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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES FOR SOME ELLIPTIC OPERATORS WITH SIMPLE REMAINDER ESTIMATES

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ABSTRACT. We are interested in remainder estimates in the Weyl formula for the asymptotic number of eigenvalues of certain elliptic operators on $\mathbb{R}^d$ and on a smooth compact manifold without boundary. The main aim of this paper is to compare spectral asymptotics of operators with irregular coefficients and certain classes of smoothed operators for which the Weyl formula is derived by means of elementary pseudodifferential calculus.

The remainder estimates are obtained here essentially with an exponent less than one half of the optimal exponent known in the case of smooth coefficients. The presentation is self-contained (we do not require any knowledge of the subject) and will be continued in a subsequent paper, where sharper remainder estimates will be proved.

KEYWORDS: *Spectral asymptotics, Weyl formula, self-adjoint elliptic operators with irregular coefficients, remainder estimates.*

MSC (2000): 35P20.

1. INTRODUCTION

The aim of this paper is to present a very simple approach of studying the asymptotic behaviour of some elliptic differential operators with coefficients which are not necessarily smooth. The main tool we use in the proof is the formula for the symbol of the composition of two pseudodifferential operators (exposed e.g. in Chapter 1 of the book of M.A. Shubin ([37]) or in Chapter 2 of the book of H. Kumano-Go ([24])) and moreover, concerning the operators on a manifold, the fact that the Hörmander class of pseudodifferential operators of type δ, ρ is well defined if $0 \leqslant \delta < \rho \leqslant 1$, $\rho + \delta \geqslant 1$.

Let us note here that the asymptotic distribution of eigenvalues is one of the important problems of the spectral theory of partial differential operators and since the pioneer work of H. Weyl (cf. [43]) concerning vibrations of membranes and elastic bodies, a lot of papers have investigated various questions concerning the accuracy of the remainder estimates in the asymptotic formulas and their validity for diverse classes of differential operators in various situations, e.g. differential operators with irregular coefficients.

Consider a differential operator A on $\mathbb{R}^d$ of degree $m \in 2\mathbb{N}$, of the form

$$A = \sum_{|\alpha|,\, |\beta| \leqslant m/2} \mathrm{D}^\alpha (a_{\alpha,\beta}(x) \mathrm{D}^\beta) \tag{1.1}$$

where the coefficients $a_{\alpha,\beta}$ are measurable, locally bounded on $\mathbb{R}^d$ and the notation (1.1) means that A is a sesquilinear form on $C_0^\infty(\mathbb{R}^d)$ given by

$$A[\varphi, \psi] = \sum_{|\alpha|,\, |\beta| \leqslant m/2} (a_{\alpha,\beta}(x) \mathrm{D}^\beta \varphi, \mathrm{D}^\alpha \psi) \tag{1.1$'$}$$

for $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$, where $(\psi_1, \psi_2) = \int \psi_1(x) \overline{\psi_2(x)}\, \mathrm{d}x$ is the scalar product of $L^2(\mathbb{R}^d)$.

Further on we assume $a_{\beta,\alpha}(x) = \overline{a_{\alpha,\beta}(x)}$ for every $|\alpha|$, $|\beta| \leqslant m/2$, i.e. A is a quadratic form. We say that A is globally elliptic of degree m if there are constants C, $c > 0$ such that

$$c(|x| + |\xi|)^m - C \leqslant a_0(x, \xi) \leqslant C(1 + |x| + |\xi|)^m \tag{1.2}$$

where

$$a_0(x, \xi) = \sum_{|\alpha|,\, |\beta| \leqslant m/2} a_{\alpha,\beta}(x) \xi^{\alpha+\beta}.$$

If A is globally elliptic of degree m, then the closed, bounded from below quadratic form being the extension of A, defines a self-adjoint operator in $L^2(\mathbb{R}^d)$ which will be denoted also by the letter A. The resolvent of A is compact and its spectrum is formed by a sequence of eigenvalues of finite multiplicities with no finite accumulation point. If $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ is the sequence of eigenvalues of A where each multiple eigenvalue is repeated as many times as its multiplicity, then $N_A(\lambda) = \max\{n \in \mathbb{N} : \lambda_n \leqslant \lambda\}$ denotes the corresponding counting function.

To describe the regularity properties of coefficients we fix $0 < r \leqslant 1$ and define

$$\begin{aligned}\mathcal{A}^r(\mathbb{R}^d) = \Big\{ & a \in L^\infty(\mathbb{R}^d) : \\ & \exists\, C > 0,\ \sup_{|y| \leqslant 1} |a(x+y) - a(x)| \leqslant C \langle x \rangle^{-r} \text{ for all } x \in \mathbb{R}^d \Big\}\end{aligned} \tag{1.3}$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. We make the following

Regularity Hypotheses. Let $0 < r \leqslant 1$. We assume that for $|\alpha + \beta| \leqslant m - 1$ the coefficient $a_{\alpha,\beta}$ may be written in the following way

$$a_{\alpha,\beta}(x) = \langle x \rangle^{m - |\alpha+\beta|} a^1_{\alpha,\beta}(x) + \langle x \rangle^{m - r - |\alpha+\beta|} a^2_{\alpha,\beta}(x) \tag{1.4}$$

with

$$a^1_{\alpha,\beta} \in \mathcal{A}^r(\mathbb{R}^d) \quad \text{and} \quad a^2_{\alpha,\beta} \in L^\infty(\mathbb{R}^d),$$

and for $|\alpha + \beta| = m$ the coefficients $a_{\alpha,\beta}$ are constant.

We have

Theorem 1.1. *Let A be globally elliptic of degree $m \in 2\mathbb{N}$ and the regularity hypotheses hold for a given $0 < r \leqslant 1$. If $\mu < r/m$ and $C > 0$, then*

$$N_A(\lambda) = N_{a_0}(\lambda) + \mathrm{O}(N_{a_0}(\lambda + C\lambda^{1-\mu}) - N_{a_0}(\lambda - C\lambda^{1-\mu})) + \mathrm{O}(1) \tag{1.5}$$

where a_0 is as in (1.2) *and*

$$N_{a_0}(\lambda) = (2\pi)^{-d} \int\limits_{a_0(x,\xi) < \lambda} \mathrm{d}x \mathrm{d}\xi.$$

Remark 1.2. It is easy to see that under our hypotheses $c_1 \lambda^{2d/m} \leqslant N_{a_0}(\lambda) \leqslant C_1 \lambda^{2d/m}$ holds for certain constants $C_1, c_1 > 0$. If we assume moreover that $\partial_{x_j} a_{\alpha,\beta}$ are L^1_{loc} and

$$|\nabla_{x,\xi} a_0(x,\xi)| \geqslant c_2 (|x| + |\xi|)^{m-1} - C_2 \tag{1.6}$$

for certain $C_2, c_2 > 0$, then it can be easily shown (cf. [37], Proposition 28.3) that

$$N_{a_0}(\lambda + C\lambda^{1-\mu}) - N_{a_0}(\lambda - C\lambda^{1-\mu}) = \mathrm{O}(\lambda^{-\mu}) N_{a_0}(\lambda) = \mathrm{O}(\lambda^{-\mu + 2d/m}).$$

Consider now differential operators on a smooth manifold M of dimension d. We shall say that a quadratic form A on $C_0^\infty(M)$ is *local* if $A[\varphi, \psi] = 0$ for every $\varphi, \psi \in C_0^\infty(M)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. For $0 < r \leqslant 1$ we denote

$$\begin{aligned}\mathcal{B}^r(\mathbb{R}^d) = \{ & a \in L^\infty(\mathbb{R}^d) : \\ & \exists\, C > 0,\ |a(x) - a(y)| \leqslant C|x - y|^r \text{ for all } x, y \in \mathbb{R}^d \}.\end{aligned} \tag{1.7}$$

Further on we assume that M is compact and $\mathrm{d}y$ is a smooth density defining the scalar product of $L^2(M)$. We say that Ξ is an *atlas* of M if every $\chi \in \Xi$ is a smooth diffeomorphism of an open set $\mathcal{U}_\chi \subset \mathbb{R}^d$ onto its image in M and $\{\chi(\mathcal{U}_\chi)\}_{\chi\in\Xi}$ is a covering of M.

We say that A is *elliptic of degree* $m \in 2\mathbb{N}$ on M if A is a local quadratic form on $C^\infty(M)$ and there is an atlas Ξ such that every $\chi \in \Xi$ gives the density $\mathrm{d}y$ on $\chi(\mathcal{U}_\chi)$ as the image of the Lebesgue measure dx on $\mathcal{U}_\chi$ and $A[\varphi,\psi] = A_\chi[\varphi\circ\chi, \psi\circ\chi]$ for $\varphi, \psi \in C_0^\infty(\chi(\mathcal{U}_\chi))$ with

$$A_\chi = \sum_{|\alpha|,\,|\beta|\leqslant m/2} \mathrm{D}^\alpha(a_{\alpha,\beta,\chi}(x)\mathrm{D}^\beta) \tag{1.8}$$

where $a_{\alpha,\beta,\chi} \in L^\infty(\mathbb{R}^d)$ and A_χ is elliptic of degree $m \in 2\mathbb{N}$, i.e. there is $c_\chi > 0$ such that

$$a_{m,\chi}(x,\xi) = \sum_{|\alpha|=|\beta|=m/2} a_{\alpha,\beta,\chi}(x)\xi^{\alpha+\beta} \geqslant c_\chi|\xi|^m. \tag{1.2$'$}$$

If A is elliptic of degree m on M, then the closed, bounded from below quadratic form being the extension of A, defines a self-adjoint operator in $L^2(M)$. Since the embedding of $H^s(M)$ into $L^2(M)$ is compact for every $s > 0$, the resolvent of A is compact and as before $N_A(\lambda)$ denotes the corresponding counting function. We have

THEOREM 1.3. *Let A be elliptic of degree $m \in 2\mathbb{N}$ on a compact manifold of dimension d. Let $0 < r < 1$ and assume that the coefficients $a_{\alpha,\beta,\chi} \in \mathcal{B}^r(\mathbb{R}^d)$ if $|\alpha| = |\beta| = m/2$. Then there is a constant c_A such that for $\mu < \frac{r}{m(r+1)}$ we have*

$$N_A(\lambda) = c_A\lambda^{d/m}(1 + \mathrm{O}(\lambda^{-\mu})). \tag{1.9}$$

Since our intention is a self-contained, easy and detailed presentation, we have formulated our statements in a simple way and without looking for possible generalisations. In particular, the statement of regularity hypotheses has been chosen to make extremely simple the idea of replacing the study of A by the study of a smooth differential or pseudodifferential operator $\widetilde{A}$ obtained from A by a very simple smoothing procedure. Afterwards, the asymptotic formula for the smoothed operator is obtained using the Tauberian idea of [15], based on a study of the counting function $N_P(\lambda)$ for the power $P = \widetilde{A}^\mu$ via its Fourier transform

$$u(t) = \int \mathrm{e}^{-\mathrm{i}t\lambda}\,\mathrm{d}N_P(\lambda). \tag{1.10}$$

The paper is organized as follows. In Section 2 we consider the properties of the differential operator $\widetilde{A}$ obtained from A by smoothing the coefficients. In particular, the powers $\widetilde{A}^{\mu}$ are described as elements of suitable classes of pseudodifferential operators. In Section 3 we check that the asymptotic formula for A follows from the analogous formula for $\widetilde{A}$. In Section 4 we describe an approximation of $\mathrm{e}^{-\mathrm{i}tP}$, allowing to obtain the asymptotic behaviour of $u(t)$ [defined by (1.10)] and to complete the proof of Theorem 1.1 by applying a version of the Tauberian theorem described in Section 5.

Further on, we consider the case of irregular top order coefficients. In this case the smoothing procedure is more complicated: every irregular coefficient should be replaced by a suitable pseudodifferential operator from Hörmander's class of type 1, δ. In Section 6 we describe the suitable classes of operators in $\mathbb{R}^d$ and show the asymptotic formula for the spectral function with a simple remainder estimate. To be self-contained, we complete the proof of Theorem 1.3 in Section 7 describing how the statements from Section 6 may be translated in the language of local coordinates.

At the end of this introduction we would like to give an indication about the place of results presented here against the background of known results in the subject. In particular, we would like to mention here L. Hörmander's development of the theory of Fourier integral operators in [15] allowing to prove that in the case of smooth coefficients (1.9) holds with $\mu = 1/m$, which is in general the best possible value of the exponent μ (if no hypotheses on the Hamiltonian flow are considered). The best possible value of the exponent μ in (1.5) is $\mu = 2/m$, and in the case of smooth coefficients A. Mohamed ([30]) proved this estimate under very weak additional hypotheses. Therefore, in the case of smooth coefficients, using the results of Theorem 1.1 and 1.3 we obtain remainder estimates with any value of μ between 0 and one half of the best possible value.

In the subsequent paper [46] we show that the approximation of $u(t)$ by $u_N(t)$ described below in Section 4 allows to obtain in fact the remainder estimates with any value of μ between 0 and the best possible value. In particular, (1.9) holds with $\mu < r/m$ if the coefficients are Hölder continuous with exponent r, where $0 < r \leqslant 1$ (cf. also [44], [45]).

More detailed comments and references to other papers and results in the area of spectral asymptotics are given in Section 8.

2. CLASSES OF SYMBOLS AND OPERATORS

If $\mathcal{X}, \mathcal{X}'$ are Banach spaces, then $B(\mathcal{X}, \mathcal{X}')$ denotes the Banach space of bounded linear operators $\mathcal{X} \to \mathcal{X}'$ and $B(\mathcal{X}, \mathcal{X}) = B(\mathcal{X})$. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and define $\Psi_{\mathrm{gl}}^{-\infty}$ as the set of integral operators with a kernel belonging to $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, i.e. the operators which can be extended to linear continuous operators $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. Further on, we write simply $\mathcal{S}$, L^2, $\mathcal{A}^r$, instead of $\mathcal{S}(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, $\mathcal{A}^r(\mathbb{R}^d)$, denoting by $\|\cdot\|$ the norm of L^2 or $B(L^2)$.

