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LINEARIZATION, COMPLETENESS, AND SPECTRAL ASYMPTOTICS FOR CERTAIN RATIONAL AND MEROMORPHIC OPERATOR FUNCTIONS

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Abstract. In this paper rational operator functions of type

$$L(\lambda) := I - \sum_{k=0}^{n} \lambda^k A_k + \sum_{k=1}^{m} \frac{1}{\lambda - a_k} H_k$$

are considered. With the aid of a linearization of L results on the completeness of the eigenvectors and associated vectors, and spectral asymptotics are given. The results are extended to certain meromorphic operator functions.

Keywords: *Operator function, linearization, completeness, spectral asymptotics.*

MSC (2000): 47A56.

1. INTRODUCTION

In the first part of this paper we consider rational operator functions of type

$$L(\lambda) := I - \sum_{k=0}^{n} \lambda^k A_k + \sum_{k=1}^{m} \frac{1}{\lambda - a_k} H_k, \quad \lambda \in \Omega, \tag{1.1}$$

$\Omega := \mathbb{C} \setminus \{a_1, a_2, \ldots, a_m\}$, where the coefficients A_k and H_k act on a Banach space X and satisfy some further conditions. In Section 2 we linearize the operator function in the sense of [10] by using a linearization of [14] for the polynomial part of (1.1), and by using a refinement of a linearization of the rational part of (1.1) given in [19]. This means that there exist operator functions $\mathfrak{E}, \mathfrak{F} : \Omega \to B(X^K)$ with invertible values such that

$$L(\lambda) \oplus \mathfrak{I}_{X^{K-1}} = \mathfrak{E}(\lambda)(\mathfrak{I}_{X^K} - \mathfrak{T} - \lambda\mathfrak{H})\mathfrak{F}(\lambda), \quad \lambda \in \Omega.$$

From this representation basic spectral properties of L in Ω are derived. Furthermore with the aid of a further linearization of the same type for an operator function connected with L we prove that the eigenvalues $\lambda \in \{a_1, a_2, \ldots, a_m\}$ of L and $\mathfrak{L}(\lambda) := \mathfrak{I}_{X^K} - \mathfrak{T} - \lambda\mathfrak{H}$, their geometric multiplicities, their partial null multiplicities, and their null multiplicities coincide; the connections between the eigenvectors and Jordan chains of L and $\mathfrak{L}$ (which are used in the subsequent section) are described in detail. In Section 3 we use the results of Section 2 to prove in the Hilbert space case that the eigenvectors and associated vectors corresponding to eigenvalues $\lambda \in \mathbb{C}$ of L are complete, and we give some asymptotics of the eigenvalues. In Section 4 we consider meromorphic operator functions of type

$$L_\infty(\lambda) := I - \sum_{k=0}^{n} \lambda^k A_k + \sum_{k=1}^{\infty} \frac{1}{\lambda - a_k} H_k, \quad \lambda \in \Omega^\infty, \tag{1.2}$$

$\Omega^\infty := \mathbb{C} \setminus \{a_1, a_2, \ldots\}$, where the coefficients A_k and H_k again act on a Banach space X and additionally satisfy some further conditions. We extend the linearization of the rational operator function L of Section 2 to the meromorphic operator function L_∞ under certain convergence conditions, and we establish analogous results for L_∞ which correspond to the results for L in Section 2. In Section 5 by using the results of Section 4 also the completeness of the eigenvectors and associated vectors corresponding to eigenvalues $\lambda \in \mathbb{C}$ of L_∞ is shown under certain convergence conditions. Furthermore we give some asymptotics of the eigenvalues.

Starting with the paper of Keldysh ([13]) there are many contributions to the completeness of eigenvectors and associated vectors and spectral asymptotics of operator polynomials (especially Keldysh pencils) and other operator functions (cf. [18]). Also rational and meromorphic operator functions are considered (cf. [1], [2], [3], [4], [5], [6], [7], [17], [19], [20], [21], [22], [23]). But our results differ from the others by our method of linearization and the consideration of the eigenvalues a_j in the sense of [12], [8], [9] under the poles of the rational and meromorphic operator functions. We also extend a result in [19] on rational operator functions of type (1.1).

2. LINEARIZATION (THE RATIONAL CASE)

Let X be a (complex) Banach space with norm $\|\cdot\|$. We denote by $B(X)$ the Banach space of all linear and bounded operators on X. The identity operator on X is denoted by I. Furthermore we denote by $B_\infty(X)$ the Banach space of all linear and compact operators on X. For $A \in B(X)$ let $N(A)$ denote the null space, and let $R(A)$ denote the range of A. The resolvent set of A is defined by $\rho(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ has an inverse in } B(X)\}$, and the spectrum $\sigma(A)$ is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. $\lambda \in \mathbb{C}$ is called an eigenvalue of A, if there is $f \neq 0$ such that $Af = \lambda f$. For an operator function $\widehat{L} : \widehat{\Omega} \subset \mathbb{C} \to B(X)$ we denote by $\rho(\widehat{L})$ the resolvent set of $\widehat{L}$ defined by $\rho(\widehat{L}) := \{\lambda \in \widehat{\Omega} \mid \widehat{L}(\lambda) \text{ has an inverse in } B(X)\}$, and we denote by $\sigma(\widehat{L})$ the spectrum of $\widehat{L}$ defined by $\sigma(\widehat{L}) := \widehat{\Omega} \setminus \rho(\widehat{L})$. Furthermore the notions eigenvalue, null multiplicity, pole multiplicity and multiplicity of an eigenvalue of $\widehat{L}$ are used as in [9], [12], [8].

Let integers $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$ be given. Let $A_k \in B(X)$, $k = 0, 1, \ldots, n$, $0 \neq H_k \in B(X)$, $k = 1, 2, \ldots, m$, and let $0 \neq a_k \in \mathbb{C}$, $a_k \neq a_j$ for $k \neq j$, $k, j = 1, 2, \ldots, m$, be given. Then the operator function $L : \Omega \to B(X)$ is considered, defined by (1.1). We assume that with $T_0 := A_0$, $T_n := I$, there exist operators $T_1, \ldots, T_{n-1}$, $B_1, \ldots, B_n \in B(X)$ such that

$$A_k = T_k B_k \cdots B_1, \quad k = 1, 2, \ldots, n. \tag{2.1}$$

Such decompositions exist always by taking $B_k := I$, $T_k := A_k$, $k = 1, \ldots, n-1$, and $B_n := A_n$. Under the assumption that the polynomial part of L is a Keldysh pencil, such decompositions are assumed, where all B_k are equal (cf. [18]).

Furthermore we assume that for $k = 1, \ldots, m$ there exist closed subspaces N_k and Z_k of X such that

$$X = N_k \oplus Z_k, \quad N_k \subset N(H_k). \tag{2.2}$$

There exist always such decompositions under the assumption that the operators H_k are of finite rank, where in addition the subspaces Z_k are finite dimensional, and N_k can be chosen equal to $N(H_k)$. Let P_k be the continuous projection from X onto Z_k along N_k. Then we have $H_k P_k = H_k$, and the operator $I - \alpha P_k$ is bijective for each $\alpha \in \mathbb{C} \setminus \{1\}$.

Let $K := n + m$ for $n \geqslant 1$, and let $K := 1 + m$ for $n = 0$. Let

$$X^K := X \oplus \cdots \oplus X \oplus Z_1 \oplus \cdots \oplus Z_m$$

the Banach space of the direct sum of n copies of X and of $Z_1, \dots, Z_m$ endowed with the norm

$$\|(f_1, \dots, f_K)^T\| := \left(\sum_{k=1}^{K} \|f_k\|^2 \right)^{\frac{1}{2}}.$$

We define linear operators $\mathfrak{T}, \mathfrak{H} : X^K \to X^K$ by the following operator matrices

$$\mathfrak{T} := \begin{pmatrix} T_0 & T_1 & \cdots & T_{n-1} & H_1 & \cdots & H_m \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \\ \frac{1}{a_1}P_1 & & & & & & \\ \vdots & & & & & & \\ \frac{1}{a_m}P_m & & & & & & \end{pmatrix}, \quad n \geqslant 2, \tag{2.3}$$

$$\mathfrak{T} := \begin{pmatrix} T_0 & H_1 & \cdots & H_m \\ \frac{1}{a_1}P_1 & & & \\ \vdots & & & \\ \frac{1}{a_m}P_m & & & \end{pmatrix}, \quad n = 0, 1,$$

$$\mathfrak{H} := \begin{pmatrix} 0 & 0 & \cdots & 0 & B_n & & & & \\ B_1 & 0 & & 0 & 0 & & & & \\ & B_2 & & & & & & & \\ & & \ddots & & \vdots & & & & \\ & & & B_{n-1} & 0 & & & & \\ & & & & 0 & \frac{1}{a_1}P_1 & 0 & \cdots & 0 \\ & & & & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & & & 0 \\ & & & & 0 & 0 & \cdots & 0 & \frac{1}{a_m}P_m \end{pmatrix}, \quad n \geqslant 2, \tag{2.4}$$

$$\mathfrak{H} := \operatorname{diag}\left(B_1, \frac{1}{a_1}P_1, \dots, \frac{1}{a_m}P_m \right), \quad n = 0, 1,$$

where $B_1 = 0$ if $n = 0$, respectively (cf. [14] for this type of linearization of the polynomial part of L, and also [18] for the case that all B_k are equal, and cf. [19] for a similar type of linearization of the rational part of L).

PROPOSITION 2.1. *Let* $\mathfrak{T}, \mathfrak{H}$ *be given by* (2.3), (2.4), *respectively. Then we have:*

(i) $\mathfrak{T}, \mathfrak{H} \in B(X^K)$.

(ii) *Let* $T_k \in B_\infty(X), 0 \leqslant k \leqslant n-1$, $B_k \in B_\infty(X)$, $1 \leqslant k \leqslant n$, *and let the operators* H_k *be of finite rank,* $1 \leqslant k \leqslant m$. *Then* $\mathfrak{T}, \mathfrak{H} \in B_\infty(X^K)$.

