0$ such that $N \neq 0$ and $N \neq X$. If $\Phi(L_x^N) = R_{g(x)}^N$, we can find $M \in \mathcal{N}_0$ such that $N \subset M$ and $M_-^\perp \neq 0$, then $\Phi(L_x^M) \subseteq \Phi(L_x^N)$, hence $R_{g(x)}^M \subseteq R_{g(x)}^N$. Thus $M \subseteq N$. This is a contradiction.

In the following part, we always assume $\mathcal{N}$ is nontrivial.

LEMMA 3.2. *Let Φ be an injective linear map on $\operatorname{Alg}\mathcal{N}$ such that $\{\Phi(x\otimes f) : x\otimes f \in \operatorname{Alg}\mathcal{N}\} = \{y\otimes g : y\otimes g \in \operatorname{Alg}\mathcal{N}\}$. Then there exist two bijective linear maps A from $\bigcup\{N \in \mathcal{N}_0,\ N_- \neq X\}$ onto itself and B from $\bigcup\{N_-^\perp : N \in \mathcal{N}_0,\ N \neq 0\}$ onto itself, such that $\Phi(x\otimes f) = Ax\otimes Bf$ for every rank-1 operator $x\otimes f \in \operatorname{Alg}\mathcal{N}$.*

Proof. Under the condition of Lemma 3.2, only the case (i) of Lemma 2.1 occurs by Lemma 2.2 and Lemma 3.1. To prove Lemma 3.2, we need only to show A_N is a bijective map from N onto N and B_N is a bijective map from $N_-^\perp$ onto $N_-^\perp$ for every $N \in \mathcal{N}_0$, where A_N and $B_{N_-^\perp}$ are as in Lemma 2.2. First, the range of A_N and the range of $B_{N_-^\perp}$ are in N and $N_-^\perp$ respectively, by Lemma 3.1 and the definitions of A_N and $B_{N_-^\perp}$. Second, note that Φ is bijective when Φ is restricted to the set of all rank-1 operators in $\operatorname{Alg}\mathcal{N}$, it is easily verified that A_N and $B_{N_-^\perp}$ are bijective by Lemma 3.1 and the construction of A_N and $B_{N_-^\perp}$.

Note that $\bigvee\{N : N \in \mathcal{N}_0$ and $N_- \neq X\} = X$ and $\bigvee\{N_-^\perp : N \in \mathcal{N}_0$ and $N \neq 0\} = X^*$, we have the following theorems analogous to Theorem 2.3 and Theorem 2.6.

THEOREM 3.3. *Φ is an weakly continuous rank-1 preserving injective linear map on $\operatorname{Alg}\mathcal{N}$ such that $\{\Phi(x\otimes f) : x\otimes f \in \operatorname{Alg}\mathcal{N}\} = \{y\otimes g : y\otimes g \in \operatorname{Alg}\mathcal{N}\}$ if and only if there exist invertible operators A and C in $\operatorname{Alg}\mathcal{N}$ such that $\Phi(T) = ATC$.*

THEOREM 3.4. *Φ is an weakly continuous rank preserving surjective linear map if and only if there exist invertible operators A and C in $\operatorname{Alg}\mathcal{N}$ such that*

$\Phi(T) = ATC$. Here A and C are unique, respectively, in the sense that A and αA ($\alpha \in \mathbb{C}$ and $\alpha \neq 0$) are regarded as the same.

REMARK 3.5. If we omit the condition "Φ is surjective", then the conclusion of Theorem 3.4 will not hold. For example, let H be a separable complex Hilbert space, $\{e_1, e_2, \dots, e_n, \dots\}$ be an orthogonal basis of H. Let $N_0 = 0$, $N_1 = \operatorname{span}\{e_1\}, \dots, N_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}, \dots, \mathcal{N} = \{N_0, N_1, \dots, N_n, \dots, H\}$, then $\mathcal{N}$ is a nest of H. The associated nest algebra

$$\operatorname{Alg}\mathcal{N} = \left\{ T = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ & x_{22} & x_{23} & \cdots \\ & & x_{33} & \cdots \\ & & & \cdots \end{bmatrix};\ x_{ij} \in \mathbb{C},\ 1 \leqslant i \leqslant j < \infty, \text{ and } T \text{ is bounded} \right\}.$$

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Define a map on $\operatorname{Alg}\mathcal{N}$ by $\Phi(T) = ATB$. Then Φ is a rank preserving map, but ϕ is not surjective. And Φ is of the form $\Phi(T) = ATB$, but A is not in $\operatorname{Alg}\mathcal{N}$.

REMARK 3.6. All of the theorems in Section 2 and Section 3 hold when X is reflexive and Φ is strongly continuous.

4. LOCAL AUTOMORPHISMS

In [6] the authors obtained the description of surjective local automorphisms of $B(X)$. Here, when $\mathcal{N}$ is non-trival, we can easily get the description of surjective local automorphisms on $\operatorname{Alg}\mathcal{N}$ by Theorem 3.1. Since an automorphism on $\operatorname{Alg}\mathcal{N}$ is inner (see [10]), then a surjective local automorphism is a surjective rank preserving map, and maps identity into identity. Hence we obtain the following theorem:

THEOREM 4.1. *Φ is an weakly continuous surjective local automorphism on $\operatorname{Alg}\mathcal{N}$ if and only if there exists an invertible element A of $\operatorname{Alg}\mathcal{N}$ such that $\Phi(T) = ATA^{-1}$. Furthermore, all weakly coutinuous surjective local automorphisms on $\operatorname{Alg}\mathcal{N}$ are inner automorphisms.*

COROLLARY 4.2. *If* $\dim X < \infty$*, then the set of all automorphisms on* $\mathrm{Alg}\,\mathcal{N}$ *is algebraically reflexive.*

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