If $p \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is polynomially bounded, then $p(x, \mathrm{D})$ is the associated pseudo-differential operator defined as a linear operator on $\mathcal{S}$ given by

$$p(x,\mathrm{D})\varphi = (2\pi)^{-d} \int \mathrm{e}^{\mathrm{i}x\xi} p(x,\xi) \widehat{\varphi}(\xi) \, \mathrm{d}\xi. \tag{2.1}$$

For $s \in \mathbb{R}$, the global Sobolev space is the completion of $\mathcal{S}$ in the norm

$$\|\varphi\|_{H_s^s} = \|\Lambda^s(x,\mathrm{D})\varphi\| \quad \text{where} \quad \Lambda^s(x,\xi) = (1+|x|^2+|\xi|^2)^{s/2}. \tag{2.2}$$

Clearly $P \in \Psi_{\mathrm{gl}}^{-\infty}$ if and only if P extends to a bounded operator $H_{-s}^{-s} \to H_s^s$ for every $s \in \mathbb{R}$. Using all the time the notation $\Lambda(x,\xi) = (1+|x|^2+|\xi|^2)^{1/2}$, introduce the metric

$$\mathrm{g} = |\mathrm{d}x|^2 + \Lambda(x,\xi)^{-2} |\mathrm{d}\xi|^2 \tag{2.3}$$

on $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ and denote by S_{g}^m the class of functions $p \in C^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ such that

$$|p_{(\alpha')}^{(\alpha)}(x,\xi)| \leqslant C_{\alpha,\alpha'} \Lambda(x,\xi)^{m-|\alpha|} \quad \text{for every } \alpha, \alpha' \in \mathbb{N}^d, \tag{2.4}$$

where $p_{(\alpha')}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha \partial_x^{\alpha'} p(x,\xi)$. We denote by Ψ_{g}^m the class of linear operators P on $\mathcal{S}$ such that $P - p(x,\mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ with $p \in S_{\mathrm{g}}^m$ called a symbol of P.

Note that if $p(x,\mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ then $p(x,\xi) = \mathrm{e}^{\mathrm{i}x\xi} p(x,\mathrm{D})(\mathrm{e}^{-\mathrm{i}x\xi})$ belongs to $\mathcal{S}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$. Also $p \in S_{\mathrm{g}}^m, q \in S_{\mathrm{g}}^{m'} \Rightarrow pq \in S_{\mathrm{g}}^{m+m'}$ and there is $p \diamond q \in S_{\mathrm{g}}^{m+m'}$ satisfying

$$p(x,\mathrm{D})q(x,\mathrm{D}) - (p \diamond q)(x,\mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty} \quad \text{with} \quad p \diamond q \cong \sum_\alpha \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} p^{(\alpha)} q_{(\alpha)} \tag{2.5}$$

[where $p^{(\alpha)}(x,\xi) = \partial_\xi^\alpha p(x,\xi)$, $p_{(\alpha)}(x,\xi) = \partial_x^\alpha p(x,\xi)$], i.e. $p(x,\mathrm{D})q(x,\mathrm{D}) \in \Psi_{\mathrm{g}}^{m+m'}$. If $p \in S_{\mathrm{g}}^m$ then $\Lambda^s(x,\mathrm{D})p(x,\mathrm{D})\Lambda^{-m-s}(x,\mathrm{D}) \in \Psi_{\mathrm{g}}^0$ extends to a bounded operator on L^2, i.e. every $P \in \Psi_{\mathrm{g}}^m$ extends to a bounded operator $H_{m+s}^{m+s} \to H_s^s$ for every

$s \in \mathbb{R}$, hence extends to a continuous operator on $\mathcal{S}'$. The adjoint of $P \in \Psi_{\mathrm{g}}^m$ belongs to Ψ_{g}^m and has the symbol

$$p^* \cong \sum_\alpha \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} p_{(\alpha)}^{(\alpha)}. \tag{2.6}$$

Identifying $(P^*)^*$ and P we get $(p^*)^* - p \in \mathcal{S}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$. If $P - p(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ then P may be chosen symmetric [i.e. $(P\varphi, \psi) = (\varphi, P\psi)$ for $\varphi, \psi \in \mathcal{S}$] if and only if $p^* - p \in \mathcal{S}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$.

If $m < 0$ then $\Lambda^m(x, \mathrm{D})$ is compact on L^2 and consequently every $P \in \Psi_{\mathrm{g}}^m$ is compact if $m < 0$. A symbol $p \in S_{\mathrm{g}}^m$ is called *globally elliptic of degree* $m > 0$ if there are constants $C_0, c_0 > 0$ such that $|p(x, \xi)| \geqslant c_0 \Lambda(x, \xi)^m - C_0$ and the operator $P \in \Psi_{\mathrm{g}}^m$ is called *globally elliptic of degree* m if $P - p(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ with p globally elliptic of degree m.

Definition 2.1. (i) If $m \in \mathbb{R}$, then S_0^m denotes the set of functions $a \in C^\infty(\mathbb{R}^d)$ satisfying $|\partial^\alpha a(x)| \leqslant C_\alpha \langle x \rangle^m$ for every $\alpha \in \mathbb{N}^d$ and for $0 \leqslant r \leqslant 1$ we set

$$S_0^m[r] = \{a \in S_0^m : \partial^\alpha a \in S_0^{m-r} \text{ if } |\alpha| = 1\}. \tag{2.7}$$

(ii) If $m \in \mathbb{R}$, $0 \leqslant r \leqslant 1$, then we set

$$S_{\mathrm{g}}^m[r] = \{p \in S_{\mathrm{g}}^m : p_{(\alpha)} \in S_{\mathrm{g}}^{m-r} \text{ if } |\alpha| = 1\} \tag{2.8}$$

and denote by $\Psi_{\mathrm{g}}^m[r]$ the class of operators $P \in \Psi_{\mathrm{g}}^m$ such that $P - p(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ with $p \in S_{\mathrm{g}}^m[r]$. We note that

$$m' \leqslant m - r \Rightarrow S_0^{m'} \subset S_0^m[r], \quad S_{\mathrm{g}}^{m'} \subset S_{\mathrm{g}}^m[r] \tag{2.9}$$

and using $\partial_x(pq) = (\partial_x p)q + p\partial_x q$ we check easily that

$$p \in S_{\mathrm{g}}^m[r], q \in S_{\mathrm{g}}^{m'}[r] \Rightarrow pq \in S_{\mathrm{g}}^{m+m'}[r], \quad p \diamond q - pq \in S_{\mathrm{g}}^{m+m'-1-r}, \tag{2.10}$$

i.e. $p(x, \mathrm{D})q(x, \mathrm{D}) \in \Psi_{\mathrm{g}}^{m+m'}[r]$, $p(x, \mathrm{D})q(x, \mathrm{D}) - (pq)(x, \mathrm{D}) \in \Psi_{\mathrm{g}}^{m+m'-1-r}$. If $p \in S_{\mathrm{g}}^m[r]$ is real, then $p^* - p \in S_{\mathrm{g}}^{m-1-r}$. If $P - p(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ and $P \in \Psi_{\mathrm{g}}^m[r]$ is symmetric, then $\operatorname{Im} p \in S_{\mathrm{g}}^{m-1-r}$ and when P is moreover globally elliptic of degree $m > 0$, then $\operatorname{Re} p$ is globally elliptic and we can always choose p such that $\operatorname{Re} p \geqslant 1$.

PROPOSITION 2.2. (i) *If* $P \in \Psi_{\mathrm{g}}^m[r]$ *is symmetric and globally elliptic of degree* $m > 0$*, then* P *is bounded from below, its extension to* H_m^m *is self-adjoint and its resolvent is compact.*

(ii) *Assume moreover that the least eigenvalue of* P *is greater than* 1*, i.e.* $P \geqslant I$*. Using the same letter* P *to denote the associated self-adjoint operator, we have* $P^\theta \in \Psi_{\mathrm{g}}^{m\theta}[r]$ *if* $\theta = j/2^n$ *with* $j, n \in \mathbb{Z}$*. If* $P - p(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ *and* $\operatorname{Re} p \geqslant 1$*, then* $(\operatorname{Re} p)^\theta \in S_{\mathrm{g}}^{m\theta}[r]$ *and* $P^\theta - (\operatorname{Re} p)^\theta(x, \mathrm{D}) \in \Psi_{\mathrm{g}}^{m\theta-1-r}$*.*

The proof of (i) follows the standard construction of a parametrix for an elliptic operator (cf. e.g. [37], Chapter 1 or [24] Chapter 2). Due to the composition properties (2.10), it suffices to prove (ii) for $\theta = -1$ and $\theta = 1/2$. In Appendix we check that the construction of the parametrix gives $P^{-1} \in S_{\mathrm{g}}^{-m}[r]$ and the standard construction of the square root (cf. e.g. [37], Section 1.6.2) gives $P^{1/2} \in \Psi_{\mathrm{g}}^{m/2}[r]$. We note that it is possible to show (ii) for every $\theta \in \mathbb{R}$ using the representation of P^θ by Cauchy integrals as described e.g. in [24], Chapter 8.

LEMMA 2.3. *Let* $\gamma \in C_0^\infty(\mathbb{R}^d)$ *be such that* $\int \gamma = 1$ *and* $\operatorname{supp} \gamma \subset \{|y| \leqslant 1\}$*. If* $a \in \mathcal{A}^r$*, then the convolution* $a * \gamma \in S_0^0[r]$ *and* $|(a - a * \gamma)(x)| \leqslant C\langle x\rangle^{-r}$*.*

Proof. Indeed, since $\operatorname{supp} \gamma \subset \{|y| \leqslant 1\}$ we get the desired estimate of

$$(a * \gamma - a)(x) = \int (a(x-y) - a(x))\gamma(y)\,\mathrm{d}y$$

from the definition of $\mathcal{A}^r$ and $a * \gamma \in S_0^0[r]$ follows by using $\int \gamma^{(\alpha)}(y)\,\mathrm{d}y = 0$ for $|\alpha| \geqslant 1$ in

$$(a * \gamma)^{(\alpha)}(x) = \int a(x-y)\gamma^{(\alpha)}(y)\,\mathrm{d}y = \int (a(x-y) - a(x))\gamma^{(\alpha)}(y)\,\mathrm{d}y. \quad \blacksquare$$

Let A be the quadratic form (1.1) satisfying the hypotheses of Theorem 1.1.

DEFINITION OF THE SMOOTHED OPERATOR $\widetilde{A}$. For a given sesquilinear form Q, its symmetrization is denoted $Q + \mathrm{hc}$, i.e. $(Q+\mathrm{hc})[\varphi,\psi] = \frac{1}{2}(Q[\varphi,\psi] + \overline{Q[\psi,\varphi]})$. Set

$$\widetilde{A} = \sum_{|\alpha|,|\beta| \leqslant m/2} \mathrm{D}^\alpha(\widetilde{a}^0_{\alpha,\beta}(x)\mathrm{D}^\beta) + \mathrm{hc} \tag{2.11}$$

with

$$\widetilde{a}^0_{\alpha,\beta}(x) = \langle x\rangle^{m-|\alpha+\beta|}(a^1_{\alpha,\beta} * \gamma)(x),$$

where γ is as in Lemma 2.3 [note that $\widetilde{a}^0_{\alpha,\beta} = a_{\alpha,\beta} = \mathrm{const}$ if $|\alpha+\beta| = m$]. Setting

$$\widetilde{a}_0(x,\xi) = \sum_{|\alpha|,|\beta| \leqslant m/2} \widetilde{a}^0_{\alpha,\beta}(x)\xi^{\alpha+\beta} \tag{2.12}$$

we have $\widetilde{a}^0_{\alpha,\beta} \in S_0^{m-|\alpha+\beta|}[r]$ and $\widetilde{a}_0 \in S_{\mathrm{g}}^m[r]$. Clearly $\widetilde{A}$ is a differential operator

$$\widetilde{A} = \sum_{|\alpha|\leqslant m} \widetilde{a}_\alpha(x)\mathrm{D}^\alpha = \widetilde{a}(x,\mathrm{D}) \in \Psi_{\mathrm{g}}^m[r],$$

because $\widetilde{a}(x,\xi) = \sum_{|\alpha|\leqslant m} \widetilde{a}_\alpha(x)\xi^\alpha$ satisfies $\widetilde{a} - \widetilde{a}_0 \in S_{\mathrm{g}}^{m-1-r}$, hence $\widetilde{a} \in S_{\mathrm{g}}^m[r]$.

3. comparison of A and $\widetilde{A}$

We keep the notations of Section 2 concerning the smooth operator $\widetilde{A}$ associated with the quadratic form A which satisfies the hypotheses of Theorem 1.1. The proof of Theorem 1.1 is based on the fact that the asymptotic behaviour of $N_A(\lambda)$ and $N_{\widetilde{A}}(\lambda)$ are similar as well as the asymptotic behaviour of $N_{a_0}(\lambda)$ and $N_{\mathrm{Re}\,\widetilde{a}_0}(\lambda)$. Here $N_P(\lambda)$ denotes always the counting function of a self-adjoint, bounded from below operator P with compact resolvent, and, for a given real function p_0 on $\mathbb{R}^{2d}$, we denote

$$N_{p_0}(\lambda) = (2\pi)^{-d} \int\limits_{p_0(x,\xi)<\lambda} \mathrm{d}x\,\mathrm{d}\xi. \tag{3.1}$$

More precisely we are going to show

Proposition 3.1. *In order to prove Theorem* 1.1*, it suffices to prove that* (1.5) *holds with A and a_0 replaced by $\widetilde{A}$ and* $\mathrm{Re}\,\widetilde{a}_0$.

To compare $N_{a_0}(\lambda)$ and $N_{\mathrm{Re}\,\widetilde{a}_0}(\lambda)$ we note that Lemma 2.3 and the definition of $\widetilde{a}_0$ give

$$|(a_0 - \widetilde{a}_0)(x,\xi)| \leqslant C \sum_{|\alpha+\beta|\leqslant m-1} \langle x\rangle^{m-r-|\alpha+\beta|}\langle \xi\rangle^{|\alpha+\beta|} \leqslant C_1\Lambda(x,\xi)^{m-r}, \tag{3.2}$$

hence $\widetilde{a}_0$ is globally elliptic, i.e. the quadratic form $\widetilde{A}$ is bounded from below, closed on $H^{m/2}_{m/2}$ and defines a self-adjoint operator with compact resolvent in L^2. Since replacing A by $A + cI$ does not change the form of (1.5), we may assume further on that $A \geqslant I$, $\widetilde{A} \geqslant I$, $a_0 \geqslant 1$ and $\mathrm{Re}\,\widetilde{a}_0 \geqslant 1$. Then (3.2) implies

$$|a_0^\mu - (\mathrm{Re}\,\widetilde{a}_0)^\mu| \leqslant C \tag{3.3}$$

if $\mu \leqslant r/m$, hence

$$N_{a_0^\mu}(\lambda^\mu - C_0) \leqslant N_{(\mathrm{Re}\,\widetilde{a}_0)^\mu}(\lambda^\mu) \leqslant N_{a_0^\mu}(\lambda^\mu + C_0), \tag{3.4}$$

and using $N_{p_0^\mu}(\lambda^\mu) = N_{p_0}(\lambda)$, $(\lambda^\mu + C_0)^{1/\mu} = \lambda(1 + C_0\lambda^{-\mu})^{1/\mu} = \lambda + \mathrm{O}(\lambda^{1-\mu})$, we obtain

$$N_{a_0}(\lambda - C_1\lambda^{1-\mu}) \leqslant N_{\mathrm{Re}\,\widetilde{a}_0}(\lambda) \leqslant N_{a_0}(\lambda + C_1\lambda^{1-\mu}). \tag{3.5}$$

A similar comparison of $N_A(\lambda)$ and $N_{\widetilde{A}}(\lambda)$ results from

PROPOSITION 3.2. *If $\mu < r/m$, then $A^\mu - \widetilde{A}^\mu$ extends to a bounded operator on L^2.*

Clearly, if Proposition 3.2 holds, then $\widetilde{A}^\mu - C_0 I \leqslant A^\mu \leqslant \widetilde{A}^\mu + C_0 I$ and

$$N_{\widetilde{A}^\mu}(\lambda^\mu - C_0) \leqslant N_{A^\mu}(\lambda^\mu) \leqslant N_{\widetilde{A}^\mu}(\lambda^\mu + C_0) \tag{3.6}$$

due to the min-max principle (cf. [33]). Then using $N_{P^\mu}(\lambda^\mu) = N_P(\lambda)$ as before we find

$$N_{\widetilde{A}}(\lambda - C_1\lambda^{1-\mu}) \leqslant N_A(\lambda) \leqslant N_{\widetilde{A}}(\lambda + C_1\lambda^{1-\mu}). \tag{3.7}$$

i.e. Proposition 3.2 implies Proposition 3.1.