Now we define operator functions $\mathfrak{E}, \mathfrak{F} : \Omega \to B(X^K)$ by

(2.5)
$$\mathfrak{F}(\lambda) := \begin{pmatrix} I &&&&&& \\ \lambda B_1 & I &&&&& \\ \lambda^2 B_2 B_1 & \lambda B_2 & I &&&& \\ \vdots && \ddots & \ddots &&& \\ \lambda^{n-1} B_{n-1} \cdots B_1 && \cdots & \lambda B_{n-1} \quad I &&& \\ \frac{-1}{\lambda - a_1} P_1 &&&& \frac{-a_1}{\lambda - a_1} P_1 && \\ \vdots &&&&& \ddots & \\ \frac{-1}{\lambda - a_m} P_m &&&&&& \frac{-a_m}{\lambda - a_m} P_m \end{pmatrix},$$

(2.6)
$$\mathfrak{E}(\lambda) := \begin{pmatrix} I & \sum_{k=1}^{n} \lambda^{k-1} T_k B_k \cdots B_2 & \cdots & T_{n-1} + \lambda B_n & \frac{-a_1}{\lambda - a_1} H_1 & \cdots & \frac{-a_m}{\lambda - a_m} H_m \\ & I &&&&& \\ && \ddots &&&& \\ &&& I &&& \\ &&&& P_1 && \\ &&&&& \ddots & \\ &&&&&& P_m \end{pmatrix}.$$

Let X^{K-1} denote the space

$$X^{K-1} := X \oplus \cdots \oplus X \oplus Z_1 \cdots \oplus Z_m$$

with $n-1$ copies of X, and let $\mathfrak{I}_{X^{K-1}}, \mathfrak{I}_{X^K}$ denote the identity operators in X^{K-1}, X^K respectively.

Theorem 2.2. *Let* $L, \mathfrak{T}, \mathfrak{H}, \mathfrak{E}, \mathfrak{F}$ *be given by* (2.3), (2.4), (2.5), (2.6) *respectively. Then for each* $\lambda \in \Omega$ *the operators* $\mathfrak{E}(\lambda)$, $\mathfrak{F}(\lambda)$ *are invertible, and we have*

$$L(\lambda) \oplus \mathfrak{I}_{X^{K-1}} = \mathfrak{E}(\lambda)(\mathfrak{I}_{X^K} - \mathfrak{T} - \lambda \mathfrak{H}) \mathfrak{F}(\lambda), \quad \lambda \in \Omega. \tag{2.7}$$

Let $\mathfrak{L} : \mathbb{C} \to B(X^K)$ be the operator function defined by

$$\mathfrak{L}(\lambda) := \mathfrak{I}_{X^K} - \mathfrak{T} - \lambda \mathfrak{H}, \quad \lambda \in \mathbb{C}. \tag{2.8}$$

Then from Theorem 2.2 it follows:

COROLLARY 2.3. $\rho(L) = \rho(\mathfrak{L}) \cap \Omega$, $\sigma(L) = \sigma(\mathfrak{L}) \cap \Omega$.

We denote by $\pi : X^K \to X$ the canonical projection of X^K onto its first coordinate space

$$\pi(f_1, f_2, \ldots, f_K)^T = f_1, \tag{2.9}$$

and we denote by $\overline{\pi}$ the canonical projection of X^K onto X^{K-1}

$$\overline{\pi}(f_1, f_2, \ldots, f_K)^T = (f_2, \ldots, f_K)^T. \tag{2.10}$$

PROPOSITION 2.4. (i) *Let $\lambda_0 \in \Omega$ be an eigenvalue of L, and let $0 \neq f \in X$ be a corresponding eigenvector. Then λ_0 is an eigenvalue of $\mathfrak{L}$, and $\mathfrak{F}(\lambda_0)(f, 0, \ldots, 0)^T \neq 0$ is a corresponding eigenvector.*

(ii) *Let $\lambda_0 \in \Omega$ be an eigenvalue of $\mathfrak{L}$, and let $0 \neq \mathfrak{f} = (f_1, f_2, \ldots, f_K)^T$ be a corresponding eigenvector. Then $f_1 \neq 0$ and λ_0 is an eigenvalue of L. Furthermore $\pi\mathfrak{F}(\lambda_0)^{-1}\mathfrak{f} = f_1 \neq 0$ is a corresponding eigenvector.*

(iii) *The (geometric) multiplicities of the eigenvalues $\lambda_0 \in \Omega$ of L and $\mathfrak{L}$ coincide.*

(iv) *Let $f_0, f_1, \ldots, f_r \in X$, $f_0 \neq 0$, be a Jordan chain of L corresponding to the eigenvalue $\lambda_0 \in \Omega$ of L. Then the vectors $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_r \in X^K$,*

$$\mathfrak{g}_j = \sum_{k=0}^{j} \frac{1}{k!} \mathfrak{F}^{(k)}(\lambda_0) \begin{pmatrix} f_{j-k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad j = 0, \ldots, r,$$

are a Jordan chain of $\mathfrak{L}$ corresponding to the eigenvalue λ_0 of $\mathfrak{L}$.

(v) *Let $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_r \in X^K$, $\mathfrak{g}_0 \neq 0$, be a Jordan chain of $\mathfrak{L}$ corresponding to the eigenvalue $\lambda_0 \in \Omega$ of $\mathfrak{L}$, and let*

$$\mathfrak{f}_0 := \mathfrak{F}(\lambda_0)^{-1}\mathfrak{g}_0,$$
$$\mathfrak{f}_j := \mathfrak{F}(\lambda_0)^{-1} \left[\mathfrak{g}_j - \sum_{\nu=0}^{j-1} \frac{1}{(j-\nu)!} \mathfrak{F}^{(j-\nu)}(\lambda_0)\mathfrak{f}_\nu \right], \quad j = 1, \ldots, r.$$

Then we have

$$\overline{\pi}\mathfrak{f}_j = 0, \quad j = 0, \ldots, r.$$

Furthermore the vectors

$$\pi\mathfrak{f}_0 = g_{1,0}, \pi\mathfrak{f}_1 = g_{1,1}, \ldots, \pi\mathfrak{f}_r = g_{1,r}$$

are a Jordan chain of L corresponding to the eigenvalue λ_0 of L.

(vi) *The partial null multiplicities and the null multiplicities of the eigenvalues $\lambda_0 \in \Omega$ of L and $\mathfrak{L}$ coincide.*

Proof. Using formula (2.7) the proof is analogous to a corresponding proof in [19] and therefore is omitted. ∎

Now we prove that the results of the Proposition 2.4 (iii), (vi) are also true for the eigenvalues $\lambda \in \{a_1, \dots, a_m\}$ of L and $\mathfrak{L}$. We proceed in a similar way as in [16], [15].

For $j \in \{1, \dots, m\}$ we define operator functions $L_j : \Omega \cup \{a_j\} \to B(X)$ and $G_j : \Omega \cup \{a_j\} \to B(X \oplus Z_j)$ by

$$L_j(\lambda) := I - \sum_{k=0}^{n} \lambda^k A_k + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{\lambda - a_k} H_k, \tag{2.11}$$

and

$$G_j(\lambda) := \begin{pmatrix} L_j(\lambda) & -H_j \\ -\frac{1}{a_j} P_j & \frac{a_j - \lambda}{a_j} P_j \end{pmatrix}. \tag{2.12}$$

Let

$$X_j^K := X \oplus Z_j \oplus X \oplus \cdots \oplus X \oplus Z_1 \oplus \cdots \oplus Z_{j-1} \oplus X \oplus Z_{j+1} \oplus \cdots \oplus Z_m.$$

We denote by $\pi_j : X_j^K \to X \oplus Z_j$ the canonical projection from X_j^K onto the first two coordinate spaces

$$\pi_j (f_1, \dots, f_K)^T := (f_1, f_2)^T,$$

and we denote by $\overline{\pi}_j$ the canonical projection of X_j^K onto

$$X_j^{K-2} := X \oplus \cdots \oplus X \oplus Z_1 \oplus \cdots \oplus Z_{j-1} \oplus X \oplus Z_{j+1} \oplus \cdots \oplus Z_m.$$

THEOREM 2.5. *Let $j \in \{1, \dots, m\}$ be given. There exist operator functions $\widehat{\mathfrak{E}}_j : \Omega \cup \{a_j\} \to B(X^K, X_j^K), \widehat{\mathfrak{F}}_j : \Omega \cup \{a_j\} \to B(X_j^K, X^K)$ such that for each $\lambda \in \Omega \cup \{a_j\}$ the operators $\widehat{\mathfrak{E}}_j(\lambda), \widehat{\mathfrak{F}}_j(\lambda)$ are invertible, and we have*

$$G_j(\lambda) \oplus \mathfrak{I}_{X_j^{K-2}} = \widehat{\mathfrak{E}}_j(\lambda)(\mathfrak{I}_{X^K} - \mathfrak{T} - \lambda \mathfrak{H}) \widehat{\mathfrak{F}}_j(\lambda), \quad \lambda \in \Omega \cup \{a_j\}. \tag{2.13}$$

Proof. Let $\mathfrak{F}_j : \Omega \cup \{a_j\} \to B(X^K)$ be defined by

$$\big(\mathfrak{F}_j(\lambda)(f_1, \dots, f_K)^T\big)_k := \begin{cases} \sum\limits_{\kappa=1}^{k-1} \lambda^{k-\kappa} B_{k-1} \cdots B_\kappa f_\kappa + f_k, & k = 1, \dots, n, \\ \frac{-1}{\lambda - a_{k-n}} P_{k-n} f_1 + \frac{-a_{k-n}}{\lambda - a_{k-n}} P_{k-n} f_k, & k = n+1, \dots, n+m, k \neq n+j, \\ P_{k-n} f_k, & k = n+j, \end{cases}$$

and let $\mathfrak{E}_j(\lambda) : \Omega \cup \{a_j\} \to B(X^K)$ be defined by

$$\big(\mathfrak{E}_j(\lambda)(f_1, \dots, f_K)^T\big)_k := \begin{cases} f_1 + \sum\limits_{\kappa=2}^{n} \sum\limits_{\nu=\kappa-1}^{n} \lambda^{\nu-\kappa+1} T_\nu B_\nu \cdots B_\kappa f_\kappa + \sum\limits_{\substack{\kappa=n+1 \\ \kappa \neq n+j}}^{n+m} \frac{-a_{\kappa-n}}{\lambda - a_{\kappa-n}} H_{\kappa-n} f_\kappa, & k = 1, \\ f_k, \quad k = 2, \dots, n, \\ P_{k-n} f_k, \quad k = n+1, \dots, n+m. \end{cases}$$

Then we have

$$\mathfrak{E}_j(\lambda)(\mathfrak{I}_{X^K} - \mathfrak{T} - \lambda \mathfrak{H})\mathfrak{F}_j(\lambda) = \begin{pmatrix} L_j(\lambda) & & & & & & & -H_j & & & \\ & I & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & I & & & & & & & \\ & & & & P_1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & P_{j-1} & & & & \\ -\frac{1}{a_j} P_j & & & & & & & \frac{a_j - \lambda}{a_j} P_j & & & \\ & & & & & & & & P_{j+1} & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & P_m \end{pmatrix}.$$

Let $\mathfrak{P}_1 \in B(X^K, X_j^K)$ be defined by

$$\mathfrak{P}_1 := \begin{pmatrix} I & & & & & & & & & & & \\ & 0 & & & & & & & P_j & & & \\ & & I & & & & & & & & & \\ & & & \ddots & & & & & & & & \\ & & & & I & & & & & & & \\ & & & & & P_1 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & P_{j-1} & & & & \\ & I & & & & & & & 0 & & & \\ & & & & & & & & & P_{j+1} & & \\ & & & & & & & & & & \ddots & \\ & & & & & & & & & & & P_m \end{pmatrix}.$$

$\mathfrak{P}_1$ originates from the identity operator in X^K by changing the second row with the $(n+j)$-th row. We set $\widehat{\mathfrak{E}}_j(\lambda) := \mathfrak{P}_1 \mathfrak{E}_j(\lambda)$. Furthermore let $\mathfrak{P}_2 \in B(X_j^K, X^K)$ be defined by $\mathfrak{P}_2 := \mathfrak{P}_1^T$. $\mathfrak{P}_2$ originates from the identity operator in X^K by changing the second column with the $(n+j)$-th column. We set $\widehat{\mathfrak{F}}_j(\lambda) := \mathfrak{F}_j(\lambda)\mathfrak{P}_2$. Then the assertion follows from the formula for $\mathfrak{E}_j(\lambda)(\mathfrak{I}_{X^K} - \mathfrak{T} - \lambda\mathfrak{H})\mathfrak{F}_j(\lambda)$ just proved. ∎

Theorem 2.5 implies that $a_j \in \{a_1, \dots, a_m\}$ is an eigenvalue of G_j if and only if a_j is an eigenvalue of $\mathfrak{L}$. Moreover the geometric multiplicities, partial null multiplicities, and null multiplicities coincide. In detail we have the following proposition.

PROPOSITION 2.6. (i) *Let $a_j \in \{a_1, \dots, a_m\}$ be an eigenvalue of G_j, and let $0 \neq (f_1, f_2)^T \in X \oplus Z_j$ be a corresponding eigenvector. Then a_j is an eigenvalue of $\mathfrak{L}$ and $\widehat{\mathfrak{F}}_j(a_j)(f_1, f_2, 0, \dots, 0)^T \neq 0$ is a corresponding eigenvector.*

(ii) *Let $a_j \in \{a_1, \dots, a_m\}$ be an eigenvalue of $\mathfrak{L}$, and let $0 \neq \mathfrak{f} = (f_1, \dots, f_K)^T$ be a corresponding eigenvector. Then a_j is an eigenvalue of G_j, and $\pi_1\widehat{\mathfrak{F}}_j(a_j)^{-1}\mathfrak{f} = (f_1, P_j f_j)^T \neq 0$ is a corresponding eigenvector.*

(iii) *The (geometric) multiplicities of the eigenvalues a_j of G_j and $\mathfrak{L}$ coincide.*

(iv) *Let $(f_{1,0}, f_{2,0}), (f_{1,1}, f_{2,1}), \dots, (f_{1,r}, f_{2,r}) \in X \oplus Z_j$, $(f_{1,0}, f_{2,0}) \neq 0$, be a Jordan chain of G_j corresponding to the eigenvalue a_j of G_j. Then the vectors $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_r \in X^K$,*

$$\mathfrak{g}_k := \widehat{\mathfrak{F}}_j(a_j)\begin{pmatrix} f_{1,k} \\ f_{2,k} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{1}{1!}\widehat{\mathfrak{F}}_j'(a_j)\begin{pmatrix} f_{1,k-1} \\ f_{2,k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \frac{1}{k!}\widehat{\mathfrak{F}}_j^{(k)}(a_j)\begin{pmatrix} f_{1,0} \\ f_{2,0} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$k = 0, 1, \dots, r$, are a Jordan chain of $\mathfrak{L}$ corresponding to the eigenvalue a_j of $\mathfrak{L}$.

(v) *Let $\mathfrak{g}_0, \mathfrak{g}_1, \dots, \mathfrak{g}_r \in X^K (\mathfrak{g}_0 \neq 0)$ be a Jordan chain of $\mathfrak{L}$ corresponding to the eigenvalue a_j of $\mathfrak{L}$, and let*

$$\mathfrak{f}_0 := \widehat{\mathfrak{F}}_j(a_j)^{-1}\mathfrak{g}_0,$$
$$\mathfrak{f}_k := \widehat{\mathfrak{F}}_j(a_j)^{-1}\left[\mathfrak{g}_k - \sum_{\nu=0}^{k-1}\frac{1}{(k-\nu!)}\widehat{\mathfrak{F}}_j^{(k-\nu)}(a_j)\mathfrak{f}_\nu\right], \quad k = 1, \dots, r.$$

Then we have

$$\overline{\pi}_j\mathfrak{f}_k = 0, \quad k = 0, 1, \dots, r.$$

Furthermore

$$\pi_j\mathfrak{f}_0, \pi_j\mathfrak{f}_1, \dots, \pi_j\mathfrak{f}_r$$

is a Jordan chain of G_j corresponding to the eigenvalue a_j of G_j.

(vi) *The partial null multiplicities and the null multiplicities of the eigenvalues a_j of G_j and $\mathfrak{L}$ coincide.*

Proof. Using formula (2.13) the proof is analogous to the proof of Proposition 2.4 and therefore is omitted. ∎

In the following proposition we characterize eigenvalues a_j and corresponding Jordan chains of L. These conditions are used in the proof of the theorem which then follows.

PROPOSITION 2.7. (i) *$a_j \in \{a_1, \ldots, a_m\}$ is an eigenvalue of L and $f_0 \neq 0$ a corresponding eigenvector if and only if there exists a vector $f_1 \in Z_j$ such that*

$$H_j f_0 = 0,$$

$$\left(I - \sum_{k=0}^{n} a_j^k A_k + \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{1}{a_j - a_k} H_k \right) f_0 = H_j f_1.$$

(ii) *Let $a_j \in \{a_1, \ldots, a_m\}$ be an eigenvalue of L. Then the vectors $f_0, f_1, \ldots, f_r \in X$, $f_0 \neq 0$, are a Jordan chain of L of length $r+1$ corresponding to the eigenvalue a_j of L if and only if there exists a vector $f_{r+1} \in Z_j$ such that*

$$H_j f_0 = 0,$$

$$\sum_{l=0}^{k} \frac{1}{(k-l)!} L_j^{(k-l)}(a_j) f_l = H_j(-f_{k+1}), \quad k = 0, 1, \ldots, r.$$

Proof. The proof is analogous to the proof of Lemma 2 in [16], and therefore is omitted. ∎

The proof of the next proposition is similar to the proof of Theorem 3 in [16]. We use the characterization of Proposition 2.7. First we prove that the geometric multiplicities of the eigenvalues a_j of L and G_j coincide, then we prove that the null multiplicities of the eigenvalues a_j of L and G_j coincide.

PROPOSITION 2.8. *Let $a_j \in \{a_1, \ldots, a_m\}$. Assume that $N_j = N(H_j)$.*

(i) *Let a_j be an eigenvalue of L, and let $f_0 \neq 0$ be a corresponding eigenvector. Then a_j is an eigenvalue of G_j and $0 \neq (f_0, f_1)^T \in X \oplus Z_j$ is a corresponding eigenvector, where $f_1 \in Z_j$ is the vector which exists according to Proposition 2.7 (i) corresponding to f_0.*

(ii) *Let a_j be an eigenvalue of G_j, and let $0 \neq (f_0, g_0)^T \in X \oplus Z_j$ be a corresponding eigenvector. Then $f_0 \neq 0$, and a_j is an eigenvalue of L and f_0 is a corresponding eigenvector.*

(iii) *The geometric multiplicities of the eigenvalues* a_j *of* G_j *and* L *coincide.*

(iv) *Let* $0 \neq f_0, f_1, \ldots, f_r$ *be a Jordan chain of* L *corresponding to the eigenvalue* a_j *of* L. *Then there exist vectors* $g_0, g_1, \ldots, g_r \in Z_j$ *such that*

$$0 \neq (f_0, g_0)^T, (f_1, g_1)^T, \ldots, (f_r, g_r)^T$$

is a Jordan chain of G_j *corresponding to the eigenvalue* a_j *of* G_j.

(v) *Let* $0 \neq (f_0, g_0)^T, (f_1, g_1)^T, \ldots, (f_r, g_r)^T \in X \oplus Z_j$ *be a Jordan chain of* G_j *corresponding to the eigenvalue* a_j *of* G_j. *Then* $0 \neq f_0, f_1, \ldots, f_r$ *is a Jordan chain of* L *corresponding to the eigenvalue* a_j *of* L.