The end of this section is devoted to the proof of Proposition 3.2. We have

LEMMA 3.3. *The self-adjoint operator $(\widetilde{A} + \lambda)^{-1/2}$ is bounded $L^2 \to H^{m/2}_{m/2}$ for every $\lambda \geqslant 0$ and denote by $B(\lambda)$ the bounded form on L^2 given by*

$$B(\lambda)[\varphi, \psi] = (A - \widetilde{A})[(\widetilde{A} + \lambda)^{-1/2}\varphi, (\widetilde{A} + \lambda)^{-1/2}\psi]. \tag{3.8}$$

Then $\|B(\lambda)\| \leqslant C(1 + \lambda)^{-\theta}$ if $\theta < r/m$.

Proof. For $|\alpha + \beta| = m$ we have $\widetilde{a}^0_{\alpha,\beta} = a_{\alpha,\beta} = \text{const}$ and for $|\alpha + \beta| \leqslant m - 1$ the functions

$$b_{\alpha,\beta}(x) = (a_{\alpha,\beta} - \widetilde{a}^0_{\alpha,\beta})(x)\langle x\rangle^{-(m-r-|\alpha+\beta|)}$$

are bounded on $\mathbb{R}^d$. If $|\alpha| \leqslant m/2$, $|\beta| \leqslant m/2 - 1$ and $\psi \in H^{m/2}_{m/2}$, then

$$|(b_{\alpha,\beta}(x)\langle x\rangle^{m/2-|\alpha|}\mathrm{D}^\alpha\psi, \langle x\rangle^{m/2-r-|\beta|}\mathrm{D}^\beta\psi)|$$

$$\leqslant C\|\langle x\rangle^{m/2-|\alpha|}\mathrm{D}^\alpha\psi\|\;\|\langle x\rangle^{m(1/2-r/m)-|\beta|}\mathrm{D}^\beta\psi\|,$$

hence assuming $\theta < r/m$ we may write

$$|(A - \widetilde{A})[\psi, \psi]| \leqslant C_1\|\psi\|_{H^{m/2}_{m/2}}\|\psi\|_{H^{m/2-\theta}_{m/2-\theta}} \leqslant C_2\|\widetilde{A}^{1/2}\psi\|\;\|\widetilde{A}^{1/2-\theta}\psi\|, \tag{3.9}$$

where the last inequality is a consequence of Proposition 2.2 that guarantees

$$c_\theta \|\varphi\|_{H^{m\theta}_{m\theta}} \leqslant \|\widetilde{A}^\theta \varphi\| \leqslant C_\theta \|\varphi\|_{H^{m\theta}_{m\theta}}, \tag{3.10}$$

with certain constants $C_\theta, c_\theta > 0$ and $\theta = j/2^n$, $j, n \in \mathbb{Z}$. Since

$$\|\widetilde{A}^{1/2-\theta}(\widetilde{A}+\lambda)^{-1/2}\| = \sup_{\lambda' \geqslant 1} |\lambda'^{1/2-\theta}(\lambda'+\lambda)^{-1/2}| \leq \bar{C}_\theta (1+\lambda)^{-\theta},$$

for $0 \leqslant \theta \leqslant 1/2$, using $\psi = (\widetilde{A}+\lambda)^{-1/2}\varphi$ in (3.9), for $\theta < r/m$ we have

$$|B(\lambda)[\varphi,\varphi]| \leqslant C\|\widetilde{A}^{1/2}(\widetilde{A}+\lambda)^{-1/2}\varphi\| \, \|\widetilde{A}^{1/2-\theta}(\widetilde{A}+\lambda)^{-1/2}\varphi\| \leqslant (1+\lambda)^{-\theta}\|\varphi\|^2. \quad ∎$$

Lemma 3.4. *If $B(\lambda)$ is given by* (3.8) *and $\|B(\lambda)\| \leqslant C(1+\lambda)^{-\theta}$, $\theta > 0$, then*

$$\|(A+\lambda)^{-1} - (\widetilde{A}+\lambda)^{-1}\| \leqslant \widetilde{C}(1+\lambda)^{-1-\theta} \quad \textit{for } \lambda \geqslant 0. \tag{3.11}$$

Proof. Setting $\psi = (\widetilde{A}+\lambda)^{1/2}\varphi$ in

$$(A+\lambda I)[(\widetilde{A}+\lambda)^{-1/2}\psi, (\widetilde{A}+\lambda)^{-1/2}\psi] = ((I+B(\lambda))\psi, \psi)$$

we have

$$(A+\lambda)^{-1} = (\widetilde{A}+\lambda)^{-1/2}(I+B(\lambda))^{-1}(\widetilde{A}+\lambda)^{-1/2}$$

and

$$(A+\lambda)^{-1} - (\widetilde{A}+\lambda)^{-1} = (\widetilde{A}+\lambda)^{-1/2}((I+B(\lambda))^{-1} - I)(\widetilde{A}+\lambda)^{-1/2}. \tag{3.12}$$

To complete the proof we estimate the norm of (3.12) using $\|(\widetilde{A}+\lambda)^{-1/2}\| \leqslant (1+\lambda)^{-1/2}$ and the fact that for $\lambda > \lambda_0$ one has $\|B(\lambda)\| \leqslant 1/2$ and

$$\|(I+B(\lambda))^{-1} - I\| \leqslant \frac{\|B(\lambda)\|}{1-\|B(\lambda)\|} \leqslant 2C(1+\lambda)^{-\theta}. \quad ∎$$

To complete the proof of Proposition 3.2 it remains to show

Lemma 3.5. *Assume that* (3.11) *holds for a certain $\theta \leqslant 1/2$. Then $A^\mu - \widetilde{A}^\mu$ extends to a bounded operator on L^2 if $\mu < \theta$.*

Proof. If $0 < \mu < 1$, $a \geqslant 1$, then $\Upsilon_a = \int_0^\infty \mathrm{d}\lambda\, \lambda^{\mu-1} a(a+\lambda)^{-1} = \Upsilon_1 a^\mu$.

If $\mu < 1/2$ and $\varphi \in H^{m/2}_{m/2} = D(A^{1/2}) \subset D(A^\mu)$, then $\mu - 3/2 < -1$ and

$$\lambda^{\mu-1}\|A(A+\lambda)^{-1}\varphi\| \leqslant \lambda^{\mu-1}\|A^{1/2}(A+\lambda)^{-1}\| \, \|A^{1/2}\varphi\| \leqslant C\lambda^{\mu-3/2}\|A^{1/2}\varphi\|$$

is integrable on $[1;\infty[$ with respect to λ, allowing to write

$$\Upsilon_1 A^\mu \varphi = \int_0^\infty \mathrm{d}\lambda\, \lambda^{\mu-1} A(A+\lambda)^{-1}\varphi$$

and an analogous formula holds with $\widetilde{A}$ instead of A. Using (3.11) and the equality $A(A+\lambda)^{-1} - \widetilde{A}(\widetilde{A}+\lambda)^{-1} = \lambda((\widetilde{A}+\lambda)^{-1} - (A+\lambda)^{-1})$, we have for $\varphi \in H^{m/2}_{m/2}$ the estimate

$$\begin{aligned}
\Upsilon_1 \|(A^\mu - \widetilde{A}^\mu)\varphi\| &\leqslant \int_0^\infty \mathrm{d}\lambda\, \lambda^{\mu-1} \|A(A+\lambda)^{-1}\varphi - \widetilde{A}(\widetilde{A}+\lambda)^{-1}\varphi\| \\
&\leqslant C_1 \left(\int_0^1 \mathrm{d}\lambda\, \lambda^{\mu-1} + \int_1^\infty \mathrm{d}\lambda\, (1+\lambda)^{\mu-1-\theta} \right) \|\varphi\| \\
&\leqslant C_2 \|\varphi\|. \quad \blacksquare
\end{aligned}$$

4. CONSTRUCTION OF A PARAMETRIX FOR e^{-itP}

In this section we consider a self-adjoint operator $P \in \Psi^{m'}_{\mathrm{g}}[r]$ such that $P - p(x, \mathrm{D}) \in \Psi^{-\infty}_{\mathrm{gl}}$ and

$$p_0(x,\xi) = \operatorname{Re} p(x,\xi) \geqslant c_0 \Lambda(x,\xi)^{c_0} \quad \text{with} \quad c_0 > 0. \tag{4.1}$$

Assuming $m' < r$ we construct an approximation of $\mathrm{e}^{-\mathrm{i}tP}$ for $t \in \mathbb{C}$, $\operatorname{Im} t < 0$, by operators

$$Q_N(t) = (q_N(t)\mathrm{e}^{-\mathrm{i}tp_0})(x, \mathrm{D}), \quad q_N(t) \in S^0_{\mathrm{g}}, \tag{4.2}$$

chosen such that $Q_N(0) = I$ and $\frac{\mathrm{d}}{\mathrm{d}t} Q_N(t) + Q_N(t)\mathrm{i}P$ has a sufficiently regular integral kernel for N large enough. We start by

PROPOSITION 4.1. *Let $N \in \mathbb{N}$ and define $\mathcal{P}_N : C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) \to C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ by*

$$\mathcal{P}_N : a \mapsto \mathcal{P}_N a = \sum_{|\alpha| \leqslant N} \frac{(-\mathrm{i})^{|\alpha|}}{\alpha!} (\bar{p}_{(\alpha)} a)^{(\alpha)}. \tag{4.3}$$

If $m' < r$ then there are $q_{N,j} \in S^{(j-1)(m'-r)}_{\mathrm{g}}$ for $1 \leqslant j \leqslant N$, $\widetilde{q}^{\,0}_{n,j} \in S^{j(m'-r)}_{\mathrm{g}}$ for $n \leqslant j \leqslant N+n$, such that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\,\mathcal{P}_N \right) (q_n(t)\mathrm{e}^{-\mathrm{i}tp_0}) = \widetilde{q}^{\,0}_n(t)\mathrm{e}^{-\mathrm{i}tp_0} \tag{4.4}$$

with

$$q_n(t) = 1 + \sum_{1 \leqslant j \leqslant n} t^j q_{N,j}, \quad \widetilde{q}_n^0(t) = \sum_{n \leqslant j \leqslant n+N} t^j \widetilde{q}_{n,j}^0.$$

Proof. If $q \in S_{\mathrm{g}}^m$, then

$$\mathrm{e}^{\mathrm{i}tp_0} \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\,\mathcal{P}_N \right) (q\mathrm{e}^{-\mathrm{i}tp_0}) = \sum_{0 \leqslant j \leqslant N} t^j \widetilde{q}_j, \tag{4.5}$$

where

$$\widetilde{q}_0 = -\mathrm{i}p_0 q + \mathrm{i}\,\mathcal{P}_N q = \mathrm{i}(\bar{p} - p_0)q - \sum_{1 \leqslant |\alpha| \leqslant N} \frac{(-\mathrm{i})^{|\alpha|+1}}{\alpha!} (\bar{p}_{(\alpha)} q)^{(\alpha)},$$

$$\widetilde{q}_j = \sum_{\substack{\alpha_0 + \cdots + \alpha_j | \leqslant N \\ \alpha_k \neq 0 \text{ if } 1 \leqslant k \leqslant j}} c_{\alpha_0, \ldots, \alpha_j} (\bar{p}_{(\alpha_0 + \cdots + \alpha_j)} q)^{(\alpha_0)} p_0^{(\alpha_1)} \cdots p_0^{(\alpha_j)} \quad \text{for } 1 \leqslant j \leqslant N. \tag{4.6}$$

Since P is symmetric we have $\bar{p} - p_0 \in S_{\mathrm{g}}^{m'-r-1}$ and $\widetilde{q}_0 \in S_{\mathrm{g}}^{m+m'-r-1}$. It is easy to check that $\widetilde{q}_j \in S_{\mathrm{g}}^{m+m'-r+j(m'-1)}$ for $1 \leqslant j \leqslant N$, hence setting $q_0(t) = 1$ we can see that the statement of Proposition 4.1 holds for $n = 0$ [because $r \leqslant 1 \Rightarrow j(m'-1) \leqslant j(m'-r)$]. Assume now that we have found $q_{N,j}$ for $1 \leqslant j \leqslant n$, such that the statement of Proposition 4.1 holds for a certain $n < N$ and we are going to find $q_{N,n+1}$ such that the statement holds for $n+1$. Since

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}tp_0} \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\,\mathcal{P}_N \right) ((q_n(t) + t^{n+1} q_{N,n+1}) \mathrm{e}^{-\mathrm{i}tp_0}) \\ &\qquad = \widetilde{q}_n^0(t) + (n+1)t^n q_{N,n+1} + t^{n+1} \mathrm{e}^{\mathrm{i}tp_0} \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\,\mathcal{P}_N \right) (q_{N,n+1} \mathrm{e}^{-\mathrm{i}tp_0}), \end{aligned}$$

we have

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}tp_0} \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\,\mathcal{P}_N \right) (q_{n+1}(t) \mathrm{e}^{-\mathrm{i}tp_0}) \\ &\qquad = t^n (\widetilde{q}_{n,n}^0 + (n+1) q_{N,n+1}) + \sum_{0 \leqslant j \leqslant N} t^{n+j+1} \widetilde{q}_{n+1,n+j+1}^0 \end{aligned}$$

and to cancel the term with t^n we take $q_{N,n+1} = -\widetilde{q}_{n,n}^0/(n+1)$, which belongs to $S_{\mathrm{g}}^{n(m'-r)}$ due to the hypothesis concerning $\widetilde{q}_n^0(t)$. Hence, setting $q = q_{N,n+1}$ in (4.5)–(4.6), we can see that $(n+1)(m'-r) + j(m'-1) \leqslant (n+j+1)(m'-r)$ implies $\widetilde{q}_{n+1,k}^0 \in S_{\mathrm{g}}^{k(m'-r)}$ for $n+1 \leqslant k \leqslant N+n+1$. ∎