(vi) *The partial null multiplicities and the null multiplicities of the eigenvalues* a_j *of* L *and* G_j *coincide.*

Proof. (i) Let a_j be an eigenvalue of L and $f_0 \neq 0$ a corresponding eigenvector. Proposition 2.7 (i) implies that there exists a vector $f_1 \in Z_j$ such that

$$H_j f_0 = 0,$$
$$L_j(a_j) f_0 = H_j f_1;$$

this implies

$$P_j f_0 = 0 \quad \text{and} \quad -\frac{1}{a_j} P_j f_0 = 0,$$
$$L_j(a_j) f_0 - H_j f_1 = 0$$

or

$$\begin{pmatrix} L_j(a_j) & -H_j \\ -\frac{1}{a_j} P_j & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = 0.$$

That is, the vector $0 \neq (f_0, f_1)^T \in X \oplus Z_j$ is an eigenvector of G_j corresponding to the eigenvalue a_j of G_j.

(ii) Let a_j be an eigenvalue of G_j and $0 \neq (f_0, g_0)^T \in X \oplus Z_j$ a corresponding eigenvector:

$$\begin{pmatrix} L_j(a_j) & -H_j \\ -\frac{1}{a_j} P_j & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = 0$$

or

$$L_j(a_j) f_0 - H_j g_0 = 0,$$
$$-\frac{1}{a_j} P_j f_0 = 0.$$

The assumption $f_0 = 0$ implies $H_j g_0 = 0$ or $g_0 \in N(H_j)$. Since $g_0 \in Z_j$ this implies $g_0 = 0$. Thus we have a contradiction to $0 \neq (f_0, g_0)^T$. Therefore $f_0 \neq 0$ and

$$H_j f_0 = 0 \quad (\text{since } P_j f_0 = 0),$$

$$L_j(a_j)f_0 = H_j g_0.$$

From Proposition 2.7 (i) then it follows that a_j is an eigenvalue of G_j, and $f_0 \neq 0$ is a corresponding eigenvector.

(iii) This follows from (i) and (ii).

(v) The assumption is equivalent to

$$\sum_{l=0}^{k} \frac{1}{(k-l)!} G_j^{(k-l)}(a_j) \begin{pmatrix} f_l \\ g_l \end{pmatrix} = 0, \quad k = 0, 1, \dots, r.$$

Since $g_0, g_1, \dots, g_r \in Z_j$ this is equivalent to

$$\begin{cases} \sum_{l=0}^{k} \frac{1}{(k-l)!} L_j^{(k-l)}(a_j) f_l = H_j g_k, & k = 0, 1, \dots, r; \\ 0 \neq f_0 \in N(H_j); \\ f_k + g_{k-1} \in N(H_j), & k = 1, 2, \dots, r. \end{cases} \tag{2.14}$$

We have considered the case $r = 0$ in (i)–(iii). Thus we can assume that $r \geqslant 1$. Then from (2.14) it follows for $k = 0, 1, \dots r - 1$ that

$$H_j g_k = H_j(-f_{k+1}). \tag{2.15}$$

Furthermore we define a vector $f_{r+1} \in Z_j$ by

$$-f_{r+1} = g_r.$$

Thus

$$H_j g_r = H_j(-f_{r+1}),$$

and equation (2.15) is valid for $k = 0, 1, \dots, r$. From Proposition 2.7 (ii) now it follows that $f_0, f_1, \dots, f_r$ is a Jordan chain of L corresponding to the eigenvalue a_j of L.

(iv) Let $f_{r+1} \in Z_j$ be the vector described in Proposition 2.7 (ii). Then we can determine successively vectors $g_0, g_1 \dots, g_r \in Z_j$ such that

$$H_j g_k = H_j(-f_{k+1})$$

for $k = 0, 1, \dots, r$. For this we set

$$\begin{cases} h_0 := -f_1, \\ h_k := -\sum_{l=0}^{k-1} h_l - \sum_{l=0}^{k} f_{l+1}, & k = 1, \dots, r, \end{cases} \tag{2.16}$$

and then

$$g_k := P_j h_k \in Z_j, \quad k = 0, 1, \ldots, r.$$

Now we show that we have $f_k + g_{k-1} \in N(H_j)$, $k = 1, \ldots, r$. For this it suffices to show that

$$f_k + h_{k-1} \in N(H_j) \tag{2.17}$$

for $k = 1, \ldots, r$. For $k = 1$ we have $f_1 + h_0 = 0 \in N(H_j)$, and for any $1 < k \leqslant r$ thus we have successively from (2.16) that

$$f_k + h_{k-1} = 0 \in N(H_j)$$

for $k = 1, \ldots, r$ and therefore (2.17) is satisfied. In conclusion the vectors $f_0, \ldots, f_r$, $g_0, \ldots, g_r$ satisfy (2.14), and a_j is an eigenvalue of G_j with $(f_0, g_0)^T, (f_1, g_1)^T, \ldots,$ $(f_r, g_r)^T$ as corresponding Jordan chain.

(vi) This follows from (iv) and (v). ∎

Combining Proposition 2.6 (i)–(iii) and Proposition 2.8 (i)–(iii), and combining Proposition 2.6 (iv)–(vi) and Proposition 2.8 (iv)–(vi) we now can prove that the (geometric) multiplicities of the eigenvalues a_j of L and $\mathfrak{L}$ coincide.

Theorem 2.9. *Let $a_j \in \{a_1, \ldots, a_m\}$. Assume that $N_j = N(H_j)$.*

(i) *Let a_j be an eigenvalue of L, and let $f_0 \neq 0$ be a corresponding eigenvector. Then a_j is an eigenvalue of $\mathfrak{L}$ and there exists a vector $f_1 \in Z_j$ (according to Proposition* 2.7 (i)*) such that $\widehat{\mathfrak{F}}_j(a_j)(f_0, f_1, 0, \ldots, 0)^T \neq 0$ is a corresponding eigenvector.*

(ii) *Let a_j be an eigenvalue of $\mathfrak{L}$, and let $0 \neq \mathfrak{f} = (f_1, f_2, \ldots, f_K)^T$ be a corresponding eigenvector. Then a_j is an eigenvalue of L and $f_1 \neq 0$ is a corresponding eigenvector.*

(iii) *The geometric multiplicities of the eigenvalues a_j of L and $\mathfrak{L}$ coincide.*

(iv) *Let $0 \neq f_0, f_1, \ldots, f_r$ be a Jordan chain of L corresponding to the eigenvalue a_j of L. Then there exist vectors $g_0, g_1, \ldots, g_r \in Z_j$ such that the vectors $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_r \in X^K$ defined by*

$$\mathfrak{g}_k = \widehat{\mathfrak{F}}_j(a_j) \begin{pmatrix} f_k \\ g_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \frac{1}{k!} \widehat{\mathfrak{F}}_j^{(k)}(a_j) \begin{pmatrix} f_0 \\ g_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad k = 0, \ldots, r,$$

are a Jordan chain of $\mathfrak{L}$ corresponding to the eigenvalue a_j of $\mathfrak{L}$.

(v) *Let* $\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_r \in X^K (\mathfrak{g}_0 \neq 0)$ *be a Jordan chain of* $\mathfrak{L}$ *corresponding to the eigenvalue* a_j *of* $\mathfrak{L}$*. Then* $g_{1,0}, g_{1,1}, \ldots, g_{1,r} \in X (g_{1,0} \neq 0)$ *is a Jordan chain of* L *corresponding to the eigenvalue* a_j *of* L*.*

(vi) *The partial null multiplicities and the null multiplicities of the eigenvalues* a_j *of* L *and* $\mathfrak{L}$ *coincide.*

Now we describe the spectrum of L under the conditions of Proposition 2.1 (ii).

THEOREM 2.10. *Let* $T_k \in B_\infty(X), 0 \leqslant k \leqslant n-1, B_k \in B_\infty(X), 1 \leqslant k \leqslant n$, *and let the operators* H_k *be of finite rank,* $1 \leqslant k \leqslant m$. *Let* $\rho(L) \neq \emptyset$. *Then the spectrum* $\sigma(L)$ *of* L *consists of eigenvalues of finite multiplicity with infinity as their only possible limit point. Furthermore the eigenvalues* $a_j \in \{a_1, a_2, \ldots, a_m\}$ *of* L *have (finite null multiplicity and finite pole multiplicity and consequently) finite multiplicity.*

Proof. From Corollary 2.3 we have $\sigma(L) = \sigma(\mathfrak{L}) \cap \Omega$ and $\rho(L) = \rho(\mathfrak{L}) \cap \Omega \neq \emptyset$. Thus the assertion follows from Theorem 12.9 in [18] (since $\mathfrak{T}, \mathfrak{H} \in B_\infty(X^K)$) and Corollary 2.3, Proposition 2.4, Theorem 2.9, since the pole multiplicities of the eigenvalues $a_j \in \{a_1, \ldots, a_m\}$ are equal to $\dim R(H_j)$. ∎

3. COMPLETENESS AND SPECTRAL ASYMPTOTICS (THE RATIONAL CASE)

In this section we use the results from Section 2 to prove in the case of a Hilbert space some asymptotic results for the eigenvalues and the completeness of the eigenvectors and associated vectors of the rational operator function L.

Throughout this section let X be a (complex, separable) Hilbert space with inner product $(\cdot, \cdot)$. The inner product in X^K then is defined by

$$(\mathfrak{f}, \mathfrak{g}) := \sum_{k=1}^{K} (f_k, g_k).$$

In this case we have

$$X = N(H_k) \oplus N(H_k)^\perp$$

for each $k \in 1, \ldots, m$, and we choose P_k to be the orthogonal projection onto $N(H_k)^\perp$. We denote by $B_p(X)$, $0 < p < \infty$, the von Neumann-Schatten classes. We suppose throughout this section that $n \geqslant 1$.

PROPOSITION 3.1. (i) *Assume* $\ker B_k = \{0\}, k = 1, \dots, n$. *Then* $\ker \mathfrak{H} = \{0\}$.

(ii) *Let for* $n = 1$ *the operator* B_1 *be normal, and let for* $n \geqslant 2$

$$B_1 B_1^* = B_2^* B_2, \dots, B_{n-1} B_{n-1}^* = B_n^* B_n, B_n B_n^* = B_1^* B_1.$$

Then $\mathfrak{H}$ *is normal.*

(iii) *Let for* $n = 1$ *the operator* $B_1 = A_1$ *be selfadjoint, and let for* $n \geqslant 2$ *the operator* A_n *and the operators*

$$\begin{array}{rcr} B_1 B_n B_{n-1} & \cdots & B_2, \\ B_2 B_1 B_n B_{n-1} & \cdots & B_3, \\ & & \cdots \\ B_{n-1} B_{n-2} & \cdots & B_1 B_n \end{array}$$

be selfadjoint. Furthermore let $a_j^n \in \mathbb{R}, j = 1, \dots, m$. *Then* $\mathfrak{H}^n$ *is selfadjoint.*

(iv) *Let* $B_k \in B_p(X)$ *for some* $p \in]0, \infty[$ *for* $k = 1, \dots, n$, *and let* H_k, $k = 1, \dots, m$, *be of finite rank. Then* $\mathfrak{H} \in B_p(X^K)$.

Proof. This follows from the definition of $\mathfrak{H}$. ∎

Assume that $B_1 = \cdots = B_n =: B$, and that B is normal. Then the assumptions of Proposition 3.1 (ii) are satisfied, and thus $\mathfrak{H}$ is normal. Assume that all B_k commute, that A_n is selfadjoint, and that $a_j^n \in \mathbb{R}$, $j = 1, \dots, m$. Then the assumptions of Proposition 3.1 (iii) are satisfied, and thus $\mathfrak{H}^n$ is selfadjoint. Therefore it follows that under the assumption that the polynomial part of L is a Keldysh pencil all conditions of Proposition 3.1 (i), (ii), (iii) are satisfied if additionally $a_j^n \in \mathbb{R}$, $j = 1, \dots, m$.

Now we have the following theorem.

THEOREM 3.2. *Let* $T_0, T_1, \dots, T_{n-1}, B_1, \dots, B_n \in B_\infty(X)$, *and let the operators* $H_1, \dots, H_m$ *be of finite rank.*

(i) *Let the operators* $B_1, \dots, B_n$ *satisfy the assumptions of Proposition* 3.1 (i), (ii) *with* $B_1 = A_1$ *selfadjoint (for* $n = 1$*), and* A_n *and the operators*

$$\begin{array}{rcr} B_1 B_n B_{n-1} & \cdots & B_2, \\ B_2 B_1 B_n B_{n-1} & \cdots & B_3, \\ & & \cdots \\ B_{n-1} B_{n-2} & \cdots & B_1 B_n \end{array}$$

selfadjoint (for $n \geqslant 2$*), respectively. Then for any* $\delta > 0$ *there are only finitely many eigenvalues of* L *outside the angles*

$$\left\{ \lambda : \left| \arg \lambda - \pi \frac{k}{n} \right| < \delta \right\}, \quad k = 0, 1, \dots, 2n - 1.$$

(ii) *Let the assumptions of* (i) *be satisfied, and let* $B_k \in B_p(X)$ *for some* $p \in]0, \infty[$ *for* $k = 1, \dots, n$. *Then the system of eigenvectors and associated vectors corresponding to the eigenvalues* $\lambda \in \mathbb{C}$ *of* L *is complete in* X.

Proof. We consider only the cases $n \geqslant 2$, since a similar reasoning is valid for the case $n = 1$. (i) For the operator function $\mathfrak{L}(\lambda) := \mathfrak{I}_{X^K} - \mathfrak{T} - \lambda\mathfrak{H}$ the operators $\mathfrak{T}$ and $\mathfrak{H}$ are compact with $\ker \mathfrak{H} = \{0\}$ and $\mathfrak{H}$ normal. Let $\mathfrak{G} := \mathfrak{H}^{-1}$ and $\mathfrak{B} := \mathfrak{T}\mathfrak{H}^{-1}$. Then the operator $\mathfrak{G}$ is normal with compact resolvent, while $\mathfrak{B}$ is compact relative to $\mathfrak{G}$. Furthermore the eigenvalues of $L, \mathfrak{L}$, and $\mathfrak{G} - \mathfrak{B}$ coincide and have the same null multiplicities. This follows from Proposition 2.4, Theorem 2.9, and Lemmma 15.2 in [18]. For the spectrum $\sigma(\mathfrak{H})$ we have

$$\sigma(\mathfrak{H}) = \sigma\left(\begin{pmatrix} 0 & \cdots & 0 & B_n \\ B_1 & & & \\ & \ddots & & \\ & & B_{n-1} & 0 \end{pmatrix}\right) \cup \bigcup_{k=1}^{m} \left\{\frac{1}{a_k}\right\}.$$

Since $\begin{pmatrix} 0 & \cdots & 0 & B_n \\ B_1 & & & \\ & \ddots & & \\ & & B_{n-1} & 0 \end{pmatrix}^n$ is selfadjoint in the Hilbert space X^n, $\sigma(\mathfrak{H})$ lies on the rays

$$\arg\lambda = \pi\frac{k}{n}, \quad k = 0, 1, \dots, 2n-1,$$
$$\arg\lambda = \arg\frac{1}{a_k}, \quad k = 1, \dots, m.$$

Then it follows from Lemma 15.3 in [18], Proposition 2.4, and Theorem 2.9 that for any $\delta > 0$ there are only finitely many eigenvalues of L outside the angles

$$\left\{\lambda : \left|\arg\lambda - \pi\frac{k}{n}\right| < \delta\right\}, \quad k = 0, 1, \dots, 2n-1,$$
$$\{\lambda : |\arg\lambda - \arg a_k| < \delta\}, \quad k = 1, \dots, m.$$

If all the values a_k, $k = 1, \dots, m$, lie on the rays $\arg\lambda = k\pi/n$, $k = 0, \dots, 2n-1$, the assertion is obviously true. Thus let $k_1, \dots, k_l \in \{1, \dots, m\}$ be such that

$$\arg a_{k_1} = \cdots = \arg a_{k_l},$$
$$\arg a_j \neq \arg a_{k_1}, \quad j \neq k_1, \dots, k_l,\ j \in \{1, \dots, m\},$$
$$\arg a_{k_1} \neq \pi\frac{k}{n}, \quad k = 0, \dots, 2n-1.$$

For $0 \neq a \in \mathbb{C}$ and $\delta > 0$ denote by $\Omega(a, \delta)$ the angle $\Omega(a, \delta) := \{\lambda : |\arg\lambda - \arg a| < \delta\}$. Let $\delta_{k_1} > 0$ be such that $a_j \notin \Omega(a_{k_1}, \delta_{k_1})$, $j \neq k_1, \dots, k_l$, $j = 1, \dots, m$, and that

$$\Omega(a_{k_1}, \delta_{k_1}) \cap \left\{\lambda : \arg\lambda = k\frac{\pi}{n}\right\} = \emptyset, \quad k = 0, \dots, 2n-1.$$

Let $0 < \delta_2 < \delta_{k_1}$. Then the spectrum of L in $\Omega(a_{k_1}, \delta_2)$ is finite. This follows from Theorem 8.2 and Remarks 8.6 and 8.7 in [18], since $a_{k_1}, \dots, a_{k_l}$ are eigenvalues of finite multiplicity of the operator $\mathfrak{G} = \mathfrak{H}^{-1}$. This argument can be applied to all a_k. From this the assertion follows.

(ii) We use the notations of the proof of (i). The operator $\mathfrak{T}$ is compact and the operator $\mathfrak{H} \in B_p(X^K)$ with $\ker\mathfrak{H} = \{0\}$ is normal. In view of Proposition 2.4 and Theorem 2.9 it suffices to establish that the eigenvectors and associated vectors of $\mathfrak{L}$ are complete in X^K. The operator $\mathfrak{G}$ is normal and $\mathfrak{G}^{-1} = \mathfrak{H} \in B_p(X^K)$, while $\mathfrak{B}$ is compact relative to $\mathfrak{G}$. Since the spectrum $\sigma(\mathfrak{H})$ of $\mathfrak{H}$ lies on the rays

$$\arg\lambda = \pi\frac{k}{n}, \quad k = 0, 1, \dots, 2n-1,$$
$$\arg\lambda = \arg\frac{1}{a_k}, \quad k = 1, \dots, m,$$

the spectrum of $\mathfrak{G}$ lies on the rays

$$\arg\lambda = \pi\frac{k}{n}, \quad k = 0, 1, \dots, 2n-1,$$
$$\arg\lambda = \arg a_k, \quad k = 1, \dots, m,$$

Thus, by Theorem 4.3 in [18], the system of root vectors of $\mathfrak{G} - \mathfrak{B}$ is complete in X^K. Thus, the set $\mathfrak{N}$ of all vectors of the form $\mathfrak{H}\mathfrak{g}$, where $\mathfrak{g}$ is a root vector of $\mathfrak{G} - \mathfrak{B}$, is complete in $R(\mathfrak{H})$. Since $\ker\mathfrak{H}^* = \{0\}$ (since $\mathfrak{H}$ is normal and $\ker\mathfrak{H} = \{0\}$), $R(\mathfrak{H})$ is dense in X^K and thus $\mathfrak{N}$ is complete in X^K. According to Lemma 15.1 in [18], $\mathfrak{N}$ coincides with the set of all eigenvectors and associated vectors of $\mathfrak{L}$. ∎

Denote by $N_k(r, L)$ for $r > 0$ and $k = 0, \dots, 2n-1$ the sum of the null multiplicities of the eigenvalues of L in the sector

$$\Delta_k(r) := \left\{\lambda : \left|\arg\lambda - \pi\frac{k}{n}\right| < \pi/2n, |\lambda| < r\right\},$$

denote by $N^+(r)$ (respectively, $N^-(r)$) the sum of the null multiplicities of the eigenvalues of A_n in $]r^{-n}, \infty[$ (respectively, $]-\infty, -r^{-n}[$), denote by $N_k(r, (a_k))$ the sum of $\dim R(H_k)$ for which a_k is in $\{\lambda : \arg\lambda = \pi k/n, |\lambda| < r\}$, $k = 0, \dots, 2n-1$, and let $N_k^+(r) := N^+(r) + N_k(r, (a_k))$, $k = 0, \dots, 2n-1$. In the next theorem we compare the functions $N_{2k}(r, L)$ with $N_{2k}^+(r)$, and $N_{2k+1}(r, L)$ with $N_{2k-1}^-(r)$. The results are formulated only for $N_{2k}(r, L)$, since the formulations are analogous for $N_{2k+1}(r, L)$. For functions $\varphi, \psi :]0, \infty[\to \mathbb{R}$ we write $\varphi(r) \sim \psi(r)$ if $\lim_{r\to\infty} \varphi(r)/\psi(r) = 1$.