LEMMA 4.2. *Assume $m' \leqslant 1+r$. Then for every $s \in \mathbb{R}$ there is a constant $C_s > 0$ such that*

$$\|\mathrm{e}^{-\mathrm{i}tP}\|_{B(H_s^s)} \leqslant (2+|t|)^{C_s} \qquad \text{for } \operatorname{Im} t \leqslant 0. \tag{4.7}$$

Proof. Due to the duality of H_{-s}^{-s} and H_s^s, it suffices to consider $s \geqslant 0$. Clearly (4.7) holds for $s = 0$. Assume now that (4.7) holds for a given $s \geqslant 0$, fix $0 < \kappa \leqslant 1 + r - m'$ and denote $[0;\, t] = \{\lambda t \in \mathbb{C} : 0 \leqslant \lambda \leqslant 1\}$. Then for $\varphi \in H_s^s$ it is easy to estimate the H_s^s-norm of

$$\Lambda^\kappa(x,\mathrm{D})\mathrm{e}^{-\mathrm{i}tP}\Lambda^{-\kappa}(x,\mathrm{D})\varphi = \mathrm{e}^{-\mathrm{i}tP}\Lambda^\kappa(x,\mathrm{D})\Lambda^{-\kappa}(x,\mathrm{D})\varphi - \int\limits_{[0;\,t]} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}(t-\tau)P}[\Lambda^\kappa(x,\mathrm{D}),\, \mathrm{i}P]\mathrm{e}^{-\mathrm{i}\tau P}\Lambda^{-\kappa}(x,\mathrm{D})\varphi,$$

because $[\Lambda^\kappa(x,\mathrm{D}),\, \mathrm{i}P] \in S_{\mathrm{g}}^{\kappa+m'-1-r} \subset B(H_s^s)$ when $\kappa \leqslant 1+r-m'$. Therefore

$$\|\mathrm{e}^{-\mathrm{i}tP}\|_{B(H_{s+\kappa}^{s+\kappa})} \leqslant C_{s,k}\|\Lambda^\kappa(x,\mathrm{D})\mathrm{e}^{-\mathrm{i}tP}\Lambda^{-\kappa}(x,\mathrm{D})\|_{B(H_s^s)},$$

and assuming (4.7) for a given $s \geqslant 0$ we obtain (4.7) with $s+\kappa$ instead of s. ∎

PROPOSITION 4.3. *Let $m' < r$ and $n \in \mathbb{N}$. If $N \in \mathbb{N}$ is large enough then there exists $C_{n,N} > 0$ such that for $\operatorname{Im} t < 0$, $0 \leqslant k \leqslant n$,*

$$\Big\|\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\mathrm{e}^{-\mathrm{i}tP} - Q_N(t))\Big\|_{B(H_{-n}^{-n}, H_n^n)} \leqslant (2+|t|)^{C_{n,N}}. \tag{4.8}$$

Proof. For $\operatorname{Im} t < 0$ and $q(t) \in C^\infty(\mathbb{R}^{3d})$ polynomially bounded we denote

$$\mathcal{K}_t(q,x,x') = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}(x-x')\xi - \mathrm{i}tp_0(x,\xi)}q(t)(x,\xi,x')\,\mathrm{d}\xi. \tag{4.9}$$

If $a_N(t)(x,\xi,x') = q_N(t)(x,\xi)\overline{p(x',\xi)}$ then

$$(Q_N(t)p(x,\mathrm{D})^*\varphi)(x) = \int \mathcal{K}_t(a_N,x,x')\varphi(x')\,\mathrm{d}x' \tag{4.10}$$

and standard integrations by parts associated with Taylor expansion of $\overline{p(x',\xi)}$ in $x' = x$ give

$$\Big[\Big(\frac{\mathrm{d}}{\mathrm{d}t}Q_N(t) + \mathrm{i}Q_N(t)p(x,\mathrm{D})^*\Big)\varphi\Big](x) = \int \mathcal{K}_t(\widetilde{q}_N^0 + \widetilde{q}_N^1, x, x')\varphi(x')\,\mathrm{d}x' \tag{4.10$'$}$$

with $\widetilde{q}_N^0$ given by (4.4) and

$$\widetilde{q}_N^1(t)(x,\xi,x') = \mathrm{e}^{\mathrm{i}tp_0(x,\xi)}(N+1)\int\limits_0^1 \mathrm{d}\tau\,(1-\tau)^N\widetilde{q}_N^2(t,\tau)(x,\xi,x') \tag{4.11}$$

with

$$\widetilde{q}_N^2(t,\tau)(x,\xi,x') = \sum_{|\alpha|=N+1} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha((q_N(t)\mathrm{e}^{-\mathrm{i}tp_0})(x,\xi)\bar{p}_{(\alpha)}(x+\tau(x'-x),\xi)).$$

Since

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}tP} - Q_N(t) &= \int\limits_{[0;\,t]} \mathrm{d}\tau \, \frac{\mathrm{d}}{\mathrm{d}\tau}(Q_N(\tau)\mathrm{e}^{-\mathrm{i}(t-\tau)P}) \\ &= \int\limits_{[0;\,t]} \mathrm{d}\tau \left(\frac{\mathrm{d}}{\mathrm{d}\tau}Q_N(\tau) + Q_N(\tau)\mathrm{i}P\right)\mathrm{e}^{-\mathrm{i}(t-\tau)P}, \end{aligned}$$

it remains to prove that for every $n_0 \in \mathbb{N}$ there is $N \in \mathbb{N}$ large enough to guarantee that for $\alpha \in \mathbb{N}^{2d+1}$ satisfying $|\alpha| \leqslant n_0$ one has

$$|\partial_{t,x,x'}^\alpha \mathcal{K}_t(\widetilde{q}_N^0 + \widetilde{q}_N^1, x, x')| \leqslant (2+|t|)^C (1+|x|+|x'|)^{-n_0}. \tag{4.12}$$

Since $m' < r \leqslant 1$, for every $N_0 \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that

$$|q(t)(x,\xi,x')| \leqslant (2+|t|)^{C_0}(1+|x'|)^{m'}\Lambda(x,\xi)^{-2N_0}, \tag{4.13}$$

$$\begin{aligned} |\mathcal{K}_t(q,x,x')| &\leqslant \int |q(t)(x,\xi,x')| \, \mathrm{d}\xi \\ &\leqslant (2+|t|)^{\bar{C}_0}(1+|x'|)^{m'}(1+|x|)^{d+1-2N_0} \end{aligned} \tag{4.14}$$

if $q = \widetilde{q}_N^0 + \widetilde{q}_N^1$. Replacing q by q_j^0 of the form

$$q_j^0(t)(x,\xi,x') = (-x_j + \partial_{\xi_j}p_0(x,\xi))q(t)(x,\xi,x') + \mathrm{i}\partial_{\xi_j}q(t)(x,\xi,x') \tag{4.15}$$

in (4.9), we estimate easily

$$|\mathcal{K}_t(q_j^0,x,x')| \leqslant (2+|t|)^{C_1}(1+|x'|)^{m'}(1+|x|)^{d+2-2N_0}. \tag{4.16}$$

Since q_j^0 given by (4.15) satisfies $\mathrm{i}\partial_{\xi_j}(q(t)\mathrm{e}^{\mathrm{i}x\xi - itp_0(x,\xi)}) = q_j^0(t)\mathrm{e}^{\mathrm{i}x\xi - itp_0(x,\xi)}$ and we have $x_j'\mathrm{e}^{\mathrm{i}x'\xi} = -\mathrm{i}\partial_{\xi_j}\mathrm{e}^{\mathrm{i}x'\xi}$, the integration by parts in (4.9) gives $x_j'\mathcal{K}_t(q,x,x') = \mathcal{K}_t(q_j^0,x,x')$, hence due to the estimates (4.16) for $j = 1,\dots,d$, we obtain

$$|\mathcal{K}_t(q,x,x')| \leqslant (2+|t|)^{\bar{C}_1}(1+|x'|)^{m'-1}(1+|x|)^{d+2-2N_0}. \tag{4.17}$$

If $0 \leqslant k \leqslant N_0$ then we may repeat the above reasoning k times and obtain

$$|\mathcal{K}_t(q,x,x')| \leqslant (2+|t|)^{\bar{C}_k}(1+|x'|)^{m'-k}(1+|x|)^{d+1+k-2N_0}. \tag{4.18}$$

Calculating q_α such that $\partial_{t,x,x'}^\alpha \mathcal{K}_t(q,x,x') = \mathcal{K}_t(q_\alpha,x,x')$ we find similar decay estimates if only N_0 is large enough. ∎

5. END OF THE PROOF OF THEOREM 1.1

Setting $P = \widetilde{A}^\mu$ and using Proposition 2.2, we can see that P satisfies the hypotheses from the beginning of Section 4 with $m' = m\mu$. Our aim is to prove Theorem 1.1 with $\widetilde{A}$ instead of A (cf. Proposition 3.1) and it suffices to prove

$$N_P(\lambda) = N_{p_0}(\lambda) + \mathrm{O}(N_{p_0}(\lambda + C_0) - N_{p_0}(\lambda - C_0)) + \mathrm{O}(1), \tag{5.1}$$

where as in Section 4, $P - p(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$ and $p_0 = \mathrm{Re}\, p \geqslant 1$. Indeed, we may interchange a_0 and $\mathrm{Re}\, \widetilde{a}_0$ in (3.5), replace a_0 by $\widetilde{a}$ in (3.2) and by $p_0^{1/\mu}$ in (3.3) due to Proposition 2.2. Hence replacing λ by λ^μ in (5.1) and using $N_{\widetilde{A}}(\lambda) = N_P(\lambda^\mu)$, $(\lambda^\mu + C_0)^{1/\mu} = \lambda + \mathrm{O}(\lambda^{1-\mu})$, $N_{p_0}(\lambda^\mu) = N_{p_0^{1/\mu}}(\lambda)$ as in Section 2, we get (1.5) with $\widetilde{A}$ and $\mathrm{Re}\, \widetilde{a}_0$ instead of A and a_0.

Since $\mu < r/m$ implies $m' = m\mu < r$, it remains to prove

THEOREM 5.1. *If P is as at the beginning of Section 4 and $m' < r$, then (5.1) holds.*

To prove Theorem 5.1 we introduce $\chi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \chi \subset [-c; c]$, $\int \chi = 1$, $\chi \geqslant 0$ and χ is strictly positive on $[-c/2; c/2]$, where $c > 0$ is fixed. We shall show that

$$N_P(\lambda) - (\chi * N_P)(\lambda) = \mathrm{O}(\nu(\lambda, 2c)) + \mathrm{O}(1), \tag{5.2}$$

$$N_{p_0}(\lambda) - (\chi * N_{p_0})(\lambda) = \mathrm{O}(\nu(\lambda, c)) + \mathrm{O}(1), \tag{5.3}$$

where we have denoted $\nu(\lambda, C) = N_{p_0}(\lambda + C) - N_{p_0}(\lambda - C)$ for $C > 0$.

Due to the hypothesis (4.1) for every $n \in \mathbb{N}$ there exists C_n such that

$$\Lambda(x, \xi)^n |\mathrm{e}^{-\mathrm{i}tp_0(x,\xi)}| \leqslant (2 + |\mathrm{Im}\, t|^{-1})^{C_n} \quad \text{for } \mathrm{Im}\, t < 0, \tag{5.4}$$

and a reasoning analogous to the proof of Proposition 4.3 allows to estimate

$$\|Q_N(t)\|_{B(H^{-n}_{-n}, H^n_n)} \leqslant (2 + |\mathrm{Im}\, t|^{-1} + |t|)^{C_{N,n}} \quad \text{for } \mathrm{Im}\, t < 0. \tag{5.5}$$

Due to Proposition 4.3, estimates (5.5) still hold if $Q_N(t)$ is replaced by $\mathrm{e}^{-\mathrm{i}tP}$. If $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ is the sequence of eigenvalues (counted with multiplicities) of P, then setting

$$u(t) = \sum_{j \geqslant 1} \mathrm{e}^{-\mathrm{i}t\lambda_j} = \mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}tP} = \int \mathrm{e}^{-\mathrm{i}t\lambda}\, \mathrm{d}N_P(\lambda) \tag{5.6}$$

and using the fact that the embedding of H^s_s into H^{-s}_{-s} is of trace class if $s > d$, we obtain

$$|u(t)| \leqslant (2 + |\operatorname{Im} t|^{-1} + |t|)^C \quad \text{for } \operatorname{Im} t < 0. \tag{5.7}$$

Therefore (cf. e.g. [17], Theorem 3.1.11), it is possible to define the boundary value of u on $\mathbb{R}$, being a distribution $\mathcal{S}'(\mathbb{R})$ such that

$$u(t) = \mathcal{S}'(\mathbb{R})\text{-}\lim_{\varepsilon \downarrow 0} u(t - \mathrm{i}\varepsilon) \quad \text{for } t \in \mathbb{R}. \tag{5.8}$$

Defining on $\{\operatorname{Im} t < 0\}$ the holomorphic function

$$u_N(t) = \operatorname{Tr} Q_N(t) = \int \mathrm{e}^{-\mathrm{i}tp_0(y,\xi)} q_N(t)(y,\xi) \, \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d}, \tag{5.9}$$

we obtain as before the existence of the boundary value on $\mathbb{R}$ in $\mathcal{S}'(\mathbb{R})$,

$$u_N(t) = \mathcal{S}'(\mathbb{R})\text{-}\lim_{\varepsilon \downarrow 0} u_N(t - \mathrm{i}\varepsilon) \quad \text{for } t \in \mathbb{R}, \tag{5.10}$$

and Proposition 4.3 implies

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} (u - u_N)(t) \right| \leqslant (2 + |t|)^{C_{N,K}} \quad \text{for } 0 \leqslant k \leqslant K,\ t \in \mathbb{R}. \tag{5.11}$$

Introducing $w_N \in C^\infty(\mathbb{R})$ given by

$$w_N(\lambda) = \sum_{0 \leqslant k \leqslant N} \int q_{N,k}(y,\xi)(-\mathrm{i})^k \chi^{(k)}(\lambda - p_0(y,\xi)) \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d} \tag{5.12}$$

we check easily that

$$\begin{aligned} \int \mathrm{e}^{-\mathrm{i}t\lambda} w_N(\lambda) \, \mathrm{d}\lambda &= \sum_{0 \leqslant k \leqslant N} \int q_{N,k}(y,\xi) \mathrm{e}^{-\mathrm{i}tp_0(y,\xi)} t^k \widehat{\chi}(t) \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d} \\ &= \widehat{\chi}(t) u_N(t) \end{aligned}$$

holds if $\operatorname{Im} t < 0$, i.e. taking the boundary value on $\mathbb{R}$ we have $\widehat{w}_N = \widehat{\chi} u_N$ and due to (5.11),

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} (\widehat{\chi} u - \widehat{\chi} u_N)(t) \right| = \left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathcal{F}_{\lambda \to t} ((\chi * N_P)' - w_N) \right| \leqslant C_n (1 + |t|)^{-n} \tag{5.13}$$

holds for every $t \in \mathbb{R}$, $n \in \mathbb{N}$ and $0 \leqslant k \leqslant K$, implying $|((\chi * N_P)' - w_N)(\lambda)| \leqslant C(1 + |\lambda|)^{-2}$, hence

$$\int_{-\infty}^{\lambda} ((\chi * N_P)' - w_N)(\lambda') \, \mathrm{d}\lambda' = \mathrm{O}(1). \tag{5.14}$$

To complete the proof of Theorem 5.1, it suffices to show

$$\int_{-\infty}^{\lambda} (w_0 - w_N)(\lambda')\,\mathrm{d}\lambda' = \mathrm{O}(\nu(\lambda, c)), \tag{5.15}$$

where

$$w_0(\lambda) = \int \chi(\lambda - p_0(y,\xi)) \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d} = \int \chi(\lambda - \lambda')\,\mathrm{d}N_{p_0}(\lambda') = (\chi * N_{p_0})'(\lambda).$$

Indeed, (5.14) and (5.15) allow to estimate

$$(\chi * (N_P - N_{p_0}))(\lambda) = \int_{-\infty}^{\lambda} ((\chi * N_P)' - w_0)(\lambda')\,\mathrm{d}\lambda' = \mathrm{O}(\nu(\lambda, c)) + \mathrm{O}(1) \tag{5.16}$$

and the proof of Theorem 5.1 follows from (5.2) and (5.3).