THEOREM 3.3. *Let the assumptions of Theorem* 3.2. (i) *be satisfied.*

(i) *Assume that the positive spectrum of A_n is finite. Then the spectrum of L in the angles*

$$\Omega_{2k} := \left\{\lambda : \left|\arg\lambda - 2\pi\frac{k}{n}\right| < \frac{\pi}{2n}\right\}, \quad k = 0, \dots, n-1,$$

is also finite.

(ii) *Assume that the positive spectrum of A_n is infinite and*

$$\liminf_{r\to\infty} \frac{\log N_{2k}^{+}(r)}{\log r} < \infty.$$

Then

$$\liminf_{r\to\infty} \left|\frac{N_{2k}(r,L)}{N_{2k}^{+}(r)} - 1\right| = 0, \quad k = 0, \dots, n-1.$$

(iii) *Assume that the positive spectrum of A_n is infinite and*

$$\liminf_{r\to\infty,\varepsilon\to 0} \frac{N_{2k}^{+}(r(1+\varepsilon))}{N_{2k}^{+}(r)} = 1.$$

Then

$$N_{2k}(r,L) \sim N_{2k}^{+}(r), \quad k = 0, \dots, n-1.$$

(iv) *Assume that $N_{2k}^{+}(r) \sim ar^{\gamma}$ for $a, \gamma > 0$. Then $N_{2k}(r,L) \sim ar^{\gamma}$, $k = 0, \dots, n-1$.*

Proof. We use the notations of the proof of Theorem 3.2 (i). The eigenvalues of $L_0(\lambda) := I - \lambda^n A_n$ are eigenvalues of $\mathfrak{G}$ with the same null multiplicities. The remaining eigenvalues of $\mathfrak{G}$ are the values $a_k, k = 1, \dots, m$, with finite null multiplicity (equal to $\dim R(H_k)$). Thus (i) follows from Theorem 8.2 (and Remarks 8.6 and 8.7) in [18], (ii) follows from Theorem 8.3 (and Remarks 8.6 and 8.7) in [18], (iii) follows from Theorem 8.4 (and Remarks 8.6 and 8.7) in [18]. (iv) is a corollary to (iii). ∎

Thus we have for L similar results as for a Keldysh pencil (cf. [18], Section 15.3).

4. LINEARIZATION (THE MEROMORPHIC CASE)

In this section first let X be a (complex) Banach space. Let $A_k \in B(X)$, $k = 0, 1, \ldots, n$, $0 \neq H_k \in B(X)$, $k = 1, 2, \ldots$, and $0 \neq a_k \in \mathbb{C}$, $k = 1, 2, \ldots$, $a_k \neq a_j$ for $k \neq j$, with

$$\lim_{k \to \infty} |a_k| = \infty \tag{4.1}$$

be given. Then the operator function $L : \Omega^\infty \to B(X)$ is considered, defined by

$$L_\infty(\lambda) := I - \sum_{k=0}^{n} \lambda^k A_k + \sum_{k=1}^{\infty} \frac{1}{\lambda - a_k} H_k, \quad \lambda \in \Omega^\infty,$$

where

$$\sum_{k=1}^{\infty} \frac{1}{\lambda - a_k} H_k$$

is compact convergent in $\mathbb{C}$. In this section we extend the results of Section 2 on rational operator functions to this type of meromorphic operator functions. But we only sketch the procedure. We assume that (2.1) and (2.2) (for $k = 1, 2, \ldots$) are satisfied. Let

$$X^\infty := \left\{ (f_1, f_2, \ldots)^T \mid f_1, \ldots, f_n \in X, f_{n+1} \in Z_1, f_{n+2} \in Z_2, \ldots, \sum_{k=1}^{\infty} \|f_k\|^2 < \infty \right\}.$$

Then X^∞ endowed with the norm

$$\|\mathfrak{f}\|_\infty := \|(f_1, f_2, \ldots)^T\|_\infty := \left(\sum_{k=1}^{\infty} \|f_k\|^2 \right)^{\frac{1}{2}}$$

is a Banach space. We consider for $n \geqslant 2$ the infinite operator matrices

$$\mathfrak{T}_\infty := \begin{pmatrix} T_0 & T_1 & \cdots & T_{n-1} & H_1 & H_2 & \cdots \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \\ \frac{1}{a_1} P_1 & & & & & & \\ \frac{1}{a_2} P_2 & & & & & & \\ \vdots & & & & & & \end{pmatrix}, \tag{4.2}$$

$$\mathfrak{H}_\infty := \begin{pmatrix} 0 & 0 & \cdots & 0 & B_n & & & \\ B_1 & 0 & & 0 & 0 & & & \\ & B_2 & & & & & & \\ & & \ddots & & \vdots & & & \\ & & & B_{n-1} & 0 & & & \\ & & & & & \frac{1}{a_1}P_1 & & \\ & & & & & & \frac{1}{a_2}P_2 & \\ & & & & & & & \ddots \end{pmatrix}, \tag{4.3}$$

and with the corresponding modifications for the cases $n = 0, 1$ as in Section 2.

PROPOSITION 4.1. *Assume that*

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \|P_k\|^2 < \infty, \tag{4.4}$$

$$\sum_{k=1}^{\infty} \|H_k\|^2 < \infty. \tag{4.5}$$

(i) *Then for* $\mathfrak{T}_\infty, \mathfrak{H}_\infty$ *given by* (4.2), (4.3), *respectively, we have* $\mathfrak{T}_\infty, \mathfrak{H}_\infty \in B(X^\infty)$.

(ii) *Let* $T_k \in B_\infty(X)$, $0 \leqslant k \leqslant n-1$, $B_k \in B_\infty(X)$, $1 \leqslant k \leqslant n$, *and let the operators* H_k *be of finite rank,* $k = 1, 2, \ldots$. *Then* $\mathfrak{T}_\infty, \mathfrak{H}_\infty \in B_\infty(X^\infty)$.

Proof. (i) We have for $\mathfrak{f} \in X^\infty$

$$\left\| \sum_{k=0}^{n-1} T_k f_{k+1} + \sum_{k=1}^{\infty} H_k f_{n+k} \right\|^2 \leqslant \left(\sum_{k=0}^{n-1} \|T_k\|^2 + \sum_{k=1}^{\infty} \|H_k\|^2 \right) \|\mathfrak{f}\|_\infty^2,$$

and

$$\sum_{k=1}^{\infty} \left\| \frac{1}{a_k} P_k f_1 \right\|^2 \leqslant \sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \|P_k\|^2 \|\mathfrak{f}\|_\infty^2.$$

From this it follows that $\mathfrak{T}_\infty \in B(X^\infty)$.

We have for $\mathfrak{f} \in X^\infty$

$$\sum_{k=1}^{n} \|B_k f_k\|^2 + \sum_{k=1}^{\infty} \left\| \frac{1}{a_k} P_k f_{n+k} \right\|^2 \leqslant \left(\max_{k=1,\ldots,n} \|B_k\|^2 + \sup_{k=1,2,\ldots} \frac{1}{|a_k|^2} \|P_k\|^2 \right) \|\mathfrak{f}\|^2.$$

From this it follows that $\mathfrak{H}_\infty \in B(X^\infty)$.

(ii) We approximate $\mathfrak{T}_\infty, \mathfrak{H}_\infty$ by finite sections in the following way: Let $m \in \mathbb{N}$ be given, then let

$$\mathfrak{T}_{\infty,m} := \begin{pmatrix} T_0 & T_1 & \cdots & T_{n-1} & H_1 & \cdots & H_m & 0 & \cdots \\ 0 & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & \\ \frac{1}{a_1}P_1 & & & & & & & & \\ \vdots & & & & & & & & \\ \frac{1}{a_m}P_m & & & & & & & & \\ 0 & & & & & & & & \\ \vdots & & & & & & & & \end{pmatrix}, \tag{4.6}$$

$$\mathfrak{H}_{\infty,m} := \begin{pmatrix} 0 & 0 & \cdots & 0 & B_n & & & & \\ B_1 & 0 & & 0 & 0 & & & & \\ & B_2 & & & & & & & \\ & & \ddots & & \vdots & & & & \\ & & & B_{n-1} & 0 & & & & \\ & & & & & \frac{1}{a_1}P_1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \frac{1}{a_m}P_m & \\ & & & & & & & & 0 \\ & & & & & & & & & \ddots \end{pmatrix}. \tag{4.7}$$

Thus we have for $\mathfrak{f} \in X^\infty$

$$\|\mathfrak{T}_\infty \mathfrak{f} - \mathfrak{T}_{\infty,m}\mathfrak{f}\|^2 \leqslant \left(\sum_{k=m+1}^{\infty} \|H_k\|^2 + \sum_{k=m+1}^{\infty} \frac{1}{|a_k|^2} \|P_k\|^2 \right) \|\mathfrak{f}\|_\infty^2 \to 0$$

for $m \to \infty$. Since $\mathfrak{T}_{\infty,m} \in B_\infty(X^\infty)$, this implies $\mathfrak{T}_\infty \in B_\infty(X^\infty)$. Furthermore we have for $\mathfrak{f} \in X^\infty$

$$\|\mathfrak{H}_\infty \mathfrak{f} - \mathfrak{H}_{\infty,m}\mathfrak{f}\|^2 \leqslant \sup_{k=m+1,\ldots} \frac{1}{|a_k|^2} \|P_k\|^2 \|\mathfrak{f}\|_\infty^2 \to 0 \text{ for } m \to \infty.$$

Since $\mathfrak{H}_{\infty,m} \in B_\infty(X^\infty)$, this implies $\mathfrak{H}_\infty \in B_\infty(X^\infty)$. ∎

Now we define operator functions $\mathfrak{E}_\infty, \mathfrak{F}_\infty : \Omega^\infty \to B(X^\infty)$ by

(4.8)

$$\mathfrak{F}_\infty(\lambda) := \begin{pmatrix} I & & & & & & & \\ \lambda B_1 & I & & & & & & \\ \lambda^2 B_2 B_1 & \lambda B_2 & I & & & & & \\ \vdots & & \ddots & \ddots & & & & \\ \lambda^{n-1} B_{n-1}\cdots B_1 & & \cdots & \lambda B_{n-1} & I & & & \\ \frac{-1}{\lambda-a_1}P_1 & & & & & \frac{-a_1}{\lambda-a_1}P_1 & & \\ \frac{-1}{\lambda-a_2}P_2 & & & & & & \frac{-a_2}{\lambda-a_2}P_2 & \\ \vdots & & & & & & & \ddots \end{pmatrix},$$

(4.9)

$$\mathfrak{E}_\infty(\lambda) := \begin{pmatrix} I & \sum_{k=1}^{n} \lambda^{k-1} T_k B_k \cdots B_2 & \cdots & T_{n-1} + \lambda B_n & \frac{-a_1}{\lambda-a_1}H_1 & \frac{-a_2}{\lambda-a_2}H_2 & \cdots \\ & I & & & & & \\ & & \ddots & & & & \\ & & & I & & & \\ & & & & P_1 & & \\ & & & & & P_2 & \\ & & & & & & \ddots \end{pmatrix}.$$

We denote by π_∞ the canonical projection from X^∞ onto its first coordinate space

$$\pi_\infty(f_1, f_2, \ldots)^T = f_1.$$

THEOREM 4.2. *Let* $L_\infty, \mathfrak{T}_\infty, \mathfrak{H}_\infty, \mathfrak{E}_\infty, \mathfrak{F}_\infty$ *as above and let* (4.4), (4.5) *be fulfilled. Then for each* $\lambda \in \Omega^\infty$ *we have* $\mathfrak{E}_\infty(\lambda), \mathfrak{F}_\infty(\lambda) \in B(X^\infty)$*, the operators* $\mathfrak{E}_\infty(\lambda), \mathfrak{F}_\infty(\lambda)$ *are invertible, and*

$$L_\infty(\lambda) \oplus \mathfrak{I}_{\ker \pi_\infty} = \mathfrak{E}_\infty(\lambda)(\mathfrak{I}_{X^\infty} - \mathfrak{T}_\infty - \lambda \mathfrak{H}_\infty)\mathfrak{F}_\infty(\lambda), \quad \lambda \in \Omega^\infty.$$

Proof. For $\mathfrak{f} \in X^\infty$ we have

$$\sum_{k=1}^{\infty} \left\| \frac{-1}{\lambda - a_k} P_k f_1 \right\|^2 \leqslant \sum_{k=1}^{\infty} \frac{1}{|\lambda - a_k|^2} \|P_k\|^2 \|\mathfrak{f}\|^2 < \infty$$

since the series $\sum_{k=1}^{\infty} \frac{1}{|\lambda - a_k|^2} \|P_k\|^2$ converges. Furthermore we have $P_k | Z_k = I_{Z_k}$. Thus

$$\sum_{k=1}^{\infty} \left\| \frac{-a_k}{\lambda - a_k} P_k f_{n+k} \right\|^2 = \sum_{k=1}^{\infty} \frac{1}{\left|1 - \frac{\lambda}{a_k}\right|^2} \|f_{k+n}\|^2 \leqslant \sup_{k=1,2,\ldots} \frac{1}{\left|1 - \frac{\lambda}{a_k}\right|^2} \|\mathfrak{f}\|^2 < \infty$$

since $|a_k| \to \infty$. From this it follows that $\mathfrak{F}_\infty(\lambda) \in B(X^\infty)$. Furthermore $\mathfrak{F}_\infty(\lambda)$ is injective with

$$\mathfrak{F}_\infty(\lambda)^{-1} = \begin{pmatrix} I & & & & & & \\ -\lambda B_1 & I & & & & & \\ & \ddots & \ddots & & & & \\ & & -\lambda B_{n-1} & I & & & \\ -\frac{1}{a_1} P_1 & & & & -\frac{\lambda - a_1}{a_1} P_1 & & \\ -\frac{1}{a_2} P_2 & & & & & -\frac{\lambda - a_2}{a_2} P_2 & \\ \vdots & & & & & & \ddots \end{pmatrix}.$$

Since for $\mathfrak{f} \in X^\infty$ we have ($P_k|Z_k = I_{Z_k}$ and $|a_k| \to \infty$)

$$\sum_{k=1}^{\infty} \left\| -\frac{1}{a_k} P_k f_1 \right\|^2 \leqslant \sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \|P_k\|^2 \|\mathfrak{f}\|_\infty^2 < \infty,$$

$$\sum_{k=1}^{\infty} \left\| -\frac{\lambda - a_k}{a_k} P_k f_{k+n} \right\|^2 = \sum_{k=1}^{\infty} \left| 1 - \frac{\lambda}{a_k} \right|^2 \|f_{k+n}\|^2 \leqslant \sup_{k=1,2,\ldots} \left| 1 - \frac{\lambda}{a_k} \right|^2 \|\mathfrak{f}\|_\infty^2 < \infty,$$

thus it follows that $\mathfrak{F}_\infty(\lambda)^{-1} \in B(X^\infty)$.

For $\mathfrak{f} \in X^\infty$ we have

$$\left\| \sum_{k=1}^{\infty} \frac{-a_k}{\lambda - a_k} H_k f_{n+k} \right\|^2 \leqslant \sup_{k=1,2,\ldots} \frac{1}{\left| 1 - \frac{\lambda}{a_k} \right|^2} \sum_{k=1}^{\infty} \|H_k\|^2 \|\mathfrak{f}\|_\infty^2 < \infty,$$

and since $P_k|Z_k = I_{Z_k}$ it follows that $\mathfrak{E}(\lambda) \in B(X^\infty)$. $\mathfrak{E}(\lambda)$ is injective with

$$\mathfrak{E}_\infty(\lambda)^{-1} = \begin{pmatrix} I & -\sum_{k=1}^{n} \lambda^{k-1} T_k B_k \cdots B_2 & \cdots & -T_{n-1} - \lambda B_n & \frac{a_1}{\lambda - a_1} H_1 & \frac{a_2}{\lambda - a_2} H_2 & \cdots \\ & I & & & & & \\ & & \ddots & & & & \\ & & & I & & & \\ & & & & P_1 & & \\ & & & & & P_2 & \\ & & & & & & \ddots \end{pmatrix},$$

and therefore $\mathfrak{E}_\infty(\lambda)^{-1} \in B(X^\infty)$. ∎

In the following we assume that the conditions (4.4), (4.5) are fulfilled. Let $\mathfrak{L}_\infty : \mathbb{C} \to B(X^\infty)$ be the operator function defined by

$$\mathfrak{L}_\infty(\lambda) := \mathfrak{I}_{X^\infty} - \mathfrak{T}_\infty - \lambda \mathfrak{H}_\infty, \quad \lambda \in \mathbb{C}.$$

For L_∞ and $\mathfrak{L}_\infty$ there are valid the analogues of Corollary 2.3, Propositions 2.4, 2.6, 2.7, 2.8, and Theorems 2.5, 2.9, 2.10, which we do not formulate explicitly.

5. COMPLETENESS AND SPECTRAL ASYMPTOTICS (THE MEROMORPHIC CASE)

In this section we use the results from Section 4 to prove in the case of a Hilbert space some asymptotic results for the eigenvalues and the completeness of the eigenvectors and associated vectors which extend those of Section 3 for the rational operator function L to the meromorphic operator function L_∞.

Throughout this section let X be a (complex, separable) Hilbert space with inner product $(\cdot,\cdot)$. The inner product in X^∞ is defined by

$$(\mathfrak{f},\mathfrak{g})_\infty := \sum_{k=1}^{\infty}(f_k, g_k).$$

In this case again we have $X = N(H_k) \oplus N(H_k)^\perp$ for each $k \in \mathbb{N}$, and we choose P_k to be the orthogonal projection onto $N(H_k)^\perp$. We suppose throughout this section that $n \geqslant 1$.

Under the condition (4.4) for $\mathfrak{H}_\infty$ there is also valid the analogue of Proposition 3.1 (i), (ii), (iii) if $a_j^n \in \mathbb{R}$, $j = 1, 2, \ldots$. Only Proposition 3.1 (iv) must be modified.

PROPOSITION 5.1. *Let $B_k \in B_p(X)$ for some $p \in [1,\infty[$ for $k = 1,2,\ldots$. Furthermore let $H_k, k \in \mathbb{N}$, be of finite rank, and let*

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|} \dim R(H_k)^{\frac{1}{p}} < \infty.$$

Then $\mathfrak{H}_\infty \in B_p(X^\infty)$.

Proof. The operator $\mathfrak{H}_{\infty,m}$ given by (4.7) is an element of $B_p(X^\infty)$ for each $m \in \mathbb{N}$. Furthermore

$$\|\mathfrak{H}_\infty - \mathfrak{H}_{\infty,m}\|_{\infty,p} \leqslant \sum_{k=m+1}^{\infty} \frac{1}{|a_k|} \dim R(H_k)^{\frac{1}{p}} \to 0$$

for $m \to \infty$. From this the assertion follows. ∎

THEOREM 5.2. *Let for the polynomial part of L_∞ the assumptions of Theorem* 3.2 (i) *be satisfied. Furthermore let*

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^2} < \infty, \quad \sum_{k=1}^{\infty} \|H_k\|^2 < \infty.$$

Assume that (without loss of generality) the numbers $1/a_k$, $k = 1, 2, \ldots$, *lie on the finitely many rays*

$$\arg\lambda = \arg\frac{1}{a_1}, \ldots, \arg\lambda = \arg\frac{1}{a_m},$$

where $\arg 1/a_k, k = 1, \ldots, m$, *are pairwise distinct.*

(i) *For any* $\delta > 0$ *there are only finitely many eigenvalues of* L_∞ *outside the angles*

$$\{\lambda : |\arg\lambda - \pi\frac{k}{n}| < \delta\}, \quad k = 0, 1, \ldots, 2n-1, \tag{5.1}$$

$$\{\lambda : |\arg\lambda - \arg a_k| < \delta\}, \quad k = 1, \ldots, m. \tag{5.2}$$

(ii) *Assume additionally that on the ray*

$$\arg\lambda = \arg a_{k_0}, \quad k_0 \in \{1, \ldots, m\},$$

there are only finitely many a_k *and* $\arg a_{k_0} \neq \pi k/n, k = 0, \ldots, 2n-1$. *Then the assertion of* (i) *is valid with* $k \neq k_0$ *in* (5.2).