Proof of (5.3). It suffices to note that

$$\begin{aligned} N_{p_0}(\lambda) - (\chi * N_{p_0})(\lambda) &= \int_{\lambda - c}^{\lambda + c} \chi(\lambda - \lambda')(N_{p_0}(\lambda) - N_{p_0}(\lambda'))\,\mathrm{d}\lambda' \\ &= \mathrm{O}(\nu(\lambda, c)). \quad \blacksquare \end{aligned} \tag{5.17}$$

Proof of (5.15). The left hand side of (5.15) is equal to

$$\sum_{1 \leqslant k \leqslant N} \int q_{N,k}(y,\xi)(-\mathrm{i})^k \chi^{(k-1)}(\lambda - p_0(y,\xi)) \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d}, \tag{5.18}$$

and using $|q_{N,k}(y,\xi)| \leqslant C$, we estimate the absolute value of (5.18) by

$$\begin{aligned} &\sum_{1 \leqslant k \leqslant N} C \int |\chi^{(k-1)}(\lambda - p_0(y,\xi))| \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d} \\ &\quad = \sum_{1 \leqslant k \leqslant N} C \int_{\lambda - c}^{\lambda + c} |\chi^{(k-1)}(\lambda - \lambda')|\,\mathrm{d}N_{p_0}(\lambda') = \mathrm{O}(\nu(\lambda, c)). \quad \blacksquare \end{aligned}$$

Proof of (5.2). Estimating $|w_N(\lambda)|$ by

$$\begin{aligned} \sum_{0 \leqslant k \leqslant N} C \int |\chi^{(k)}(\lambda - p_0(y,\xi))| \frac{\mathrm{d}y\mathrm{d}\xi}{(2\pi)^d} &= \sum_{0 \leqslant k \leqslant N} C \int |\chi^{(k)}(\lambda - \lambda')|\,\mathrm{d}N_{p_0}(\lambda') \\ &= \mathrm{O}(\nu(\lambda, c)), \end{aligned}$$

we obtain $(\chi * N_P)'(\lambda) = w_N(\lambda) + \mathrm{O}(1) = \mathrm{O}(\nu(\lambda, c)) + \mathrm{O}(1)$ and due to $\chi(\lambda) \geqslant 1/C_0 > 0$ for $|\lambda| \leqslant c/2$,

$$\begin{aligned} N_P(\lambda + c/2) - N_P(\lambda - c/2) &\leqslant \int_{\lambda-c/2}^{\lambda+c/2} C_0\, \chi(\lambda - \lambda')\, \mathrm{d}N_P(\lambda') \\ &\leqslant C_0(\chi * N_P)'(\lambda) = \mathrm{O}(\nu(\lambda, c)) + \mathrm{O}(1), \\ N_P(\lambda) - (\chi * N_P)(\lambda) &= \int \chi(\lambda - \lambda')(N_P(\lambda) - N_P(\lambda'))\, \mathrm{d}\lambda' \\ &\leqslant C_1(N_P(\lambda + c) - N_P(\lambda - c)) \\ &= \mathrm{O}(\nu(\lambda, 2c)) + \mathrm{O}(1). \quad \blacksquare \end{aligned}$$

6. SMOOTHING OF TOP ORDER COEFFICIENTS

For $s \in \mathbb{R}$, the Sobolev space H^s is the completion of $\mathcal{S}$ in the norm $\|\varphi\|_{H^s} = \|\langle D\rangle^s \varphi\|$, where $\langle D\rangle = (I - \Delta)^{1/2}$ and $\Psi^{-\infty}$ denotes the set of operators which can be extended to linear continuous operators $H^{-n} \to H^n$ for every $n \in \mathbb{N}$.

DEFINITION 6.1. Let $m \in \mathbb{R}$, $0 \leqslant \delta < 1$ and $0 \leqslant r \leqslant 1$. Then $S^m_{1,\delta}(r)$ denotes the class of functions $p \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ satisfying

$$\begin{aligned} &|p^{(\alpha)}(x, \xi)| \leqslant C_\alpha \langle\xi\rangle^{m-|\alpha|} \qquad \text{for } \alpha \in \mathbb{N}^d, \\ &|p^{(\alpha)}_{(\alpha')}(x, \xi)| \leqslant C_\alpha \langle\xi\rangle^{m-|\alpha|+\delta(|\alpha'|-r)} \qquad \text{for } \alpha, \alpha' \in \mathbb{N}^d,\ |\alpha'| \geqslant 1 \end{aligned} \tag{6.1}$$

and $\Psi^m_{1,\delta}(r)$ denotes the class of linear operators P such that $P - p(x, \mathrm{D}) \in \Psi^{-\infty}$ with $p \in S^m_{1,\delta}(r)$, called the symbol of P. For $r = 0$ the class $S^m_{1,\delta}(0)$ is the usual Hörmander class denoted $S^m_{1,\delta}$ and $\Psi^m_{1,\delta}(0)$ is denoted $\Psi^m_{1,\delta}$.

PROPOSITION 6.2. (i) *Let $P \in \Psi^m_{1,\delta}(r)$ be symmetric, i.e. $(P\varphi, \psi) = (\varphi, P\psi)$ for $\varphi, \psi \in \mathcal{S}$ and $P - p(x, \mathrm{D}) \in \Psi^{-\infty}$. Then $\operatorname{Im} p \in S^{m-1+\delta(1-r)}_{1,\delta}$.*

(ii) *Assume moreover that P is elliptic of degree $m > 0$, i.e. $|p(x, \xi)| \geqslant c_0|\xi|^m - C_0$ for certain constants $C_0, c_0 > 0$. Then P is bounded from below, its extension on H^m is self-adjoint and will be denoted by the same letter P.*

(iii) *If moreover $P \geqslant I$, then $P^\theta \in \Psi^{m\theta}_{1,\delta}(r)$ for $\theta = j/2^n$, $j, n \in \mathbb{Z}$. If $\operatorname{Re} p \geqslant 1$, then $(\operatorname{Re} p)^\theta \in S^{m\theta}_{1,\delta}(r)$ and $P^\theta - (\operatorname{Re} p)^\theta q(x, \mathrm{D}) \in \Psi^{m\theta-1+\delta(1-r)}_{1,\delta}$.*

The proof of Proposition 6.2 is similar to the proof of Proposition 2.2 (cf. Appendix).

Further on in this section we consider a quadratic form A on $C_0^\infty(\mathbb{R}^d)$ given by (1.1) with $a_{\alpha,\beta} \in L^\infty(\mathbb{R}^d)$ and $a_{\alpha,\beta} \in \mathcal{B}^r(\mathbb{R}^d)$ if $|\alpha+\beta| = m$, assuming that there is $c_0 > 0$ such that

$$a_m(x,\xi) = \sum_{|\alpha|=|\beta|=m/2} a_{\alpha,\beta}(x)\xi^{\alpha+\beta} \geqslant c_0|\xi|^m. \tag{6.2}$$

DEFINITION OF THE "SMOOTH" OPERATOR $\widetilde{A}$. Let $\gamma \in \mathcal{S}$ satisfy $\int \gamma = 1$. We fix $0 \leqslant \delta < 1$ and define

$$\widetilde{a}_{\alpha,\beta}(x,\xi) = (a_{\alpha,\beta} * \gamma_{\delta,\xi})(x) \quad \text{with} \quad \gamma_{\delta,\xi}(x) = \gamma(x\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d}, \tag{6.3}$$

$$\widetilde{A} = \sum_{|\alpha|=|\beta|=m/2} \mathrm{D}^\alpha(\widetilde{a}_{\alpha,\beta}(x,\mathrm{D})\mathrm{D}^\beta) + hc. \tag{6.4}$$

PROPOSITION 6.3. *If $a_{\alpha,\beta} \in \mathcal{B}^r(\mathbb{R}^d)$ and $\widetilde{a}_{\alpha,\beta}$ is given as in* (6.3)*, then $\widetilde{a}_{\alpha,\beta} \in S^0_{1,\delta}(r)$ and*

$$|a_{\alpha,\beta}(x) - \widetilde{a}_{\alpha,\beta}(x,\xi)| \leqslant C\langle\xi\rangle^{-r\delta}. \tag{6.5}$$

Proof. Let $\widetilde{a}(x,\xi) = (a * \gamma_{\delta,\xi})(x) = \int a(y)\gamma((x-y)\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d}\,\mathrm{d}y$ with $a \in \mathcal{B}^r(\mathbb{R}^d)$. Since there exist bounded functions χ_β such that

$$\begin{aligned}&\partial_x^{\alpha'}\partial_\xi^\alpha(\gamma((x-y)\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d})\\ &\quad= \sum_{\beta\leqslant\alpha} \chi_\beta(\xi)\langle\xi\rangle^{\delta(|\alpha'|+|\beta|)-|\alpha|}(x-y)^\beta\gamma^{(\alpha'+\beta)}((x-y)\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d},\end{aligned}$$

we obtain $|\widetilde{a}^{(\alpha)}_{(\alpha')}(x,\xi)| \leqslant C\langle\xi\rangle^{\delta|\alpha'|-|\alpha|}$, because we may estimate $|\widetilde{a}^{(\alpha)}_{(\alpha')}(x,\xi)|$ by

$$C'\langle\xi\rangle^{\delta|\alpha'|-|\alpha|}\sum_{\beta\leqslant\alpha}\int |x-y|^{|\beta|}\langle\xi\rangle^{\delta|\beta|}|\gamma^{(\alpha'+\beta)}((x-y)\langle\xi\rangle^\delta)|\langle\xi\rangle^{\delta d}\,\mathrm{d}y, \tag{6.6}$$

and every integral in (6.6) is bounded with respect to ξ.

If $|\alpha'| > 0$ then $\int \partial_x^{\alpha'}\partial_\xi^\alpha(\gamma((x-y)\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d})\,\mathrm{d}y = 0$, hence

$$\widetilde{a}^{(\alpha)}_{(\alpha')}(x,\xi) = \int (a(y)-a(x))\partial_x^{\alpha'}\partial_\xi^\alpha(\gamma((x-y)\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d})\,\mathrm{d}y.$$

Using $|a(x)-a(y)| \leqslant C_0|x-y|^r$, we may replace (6.6) by

$$C''\langle\xi\rangle^{\delta|\alpha'|-|\alpha|-\delta r}\sum_{\beta\leqslant\alpha}\int |x-y|^{|\beta|+r}\langle\xi\rangle^{\delta(|\beta|+r)}|\gamma^{(\alpha'+\beta)}((x-y)\langle\xi\rangle^\delta)|\langle\xi\rangle^{\delta d}\,\mathrm{d}y$$

and every integral is bounded with respect to ξ, completing the proof of $\widetilde{a} \in S^0_{1,\delta}(r)$. Introducing $b(x,\xi) = \widetilde{a}(x,\xi) - a(x)$, we have

$$b^{(\alpha)}(x,\xi) = \int (a(y) - a(x))\partial_\xi^\alpha(\gamma((x-y)\langle\xi\rangle^\delta)\langle\xi\rangle^{\delta d})\,\mathrm{d}y$$

and estimating as before we obtain

$$|b^{(\alpha)}(x,\xi)| \leqslant C_\alpha \langle\xi\rangle^{-|\alpha|-\delta r}. \qquad \blacksquare \tag{6.7}$$

Therefore we check easily that $\widetilde{A}[\varphi,\psi] = ((\widetilde{a}(x,\mathrm{D}) + R)\varphi,\psi)$ with $\widetilde{a} \in S^m_{1,\delta}(r)$ and $R \in \Psi^{-\infty}$, i.e. we may treat $\widetilde{A}$ as a pseudodifferential operator of class $\Psi^m_{1,\delta}(r)$ with the symbol $\widetilde{a}$. Moreover (6.5) implies $|a_m^\mu - (\operatorname{Re}\widetilde{a}_m)^\mu| \leqslant \text{const}$ for $\mu < r\delta/m$ if

$$\widetilde{a}_m(x,\xi) = \sum_{|\alpha|=|\beta|=m/2} \widetilde{a}_{\alpha,\beta}(x,\xi)\xi^{\alpha+\beta} \tag{6.8}$$

and $\widetilde{a}_m - \widetilde{a} \in S^{m-1+\delta(1-r)}_{1,\delta}$ implies that $\widetilde{A}$ is elliptic of degree m. Further on we assume $\widetilde{A} \geqslant I$, $\widetilde{a} \geqslant 1$ and show

Proposition 6.4. *If $\mu < \delta r$, then $\widetilde{A}^\mu - A^\mu$ extends to a bounded operator on L^2.*

Proof. Reasoning as in Section 3 it suffices to check that $\|B(\lambda)\| \leqslant C(1+\lambda)^{-\theta}$ with $\theta < \delta r$ and that (3.9) holds with the norms of H^s_s replaced by the norms H^s. The last statement follows from

Proposition 6.5. *If $0 < r \leqslant 1$, $a_{\alpha,\beta} \in \mathcal{B}^r(\mathbb{R}^d)$ and $\widetilde{a}_{\alpha,\beta}$ is given as in* (6.3), *then the difference $a_{\alpha,\beta}(x) - \widetilde{a}_{\alpha,\beta}(x,\mathrm{D})$ extends to a bounded operator $H^{-s} \to L^2$ for every $s < \delta r$.*

Proof. Let $q(x,\xi) = (a_{\alpha,\beta}(x) - \widetilde{a}_{\alpha,\beta}(x,\xi))\langle\xi\rangle^s$, let $\chi \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi(x) = 1$ if $|x| \leqslant 1$ and for $\varepsilon > 0$ set $q_\varepsilon(x,\xi) = q(x,\xi)\chi(\varepsilon\xi)$. Then

$$K_\varepsilon(x,x') = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}(x-x')\xi} q_\varepsilon(x,\xi)\,\mathrm{d}\xi$$

is the integral kernel of $q_\varepsilon(x,\mathrm{D})$ and integrating by parts we have, for $|\alpha| \geqslant 1$,

$$\begin{aligned}(2\pi)^d(x-x')^\alpha K_\varepsilon(x,x') &= \int \mathrm{e}^{\mathrm{i}(x-x')\xi}(\mathrm{i}\partial_\xi)^\alpha q_\varepsilon(x,\xi)\,\mathrm{d}\xi \\ &= \int (\mathrm{e}^{\mathrm{i}(x-x')\xi} - 1)(\mathrm{i}\partial_\xi)^\alpha q_\varepsilon(x,\xi)\,\mathrm{d}\xi\end{aligned} \tag{6.9}$$

due to $\int(\mathrm{i}\partial_\xi)^\alpha q_\varepsilon(x,\xi)\,\mathrm{d}\xi = 0$. Using $|\mathrm{e}^{\mathrm{i}(x-x')\xi} - 1| \leqslant |x-x'|^\kappa|\xi|^\kappa$ with $0 < \kappa < \delta r - s \leqslant 1$ and $|\alpha| = d$ we estimate the absolute value of (6.9) by $C\int |x - x'|^\kappa\langle\xi\rangle^{\kappa+s-d-\delta r}\,\mathrm{d}\xi$, where the integral is convergent due to $\kappa+s-\delta r < 0$. Therefore $|K_\varepsilon(x,x')| \leqslant C'|x-x'|^{-d+\kappa}$ and moreover $|K_\varepsilon(x,x')| \leqslant C_N|x-x'|^{-N}$ for all $N \in \mathbb{N}$ large enough, which completes the proof due to the Schwartz lemma. $\blacksquare$

If $E_{\widetilde{A}}(\lambda)$ denotes the spectral projector of $\widetilde{A}$, then $\widetilde{A}^n E_{\widetilde{A}}(\lambda)$ is bounded for all $n \in \mathbb{N}$, hence $E_{\widetilde{A}}(\lambda) \in \Psi^{-\infty}$ and we denote by $e_{\widetilde{A}}(\,\cdot\,,\,\cdot\,,\lambda)$ the smooth integral kernel of $E_{\widetilde{A}}$. We have

THEOREM 6.6. *If* $0 < r \leqslant 1$, $\widetilde{A} \in \Psi^m_{1,\delta}(r)$ *is elliptic, self-adjoint and* $\mu < (1-\delta)/m \leqslant (\delta r)/m$, *then*

$$\begin{aligned} e_{\widetilde{A}}(x,x,\lambda) &= e_{\operatorname{Re}\widetilde{a}_m}(x,\lambda) \\ &\quad + \mathrm{O}(e_{\operatorname{Re}\widetilde{a}_m}(x,\lambda + C\lambda^{1-\mu}) - e_{\operatorname{Re}\widetilde{a}_m}(x,\lambda - C\lambda^{1-\mu})) + \mathrm{O}(1) \end{aligned} \tag{6.10}$$

holds uniformly with respect to $x \in \mathbb{R}^d$, *where*

$$e_{\operatorname{Re}\widetilde{a}_m}(x,\lambda) = (2\pi)^{-d} \int\limits_{\operatorname{Re}\widetilde{a}_m(x,\xi)<\lambda} \mathrm{d}\xi. \tag{6.10$'$}$$

Reasoning as at the beginning of Section 5 we may reduce Theorem 6.6 to

THEOREM 6.7. *Let* $0 < r \leqslant 1$, *let* P *be self-adjoint,* $P - p(x,\mathrm{D}) \in \Psi^{-\infty}$ *with* $p \in S^{m'}_{1,\delta}(r)$ *and* $|p(x,\xi)| \geqslant c_0\langle\xi\rangle^{c_0}$ *with* $c_0 > 0$. *Then* (6.10) *holds with* $\mu = 1$ *if* $m' < 1-\delta \leqslant \delta r$ *and* $\widetilde{A}$, $\widetilde{a}_m$ *are replaced by* P *and* $p_0 = \operatorname{Re} p$.

We note first that following the construction of the parametrix of $\mathrm{e}^{-\mathrm{i}tP}$ from Section 4 we obtain

PROPOSITION 6.8. *If* P *satisfies the hypotheses of Theorem* 6.7, $N \in \mathbb{N}$ *and* $0 \leqslant n \leqslant N$, *then we can find* $q_{N,j} \in S^{(j-1)(m'-1+\delta)}_{1,\delta}$ *for* $1 \leqslant j \leqslant n$ *and* $\widetilde{q}^0_{n,j} \in S^{j(m'-1+\delta)}_{1,\delta}$ *for* $n \leqslant j \leqslant N+n$, *such that* (4.4) *holds with* $q_n(t)$, $\widetilde{q}^0_n(t)$ *expressed as in Proposition* 4.1.

Proof. It suffices to follow the proof of Proposition 4.1. If $q \in S^m_{1,\delta}$, then (4.5), (4.6) hold with $\widetilde{q}_j \in S^{m+(j+1)m'-\delta r-j(1-\delta)}_{1,\delta}$ for $1 \leqslant j \leqslant N$ and $1-\delta \leqslant \delta r \Rightarrow (j+1)m' - \delta r - j(1-\delta) \leqslant (j+1)(m'-(1-\delta))$. ∎

Now to complete the proof of Theorem 6.7, it remains to follow the Tauberian reasoning from Section 5, where instead of $u(t)$ and $u_N(t)$, we compare $u(t,x)$ and $u_N(t,x)$ given by

$$u(t,x) = \int \mathrm{e}^{-\mathrm{i}t\lambda}\,\mathrm{d}_\lambda e_P(x,x,\lambda), \quad u_N(t,x) = \int \mathrm{e}^{-\mathrm{i}tp_0(x,\xi)} q_N(t)(x,\xi)\,\frac{\mathrm{d}\xi}{(2\pi)^d}$$

using pointwise uniform estimates with respect to $x \in \mathbb{R}^d$ instead of the integration with respect to x used in Section 5.

7. PROOF OF THEOREM 1.3

Let X, Y be smooth differential manifolds. If $\chi : X \to Y$ is a smooth diffeomorphism and $y = \chi(x)$, then $\mathrm{d}\chi(x) : T_xX \to T_yY$ has the dual $\mathrm{d}\chi(x)^* : T_y^*Y \to T_x^*X$ and we define the coderivative $\mathrm{d}\chi^* : T^*Y \to T^*X$ by the equality $\mathrm{d}\chi^*(y,\eta) = (\chi^{-1}(y), \mathrm{d}\chi(\chi^{-1}(y))^*\eta)$. If $A : C_0^\infty(X) \to C^\infty(X)$, then χ_*A denotes the operator $C_0^\infty(Y) \to C^\infty(Y)$ acting according to the formula $\varphi \mapsto (A(\varphi \circ \chi)) \circ \chi^{-1}$ for $\varphi \in C_0^\infty(Y)$.

Assume now that χ is a diffeomorphism of $\mathbb{R}^d$ such that $|\partial^\alpha \chi(x)| \leqslant C_\alpha$ for $|\alpha| \geqslant 1$ and the Jacobian $|\det(\partial_{x_j}\chi_k(x))| \geqslant c_0 > 0$. If $p \in S^m_{1,\delta}(r)$, then the well known formula for the symbol of $\chi_*(p(x,\mathrm{D})) \in \Psi^m_{1,\delta}$ (cf. e.g. [37], [24] or [17]) gives

$$\chi_*(p(x,\mathrm{D})) - (p \circ \mathrm{d}\chi^*)(x,\mathrm{D}) \in \Psi^{m-1+\delta(1-r)}_{1,\delta}. \tag{7.1}$$

Since $S^{m'}_{1,\delta} \subset S^m_{1,\delta}(r)$ if $m' \leqslant m - r\delta$ and it is easy to check that $p \circ \mathrm{d}\chi^* \in S^m_{1,\delta}(r)$ if $p \in S^m_{1,\delta}(r)$, (7.1) gives $\chi_*P \in \Psi^m_{1,\delta}(r)$ if $P \in \Psi^m_{1,\delta}(r)$, allowing to define the corresponding classes of symbols and operators on a manifold as follows

Definition 7.1. (i) Let M be a smooth manifold with a smooth density $\mathrm{d}y$ and denote by $\Psi^{-\infty}(M)$ the class of smoothing operators on M, i.e. linear integral operators with a smooth kernel or in other words, the linear operators $C_0^\infty(M) \to C^\infty(M)$ having a continuous extension $\mathcal{E}'(M) \to C^\infty(M)$ [$\mathcal{E}'(M)$ is the space of distributions with compact support].

(ii) A linear operator $P : C_0^\infty(M) \to C^\infty(M)$ is called *pseudolocal* if $\operatorname{supp}\vartheta_1 \cap \operatorname{supp}\vartheta_2 = \emptyset \Rightarrow \vartheta_1 P \vartheta_2 \in \Psi^{-\infty}(M)$, where ϑ_j denotes here the operator of multiplication by $\vartheta_j \in C_0^\infty(M)$.

(iii) We define $S^m_{1,\delta}(r)(T^*M)$ as the class of $p \in C^\infty(T^*M)$ such that there is an atlas Ξ and a family $\{p_\chi\}_{\chi\in\Xi}$ of symbols $p_\chi \in S^m_{1,\delta}(r)$ satisfying $p_\chi \circ \mathrm{d}\chi^* = p$ on $T^*\chi(\mathcal{U}_\chi)$. We say that p is elliptic of degree m if it is possible to choose every p_χ elliptic of degree m.

(iv) We define $\Psi^m_{1,\delta}(r)(M)$ as the class of pseudolocal operators P such that there is an atlas Ξ and a family $\{P_\chi\}_{\chi\in\Xi}$ of operators $P_\chi \in \Psi^m_{1,\delta}(r)$ satisfying $P_\chi(\varphi \circ \chi) = (P\varphi) \circ \chi$ on $\mathcal{U}_\chi$ for every $\varphi \in C_0^\infty(\chi(\mathcal{U}_\chi))$. We say that P is elliptic of degree m if it is possible to choose every P_χ elliptic of degree m. If $P \in \Psi^m_{1,\delta}(r)(M)$, then $\sigma(P)$ is called a *principal symbol of* P if $\sigma(P) \in S^m_{1,\delta}(r)(T^*M)$ and there is an atlas Ξ such that $P_\chi - \sigma(P)_\chi(x,\mathrm{D}) \in \Psi^{m-1+\delta(1-r)}_{1,\delta}$, where $\sigma(P)_\chi$ is associated with the symbol $\sigma(P)$ as in (iii) and P_χ with P as above.

Proposition 7.2. (i) *The operator $P \in \Psi^m_{1,\delta}(r)(M)$ is elliptic of degree m if and only if $\sigma(P)$ is elliptic of degree m. If P is symmetric, i.e. $(P\varphi,\psi) = (\varphi, P\psi)$ for $\varphi,\psi \in C_0^\infty(M)$, then we may choose $\sigma(P)$ real.*

(ii) *Assume that M is compact, $P \in \Psi^m_{1,\delta}(r)(M)$ is symmetric and elliptic of degree $m > 0$. Then P is bounded from below, its extension to the Sobolev space $H^m(M)$ is self-adjoint and the resolvent is compact.*

(iii) *Assume moreover $P \geqslant I$, using the same letter P for the associated self-adjoint operator. Then $P^\theta \in \Psi^{m\theta}_{1,\delta}(r)(M)$ if $\theta = j/2^n$, $j, n \in \mathbb{Z}$ and assuming $\sigma(P) \geqslant 1$, we may take $\sigma(P^\theta) = \sigma(P)^\theta$.*

The proof is similar to the proof of Proposition 6.2 (cf. Appendix).

Further on we consider the situation assumed in Theorem 1.2, i.e. A is a local quadratic form on the compact manifold M with the density $\mathrm{d}y$, expressed in local coordinates by A_χ for $\chi \in \Xi$. If $\sigma(A) \in C(T^*M)$ is such that $\sigma(A) = a_{m,\chi} \circ \mathrm{d}\chi^*$ on $T^*\chi(\mathcal{U}_\chi)$, then $\sigma(A)$ is homogeneous of degree m and strictly positive outside the section of zero cotangent vectors. Let $\{\vartheta_j\}_{1 \leqslant j \leqslant J}$ be a partition of unity of smooth functions on M satisfying

$$\text{(7.2)} \qquad \begin{aligned} &\operatorname{supp} \vartheta_j \cap \operatorname{supp} \vartheta_k \neq \emptyset \Rightarrow \exists\, \chi(j,k) \in \Xi, \\ &\qquad\qquad \operatorname{supp} \vartheta_j \cup \operatorname{supp} \vartheta_k \subset \chi(j,k)(\mathcal{U}_{\chi(j,k)}) \end{aligned}$$

for all $1 \leqslant j, k \leqslant J$ and assume moreover that $\chi(j,k) = \chi(k,j)$. Since A is local, we have

$$\text{(7.3)} \qquad A[\varphi, \psi] = \sum_{\operatorname{supp} \vartheta_j \cap \operatorname{supp} \vartheta_k \neq \emptyset} A_{\chi(j,k)}[(\vartheta_j \varphi) \circ \chi(j,k),\ (\vartheta_k \psi) \circ \chi(j,k)].$$

DEFINITION OF THE SMOOTH OPERATOR $\widetilde{A}$. Let $\widetilde{A}_{\chi(j,k)}$ be the quadratic form on $C_0^\infty(\mathbb{R}^d)$ associated with $A_{\chi(j,k)}$ according to the procedure described in Section 6 and set

$$\text{(7.4)} \qquad \widetilde{A}[\varphi, \psi] = \sum_{\operatorname{supp} \vartheta_j \cap \operatorname{supp} \vartheta_k \neq \emptyset} \widetilde{A}_{\chi(j,k)}[(\vartheta_j \varphi) \circ \chi(j,k),\ (\vartheta_k \psi) \circ \chi(j,k)].$$

Then $\widetilde{A} \in \Psi^m_{1,\delta}(r)(M)$ and

$$\text{(7.5)} \qquad |\sigma(\widetilde{A}) - \sigma(A)| \leqslant C(1 + \sigma(A))^{1 - r\delta/m},$$

hence $\sigma(\widetilde{A})$ is elliptic of degree m. Further on we assume $A \geqslant I$, $\widetilde{A} \geqslant I$, $\sigma(\widetilde{A}) \geqslant 1$ and show

PROPOSITION 7.3. *If $\mu < \delta r/m$, then $A^\mu - \widetilde{A}^\mu$ extends to a bounded operator on L^2.*