(iii) *Let additionally be assumed that the positive spectrum of* A_n *is finite, and that only finitely many* a_k *are on the rays*

$$\arg\lambda = 2\pi\frac{k}{n}, \quad k = 0, \ldots, n-1.$$

Then the assertion of (i) *is valid with* $k \neq 2l$, $l = 0, \ldots, n-1$ *in* (5.1). *An analogous result is valid if the negative spectrum of* A_n *is finite, and only finitely many* a_k *are on the rays*

$$\arg\lambda = \frac{\pi(2k+1)}{n}, \quad k = 0, \ldots, n-1.$$

(iv) *Let additionally* $B_k \in B_p(X)$ *for some* $p \in [1, \infty[$ *for* $k = 1, \ldots, n$, *and let*

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|} \dim R(H_k)^{\frac{1}{p}} < \infty.$$

Then the system of eigenvectors and associated vectors corresponding to the eigenvalues $\lambda \in \mathbb{C}$ *of* L_∞ *is complete in* X^∞.

Proof. We consider only the cases $n \geqslant 2$, since a similar reasoning is valid for the case $n = 1$. (i) For the operator function $\mathfrak{L}_\infty(\lambda) = \mathfrak{I}_{X^\infty} - \mathfrak{T}_\infty - \lambda\mathfrak{H}_\infty$ the operator $\mathfrak{T}_\infty$ is compact, and $\mathfrak{H}_\infty$ is compact and normal with $\ker\mathfrak{H}_\infty = \{0\}$. Let $\mathfrak{G}_\infty := \mathfrak{H}_\infty^{-1}$ and $\mathfrak{B}_\infty := \mathfrak{T}_\infty\mathfrak{H}_\infty^{-1}$. Then the operator $\mathfrak{G}_\infty$ is normal with compact resolvent, while $\mathfrak{B}_\infty$ is compact relative to $\mathfrak{G}_\infty$. Furthermore the eigenvalues of $L_\infty, \mathfrak{L}_\infty$, and $\mathfrak{G}_\infty - \mathfrak{B}_\infty$ coincide and have the same null multiplicities. This follows from the analogues of Proposition 2.4, Theorem 2.9 for L_∞ and $\mathfrak{L}_\infty$, and Lemma 15.2 in [18]. For the spectrum $\sigma(\mathfrak{H}_\infty)$ we have

$$\sigma(\mathfrak{H}_\infty) = \sigma\left(\begin{pmatrix} 0 & \dots & 0 & B_n \\ B_1 & & & \\ & \ddots & & \\ & & B_{n-1} & 0 \end{pmatrix}\right) \cup \bigcup_{k=1}^{\infty} \left\{\frac{1}{a_k}\right\}.$$

Therefore from our assumptions it follows that the spectrum of $\mathfrak{H}_\infty$ lies on the finitely many rays

$$\arg\lambda = \arg\pi\frac{k}{n}, \quad k = 0, \dots 2n-1,$$
$$\arg\lambda = \arg\frac{1}{a_k}, \quad k = 1, \dots, m.$$

Thus the assertion of (i) follows from Lemma 15.3 in [18].

To prove (ii) and (iii) the same procedure as in the second part of the proof of Theorem 3.2 (i) give the assertions.

(iv) The proof is analogous to the proof of Theorem 3.2 (iii). ∎

Now we consider the asymptotics of the eigenvalues of L_∞ in one of the angles given by (5.2) in more detail.

THEOREM 5.3. *Let the assumptions of Theorem* 5.2 (i) *be satisfied. Let* $k_0 \in \{1, \dots, m\}$ *be given. Assume that on the ray* $\arg\lambda = \arg a_{k_0}$ *there are infinitely many numbers* a_k, *where* $\arg a_{k_0} \neq \pi k/n$, $k = 0, \dots, 2n-1$. *Let* $\delta_{k_0} > 0$ *be such that* $a_k \notin \Omega(a_{k_0}, \delta_{k_0})$, $k = 1, \dots, m, k \neq k_0$, *and that*

$$\Omega(a_{k_0}, \delta_{k_0}) \cap \{\lambda : \arg\lambda = \pi\frac{k}{n}\} = \emptyset, \quad k = 0, \dots, 2n-1.$$

Let $0 < \delta_1 < \delta_{k_0}$. *Denote by* $N(r, \delta_1, L_\infty)$ *the sum of the null multiplicities of the eigenvalues of* L_∞ *lying in the sector*

$$\Delta(a_{k_0}, \delta_1, r) := \{\lambda : |\arg\lambda - \arg a_{k_0}| < \delta_1, |\lambda| < r\},$$

and denote by $N(r, a_{k_0})$ *the sum of* $\dim R(H_k)$ *for which* $a_k \in \{\lambda : \arg\lambda = \arg a_{k_0}, |\lambda| < r\}$.

(i) *Assume that*

$$\liminf_{r\to\infty} \frac{\log N(r, a_{k_0})}{\log r} < \infty.$$

Then

$$\liminf_{r\to\infty}\left|\frac{N(r,\delta_1,L_\infty)}{N(r,a_{k_0})}-1\right|=0.$$

(ii) *Assume that*

$$\liminf_{r\to\infty,\varepsilon\to 0}\frac{N(r(1+\varepsilon),a_{k_0})}{N(r,a_{k_0})}=1.$$

Then

$$N(r,\delta_1,L_\infty)\sim N(r,a_{k_0}).$$

(iii) *Assume that* $N(r,a_{k_0})\sim ar^\gamma (a,\gamma>0)$. *Then* $N(r,\delta_1,L_\infty)\sim ar^\gamma$.

Proof. We use the notations of Theorem 5.2 (i). The eigenvalues of $L_0(\lambda):=I-\lambda^n A_n$ are eigenvalues of $\mathfrak{G}_\infty$ with the same null multiplicities. The remaining eigenvalues of $\mathfrak{G}_\infty$ are the values $a_k, k=1,2,\dots,$ with finite null multiplicity (equal to $\dim R(H_k)$). Thus (i) follows from Theorem 8.3 (and Remarks 8.6 and 8.7) in [18], (ii) follows from Theorem 8.4 (and Remarks 8.6 and 8.7) in [18]. (iii) is a corollary to (ii). ∎

Note that $N(r,a_{k_0})$ is equal to the sum of the pole multiplicities of the poles a_k of L_∞ in the sector $\Delta(a_{k_0},\delta_1,r)$. Thus Theorem 5.3 shows that a certain asymptotic behavior of the sums of the pole multiplicities of the poles of L_∞ on the ray $\arg\lambda=\arg a_{k_0}$ implies a certain asymptotic behavior of the sums of the null multiplicities of the eigenvalues of L_∞ in the angle $\Omega(a_{k_0},\delta_1)$.

In the next theorem, which can be proved analogous to Theorem 5.3, we consider the asymptotics of the eigenvalues of L_∞ in one of the angles given by (5.1) in more detail.

THEOREM 5.4. *Let the assumptions of Theorem* 5.2 (i) *be satisfied. Let* $k_0\in\{0,\dots,n-1\}$ *be given. Assume that for the ray* $\arg\lambda=2\pi k_0/n$, *one of the following three conditions holds:*

(i) *The positive spectrum of* A_n *is finite, and the number of* a_k *lying on this ray is infinite.*

(ii) *The positive spectrum of* A_n *is infinite, and the number of* a_k *lying on this ray is finite.*

(iii) *The positive spectrum of* A_n *is infinite, and the number of* a_k *lying on this ray is infinite.*

Denote by $N^+(r)$ *the sum of the null multiplicities of the eigenvalues of the eigenvalues of* A_n *in* $]r^{-n},\infty[$, *denote by* $N(r,2k_0,(a_k))$ *the sum of the* $\dim R(H_k)$ *for which* a_k *is in*

$$\left\{\lambda:\arg\lambda=2\pi\frac{k_0}{n},|\lambda|<r\right\},$$

and let $N^+(r, 2k_0) := N^+(r) + N(r, 2k_0, (a_k))$. *Let* $0 < \delta_{k_0} < \pi/2n$ *be such that for at most one* $l_0 \in \{1, \ldots, m\}$ *we have* $\arg a_{l_0} = 2\pi k_0/n$, *and no other ray* $\arg\lambda = \arg a_k$, $k \neq l_0, k = 1, \ldots, m$, *lies in the angle* $\{\lambda : |\arg\lambda - 2\pi k_0/n| < \delta_{k_0}\}$. *Let* $0 < \delta_1 < \delta_{k_0}$. *Denote by* $N(r, \delta_1, L_\infty)$ *the sum of the null multiplicities of* L_∞ *lying in the sector*

$$\Delta(2k_0, \delta_1, r) := \left\{\lambda : \left|\arg\lambda - 2\pi\frac{k_0}{n}\right| < \delta_1, |\lambda| < r\right\}.$$

Then the assertions of Theorem 5.3 (i), (ii), (iii) *are valid with* $N(r, a_{k_0})$, $N(r, \delta_1, L_\infty)$ *replaced by* $N^+(r, 2k_0)$, $N(r, \delta_1, L_\infty)$, *respectively.*

An analogous result is valid for a ray $\arg\lambda = (2k_0 + 1)\pi/n$.

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