Proof. Reasoning as in Section 3 it suffices to check that $\|B(\lambda)\| \leqslant C(1+\lambda)^{-\theta}$ with $\theta < \delta r$ and that (3.9) holds with the norms of H^s_s replaced by the norms $H^s(M)$. Due to (7.3), (7.4), the last statement follows from the corresponding estimate for $A_{\chi(j,k)} - \widetilde{A}_{\chi(j,k)}$ proved in Section 6. ∎

As in Section 6, the spectral projection $E_{\widetilde{A}}(\lambda) \in \Psi^{-\infty}(M)$ has a smooth kernel $e_{\widetilde{A}}(\,\cdot\,,\,\cdot\,,\lambda)$ and we have

Theorem 7.4. *If $\widetilde{A} \in \Psi^m_{1,\delta}(r)(M)$ is elliptic, self-adjoint and $\mu < (1-\delta)/m \leqslant (\delta r)/m$, then*

$$e_{\widetilde{A}}(y,y,\lambda) = e_{\sigma(\widetilde{A})}(y,\lambda) + \mathrm{O}(e_{\sigma(\widetilde{A})}(y,\lambda + C\lambda^{1-\mu}) - e_{\sigma(\widetilde{A})}(y,\lambda - C\lambda^{1-\mu})) + \mathrm{O}(1), \tag{7.6}$$

holds uniformly with respect to $y \in M$, where

$$e_{\sigma(\widetilde{A})}(\chi(x),\lambda) = (2\pi)^{-d} \int\limits_{\sigma(\widetilde{A})((\mathrm{d}\chi^*)^{-1}(x,\xi))<\lambda} \mathrm{d}\xi \quad \textit{for } x \in \mathcal{U}_\chi. \tag{7.6$'$}$$

Due to $N_{\widetilde{A}}(\lambda) = \operatorname{Tr} E_{\widetilde{A}}(\lambda) = \int\limits_M e_{\widetilde{A}}(y,y,\lambda)\,\mathrm{d}y$, Theorem 7.4 implies

$$N_{\widetilde{A}}(\lambda) = N_{\sigma(\widetilde{A})}(\lambda) + \mathrm{O}(N_{\sigma(\widetilde{A})}(\lambda + C\lambda^{1-\mu}) - N_{\sigma(\widetilde{A})}(\lambda - C\lambda^{1-\mu})) + \mathrm{O}(1)$$

with

$$N_{\sigma(\widetilde{A})}(\lambda) = \int\limits_M e_{\sigma(\widetilde{A})}(y,\lambda)\,\mathrm{d}y,$$

and as in Section 2, Proposition 7.3 and (7.5) allow to replace $N_{\widetilde{A}}(\lambda)$ and $N_{\sigma(\widetilde{A})}(\lambda)$ by $N_A(\lambda)$ and $N_{\sigma(A)}(\lambda)$, which gives Theorem 1.2 because

$$N_{\sigma(A)}(\lambda) = N_{\sigma(A)}(1)\lambda^{d/m},$$

$$N_{\sigma(A)}(\lambda + C\lambda^{1-\mu}) - N_{\sigma(A)}(\lambda - C\lambda^{1-\mu}) = \mathrm{O}(\lambda^{-\mu+d/m})$$

with $\mu < r/(r+1)$ if we take $\delta = 1/(r+1)$. As in Section 5, due to Proposition 7.2, it suffices to prove that (7.6) holds with $\mu = 1$ and with $\widetilde{A}$, $\sigma(\widetilde{A})$ replaced by P, $\sigma(P)$, where $P \in \Psi^{m'}_{1,\delta}(r)(M)$ is self-adjoint and elliptic of degree $m' < 1-\delta \leqslant \delta r$. Let Ξ be an atlas and $\{P_\chi\}_{\chi\in\Xi}$ the family of associated operators $P_\chi \in \Psi^{m'}_{1,\delta}(r)$. We fix $\chi \in \Xi$ defined on $\mathcal{U}_\chi \subset \mathbb{R}^d$ and we are going to check that for any compact set $\mathcal{C}_\chi \subset \mathcal{U}_\chi$, it is possible to find constants C_k satisfying

$$|\mathrm{D}_t^k(u(t,\chi(x)) - u_\chi(t,x))| \leqslant (2+|t|)^{C_k} \quad \text{for } x \in \mathcal{C}_\chi, \tag{7.7}$$

where

$$u(t,y) = \int \mathrm{e}^{-\mathrm{i}t\lambda}\,\mathrm{d}e_P(y,y,\lambda), \quad u_\chi(t,x) = \int \mathrm{e}^{-\mathrm{i}t\lambda}\,\mathrm{d}e_{P_\chi}(x,x,\lambda).$$

If (7.7) holds, then it still holds with $u_\chi(t,x)$ replaced by the approximation $u_{\chi,N}(t,x)$ constructed as in Section 6 (where P is replaced by P_χ), hence one may follow the same Tauberian reasoning as before with $u(t,\chi(x))$ instead of $u_\chi(t,x)$, obtaining the same asymptotic formula for $e_P(\chi(x),\chi(x),\lambda)$ as obtained for $e_{P_\chi}(x,x,\lambda)$.

Thus, to complete the proof of Theorem 7.4, it remains to prove (7.7). Let $\mathcal{U}'_\chi \subset \mathbb{R}^d$ be an open set containing $\mathcal{C}_\chi$, such that the closure of $\mathcal{U}'_\chi$ is a compact subset of $\mathcal{U}_\chi$ and let $\vartheta \in C_0^\infty(\mathcal{U}_\chi)$ be such that $\vartheta = 1$ on a neighbourhood of the closure of $\mathcal{U}'_\chi$. Then we have

LEMMA 7.5. *For $\varphi \in C_0^\infty(\mathcal{U}'_\chi)$ denote $\varphi_t = \mathrm{e}^{-\mathrm{i}tP_\chi}\varphi$ and $\widetilde{\varphi}_t = \vartheta\varphi_t$. Then for every $k,n \in \mathbb{N}$,*

$$\|\mathrm{D}_t^k(\varphi_t - \widetilde{\varphi}_t)\|_{H^n} \leqslant (2+|t|)^{C_{k,n}}\|\varphi\|_{H^{-n}}. \tag{7.8}$$

Proof. It suffices to show that for every $\vartheta_1 \in C_0^\infty(\mathcal{U}_\chi)$ such that $\vartheta = 1$ on $\operatorname{supp}\vartheta_1$ and $\vartheta_1 = 1$ on a neighbourhood of the closure of $\mathcal{U}'_\chi$, we have

$$\|(1-\vartheta_1)\varphi_t\|_{H^{s-n}} \leqslant (2+|t|)^{\bar{C}_{s,n}}\|\varphi\|_{H^{-n}} \tag{7.9}$$

for every $n \in \mathbb{N}$ and $s \geqslant 0$. Indeed, $(1-\vartheta)P_\chi^k\vartheta_1 \in \Psi^{-\infty}$ and $(-\mathrm{D}_t)^k(\varphi_t - \widetilde{\varphi}_t) = (1-\vartheta)P_\chi^k\vartheta_1\varphi_t + (1-\vartheta)P_\chi^k(1-\vartheta_1)\varphi_t$, hence (7.9) implies (7.8). However (7.9) holds for $s=0$, beacause $\mathrm{e}^{-\mathrm{i}tP_\chi}$ is uniformly bounded $H^{-n} \to H^{-n}$ for every $n \in \mathbb{N}$. To prove (7.9) for a given $s>0$ we may assume that (7.9) holds with $s+m'-1$ instead of s for every $\vartheta_2 \in C_0^\infty(\mathcal{U}_\chi)$ such that $\vartheta_2 = 1$ on a neighbourhood of the closure of $\mathcal{U}'_\chi$ and $\vartheta_1 = 1$ on $\operatorname{supp}\vartheta_2$. Now to obtain (7.9) we estimate the H^{s-n}-norm of

$$(1-\vartheta_1)\varphi_t = \mathrm{e}^{-\mathrm{i}tP_\chi}(1-\vartheta_1)\varphi_t + \mathrm{i}t\int_0^1 \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}t\tau P_\chi}[1-\vartheta_1,\ P_\chi]\varphi_{t(1-\tau)},$$

using $(1-\vartheta_1)\varphi = 0$ and $[1-\vartheta_1,\ P_\chi] \in \Psi_{1,\delta}^{m'-1}$ with $[1-\vartheta_1,\ P_\chi]\vartheta_2 \in \Psi^{-\infty}$ which guarantee

$$\begin{aligned}
\|[1-\vartheta_1,\ P_\chi]\varphi_{t(1-\tau)}\|_{H^{s-n}} &\leqslant \|[1-\vartheta_1,\ P_\chi](1-\vartheta_2)\varphi_{t(1-\tau)}\|_{H^{s-n}} \\
&\quad + \|[1-\vartheta_1,\ P_\chi]\vartheta_2\varphi_{t(1-\tau)}\|_{H^{s-n}} \\
&\leqslant C\|(1-\vartheta_2)\varphi_{t(1-\tau)}\|_{H^{s+m'-1-n}} + C\|\varphi\|_{H^{-n}} \\
&\leqslant C(2+|t|)^{\bar{C}_{s+m'-1,n}}\|\varphi\|_{H^{-n}} + C\|\varphi\|_{H^{-n}}. \quad \blacksquare
\end{aligned}$$

Let φ, φ_t, $\widetilde{\varphi}_t$ be as in Lemma 7.5 and denote $\psi = \varphi \circ \chi^{-1}$, $\psi_t = \mathrm{e}^{-\mathrm{i}tP}\psi$, $\widetilde{\psi}_t = \widetilde{\varphi}_t \circ \chi^{-1}$. Then for every $k,n \in \mathbb{N}$,

$$\|\mathrm{D}_t^k(\psi_t - \widetilde{\psi}_t)\|_{H^n(M)} \leqslant (2+|t|)^{\widetilde{C}_{k,n}}\|\varphi\|_{H^{-n}}. \tag{7.10}$$

Now (7.10) follows from (7.8) and $(\mathrm{D}_t + P_\chi)\widetilde{\varphi}_t = (\mathrm{D}_t + P_\chi)(\widetilde{\varphi}_t - \varphi_t)$ in

$$\widetilde{\psi}_t - \psi_t = \mathrm{i}\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}(t-\tau)P}(\mathrm{D}_\tau + P)\widetilde{\psi}_t,$$

because $(\mathrm{D}_\tau + P)\widetilde{\psi}_t = ((\mathrm{D}_\tau + P_\chi)\widetilde{\varphi}_\tau)\circ\chi^{-1}$ holds on $\chi(\mathcal{U}_\chi)$. Finally (7.7) follows from (7.10) and (7.8), because using a sequence of functions $\varphi_{x,\varepsilon} \in C_0^\infty(\mathcal{U}'_\chi)$ converging in $H^{-d/2-1}$ to δ_x (the delta of Dirac in x) uniformly with respect to $x \in \mathcal{C}_\chi$ when $\varepsilon \to 0$, we get the convergence of $\mathrm{e}^{-\mathrm{i}tP_\chi}\varphi_{x,\varepsilon}$ to $u_\chi(t,x)$ and $\mathrm{e}^{-\mathrm{i}tP}(\varphi_{x,\varepsilon}\circ\chi^{-1})$ to $u(t,\chi(x))$ when $\varepsilon \to 0$. ∎

8. COMMENTS

To begin this discussion we would like to cite the review [3] presenting the historical development of the theory (with over 300 references), the books [37], [25], and, concerning the connections with semiclassical approximations, cf. [20], [21], [36].

Concerning the formula (1.5) with the optimal value $\mu = 2/m$, we refer to the developments of Fourier integral operators in the analysis of u(t) [given by (1.10)] for the operators with polynomial coefficients in [11], quasi-homogeneous symbols in [13], [12] and more general situations considered in [18], [8], [38], [39], [30].

We note that the paper of Weyl [43] was based on the Laplace transform and many papers used Tauberian theorems associated with Laplace-Fourier transform (cf. e.g. [4], [5], [6]), but we should mention also the Stieltjes transform (used mainly to get results for boundary value problems cited below), the method of complex powers of Seeley (cf. [35], [37]) and the method of approximate spectral projectors of Tulovski-Shubin (cf. [37], [16], [25]). The estimates for operators in $\mathbb{R}^d$ given in [35], [37] give essentially (1.6) with $\mu < 1/m$ (similarly as our Theorem 1.1 with $r = 1$), while the development of [16] allows to get (1.5) with $\mu < 4/(3m)$.

However, all these results concern only differential operators with smooth coefficients (or pseudodifferential operators with smooth symbols) and except [39], [18], [20], [21], [30], the considered symbols satisfy estimates of the form $|\partial_\alpha p| \leqslant C_\alpha |p|^{1-|\alpha|/m}$, i.e. every derivative in x decreases the order of the symbol in all directions of $\mathbb{R}^d \times \mathbb{R}^d$, which is not the case of the operator considered in Theorem 1.1 even if the coefficients $a_{\alpha,\beta}$ are symbols of degree $m - |\alpha+\beta|$. In a subsequent paper we shall develop our approach to obtain better values of μ under stronger hypotheses on the regularity of coefficients and under the additional "microhyperbolic hypothesis" which may be expressed e.g. by (1.6). A similar "microhyperbolic hypothesis" is considered in [18], [20], [21], [38], [39] and [8], [30], where the considered condition is described in a more general form, being satisfied e.g. by quasi-homogeneous symbols. Concerning the operators with irregular coefficients on $\mathbb{R}^d$ we refer to [4], [5], [6] (and contained references), [9], [10] (where new interesting results have been recently obtained by developing the bracketing idea of [1]) and [7] (where new results are compared with historical ones for Schrödinger operators with irregular potentials).

The remainder estimates in the case of a compact manifold are similar as for boundary value problems, cf. [19], [28], [29] and concerning the historical progress in the theory we refer to [1], [26], [34], [27], [41], [42].

We note that the approximation of $\mathrm{e}^{-\mathrm{i}tP}$ from Section 4 uses a representation with pseudodifferential operators conjugate with respect to the standard one (cf. e.g. [24], Chapter 7, Section 4). The reason for this will appear in [46].

We note that the definition of classes of symbols $S^m_{1,\delta}(r)$ and the proof of Proposition 6.5 follow [22] (cf. also [23], [40], concerning the theory of Fourier integral operators and the propagation of singularities for $r = 2$).

Finally, let us mention that the analysis presented in this paper can be applied also in the case of degenerate elliptic operators and other situations of "hypoelliptic type" (cf. [44], [45]). For instance the proof of the estimate of the spectral function given in Theorem 6.7 holds for any pseudodifferential operator from Hörmander's class of type ρ, δ if its degree is less than $(\rho - \delta)/2$. Since the powers of hypoelliptic operators of type ρ, δ are still of type ρ, δ, we get the remainder estimate with $\mu < (\rho - \delta)/(2m)$ for an operator of degree m, similarly as other methods (cf. [31], [2], [32], [37] etc.) where the "microhyperbolic hypothesis" is not used.

APPENDIX

Proof of Proposition 2.2. Let $P \in \Psi^m_{\mathrm{g}}[r]$ be globally elliptic of degree $m > 0$. ∎

LEMMA A.1. *There exists $P_{-1} \in \Psi^{-m}_{\mathrm{g}}[r]$ such that $PP_{-1} - I \in \Psi^{-\infty}_{\mathrm{gl}}$.*

Proof. Let $p \in S^m_{\mathrm{g}}[r]$ be a symbol of P satisfying $|p| \geqslant 1$. Standard calculations using the ellipticity hypothesis give $1/p \in S^{-m}_{\mathrm{g}}$ and $\partial_x(1/p) = -\partial_x p/p^2 \in S^{-m-r}_{\mathrm{g}} \Rightarrow q_0 = 1/p \in S^{-m}_{\mathrm{g}}[r]$. Hence $R_0 = I - p(x,\mathrm{D})q_0(x,\mathrm{D}) \in \Psi^{-1-r}_{\mathrm{g}}$ and $I - P \sum_{0\leqslant j\leqslant N} q_0(x,\mathrm{D})R_0^j = R_0^{N+1} \in \Psi^{-(N+1)(1+r)}_{\mathrm{g}}$. It remains to take $P_{-1} = p_{-1}(x,\mathrm{D})$ with $p_{-1} \cong \sum_{j\geqslant 0} q_j$, where $q_j \in S^{-m-(1+r)j}_{\mathrm{g}} \in S^{-m}_{\mathrm{g}}[r]$ for $j \geqslant 1$ (cf. (2.9)) are such that $q_0(x,\mathrm{D})R_0^j - q_j(x,\mathrm{D}) \in \Psi^{-\infty}_{\mathrm{gl}}$. ∎

Corollary. *If $P \geqslant I$, then $P^{-1} - P_{-1} \in \Psi_{\mathrm{gl}}^{-\infty}$, hence $P^{-1} \in \Psi_{\mathrm{g}}^{-m}[r]$ and for every $k \in \mathbb{N}$,*

$$C_k^{-1}\|\varphi\|_{H_{km}^{km}} \leqslant \|P^k\varphi\| \leqslant C_k\|\varphi\|_{H_{km}^{km}}. \tag{A.1}$$

Proof. The second inequality (A.1) follows from $P^k \in \Psi_{\mathrm{g}}^{km}[r]$ and the first one from writing $\varphi = P_{-k}^* P^k\varphi + R_k\varphi$ with $P_{-k}^* \in \Psi_{\mathrm{g}}^{-km}[r]$, $R_k \in \Psi_{\mathrm{gl}}^{-\infty}$, which gives

$$\|\varphi\|_{H_{km}^{km}} \leqslant C_k'(\|P_{-k}^* P^k\varphi\|_{H_{km}^{km}} + \|R_k\varphi\|_{H_{km}^{km}}) \leqslant C_k''(\|P^k\varphi\| + \|\varphi\|) \leqslant C_k\|P^k\varphi\|.$$

To complete the proof we note that $P^k(P_{-1} - P^{-1}) = P^{k-1}(PP_{-1} - I) \in \Psi_{\mathrm{gl}}^{-\infty}$ implies that for every $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$\|(P_{-1} - P^{-1})\varphi\|_{H_{km}^{km}} \leqslant C_k\|P^k(P_{-1} - P^{-1})\varphi\| \leqslant C_{k,s}\|\varphi\|_{H_s^s}. \quad \blacksquare$$

Lemma A.2. *Assume moreover that P is symmetric. Then there exists $P_{1/2} \in \Psi_{\mathrm{g}}^{m/2}[r]$ globally elliptic of degree $m/2$, symmetric and satisfying $P - P_{1/2}^2 \in \Psi_{\mathrm{gl}}^{-\infty}$.*

Proof. Since P is symmetric, there is $p_0 \in S_{\mathrm{g}}^{m/2}[r]$ satisfying $P - p_0(x,\mathrm{D}) \in \Psi_{\mathrm{g}}^{m-1-r}[r]$, $p_0 \geqslant 1$ and standard calculations give $p_0^{1/2} \in S_{\mathrm{g}}^{m/2}$. But $\partial_x p_0^{1/2} = \partial_x p/(2p_0^{1/2}) \in S_{\mathrm{g}}^{m/2-r}$ implies $p_0^{1/2} \in S_{\mathrm{g}}^{m/2}[r]$ and $p_0^{1/2\,*} - p_0^{1/2} \in S_{\mathrm{g}}^{m/2-r-1}$.

Setting $q_0 = (p_0^{1/2} + p_0^{1/2\,*})/2$ we have $q_0 \in S_{\mathrm{g}}^{m/2}[r]$, $q_0 - p_0^{1/2} \in S_{\mathrm{g}}^{m/2-1-r}$ and $q_0^2 - p_0 = (q_0 - p_0^{1/2})(q_0 + p_0^{1/2}) \in S_{\mathrm{g}}^{m-1-r}$. Since $q_0(x,\mathrm{D})^2 - (q_0^2)(x,\mathrm{D}) \in \Psi_{\mathrm{g}}^{m-1-r}$, we have $R_0 = P - q_0(x,\mathrm{D})^2 \in \Psi_{\mathrm{g}}^{m-1-r}$.

We shall check that for $j \geqslant 1$ there is $q_j \in S_{\mathrm{g}}^{m/2-j-r} \subset S_{\mathrm{g}}^{m/2}[r]$ such that $q_j^* - q_j \in \mathcal{S}(\mathbb{R}^{2d})$ and

$$R_n = P - (q_0 + \cdots + q_n)(x,\mathrm{D})^2 \in \Psi_{\mathrm{g}}^{m-n-1-r}. \tag{A.2.n}$$

Let us assume that (A.2.n) holds and $r_n \in S_{\mathrm{g}}^{m-n-1-r}$ is such that $R_n - r_n(x,\mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$. But $q_0 + \cdots + q_n$ is globally elliptic and modifying q_n with a function $\mathcal{S}(\mathbb{R}^{2d})$ we may assume $q_0 + \cdots + q_n \geqslant 1$, hence $\widetilde{q}_{n+1} = 2r_n/(q_0 + \cdots + q_n) \in S_{\mathrm{g}}^{m/2-n-1-r}$. But

$$q_j^* - q_j \in \mathcal{S}(\mathbb{R}^{2d}) \Rightarrow \operatorname{Im} q_j \in S_{\mathrm{g}}^{m/2-j-1-r} \Rightarrow \operatorname{Im}(q_0 + \cdots + q_n)^{-1} \in S_{\mathrm{g}}^{-m/2-1-r}$$

and

$$\begin{aligned} r_n^* - r_n \in \mathcal{S}(\mathbb{R}^{2d}) &\Rightarrow \operatorname{Im} r_n \in S_{\mathrm{g}}^{m-n-2-r} \\ &\Rightarrow \operatorname{Im} \widetilde{q}_{n+1} \in S_{\mathrm{g}}^{m/2-n-2-r} \Rightarrow \widetilde{q}_{n+1}^* - \widetilde{q}_{n+1} \in S_{\mathrm{g}}^{m/2-n-2-r}. \end{aligned}$$

Hence setting $q_{n+1} = (\widetilde{q}^*_{n+1} + \widetilde{q}_{n+1})/2$, we have

$$r_n - 2q_{n+1}(q_0 + \cdots + q_n) \in S_{\mathrm{g}}^{m-n-2-r}$$

and

$$\begin{aligned} P &- (q_0 + \cdots + q_{n+1})(x, \mathrm{D})^2 \\ &= R_n + (q_0 + \cdots + q_n)(x, \mathrm{D})^2 - (q_0 + \cdots + q_{n+1})(x, \mathrm{D})^2 \\ &= R_n - 2q_{n+1}(q_0 + \cdots + q_n)(x, \mathrm{D}) + \widetilde{R}_{n+1} \in \Psi_{\mathrm{g}}^{m-n-2-r} \end{aligned}$$

with $\widetilde{R}_{n+1} \in \Psi_{\mathrm{g}}^{m-n-2-r}$, i.e. (A.2.$n+1$) holds. Taking $p_{1/2} \cong \sum_{j \geqslant 0} q_j$ we have $p_{1/2} \in S_{\mathrm{g}}^{m/2}[r]$ (cf. (2.9)), $p^*_{1/2} - p_{1/2} \in \mathcal{S}(\mathbb{R}^{2d})$ and there exists $P_{1/2}$ symmetric, satisfying $P_{1/2} - p_{1/2}(x, \mathrm{D}) \in \Psi_{\mathrm{gl}}^{-\infty}$. ∎

COROLLARY. (i) *Let P be as in Lemma* A.2. *Then there exists $R \in \Psi_{\mathrm{gl}}^{-\infty}$ such that $P + R \geqslant 0$.*

(ii) *If $P \geqslant I$, then $P^{1/2} - P_{1/2} \in \Psi_{\mathrm{gl}}^{-\infty}$, hence $P^{1/2} \in \Psi_{\mathrm{g}}^{m/2}[r]$.*

Proof. (i) It suffices to use Lemma A.2 noting that $P_{1/2}^2 \geqslant 0$.

(b) Since $P_{1/2} - I$ is globally elliptic of degree $m/2 > 0$, the assertion (a) holds also for $P_{1/2} - I$ instead of P, i.e. there is a self-adjoint $R \in \Psi_{\mathrm{gl}}^{-\infty}$ such that $P_{1/2} + R \geqslant I$. Setting $P' = (P_{1/2} + R)^2$ we have $P' \geqslant I$ and $P' - P_{1/2}^2 \in \Psi_{\mathrm{gl}}^{-\infty}$, hence $P' - P \in \Psi_{\mathrm{gl}}^{-\infty}$ and ${P'}^N - P^N \in \Psi_{\mathrm{gl}}^{-\infty}$ for every $N \in \mathbb{N}$. Consider now $f \in C(\mathbb{R})$ such that the Fourier transform $\widehat{f}$ and $\lambda \widehat{f}(\lambda)$ belong to $L^1(\mathbb{R})$, and let us check that $f(P') - f(P) \in \Psi_{\mathrm{gl}}^{-\infty}$. Indeed,

$$f(P')\varphi - f(P)\varphi = \int \frac{\mathrm{d}\lambda}{2\pi} \widehat{f}(\lambda)(\mathrm{e}^{\mathrm{i}\lambda P'} - \mathrm{e}^{\mathrm{i}\lambda P})\varphi = \int \frac{\mathrm{d}\lambda}{2\pi} \widehat{f}(\lambda)\mathrm{i}\lambda \int_0^1 \mathrm{d}\tau\, Z(\tau, \lambda)\varphi$$

where $Z(\tau, \lambda) = \mathrm{e}^{\mathrm{i}\lambda\tau P'}(P' - P)\mathrm{e}^{\mathrm{i}\lambda(1-\tau)P}$ and using (A.1) with P replaced by P' and $P + \widetilde{C}I$ with $\widetilde{C}$ large enough, we have the estimate

$$\begin{aligned} \|Z(\tau, \lambda)\|_{B(H^{-km}_{-km}, H^{km}_{km})} &\leqslant C_k \|{P'}^k Z(\tau, \lambda)(P + \widetilde{C})^k\| \\ &= C_k \|{P'}^k (P' - P)(P + \widetilde{C})^k\| \leqslant C'_k. \end{aligned}$$

To complete the proof we take $f \in C^\infty(\mathbb{R})$ such that $f(\lambda) = 0$ for $\lambda \leqslant 1/2$ and $f(\lambda) = \lambda^{1/2-N}$ for $\lambda \geqslant 1$, where $N \in \mathbb{N}$ is chosen large enough. Then $P^{1/2} + R - P^{1/2} = {P'}^N f(P') - P^N f(P) = ({P'}^N - P^N) f(P') - P^N(f(P') - f(P)) \in \Psi_{\mathrm{gl}}^{-\infty}$. ∎

Proof of Proposition 6.2. Recall that

$$p(x,\mathrm{D}) \in \Psi^{-\infty} \Leftrightarrow p \in S^{-\infty}_{1,\delta} = \bigcap_{m\in\mathbb{R}} S^m_{1,\delta}$$

and

$$p \in S^m_{1,\delta}(r),\ q \in S^{m'}_{1,\delta}(r) \Rightarrow \begin{cases} pq \in S^{m+m'}_{1,\delta}(r), \\ p \diamond q - pq \in S^{m+m'-1+\delta(1-r)}_{1,\delta}, \end{cases} \tag{A.3}$$

i.e.

$$p(x,\mathrm{D})q(x,\mathrm{D}) \in \Psi^{m+m'}_{1,\delta}(r)$$

and

$$p(x,\mathrm{D})q(x,\mathrm{D}) - (pq)(x,\mathrm{D}) \in \Psi^{m+m'-1+\delta(1-r)}_{1,\delta}$$

holds for $r = 0$ and using $m' \leqslant m - r\delta \Rightarrow S^{m'}_{1,\delta} \subset S^m_{1,\delta}(r)$ we check easily that all the properties (A.3) hold also if $0 \leqslant r \leqslant 1$. If $p \in S^m_{1,\delta}(r)$ is real, then $p^* - p \in S^{m-1+\delta(1-r)}_{1,\delta}$. If $P - p(x,\mathrm{D}) \in \Psi^{-\infty}$ with $P \in \Psi^m_{1,\delta}(r)$ being symmetric, then $\operatorname{Im} p \in S^{m-1+\delta(1-r)}_{1,\delta}$ and when P is in addition elliptic of degree $m > 0$, then $\operatorname{Re} p$ is elliptic and we can always choose p such that $\operatorname{Re} p \geq 1$. Following the proof of Lemma A.1 with $S^m_{\mathrm{g}}[r]$, $\Psi^m_{\mathrm{g}}[r]$, $\Psi^{-\infty}_{\mathrm{gl}}$, $\mathcal{S}(\mathbb{R}^{2d})$ replaced by $S^m_{1,\delta}(r)$, $\Psi^m_{1,\delta}(r)$, $\Psi^{-\infty}$, $S^{-\infty}_{1,\delta}$ we have $q_0 = 1/p \in S^{-m}_{1,\delta}(r)$, $R_0 \in \Psi^{-m-1+\delta(1-r)}_{1,\delta}$ and $q_j \in S^{-m-j(1-\delta(1-r))}_{1,\delta} \subset S^{-m}_{1,\delta}(r)$. Changing similarly the proof of Lemma A.2 we have $q_j \in S^{m/2-j(1-\delta)-\delta r}_{1,\delta} \subset S^{m/2}_{1,\delta}(r)$ and $R_n \in \Psi^{m-(n+1)(1-\delta)-\delta r}_{1,\delta}$. ∎

Proof of Proposition 7.2. It suffices to adapt the proof of Proposition 6.2 in the standard way, i.e. instead of p, q_j defined on $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$ one uses $\sigma(P)$ and q_j defined on T^*M, replacing $q_j(x,\mathrm{D})$ by a pseudodifferential operator Q_j on M such that $q_j = \sigma(Q_j)$. ∎

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